Technical Report
RAL-TR-2007-004

Proceedings of the School for Experimental High Energy Physics Students held 3 to 15 September 2006

T J Greenshaw (Editor)
M E Evans (Compiler)

June 2007

Council for the Central Laboratory of the Research Councils

Library and Information Services
CCLRC Rutherford Appleton Laboratory
Chilton Didcot
Oxfordshire OX11 0QX
UK
Tel: +44 (0)1235 445384
Fax: +44 (0)1235 446403
Email: library@rl.ac.uk

CCLRC reports are available online at:
http://www.clrc.ac.uk/Activity/ACTIVITY=Publications;SECTION=225;

ISSN 1358-6254

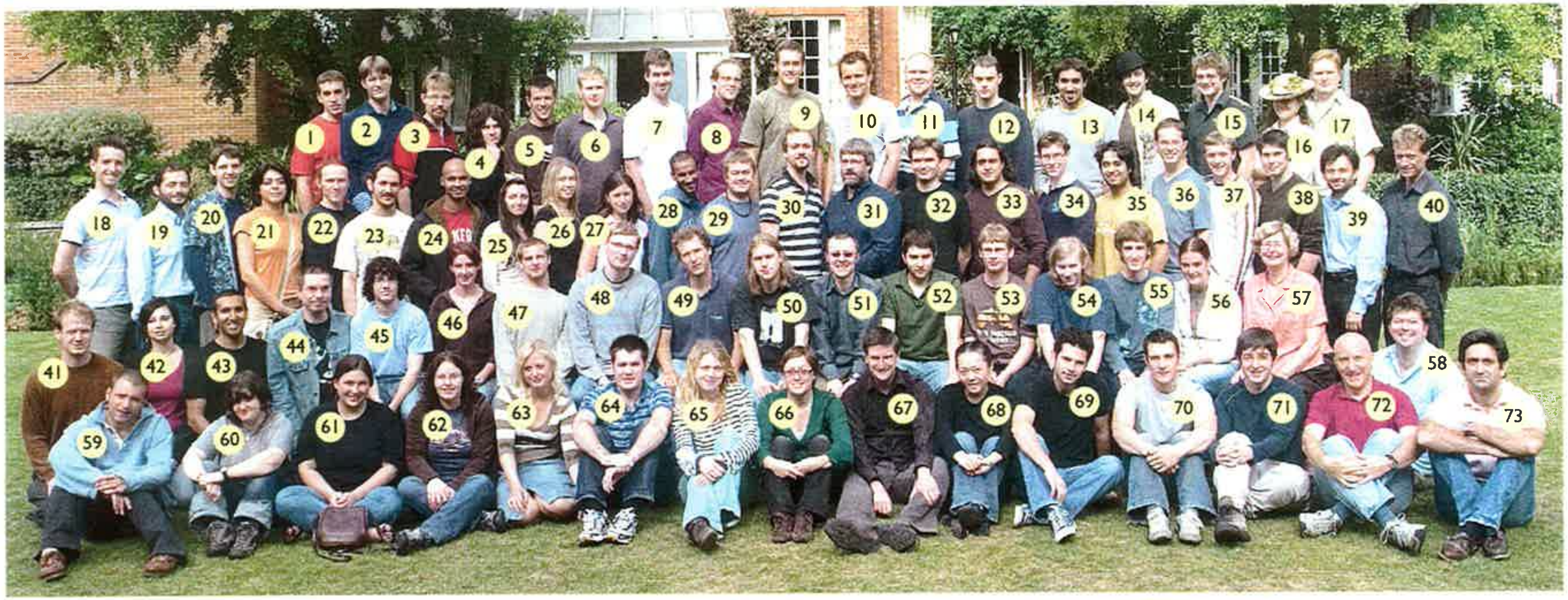
RAL-TR-2007-004
Central Laboratory of the Research Councils

Rutherford Appleton Laboratory
Chilton DIDCOT Oxon OX11 0QX

Proceedings of the School for Experimental High Energy Physics Students

Editor: T Greenshaw
Compiler: M E Evans

Rutherford Appleton Laboratory
3 - 15 September 2006



Back row: 1. Mark Grimes; 2. Christopher Lester; 3. Kostyantyn Holubyev; 4. Kenneth Lessnoff; 5. John Dale; 6. Dave Evans; 7. Ian Bingham; 8. David Auty; 9. David Wardrope; 10. David Lee; 11. Robert Pittam; 12. Alastair Dewhurst; 13. Filimon Gournaris; 14. Simon Bevan; 15. Ben Still; 16. Sacha Davidson; 17. David Miller.

Third row: 18. Ian Preston; 19. Michele Faucci Giannelli; 20. Matthew Forrest; 21. Jelena Ilic; 22. Mark Ward; 23. Franciole Marinho; 24. Michael Sigamani; 25. Catherine Wright; 26. Jennifer Watson; 27. Ellie Dobson; 28. Sudan Paramesvaran; 29. Nick Austin; 30. Joseph Walding; 31. Tim Greenshaw; 32.Hugh Skottowe; 33. Mark Pesaresi; 34. Colin McLean; 35. Konstantinos A Petridis; 36. Mike Flowerdew; 37. James Frost; 38. Andy Buckley; 39. Paolo Adragna; 40. Owe Philipsen.

Second row: 41. Lawrence Deacon; 42. Gemma Tinti; 43. Eram Rizvi; 44. Tim Adye; 45. Daniel Beecher; 46. Kim Alwyn; 47. Mark Tibbetts; 48. Laurence Carson; 49. Erik Devetak; 50. Nick Styles; 51. Rob Lambert; 52. Hakan Yilmaz; 53. Simon Head; 54. Chris Eames; 55. Jacob Linacre; 56. Christina Swinson; 57. Margaret Evans; 58. David Bailey.

Front row: 59. Paul Prichard; 60. Aileen Robertson; 61. Anna Holin; 62. Lisa Dwyer; 63. Tina Potter; 64. Sean Brisbane; 65. Lily Asquith; 66. Sarah Boutle; 67. Philip Rich; 68. Ying Ying Li; 69. Theo Christoudias; 70. Liam Daniel; 71. Chris Curtis; 72. Joel Goldstein; 73. Stefano Moretti.

SUMMER SCHOOL FOR EXPERIMENTAL HIGH ENERGY PHYSICS STUDENTS
RUTHERFORD APPLETON LABORATORY/THE COSENER'S HOUSE, ABINGDON

3 – 15 SEPTEMBER 2006

Director:	Professor T J Greenshaw	University of Liverpool
Lecturers:	Dr S Davidson	University of Lyon
	Dr D Miller	University of Glasgow
	Dr S Moretti	University of Southampton
	Dr O Philipsen	Uniersity of Münster
Tutors:	Dr T Adye	Rutherford Appleton Laboratory
	Dr D Bailey	University of Manchester
	Dr A Buckley	University of Durham
	Dr J Goldstein	University of Bristol
	Dr C Lester	University of Cambridge
	Dr E Rizvi	Queen Mary University of London
	Dr D R Tovey	University of Sheffield
School Secretary:	Mrs M E Evans	Rutherford Appleton Laboratory
Students:	Mr P Adragna	Queen Mary University of London
	Miss K E Alwyn	University of Manchester
	Ms L Asquith	University College London
	Mr N Austin	University of Liverpool
	Mr D Auty	University of Sussex
	Mr D Beecher	University College London
	Mr S Bevan	University College London
	Mr I Bingham	University of Liverpool
	Ms S Boutle	University College London
	Mr S Brisbane	University of Oxford
	Mr L Carson	University of Glasgow
	Mr T Christoudias	Imperial College London
	Mr C Curtis	University of Birmingham
	Mr J Dale	University of Oxford
	Mr L Daniel	University of Birmingham
	Mr L C Deacon	Royal Holloway University of London
	Mr E Devetak	University of Oxford
	Mr A Dewhurst	University of Lancaster
	Miss E Dobson	University of Oxford
	Miss L B Dwyer	University of Liverpool
	Mr C Eames	Imperial College London
	Mr D Evans	University of Bristol
	Mr M Faucci Giannelli	Royal Holloway University of London
	Mr M Flowerdew	University of Liverpool
	Mr M Forrest	University of Glasgow
	Mr J Frost	University of Cambridge
	Mr F Gournaris	University College London
	Mr M Grimes	University of Bristol
	Mr S Head	University of Manchester
	Ms A Holin	University College London

Students (cont'd)	Mr K Holubyev	University of Lancaster
	Ms J Ilic	University of Warwick
	Mr R Lambert	University of Edinburgh
	Mr D Lee	Imperial College London
	Mr K Lessnoff	University of Bristol
	Miss Y Y Li	University of Cambridge
	Mr J Linacre	University of Oxford
	Mr F Marinho	University of Glasgow
	Mr C McLean	University of Edinburgh
	Mr S Paramesvaran	Royal Holloway University of London
	Mr M Pesaresi	Imperial College London
	Mr K A Petridis	Imperial College London
	Mr R Pittam	University of Oxford
	Miss C Potter	Royal Holloway University of London
	Mr I Preston	University of Oxford
	Mr P Prichard	University of Liverpool
	Mr P A Rich	University of Manchester
	Miss A Robertson	University of Oxford
	Mr M Sigamani	Queen Mary University of London
	Mr H Skottowe	University of Cambridge
	Mr B Still	University of Sheffield
	Mr N Styles	University of Edinburgh
	Miss C Swinson	University of Oxford
	Mr M J Tibbetts	Imperial College London
	Miss G Tinti	University of Oxford
	Mr J J Walding	Imperial College London
	Mr M Ward	University of Sheffield
	Mr D R Wardrope	Imperial College London
	Ms J Watson	University of Edinburgh
	Ms C Wright	University of Glasgow
	Mr H Yilmaz	Imperial College London
	Mr H Zhu	University of Sheffield

RAL Summer School for Experimental High Energy Physics Students

The Cosener's House, 3rd – 15th September 2006

Preface

The 2006 RAL Summer School was attended by sixty-two experimental particle physics students, filling The Cosener's House in Abingdon to bursting point. The staff at Cosener's were welcoming and helpful, the food as always excellent and, as and when the weather allowed it, the grounds provided opportunity for football, cricket and Frisbee games; a welcome break after the lectures and tutorials.

The lectures reproduced here were given by Owe Philipsen (Quantum Field Theory), David Miller (Introduction to QED), Sacha Davidson (The Standard Model) and Stefano Moretti (Phenomenology). All the lecturers were tremendously enthusiastic and hard working, both before and during the school, and indeed afterwards in preparing the notes for these Proceedings. In addition, there were two interesting guest seminars: Andy Wolski (Cockcroft Institute) talked about the International Linear Collider (ILC) and some of the machine physics issues involved in its design and Chris Damerell (RAL) about vertex detectors and the physics they make possible, with particular reference to the ILC. After the School dinner, Ken Peach (Adams Institute) entertained us with his recollections of some of the experiments on which he has worked and some thoughts on the directions High Energy Physics is now taking.

A further highlight of the 2006 School was the visit to the Central Laser Facility (CLF) at RAL. This took place on the Saturday afternoon half-way through the School. Peter Norreys gave an introduction to the work of the CLF, with particular emphasis on their recent successes in the acceleration of protons and electrons using high intensity laser beams, and then tours of the Vulcan and Astra Gemini lasers and the Petawatt Target Area were organised. Many thanks to Peter and his team for a most entertaining and educational afternoon!

On four evenings during the school, there were poster sessions at which, over a beer or wine, the students presented their work to their peers and to the lecturers and tutors.

The tutors, Dave Bailey (Manchester), Andy Buckley (Durham), Joel Goldstein (Bristol), Chris Lester (Cambridge), Eram Rizvi (QMUL) and Dan Tovey (Sheffield) all worked tremendously hard and their efforts were much appreciated by the students, the lecturers and myself. (Dan was replaced by Tim Adye of RAL for the last four days of the School, as Dan had to attend an ATLAS meeting.)

The School Secretary, Margaret Evans, organised and administered the school superbly: thank you! The financial support of PPARC/RAL is gratefully acknowledged and I would also like to thank RAL Particle Physics Department for providing the computing facilities at the School and the RAL personnel who helped with transporting easels, text-books, stationery and the other paraphernalia needed for the School.

Last but not least, thanks to all the students who contributed so much to the 2006 School. I look forward to meeting you again at other schools, conferences and workshops!

Tim Greenshaw (Director)
Department of Physics
Liverpool University

Poster Schedule for the Rutherford HEP Summer School 2006

Tuesday 5th September

Surname	Forename	University	Experiment	Poster Title
Austin	Nick	Liverpool	ATLAS	Finding the Higgs at ATLAS in the t tbar H channel
Curtis	Chris	Birmingham	ATLAS	Detector control system for the ATLAS level-1 calorimeter trigger
Flowerdew	Mike	Liverpool	ATLAS	Forward electron identification in ATLAS
Frost	James	Cambridge	ATLAS	Searching for physics beyond the Standard Model in ATLAS
Preston	Ian	Oxford	ATLAS	JPC - PC emulation
Evans	Dave	Bristol	CMS	The Compact Muon Solenoid experiment
Pesaresi	Mark	Imperial	CMS	The APV emulator for the prevention of buffer overflows in CMS2. CMS3.
Petridis	Konstantinos	Imperial	CMS	Reconstruction of $Z \rightarrow tt \rightarrow$ t jet e and $Z \rightarrow ee$ with the single electron trigger at CMS
Wardrope	David	Imperial	CMS	Measurement of Single Electron Trigger efficiencies in CMS
Christoudias	Theo	Imperial	D0	The D0 experiment and the level-3 trigger
Daniel	Liam	Birmingham	ALICE	Strangeness at ALICE
Bevan	Simon	UCL	ACoRNE	Ultra-high-energy neutrino astronomy
Lessnoff	Kenneth	Bristol	LHCb	Excited B meson production in Pythia for the LHC
Li	Ying Ying	Cambridge	LHCb	Windows implementation of the LHCb experiment production system

Thursday 7th September

Surname	Forename	University	Experiment	Poster Title
Asquith	Lily	UCL	ATLAS	Optimization of the ATLAS high level trigger
Adragna	Paolo	QMUL	ATLAS	OHP an online histogram presenter for the ATLAS experiment 2
Head	Simon	Manchester	ATLAS	WW Scattering at ATLAS
Potter	Christina	RHUL	ATLAS	An exclusive channel – direct chargino-neutralino production at the focus point
Prichard	Paul	Liverpool	ATLAS	Discovering supersymmetry with early data at ATLAS
Wright	Catherine	Glasgow	ATLAS	Improving the fast simulation for ATLAS
Zhu	Hongbo	Sheffield	ATLAS	Analysis of the reaction Higgs $\rightarrow$ two photons
Beecher	Daniel	UCL	CDF	Direct measurement of the W width
Linacre	Jacob	Oxford	CDF	Measurement of the top quark mass and cross section in the lepton + jets channel
Rich	Philip	Manchester	D0	Standard Model Higgs searches in the hadronic tau decay channel
Alwyn	Kim	Manchester	BaBar	Detection efficiency for π^0s at BaBar
Bingham	Ian	Liverpool	BaBar	Investigating the calorimeter efficiency at BaBar
Ilic	Jelena	Warwick	BaBar	Analysis of the $B^0 \rightarrow K_s \pi^+ \pi^-$ decay
Carson	Laurence	Glasgow	LHCb	Hybrid photon detectors for the LHCb experiment
Lambert	Rob	Edinburgh	LHCb	Performance of hybrid photon detectors for the LHCb RICH

Monday 11th September

Surname	Forename	University	Experiment	Poster Title
Brisbane	Sean	Oxford	LHCb	Testing of the Level 0 LHCb RICH electronics
Dwyer	Lisa	Liverpool	LHCb	$B_s \rightarrow J/\psi\ \phi$ at LHCb
Eames	Chris	Imperial	LHCb	Rare decays at LHCb – $\Lambda_b \rightarrow \Lambda\ \mu\ \mu$
Dewhurst	Alastair	Lancaster	ATLAS	Simulation of $B_d \rightarrow J/\psi\ K^{0*}$ decays at ATLAS
Holubyev	Kostyantyn	Lancaster	D0	Measurement of CPV asymmetries in B hadron decays
Marinho	Franciole	Glasgow	LHCb	Long term testing of LHCb vertex locator sensors
McLean	Colin	Edinburgh	LHCb	Review of the $B_s \rightarrow J/\ \psi$ decay chain at the LHCb
Skottowe	Hugh	Cambridge	LHCb	Electronics and testing for the LHCb RICH detector
Styles	Nick	Edinburgh	LHCb	Test beam studies for the LHCb RICH detectors
Paramesvaran	Sudarshan	RHUL	BaBar	Measuring the energy resolution of the BaBar EMC for photons using a $\mu\ \mu\ \gamma$ sample
Sigamani	Michael	QMUL	BaBar	Determination of CKM Matrix element $\lvert V_{ub} \rvert$ using an endpoint analysis
Tibbetts	Mark	Imperial	BaBar	Investigation of simulated events used in the search for the rare decay $B^+ \rightarrow a^0 + \pi^0$
Watson	Jennifer	Edinburgh	BaBar	A measure of CP violation: $B_s \rightarrow l\ l$ physics analysis using the BaBar detector
Boutle	Sarah	UCL	ZEUS	Beauty Photoproduction at HERA II with the ZEUS experiment
Robertson	Aileen	Oxford	ZEUS	Correlations in neutral strange particles at ZEUS
Forrest	Matthew	Glasgow	ZEUS	Tuning the Barrel Electromagnetic Calorimeter response of the ZEUS Monte Carlo

Wednesday 13th September

Surname	Forename	University	Experiment	Poster Title
Dobson	Ellie	Oxford	ATLAS	Frequency Scanning Interferometry in the ATLAS detector
Dale	John	Oxford	LiCAS	Phase extraction techniques for the LiCAS reference interferometer
Giannelli	Michele	RHUL	CALICE	CALICE: full simulation study and test beam for Linear Collider detector
Yilmaz	Hakan	Imperial	CALICE	Position and angular resolution of the CALICE Ecal
Swinson	Christina	Oxford	FONT	Flux measurements with the ILC test beam at end station A, SLAC
Deacon	Lawrence	RHUL	ILC	R&D for the International Linear Collider
Gournaris	Filimon	UCL	ILC	Top quark physics and the ILC energy spectrometer
Devetak	Erik	Oxford	LCFI	Flavour tagging and vertexing at the ILC
Grimes	Mark	Bristol	LCFI	Vertexing at the International Linear Collider
Auty	David	Sussex	MINOS	Neutrino oscillations at MINOS
Hollin	Anna	UCL	MINOS	ν_e appearance measurement in the MINOS detectors
Pittam	Robert	Oxford	MINOS	The MINOS experiment and neutral current event migration
Tinti	Gemma	Oxford	MINOS	Reconstruction of neutrino events in the MINOS Near Detector
Lee	David	Imperial	Neutrino Factory	Beam density distribution measurements at the RAL front end test stand
Walding	Joseph	Imperial	SciBooNE	SciBooNE and its implications for T2K
Still	Ben	Sheffield	T2K	The changing face of the neutrino
Ward	Mark	Sheffield	T2K	T2K: photosensors for the 280m Near Detector

CONTENTS **Pages**

LECTURE COURSES

AN INTRODUCTION TO QUANTUM FIELD THEORY

By Dr O Philipsen
University of Münster

Lectures delivered at the School for Experimental High Energy Physics Students
Rutherford Appleton Laboratory, September 2006

An Introduction to Quantum Field Theory

Owe Philipsen

Institut für Theoretische Physik
Universität Münster
Wilhelm-Klemm-Str.9, 48149 Münster, Germany

Lectures presented at the School for Young High Energy Physicists,
Rutherford Appleton Laboratory, September 2006

Contents

When I became a student of Pomeranchuk in 1950 I heard from him a kind of joke that the Book of Physics had two volumes: vol.1 is "Pumps and Manometers", vol.2 is "Quantum Field Theory"

Lev Okun

0 Prologue

The development of Quantum Field Theory is surely one of the most important achievements in modern physics. Presently, all observational evidence points to the fact that Quantum Field Theory (QFT) provides a good description of all known elementary particles, as well as for particle physics beyond the Standard Model for energies ranging up to the Planck scale $\sim 10^{19}$ GeV, where quantum gravity is expected to set in and presumably requires a new and different description. Historically, Quantum Electrodynamics (QED) emerged as the prototype of modern QFT's. It was developed in the late 1940s and early 1950s chiefly by Feynman, Schwinger and Tomonaga, and is perhaps the most successful theory in physics: the anomalous magnetic dipole moment of the electron predicted by QED agrees with experiment with a stunning accuracy of one part in 10^{10}!

The scope of these lectures is to provide an introduction to the *formalism* of Quantum Field Theory, and as such is somewhat complementary to the other lectures of this school. It is natural to wonder why QFT is necessary, compelling us to go through a number of formal rather than physical considerations, accompanied by the inevitable algebra. However, thinking for a moment about the high precision experiments, with which we hope to detect physics beyond the Standard Model, it is clear that comparison between theory and experiment is only conclusive if the numbers produced by either side are "water-tight". On the theory side this requires a formalism for calculations, in which every step is justified and reproducible, irrespective of subjective intuition about the physics involved. In other words, QFT aims to provide the bridge from the building blocks of a theory to the evaluation of its predictions for experiments.

This program is best explained by restricting the discussion to the quantum theory of scalar fields. Furthermore, I shall use the Lagrangian formalism and canonical quantisation, thus leaving aside the quantisation approach via path integrals. Since the main motivation for these lectures is the discussion of the underlying formalism leading to the derivation of *Feynman rules*, the canonical approach is totally adequate. The physically relevant theories of QED, QCD and the electroweak model are covered in the lectures by Nick Evans, Sacha Davidson and Stefano Moretti.

The outline of these lecture notes is as follows: to put things into perspective, we shall review the Lagrangian formalism in classical mechanics, followed by a brief reminder of the basic principles of quantum mechanics in Section 1. Section 2 discusses the step from classical mechanics of non-relativistic point particle to a classical, relativistic theory for non-interacting scalar fields. There we will also derive the wave equation for free scalar fields, i.e. the Klein-Gordon equation. The quantisation of this field theory is done is Section 3, where also the relation of particles to the quantised fields will be elucidated. The more interesting case of interacting scalar fields is presented in Section 4: we shall introduce the S-matrix and examine its relation with the Green's functions of the theory. Finally, in Section 5 the general method of perturbation theory is presented, which serves to compute the Green functions in terms of a power series in the coupling constant. Here, Wick's Theorem is of central importance in order to understand the derivation of Feynman rules.

1 Introduction

Let us begin this little review by considering the simplest possible system in classical mechanics, a single point particle of mass m in one dimension, whose coordinate and velocity are functions of time, $x(t)$ and $\dot{x}(t) = dx(t)/dt$, respectively. Let the particle be exposed to a time-independent potential $V(x)$. It's motion is then governed by Newton's law

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = F(x), \tag{1.1}$$

where $F(x)$ is the force exerted on the particle. Solving this equation of motion involves two integrations, and hence two arbitrary integration constants to be fixed by initial conditions. Specifying, e.g., the position $x(t_0)$ and velocity $\dot{x}(t_0)$ of the particle at some initial time t_0 completely determines its motion: knowing the initial conditions and the equations of motion, we also know the evolution of the particle at all times (provided we can solve the equations of motion).

1.1 Lagrangian formalism in classical mechanics

The equation of motion in the form of Newton's law was originally formulated as an equality of two forces, based on the physical principle *actio* = *reactio*, i.e. the external force is balanced by the particle's inertia. The Lagrangian formalism allows to derive the same physics through a formal algorithm. It is formal, rather than physical, but as will become apparent throughout the lectures, it is an immensely useful tool allowing to treat all kinds of physical systems by the same methods.

To this end, we introduce the Lagrange function

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x), \tag{1.2}$$

which is a function of coordinates and velocities, and given by the difference between the kinetic and potential energies of the particle. Next, the action functional is defined as

$$S = \int_{t_0}^{t_1} dt\, L(x, \dot{x}). \tag{1.3}$$

From these expressions the equations of motion can be derived by the *Principle of least Action*: consider small variations of the particle's trajectory, cf. Fig. 1,

$$x'(t) = x(t) + \delta x(t), \qquad \delta x/x \ll 1, \tag{1.4}$$

with its initial and end points fixed,

$$\left.\begin{array}{l} x'(t_1) = x(t_1) \\ x'(t_2) = x(t_2) \end{array}\right\} \Rightarrow \delta x(t_1) = \delta x(t_2) = 0. \tag{1.5}$$

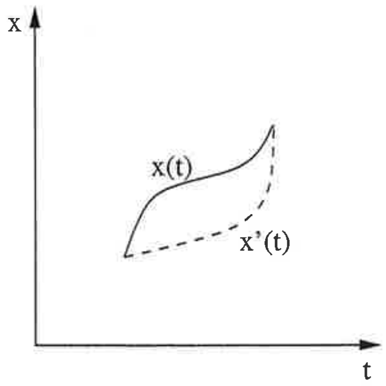


Figure 1: *Variation of particle trajectory with identified initial and end points.*

The true trajectory the particle will take is the one for which

$$\delta S = 0, \tag{1.6}$$

i.e. the action along $x(t)$ is stationary. In most systems of interest to us the stationary point is a minimum, hence the name of the principle, but there are exceptions as well (e.g. a pencil balanced on its tip). We can now work out the variation of the action by doing a Taylor expansion to leading order in the variation δx,

$$\begin{aligned} S + \delta S &= \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta\dot{x})\, dt, \qquad \delta\dot{x} = \frac{d}{dt}\delta x \\ &= \int_{t_1}^{t_2} \left\{ L(x, \dot{x}) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \dots \right\} dt \\ &= S + \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right\} \delta x\, dt, \end{aligned} \tag{1.7}$$

where we performed an integration by parts on the last term in the second line. The second and third term in the last line are the variation of the action, δS, under variations of the trajectory, δx. The second term vanishes because of the boundary conditions for the variation, and we are left with the third. Now the Principal of least Action demands $\delta S = 0$. For the remaining integral to vanish for arbitrary δx is only possible if the integrand vanishes, leaving us with the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1.8}$$

If we insert the Lagrangian of our point particle, Eq. (1.2), into the Euler-Lagrange equation we obtain

$$\begin{aligned} \frac{\partial L}{\partial x} &= -\frac{\partial V(x)}{\partial x} = F \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{d}{dt} m\dot{x} = m\ddot{x} \\ &\Rightarrow\; m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton's law)}. \end{aligned} \tag{1.9}$$

Hence, we have derived the equation of motion by the Principal of least Action and found it to be equivalent to the Euler-Lagrange equation. The benefit is that the latter can be easily generalised to other systems in any number of dimensions, multi-particle systems, or systems with an infinite number of degrees of freedom, such as needed for field theory. For example, if we now consider our particle in the full three-dimensional Euclidean space, the Lagrangian depends on all coordinate components, $L(\mathbf{x}, \dot{\mathbf{x}})$, and all of them get varied independently in implementing Hamilton's principle. As a result one obtains Euler-Lagrange equations for the components,

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0. \tag{1.10}$$

In particular, the Lagrangian formalism makes symmetries and their physical consequences explicit and thus is a convenient tool when constructing different kinds of theories based on symmetries observed (or speculated to exist) in nature.

For later purposes in field theory we need yet another, equivalent, formal treatment, the Hamiltonian formalism. In our 1-d system, we define the 'conjugate momentum' p by

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{1.11}$$

and the Hamiltonian H via

$$\begin{aligned} H(x,p) &\equiv p\dot{x} - L(x,\dot{x}) \\ &= m\dot{x}^2 - \tfrac{1}{2}m\dot{x}^2 + V(x) \\ &= \tfrac{1}{2}m\dot{x}^2 + V(x) = T + V. \end{aligned} \tag{1.12}$$

The Hamiltonian $H(x,p)$ corresponds to the total energy of the system; it is a function of the position variable x and the conjugate momentum[1] p. It is now easy to derive Hamilton's equations

$$\frac{\partial H}{\partial x} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{x}. \tag{1.13}$$

These are two equations of first order, while the Euler-Lagrange equation was a single equation of second order. Taking another derivative in Hamilton's equations and substituting one into the other, it is easy to convince oneself that the Euler-Lagrange equations and Hamilton's equations provide an entirely equivalent description of the system. Again, this generalises obviously to three-dimensional space yielding equations for the components,

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{x}_i. \tag{1.14}$$

1.2 Quantum mechanics

Having set up some basic formalism for classical mechanics, let us now move on to quantum mechanics. In doing so we shall use 'canonical quantisation', which is historically what was used first and what we shall later use to quantise fields as well. We remark, however, that one can also quantise a theory using path integrals.

[1]It should be noted that the conjugate momentum is in general not equal to $m\dot{x}$.

Canonical quantisation consists of two steps. Firstly, the dynamical variables of a system are replaced by operators, which we denote by a hat. For example, in our simplest one particle system,

$$\begin{aligned}
\text{position: } x_i &\longrightarrow \hat{x}_i \\
\text{momentum: } p_i &\longrightarrow \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \\
\text{Hamiltonian: } H &\longrightarrow \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}) = -\frac{\hbar^2\nabla^2}{2m} + V(\hat{\mathbf{x}}).
\end{aligned} \tag{1.15}$$

Secondly, one imposes commutation relations on these operators,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij} \tag{1.16}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{1.17}$$

The physical state of a quantum mechanical system is encoded in state vectors $|\psi\rangle$, which are elements of a Hilbert space $\mathcal{H}$. The hermitian conjugate state is $\langle\psi| = (|\psi\rangle)^\dagger$, and the modulus squared of the scalar product between two states gives the probability for the system to go from state 1 to state 2,

$$|\langle\psi_1|\psi_2\rangle|^2 = \text{probability for} \quad |\psi_1\rangle \rightarrow |\psi_2\rangle. \tag{1.18}$$

On the other hand physical observables O, i.e. measurable quantities, are given by the expectation values of hermitian operators, $\hat{O} = \hat{O}^\dagger$,

$$O = \langle\psi|\hat{O}|\psi\rangle, \quad O_{12} = \langle\psi_2|\hat{O}|\psi_1\rangle. \tag{1.19}$$

Hermiticity ensures that expectation values are real, as required for measurable quantities. Due to the probabilistic nature of quantum mechanics, expectation values correspond to statistical averages, or mean values, with a variance

$$(\Delta O)^2 = \langle\psi|(\hat{O} - O)^2|\psi\rangle = \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|\hat{O}|\psi\rangle^2. \tag{1.20}$$

An important concept in quantum mechanis is that of eigenstates of an operator, defined by

$$\hat{O}|\psi\rangle = O|\psi\rangle. \tag{1.21}$$

Evidently, between eigenstates we have $\Delta O = 0$. Examples are coordinate eigenstates, $\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$, and momentum eigenstates, $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, describing a particle at position $\mathbf{x}$ or with momentum $\mathbf{p}$, respectively. However, a state vector cannot be simultaneous eigenstate of non-commuting operators. This leads to the Heisenberg uncertainty relation for any two non-commuting operators $\hat{A}, \hat{B}$,

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|. \tag{1.22}$$

Finally, sets of eigenstates can be orthonormalized and we assume completeness, i.e. they span the entire Hilbert space,

$$\langle\mathbf{p}'|\mathbf{p}\rangle = \delta(\mathbf{p} - \mathbf{p}'), \quad 1 = \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}|. \tag{1.23}$$

As a consequence, an arbitrary state vector can always be expanded in terms of a set of eigenstates. In particular, the Schrödinger wave function of a particle in coordinate representation is given by $\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle$.

Having quantised our system, we now want to describe its time evolution. This can be done in different quantum pictures.

1.3 The Schrödinger picture

In this approach state vectors are functions of time, $|\psi(t)\rangle$, while operators are time independent, $\partial_t \hat{O} = 0$. The time evolution of a system is described by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},t) = \hat{H}\psi(\mathbf{x},t). \tag{1.24}$$

If at some initial time t_0 our system is in the state $\Psi(\mathbf{x},t_0)$, then the time dependent state vector

$$\Psi(\mathbf{x},t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(\mathbf{x},t_0) \tag{1.25}$$

solves the Schrödinger equation for all later times t.

The expectation value of some hermitian operator $\hat{O}$ at a given time t is then defined as

$$\langle \hat{O}\rangle_t = \int d^3x\, \Psi^*(\mathbf{x},t)\hat{O}\Psi(\mathbf{x},t), \tag{1.26}$$

and the normalisation of the wavefunction is given by

$$\int d^3x\, \Psi^*(\mathbf{x},t)\Psi(\mathbf{x},t) = \langle 1\rangle_t. \tag{1.27}$$

Since $\Psi^*\Psi$ is positive, it is natural to interpret it as the <u>probability density</u> for finding a particle at position $\mathbf{x}$. Furthermore one can derive a conserved current $\mathbf{j}$, as well as a continuity equation by considering

$$\Psi^* \times (\text{Schr.Eq.}) - \Psi \times (\text{Schr.Eq.})^*. \tag{1.28}$$

The continuity equation reads

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\mathbf{j} \tag{1.29}$$

where the density ρ and the current $\mathbf{j}$ are given by

$$\rho = \Psi^*\Psi \qquad \text{(positive)}, \tag{1.30}$$

$$\mathbf{j} = \frac{\hbar}{2im}\left(\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right) \qquad \text{(real)}. \tag{1.31}$$

Now that we have derived the continuity equation let us discuss the probability interpretation of Quantum Mechanics in more detail. Consider a finite volume V with boundary S. The integrated continuity equation is

$$\begin{aligned}\int_V \frac{\partial\rho}{\partial t}\, d^3x &= -\int_V \nabla\cdot\mathbf{j}\, d^3x \\ &= -\int_S \mathbf{j}\cdot \underline{dS}\end{aligned} \tag{1.32}$$

where in the last line we have used Gauss's theorem. Using Eq. (1.27) the lhs. can be rewritten and we obtain

$$\frac{\partial}{\partial t}\langle 1\rangle_t = -\int_S \mathbf{j}\cdot \underline{dS} = 0. \tag{1.33}$$

In other words, provided that $\mathbf{j} = 0$ everywhere at the boundary S, we find that the time derivative of $\langle 1\rangle_t$ vanishes. Since $\langle 1\rangle_t$ represents the total probability for finding the particle anywhere inside the volume V, we conclude that this probability must be conserved: particles cannot be created or destroyed in our theory. Non-relativistic Quantum Mechanics thus provides a consistent formalism to describe a single particle. The quantity $\Psi(\mathbf{x}, t)$ is interpreted as a one-particle wave function.

1.4 The Heisenberg picture

Here the situation is the opposite to that in the Schrödinger picture, with the state vectors regarded as constant, $\partial_t|\Psi_H\rangle = 0$, and operators which carry the time dependence, $\hat{O}_H(t)$. This is the concept which later generalises most readily to field theory. We make use of the solution Eq. (1.25) to the Schrödinger equation in order to *define* a Heisenberg state vector through

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi(x,t_0) \equiv e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\Psi_H(x), \tag{1.34}$$

i.e. $\Psi_H(\mathbf{x}) = \Psi(\mathbf{x}, t_0)$. In other words, the Schrödinger vector at some time t_0 is defined to be equivalent to the Heisenberg vector, and the solution to the Schrödinger equation provides the transformation law between the two for all times. This transformation of course leaves the physics, i.e. expectation values, invariant,

$$\langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \langle\Psi(t_0)|e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\Psi(t_0)\rangle = \langle\Psi_H|\hat{O}_H(t)|\Psi_H\rangle, \tag{1.35}$$

with

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \tag{1.36}$$

From this last equation it is now easy to derive the equivalent of the Schrödinger equation for the Heisenberg picture, the Heisenberg equation of motion for operators:

$$i\hbar\frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H, \hat{H}]. \tag{1.37}$$

Note that all commutation relations, like Eq. (1.16), with time dependent operators are now intended to be valid for all times. Substituting $\hat{x}, \hat{p}$ for $\hat{O}$ into the Heisenberg equation readily leads to

$$\begin{aligned}\frac{d\hat{x}_i}{dt} &= \frac{\partial\hat{H}}{\partial\hat{p}_i},\\ \frac{d\hat{p}_i}{dt} &= -\frac{\partial\hat{H}}{\partial\hat{x}_i},\end{aligned} \tag{1.38}$$

the quantum mechanical equivalent to the Hamilton equations of classical mechanics.

1.5 The quantum mechanical harmonic oscillator

Because of similar structures later in quantum field theory, it is instructive to also briefly recall the harmonic oscillator in one dimension. Its Hamiltonian is given by

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right). \tag{1.39}$$

Employing the canonical formalism we have just set up, we easily identify the momentum operator to be $\hat{p}(t) = m\partial_t\hat{x}(t)$, and from the Hamilton equations we find the equation of motion to be $\partial_t^2\hat{x} = -\omega^2\hat{x}$, which has the well known plane wave solution $\hat{x} \sim \exp i\omega t$.

An alternative path useful for later field theory applications is to introduce new operators, expressed by the old ones,

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{\hbar}{m\omega}}\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{\hbar}{m\omega}}\hat{p}\right). \tag{1.40}$$

Using the commutation relation for $\hat{x},\hat{p}$, one readily derives

$$[\hat{a},\hat{a}^\dagger] = 1, \quad [\hat{H},\hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H},\hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{1.41}$$

With the help of these the Hamiltonian can be rewritten in terms of the new operators,

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega. \tag{1.42}$$

With this form of the Hamiltonian it is easy to construct a complete basis of energy eigenstates $|n\rangle$,

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{1.43}$$

Using the above commutation relations, one finds

$$\hat{a}^\dagger\hat{H}|n\rangle = (\hat{H}\hat{a}^\dagger - \hbar\omega\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{1.44}$$

and from the last equation

$$\hat{H}\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle. \tag{1.45}$$

Thus, the state $\hat{a}^\dagger|n\rangle$ has energy $E_n + \hbar\omega$, and therefore $\hat{a}^\dagger$ may be regarded as a "creation operator" for a quantum with energy $\hbar\omega$. Along the same lines one finds that $\hat{a}|n\rangle$ has energy $E_n - \hbar\omega$, and $\hat{a}$ is an "annihilation operator".

Let us introduce a vacuum state $|0\rangle$ with no quanta excited, for which $\hat{a}|n\rangle = 0$, because there cannot be any negative energy states. Acting with the Hamiltonian on that state we find

$$\hat{H}|0\rangle = \hbar\omega/2, \tag{1.46}$$

i.e. the quantum mechanical vacuum has a non-zero energy, known as vacuum oscillation or zero point energy. Acting with a creation operator onto the vacuum state one easily finds the state with one quantum excited, and this can be repeated n times to get

$$|1\rangle = \hat{a}^\dagger|0\rangle \quad , \quad E_1 = (1+\frac{1}{2})\hbar\omega, \quad \ldots$$

$$|n\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}|n-1\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad , \quad E_n = (n+\frac{1}{2})\hbar\omega. \tag{1.47}$$

The root of the factorial is there to normalise all eigenstates to one. Finally, the "number operator" $\hat{N} = \hat{a}^\dagger \hat{a}$ returns the number of quanta in a given energy eigenstate,

$$\hat{N}|n\rangle = n|n\rangle. \tag{1.48}$$

Problems

1.1 Starting from the definition of the Hamiltonian,

$$H(x,p) \equiv p\dot{x} - L(x,\dot{x}),$$

derive Hamilton's equations

$$\frac{\partial H}{\partial x} = -\dot{p}, \qquad \frac{\partial H}{\partial p} = \dot{x}.$$

[**Hint:** the key is to keep track of what are the independent variables]

1.2 Using the Schrödinger equation for the wavefunction $\Psi(\mathbf{x},t)$,

$$\left\{-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x})\right\}\Psi(\mathbf{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t),$$

show that the probability density $\rho = \Psi^*\Psi$ satisfies the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0,$$

where

$$\mathbf{j} = \frac{\hbar}{2im}\left\{\Psi^*\nabla\Psi - (\nabla\Psi^*)\,\Psi\right\}.$$

[**Hint:** Consider $\Psi^*\times$ (Schr.Eq.) $-$ $\Psi\times$ (Schr.Eq.)*]

1.3 Let $|\psi\rangle$ be a simultaneous eigenstate of two operators $\hat{A}, \hat{B}$. Prove that this implies a vanishing commutator $[\hat{A},\hat{B}]$.

1.4 Let $\hat{O}$ be an operator in the Schrödinger picture. Starting from the definition of a Heisenberg operator,

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)},$$

derive the Heisenberg equation of motion

$$i\hbar\frac{d\hat{O}_H}{dt} = [\hat{O}_H,\hat{H}].$$

1.5 Consider the Heisenberg equation of motion for the momentum operator $\hat{p}$ of the harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right),$$

and show that it is equivalent to Newton's law for the position operator $\hat{x}$.

2 Classical Field Theory

2.1 From N-point mechanics to field theory

In the previous sections we have reviewed the Lagrangian formalism for a single point particle in classical mechanics. A benefit of that formalism is that it easily generalises to any number of particles or dimensions. Let us return to one dimension for the moment but consider an N-particle system, i.e. we have N coordinates and N momenta, $x_i(t), p_i(t), i = 1, \ldots N$. For such a system we get $2N$ Heisenberg equations,

$$-\frac{\partial H}{\partial x_i} = \frac{dp_i}{dt}, \quad \frac{\partial H}{\partial p_i} = \frac{dx_i}{dt}. \tag{2.1}$$

To make things more specific, consider a piece of a guitar string, approximated by N coupled oscillators, as in Fig. 2. Each point mass of the string can only move in the

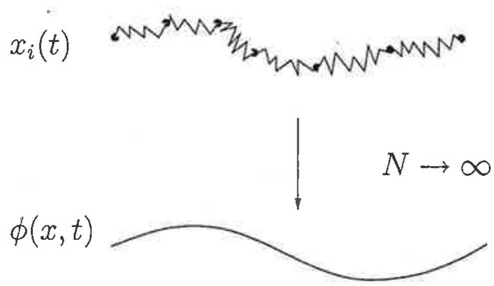


Figure 2: *From N coupled point masses to a continuous string, i.e. infinitely many degrees of freedom.*

direction perpendicular to the string, i.e. is a particle moving in one dimension. This approximation of a string gets better and better the more points we fill in between the springs, and a continuous string obtains in the limit $N \longrightarrow \infty$. The displacement of the string at some particular point x along its length is now given by a field coordinate $\phi(x,t)$. Going back to the N-point system and comparing what measures the location of a point and its displacement, we find the following "dictionary" between point mechanics and field theory:

$$\begin{array}{rcl} \text{Classical Mechanics:} & & \text{Classical Field Theory:} \\ x(t) & \longrightarrow & \phi(x,t) \\ \dot{x}(t) & \longrightarrow & \dot{\phi}(x,t) \\ i & \longrightarrow & x \\ L(x,\dot{x}) & \longrightarrow & \mathcal{L}[\phi,\dot{\phi}] \end{array} \tag{2.2}$$

In the last line we have introduced a new notation: the square brackets indicate that $L[\phi, \dot{\phi}]$ depends on the functions $\phi(x,t), \dot{\phi}(x,t)$ at every space-time point, but not on the coordinates directly. Such an object is called a "functional", as opposed to a function which depends on the coordinate variables only.

Formally the above limit of infinite degrees of freedom can also be taken if we are dealing with particles in a three-dimensional Euclidean space, for which there are N three-vectors $\mathbf{x}_i$ specifying the positions. We then obtain a field $\phi(\mathbf{x}, t)$, defined at every point in space and time.

2.2 Relativistic field theory

Before continuing to set up the formalism of field theory, we want to make it relativistic as well. Coordinates are combined into four-vectors, $x^\mu = (t, x_i)$ or $x = (t, \mathbf{x})$, whose length $x^2 = t^2 - \mathbf{x}^2$ is invariant under Lorentz transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \tag{2.3}$$

A general function transforms as $f(x) \to f'(x')$, i.e. both the function and its argument transform. A Lorentz scalar is a function which is the same in all inertial frames,

$$\phi'(x') = \phi(x) \quad \text{for all } \Lambda. \tag{2.4}$$

On the other hand a vector function transforms as

$$V'^\mu(x') = \Lambda^\mu_\nu V^\nu(x). \tag{2.5}$$

An example is the covariant derivative of a scalar field,

$$\partial^\mu \phi(x) = \frac{\partial \phi(x)}{\partial x_\mu}, \quad \partial_\mu \phi(x) = \frac{\partial \phi(x)}{\partial x^\mu}, \tag{2.6}$$

whose square evaluates to

$$(\partial^\mu \phi)(\partial_\mu \phi) = (\partial^0 \phi)^2 - (\nabla \phi)^2. \tag{2.7}$$

2.3 Action for a scalar field

We are now ready to write down the action for a relativistic scalar field. According to our dictionary, the action from point mechanics, Eq. (1.3), should go into

$$S = \int dt\, L[\phi, \dot{\phi}]. \tag{2.8}$$

However, for a relativistic theory we require Lorentz invariance of the action, and this is not obvious in the current form. The integration is over time only, rather than over the Lorentz-invariant four-volume element $d^4x = dt\, d^3x$, and so the non-invariance of the integration measure has to cancel against that of the Lagrange function in order to have an invariant action. Similar reasoning applies to the arguments of the Lagrangian. In order to have the symmetries manifest, we instead rewrite

$$S = \int d^4x\, \mathcal{L}[\phi, \partial^\mu], \quad L[\phi, \dot{\phi}] = \int d^3x\, \mathcal{L}[\phi, \partial^\mu \phi]. \tag{2.9}$$

Now everything is expressed in covariant quantities, and the action is Lorentz-invariant as soon as the newly defined Lagrangian density $\mathcal{L}$ is.

We now follow the same procedure as in point mechanics and apply the Hamiltonian principle by demanding $\delta S = 0$. For the variation of the field and its derivative we have

$$\phi \to \phi + \delta\phi, \quad \partial_\mu\phi \to \partial_\mu\phi + \delta\partial_\mu\phi, \qquad \delta\partial_\mu\phi = \partial_\mu\delta\phi. \tag{2.10}$$

Using the rule for functional differentiation, $\delta\phi(x)/\delta\phi(y) = \delta^4(x-y)$, the variation of the action then is (to first order in a Taylor expansion)

$$\begin{aligned} \delta S &= \int d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right\} \\ &= \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\bigg|}_{=0 \text{ at boundaries}} + \int d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \right\}\delta\phi. \end{aligned} \tag{2.11}$$

Again the integrand itself must vanish if $\delta S = 0$ for arbitrary variations of the field, $\delta\phi$. This yields the Euler-Lagrange equations for a classical field theory:

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = 0, \tag{2.12}$$

where in the second term a summation over the Lorentz index μ is implied.

Let us now consider the specific Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{2.13}$$

The functional derivatives yield

$$\frac{\delta\mathcal{L}}{\delta\phi} = -m^2\phi, \quad \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \partial^\mu\phi, \tag{2.14}$$

so that

$$\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \Box\phi. \tag{2.15}$$

The Euler-Lagrange equation then implies

$$(\Box + m^2)\phi(x) = 0. \tag{2.16}$$

This is the Klein-Gordon equation for a scalar field. It is the simplest relativistic wave equation and can be deduced from relativistic energy considerations. Here we have derived it from the Lagrange density following our canonical formalism, in complete analogy to point mechanics. Relativistic invariance of the equations of motion is ensured because we started from an invariant Lagrange density. This is the power of the formalism.

In keeping the analogy with point mechanics, we can define a conjugate momentum π through

$$\pi(x) \equiv \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\dot{\phi}(x)} = \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_0\phi(x))} = \partial_0\phi(x). \tag{2.17}$$

Note that the momentum variables p_μ and the conjugate momentum π are *not* the same. The word "momentum" is used only as a semantic analogy to classical mechanics. Further, we define the Hamilton function and a corresponding Hamilton density,

$$H(t) = \int d^3x\, \mathcal{H}[\phi,\pi], \quad \mathcal{H}[\phi,\pi] = \pi\dot{\phi} - \mathcal{L}. \tag{2.18}$$

For the Lagrangian density we considered, this gives

$$\mathcal{H} = \frac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right]. \tag{2.19}$$

2.4 Plane wave solution to the Klein-Gordon equation

Let us consider real solutions to Eq. (2.16), characterised by $\phi^*(x) = \phi(x)$. To find them we try an ansatz of plane waves

$$\phi(x) \propto \mathrm{e}^{i(k^0 t - \mathbf{k}\cdot\mathbf{x})}. \tag{2.20}$$

The Klein-Gordon equation is satisfied if $(k^0)^2 - \mathbf{k}^2 = m^2$ so that

$$k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}. \tag{2.21}$$

If we choose the positive branch of the square root then we can define the energy as

$$E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2} > 0, \tag{2.22}$$

and obtain two types of solutions which read

$$\phi_+(x) \propto \mathrm{e}^{i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}, \quad \phi_-(x) \propto \mathrm{e}^{-i(E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})}. \tag{2.23}$$

The general solution is a superposition of ϕ_+ and ϕ_-. Using

$$E(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x} = k^\mu k_\mu = k_\mu k^\mu = k\cdot x \tag{2.24}$$

this solution reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left(\mathrm{e}^{ik\cdot x}\alpha^*(\mathbf{k}) + \mathrm{e}^{-ik\cdot x}\alpha(\mathbf{k})\right), \tag{2.25}$$

where $\alpha(\mathbf{k})$ is an arbitrary complex coefficient. From the general solution one easily reads off that ϕ is real, i.e. $\phi = \phi^*$.

2.5 Symmetries and conservation laws

Symmetries play such a fundamental role in physics because they are related to conservation laws. This is stated in Noether's theorem. In a nutshell, Noether's theorem says that invariance of the action under a symmetry transformation implies the existence of a conserved quantity. For instance, the conservation of 3-momentum $\mathbf{p}$ is associated with translational invariance of the Lagrangian, i.e. the transformation

$$\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{a}, \quad \mathbf{a}: \text{ constant 3-vector}, \tag{2.26}$$

while the conservation of energy comes from the invariance under time translations

$$t \longrightarrow t + \tau, \quad \tau: \text{ constant time interval}. \tag{2.27}$$

Let us apply this to our relativistic field theory and consider four-translations, $x^\mu \longrightarrow x^\mu + \epsilon^\mu$. The variation of the Lagrangian is

$$\begin{aligned}\delta\mathcal{L} &= = \frac{\delta\mathcal{L}}{\delta\phi}\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu + \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\frac{\partial(\partial^\mu\phi)}{\partial x^\nu}\epsilon^\nu \\ &= \frac{\partial}{\partial x_\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu\right], \end{aligned} \tag{2.28}$$

where we have made use of the Euler-Lagrange Eqs. (2.12), to get to the last expression. If the action is to be invariant under such translations, its variation has to vanish for arbitrary ϵ^ν, which leads to

$$\frac{\partial}{\partial x_\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\right] = 0. \tag{2.29}$$

The quantity in square brackets is called the energy-momentum tensor $\Theta_{\mu\nu}$, and thus we have

$$\partial^\mu\Theta_{\mu\nu} \equiv \partial^0\Theta_{0\nu} - \partial^j\Theta_{j\nu} = 0, \tag{2.30}$$

i.e. four conservation laws (one for every value of ν). Let us look in more detail at the components of the energy-momentum tensor,

$$\begin{aligned}\Theta_{00} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}\partial_0\phi - g_{00}\mathcal{L} = \pi(x)(\partial_0\phi(x)) - \mathcal{L}, \\ \Theta_{0j} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\phi)}\partial_j\phi - g_{0j}\mathcal{L} = \pi(x)\partial_j\phi. \end{aligned} \tag{2.31}$$

The first line is nothing but the Hamiltonian density, and integrating it over space will thus be the Hamiltonian, or the energy. Its conservation can then be shown by considering

$$\begin{aligned}\frac{\partial}{\partial t}\int_V d^3x\,\Theta_{00} &= \int_V d^3x\,\partial^0\Theta_{00} \\ &= \int_V d^3x\,\partial^j\Theta_{j0} = \int_S dS_j\cdot\Theta_{0j} = 0, \end{aligned} \tag{2.32}$$

where we have used Eq. (2.30) in the second line. The Hamiltonian density is a conserved quantity, provided that there is no energy flow through the surface S which encloses the volume V. In a similar manner one can show that the 3-momentum p_j, which is related to Θ_{0j}, is conserved as well. It is then useful to define a conserved energy-momentum four-vector

$$P_\mu = \int d^3x\,\Theta_{0\mu}. \tag{2.33}$$

In analogy to point mechanics, we thus see that invariances of the Lagrangian density correspond to conservation laws. An entirely analogous procedure leads to conserved quantities like anguluar mometum and spin. Furthermore one can study so-called internal symmetries, i.e. ones which are not related to coordinate but other transformations. Examples are conservation of all kinds of charges, isospin, etc.

We have thus established the Lagrange-Hamilton formalism for classical field theory: we derived the equation of motion (Euler-Lagrange equation) from the Lagrangian and introduced the conjugate momentum. We then defined the Hamiltonian (density) and considered conservation laws by studying the energy-momentum tensor $\Theta_{\mu\nu}$.

Problems

2.1 Given the relativistic invariance of the measure d^4k, show that the integration measure

$$\frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}$$

is Lorentz-invariant, provided that $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$.

[**Hint:** Start from the Lorentz-invariant expression

$$\frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,\theta(k_0)$$

and use

$$\delta(x^2 - x_0^2) = \frac{1}{2|x|}(\delta(x - x_0) + \delta(x + x_0)).$$

What is the significance of the δ and θ functions above? If you're really keen, you may prove the relation for $\delta(x^2 - x_0^2)$.]

2.2 Verify that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,\left\{e^{ik\cdot x}a(\mathbf{k}) + e^{-ik\cdot x}b(\mathbf{k})\right\}$$

is a solution of the Klein-Gordon equation. Show that a real scalar field $\phi^*(x) = \phi(x)$ requires the condition $b(\mathbf{k}) = a^*(\mathbf{k})$.

2.3 Show that the Hamiltonian density $\mathcal{H}$ for a free scalar field is given by

$$\mathcal{H} = \frac{1}{2}\left\{(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right\}.$$

Derive the components $\hat{P}_0$, $\hat{\mathbf{P}}$ of the energy-momentum four-vector $\hat{P}^\mu$ in terms of the field operators $\hat{\phi}, \hat{\pi}$.

3 Quantum Field Theory

After many preparations, we have finally arrived at the proper subject of the lecture. In this section we shall apply the canonical quantisation formalism to field theory.

3.1 Canonical field quantisation

To lighten notation, let us follow common practice in quantum field theory and set $\hbar = c = 1$. Our starting point is the Lagrangian density for the free scalar field,

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2, \tag{3.1}$$

which led to the Klein-Gordon equation in the previous section. We have seen that in field theory the field $\phi(x)$ plays the role of the coordinates in ordinary point mechanics, and we defined a canonically conjugate momentum, $\pi(x) = \delta\mathcal{L}/\delta\dot{\phi} = \dot{\phi}(x)$. We then continue the analogy to point mechanics through the quantisation procedure, i.e. we now take our canonical variables to be operators,

$$\phi(x) \to \hat{\phi}(x), \quad \pi(x) \to \hat{\pi}(x). \tag{3.2}$$

Next we impose equal-time commutation relations on them,

$$\begin{aligned}\left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)\right] &= i\delta^3(\mathbf{x}-\mathbf{y}), \\ \left[\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)\right] &= \left[\hat{\pi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)\right] = 0.\end{aligned} \tag{3.3}$$

As in the case of quantum mechanis, the canonical variables commute among themselves, but not the canonical coordinate and momentum with each other. Note that the commutation relation is entirely analogous to the quantum mechanical case. There would be an $\hbar$, if it hadn't been set to one earlier, and the delta-function accounts for the fact that we are dealing with fields. It is one if the fields are evaluated at the same space-time point, and zero otherwise.

After quantisation, our fields have turned into field operators. Note that within the relativistic formulation they depend on time, and hence they are Heisenberg operators.

3.2 Causality and commutation relations

In the previous paragraph we have formulated commutation relations for fields evaluated at equal time, which is clearly a special case when considering fields at general x, y. The reason has to do with maintaining causality in a relativistic theory. Let us recall the light cone about an event at y, as in Fig. 3. One important postulate of special relativity states that no signal and no interaction can travel faster than the speed of light. This has important consequences about the way in which different events can affect each other. For instance, two events which are characterised by space-time points x^μ and y^μ are said to be causal if the distance $(x-y)^2$ is time-like, i.e. $(x-y)^2 > 0$. By contrast, two events characterised by a space-like separation, i.e. $(x-y)^2 < 0$, cannot affect each other, since the point x is not contained inside the light cone about y.

In non-relativistic Quantum Mechanics the commutation relations among operators indicate whether precise and independent measurements of the corresponding observables can be made. If the commutator does not vanish, then a measurement of one observable affects that of the other. From the above it is then clear that the issue of causality must be incorporated into the commutation relations of the relativistic version of our quantum theory: whether or not independent and precise measurements of two observables can be made depends also on the separation of the 4-vectors characterising the points at which these measurements occur. Clearly, events with space-like separations cannot affect each other, and hence all fields must commute,

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \left[\hat{\pi}(x), \hat{\pi}(y)\right] = \left[\hat{\phi}(x), \hat{\pi}(y)\right] = 0 \quad \text{for} \quad (x-y)^2 < 0. \tag{3.4}$$

This condition is sometimes called micro-causality. Writing out the four-components of the time interval, we see that as long as $|t'-t| < |\mathbf{x}-\mathbf{y}|$, the commutator vanishes in

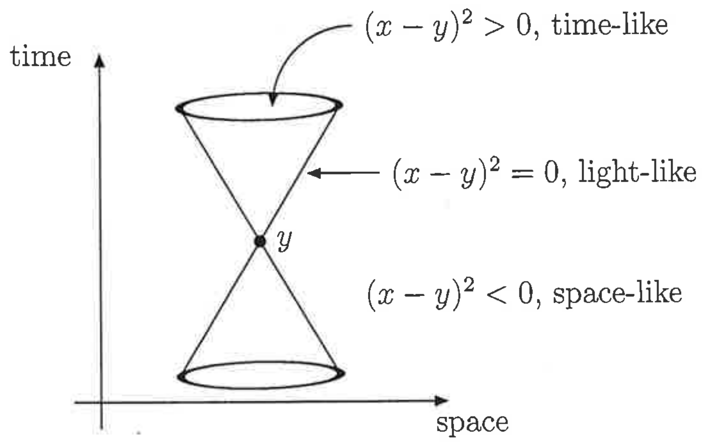


Figure 3: The light cone about y. Events occurring at points x and y are said to be time-like (space-like) if x is inside (outside) the light cone about y.

a finite interval $|t'-t|$. It also vanishes for $t'=t$, as long as $\mathbf{x}\neq\mathbf{y}$. Only if the fields are evaluated at an equal space-time point can they affect each other, which leads to the equal-time commutation relations above. They can also affect each other everywhere within the light cone, i.e. for time-like intervals. It is not hard to show that in this case

$$\begin{aligned}\left[\hat\phi(x),\hat\phi(y)\right]=[\hat\pi(x),\hat\pi(y)] &= 0, \quad \text{for} \quad (x-y)^2>0\\ \left[\hat\phi(x),\hat\pi(y)\right] &= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\left(e^{ip\cdot(x-y)}+e^{-ip\cdot(x-y)}\right).\end{aligned} \tag{3.5}$$

3.3 Creation and annihilation operators

After quantisation, the Klein-Gordon equation we derived earlier turns into an equation for operators. For its solution we simply promote the classical plane wave solution, Eq. (2.25), to operator status,

$$\hat\phi(x)=\int\frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\left(\mathrm{e}^{ik\cdot x}\hat a^\dagger(\mathbf{k})+\mathrm{e}^{-ik\cdot x}\hat a(\mathbf{k})\right). \tag{3.6}$$

Note that the complex conjugation of the Fourier coefficient turned into hermitian conjugation for an operator.

Let us now solve for the operator coefficients of the positive and negative energy solutions. In order to do so, we invert the Fourier integrals for the field and its time derivative,

$$\int d^3x\,\hat\phi(\mathbf{x},t)e^{ikx} = \frac{1}{2E}\left[\hat a(\mathbf{k})+\hat a^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{3.7}$$

$$\int d^3x\,\dot{\hat\phi}(\mathbf{x},t)e^{ikx} = -\frac{i}{2}\left[\hat a(\mathbf{k})-\hat a^\dagger(\mathbf{k})e^{2ik_0x_0}\right], \tag{3.8}$$

and then build the linear combination $iE(k)(3.7)-(3.8)$ to find

$$\int d^3x\left[iE(k)\hat\phi(\mathbf{x},t)-\dot{\hat\phi}(\mathbf{x},t)\right]e^{ikx}=i\hat a(\mathbf{k}), \tag{3.9}$$

Following a similar procedure for $\hat{a}^\dagger(k)$, and using $\hat{\pi}(x) = \dot{\hat{\phi}}(x)$ we find

$$\begin{aligned} \hat{a}(\mathbf{k}) &= \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) + i\hat{\pi}(\mathbf{x},t)\right] e^{ikx}, \\ \hat{a}^\dagger(\mathbf{k}) &= \int d^3x \left[E(k)\hat{\phi}(\mathbf{x},t) - i\hat{\pi}(\mathbf{x},t)\right] e^{-ikx}. \end{aligned} \tag{3.10}$$

Note that, as Fourier coefficients, these operators do not depend on time, even though the right hand side does contain time variables. Having expressions in terms of the canonical field variables $\hat{\phi}(x), \hat{\pi}(x)$, we can now evaluate the commutators for the Fourier coefficients. Expanding everything out and using the commutation relations Eq. (3.3), we find

$$\left[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = 0 \tag{3.11}$$

$$\left[\hat{a}(\mathbf{k}_1), \hat{a}(\mathbf{k}_2)\right] = 0 \tag{3.12}$$

$$\left[\hat{a}(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k}_2)\right] = (2\pi)^3\, 2E(\mathbf{k}_1)\delta^3(\mathbf{k}_1 - \mathbf{k}_2) \tag{3.13}$$

We easily recognise these for every $\mathbf{k}$ to correspond to the commutation relations for the harmonic oscillator, Eq. (1.41). This motivates us to also express the Hamiltonian and the energy momentum four-vector of our quantum field theory in terms of these operators. This yields

$$\begin{aligned} \hat{H} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, E(\mathbf{k})\left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right), \\ \hat{\mathbf{P}} &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\, \mathbf{k}\left(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})\right). \end{aligned} \tag{3.14}$$

We thus find that the Hamiltonian and the momentum operator are nothing but a continuous sum of excitation energies/momenta of one-dimensional harmonic oscillators! After a minute of thought this is not so surprising. We expanded the solution of the Klein-Gordon equation into a superposition of plane waves with momenta $\mathbf{k}$. But of course a plane wave solution with energy $E(\mathbf{k})$ is also the solution to a one-dimensional harmonic oscillator with the same energy. Hence, our free scalar field is simply a collection of infinitely many harmonic oscillators distributed over the whole energy/momentum range. These energies sum up to that of the entire system. We have thus reduced the problem of handling our field theory to oscillator algebra. From the harmonic oscillator we know already how to construct a complete basis of energy eigenstates, and thanks to the analogy of the previous section we can take this over to our quantum field theory.

3.4 Energy of the vacuum state and renormalisation

In complete analogy we begin again with the postulate of a vacuum state $|0\rangle$ with norm one, which is annihilated by the action of the operator a,

$$\langle 0|0\rangle = 1, \quad \hat{a}(\mathbf{k})|0\rangle = 0 \quad \text{for all} \quad \mathbf{k}. \tag{3.15}$$

Let us next evaluate the energy of this vacuum state, by taking the expectation value of the Hamiltonian,

$$E_0 = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})}\, E(\mathbf{k})\left\{\langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})|0\rangle + \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\}. \tag{3.16}$$

The first term in curly brackets vanishes, since a annihilates the vacuum. The second can be rewritten as

$$\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = \left\{\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})\right] + \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\right\}|0\rangle. \tag{3.17}$$

It is now the second term which vanishes, whereas the first can be replaced by the value of the commutator. Thus we obtain

$$E_0 = \langle 0|\hat{H}|0\rangle = \delta^3(0)\frac{1}{2}\int d^3k\, E(\mathbf{k}) = \delta^3(0)\frac{1}{2}\int d^3k\,\sqrt{\mathbf{k}^2+m^2} = \infty, \tag{3.18}$$

which means that the energy of the ground state is infinite! This result seems rather paradoxical, but it can be understood again in terms of the harmonic oscillator. Recall that the simple quantum mechanical oscillator has a finite zero-point energy. As we have seen above, our field theory corresponds to an infinite collection of harmonic oscillators, i.e. the vacuum receives an infinite number of zero point contributions, and its energy thus diverges.

This is the first of frequent occurrences of infinities in quantum field theory. Fortunately, it is not too hard to work around this particular one. Firstly, we note that nowhere in nature can we observe absolute values of energy, all we can measure are energy differences relative to some reference scale, at best the one of the vacuum state, $|0\rangle$. In this case it does not really matter what the energy of the vacuum is. This then allows us to redefine the energy scale, by always subtracting the (infinite) vacuum energy from any energy we compute. This process is called "renormalisation".

We then *define* the renormalised vacuum energy to be zero, and take it to be the expectation value of a renormalised Hamiltonian,

$$E_0^R \equiv \langle 0|\hat{H}^R|0\rangle = 0. \tag{3.19}$$

According to this recipe, the renormalised Hamiltonian is our original one, minus the (unrenormalised) vacuum energy,

$$\begin{aligned}\hat{H}^R &= \hat{H} - E_0\\ &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) - \langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle\right\}\\ &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3\,2E(\mathbf{k})}\,E(\mathbf{k})\left\{2\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})\right] - \langle 0|\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})\right]|0\rangle\right\}\end{aligned} \tag{3.20}$$

Here the subtraction of the vacuum energy is shown explicitly, and we can rewrite is as

$$\begin{aligned}\hat{H}^R &= \int\frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\\ &\quad+\frac{1}{2}\int\frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\left\{\left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{p})\right] - \langle 0|\left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{p})\right]|0\rangle\right\}.\\ &= \int\frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\,\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{H}^{\text{vac}}\,.\end{aligned} \tag{3.21}$$

The operator $\hat{H}^{\text{vac}}$ ensures that the vacuum energy is properly subtracted: if $|\psi\rangle$ and $|\psi'\rangle$ denote arbitrary N-particle states, then one can convince oneself that $\langle\psi'|\hat{H}^{\text{vac}}|\psi\rangle = 0$. In particular we now find that

$$\langle 0|\hat{H}^R|0\rangle = 0, \tag{3.22}$$

as we wanted. A simple way to automatise the removal of the vacuum contribution is to introduce *normal ordering.* Normal ordering means that all annihilation operators appear to the right of any creation operator. The notation is

$$:\hat{a}\hat{a}^\dagger: = \hat{a}^\dagger\hat{a}, \tag{3.23}$$

i.e. the normal-ordered operators are enclosed within colons. For instance

$$:\tfrac{1}{2}\left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right): \;=\; \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}). \tag{3.24}$$

It is important to keep in mind that $\hat{a}$ and $\hat{a}^\dagger$ *always* commute inside $:\cdots:$. This is true for an arbitrary string of $\hat{a}$ and $\hat{a}^\dagger$. With this definition we can write the normal-ordered Hamiltonian as

$$\begin{aligned} :\hat{H}: &= :\frac{1}{2}\int\frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\left(\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right): \\ &= \int\frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\,E(\mathbf{p})\,\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}), \end{aligned} \tag{3.25}$$

and thus have the relation

$$\hat{H}^R =: \hat{H} : + \hat{H}^{\rm vac}. \tag{3.26}$$

Hence, we find that

$$\langle\psi'|:\hat{H}:|\psi\rangle = \langle\psi'|\hat{H}^R|\psi\rangle, \tag{3.27}$$

and, in particular, $\langle 0|:\hat{H}:|0\rangle = 0$. The normal ordered Hamiltonian thus produces a renormalised, sensible result for the vacuum energy.

3.5 Fock space and particle number representation

After this lengthy grappling with the vacuum state, we can continue to construct our basis of states in analogy to the harmonic oscillator, making use of the commutation relations for the operators $\hat{a}, \hat{a}^\dagger$. In particular, we define the state $|\mathbf{k}\rangle$ to be the one obtained by acting with the operator $a^\dagger(\mathbf{k})$ on the vacuum,

$$|\mathbf{k}\rangle = \hat{a}^\dagger(\mathbf{k})|0\rangle. \tag{3.28}$$

Using the commutator, its norm is found to be

$$\begin{aligned} \langle\mathbf{k}|\mathbf{k}'\rangle &= \langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')|0\rangle = \langle 0|[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')]|0\rangle + \langle 0|\hat{a}^\dagger(\mathbf{k}')a(\mathbf{k})|0\rangle \\ &= (2\pi)^3 2E(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{3.29}$$

since the last term in the first line vanishes ($\hat{a}(\mathbf{k})$ acting on the vacuum). Next we compute the energy of this state, making use of the normal ordered Hamiltonian,

$$\begin{aligned} :\hat{H}:|\mathbf{k}\rangle &= \int\frac{d^3k'}{(2\pi)^3\,2E(\mathbf{k}')}E(\mathbf{k}')\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle \\ &= \int\frac{d^3k'}{(2\pi)^3\,2E(\mathbf{k}')}E(\mathbf{k}')(2\pi)^3 2E(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\hat{a}^\dagger(\mathbf{k})|0\rangle \\ &= E(\mathbf{k})\hat{a}^\dagger(\mathbf{k})|0\rangle = E(\mathbf{k})|\mathbf{k}\rangle, \end{aligned} \tag{3.30}$$

and similarly one finds

$$:\hat{\mathbf{P}}: |\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle. \quad (3.31)$$

Observing that the normal ordering did its job and we obtain renormalised, finite results, we may now interpret the state $|\mathbf{k}\rangle$. It is a one-particle state for a relativistic particle of mass m and momentum $\mathbf{k}$, since acting on it with the energy-momentum operator returns the relativistic one particle energy-mometum dispersion relation, $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. The $a^\dagger(\mathbf{k}), a(\mathbf{k})$ are creation and annihilation operators for particles of momentum $\mathbf{k}$.

In analogy to the harmonic oscillator, the procedure can be continued to higher states. One easily checks that

$$:\hat{P}^\mu: \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle, \quad (3.32)$$

and so the state

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = \frac{1}{\sqrt{2!}}\hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle \quad (3.33)$$

is a two-particle state (the factorial is there to have it normalised in the same way as the one-particle state), and so on for higher Fock states.

At the long last we can now see how the field in our free quantum field theory is related to particles. A particle of momentum $\mathbf{k}$ corresponds to an excited Fourier mode of a field. Since the field is a superpositon of all possible Fourier modes, one field is enough to describe all possible configurations representing one or many particles of the same kind in any desired momentum state.

Let us investigate what happens under interchange of the two particles. Since $[\hat{a}^\dagger(\mathbf{k}_1), \hat{a}^\dagger(\mathbf{k})] = 0$ for all $\mathbf{k}_1, \mathbf{k}_2$, we see that

$$|\mathbf{k}_2, \mathbf{k}_1\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle, \quad (3.34)$$

and hence the state is symmetric under interchange of the two particles. Thus, the particles described by the scalar field are bosons.

Finally we complete the analogy to the harmonic oscillator by introducing a number operator

$$\hat{N}(\mathbf{k}) = \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \quad \hat{\mathcal{N}} = \int d^3k\; \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}), \quad (3.35)$$

which gives us the number of bosons described by a particular Fock state,

$$\hat{\mathcal{N}}|0\rangle = 0, \quad \hat{\mathcal{N}}|\mathbf{k}\rangle = |\mathbf{k}\rangle, \quad \hat{\mathcal{N}}|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = n|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \quad (3.36)$$

Of course the normal-ordered Hamiltonian can now simply be given in terms of this operator,

$$:\hat{H} := \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} E(\mathbf{k})\hat{N}(\mathbf{k}), \quad (3.37)$$

i.e. when acting on a Fock state it simply sums up the energies of the individual particles to give

$$:\hat{H}: |\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = (E(\mathbf{k}_1) + \ldots E(\mathbf{k}_n))\,|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle. \quad (3.38)$$

This concludes the quantisation of our free scalar field theory. We have followed the canonical quantisation procedure familiar from quantum mechanics. Due to the infinite number of degrees of freedom, we encountered a divergent vacuum energy, which we had to renormalise. The renormalised Hamiltonian and the Fock states that we constructed describe free relativistic, uncharged spin zero particles of mass m, such as neutral pions, for example.

If we want to describe charged pions as well, we need to introduce complex scalar fields, the real and imaginary parts being necessary to describe opposite charges. For particles with spin we need still more degrees of freedom and use vector or spinor fields, which have the appropriate rotation and Lorentz transformation properties. Moreover, for fermions there is the Pauli principle prohibiting identical particles with the same quantum numbers to occupy the same state, so the state vectors have to be anti-symmetric under interchange of two particles. This is achieved by imposing anti-commutation relations, rather than commutation relations, on the corresponding field operators. Apart from these complications which account for the nature of the particles, the formalism and quantisation procedure is the same as for the simpler scalar fields, to which we shall stick for this reason.

Problems

3.1 Using the expressions for $\hat{\phi}$ and $\hat{\pi}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, show that the unequal time commutator $\left[\hat{\phi}(x), \hat{\pi}(x')\right]$ is given by

$$\left[\hat{\phi}(x), \hat{\pi}(x')\right] = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{ip\cdot(x-x')} + e^{-ip\cdot(x-x')}\right).$$

Show that for $t = t'$ one recovers the equal time commutator

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)\right] = i\delta^3(\mathbf{x} - \mathbf{x}').$$

3.2 Being time-dependent Heisenberg operators, the operators $\hat{O} = \hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}, t)$ of scalar field theory obey the Heisenberg equation

$$i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}].$$

In analogy to what you did in problem 1.5, demonstrate the equivalence of this equation with the Klein-Gordon equation.

3.3 Express the Hamiltonian

$$\hat{H} = \frac{1}{2} \int d^3x \left\{\partial_0\hat{\phi})^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right\}$$

of the quantised free scalar field theory in terms of creation and annihilation operators and show that it is given by

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})} E(\mathbf{p}) \left\{\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) + \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{p})\right\}.$$

3.3 Prove the commutator relation

$$\left[: \hat{P}^\mu :, \hat{a}^\dagger(\mathbf{k})\right] = k^\mu \hat{a}^\dagger(\mathbf{k})$$

to show that

$$: \hat{P}^\mu : \; \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\, \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle. \tag{3.39}$$

Interpret the physics of this result.

3.4 Prove by induction that

$$\int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} = n \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} .$$

[**Hint:** induction proceeds in two steps. *i)* show that the statement is true for some starting value of n; *ii)* show that if the statement holds for some general n, then it also holds for $n+1$.]

4 Interacting scalar fields

From now on we shall always discuss quantised real scalar fields. It is then convenient to drop the "hats" on the operators that we have considered up to now. So far we have only discussed free fields without any interaction between them, which we could solve exactly in terms of plane waves. As this does not make for a very interesting theory, let us now add an interaction Lagrangian $\mathcal{L}_{\text{int}}$. The full Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{4.1}$$

where

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{4.2}$$

is the free Lagrangian density discussed before. The Hamiltonian density of the interaction is related to $\mathcal{L}_{\text{int}}$ simply by

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}, \tag{4.3}$$

which follows from its definition. We shall leave the details of $\mathcal{L}_{\text{int}}$ unspecified for the moment. What we will be concerned with mostly are scattering processes, in which two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter, thereby producing a number of particles in the final state, characterised by momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$. This is schematically shown in Fig. 4. Our task is to find a description of such a scattering process in terms of the underlying quantum field theory.

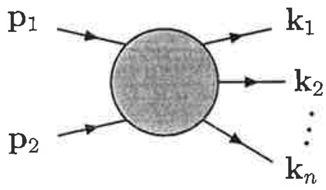


Figure 4: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into n particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ in the final state.

4.1 The S-matrix

The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short interval around some particular time t with $-\infty \ll t \ll \infty$. Long before t, the incoming particles evolve independently and freely. They are described by a field operator ϕ_{in} defined through

$$\lim_{t \to -\infty} \phi(x) = \phi_{\text{in}}(x), \tag{4.4}$$

which acts on a corresponding basis of $|\text{in}\rangle$ states. Long after the collision the particles in the final state evolve again like in the free theory, and the corresponding operator is

$$\lim_{t \to +\infty} \phi(x) = \phi_{\text{out}}(x), \tag{4.5}$$

acting on states $|\text{out}\rangle$. The fields ϕ_{in}, ϕ_{out} are the asymptotic limits of the Heisenberg operator ϕ. They both satisfy the free Klein-Gordon equation, i.e.

$$(\Box + m^2)\phi_{\text{in}}(x) = 0, \qquad (\Box + m^2)\phi_{\text{out}}(x) = 0. \tag{4.6}$$

Operators describing free fields can be expressed as a superposition of plane waves (see Eq. (3.6)). Thus, for ϕ_{in} we have

$$\phi_{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\mathbf{k})} \left(\mathrm{e}^{ik\cdot x} a_{\text{in}}^\dagger(\mathbf{k}) + \mathrm{e}^{-ik\cdot x} a_{\text{in}}(\mathbf{k}) \right), \tag{4.7}$$

with an entirely analogous expression for $\phi_{\text{out}}(x)$. Note that the operators $a^\dagger$ and a also carry subscripts "in" and "out".

We can now use the creation operators $a_{\text{in}}^\dagger$ and $a_{\text{out}}^\dagger$ to build up Fock states from the vacuum. For instance

$$\begin{aligned} a_{\text{in}}^\dagger(\mathbf{p}_1)\, a_{\text{in}}^\dagger(\mathbf{p}_2)|0\rangle &= |\mathbf{p}_1, \mathbf{p}_2; \text{in}\rangle, && (4.8) \\ a_{\text{out}}^\dagger(\mathbf{k}_1) \cdots a_{\text{out}}^\dagger(\mathbf{k}_n)|0\rangle &= |\mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\rangle. && (4.9) \end{aligned}$$

We must now distinguish between Fock states generated by $a_{\text{in}}^\dagger$ and $a_{\text{out}}^\dagger$, and therefore we have labelled the Fock states accordingly. In eqs. (4.8) and (4.9) we have assumed that there is a stable and unique vacuum state:

$$|0\rangle = |0; \text{in}\rangle = |0; \text{out}\rangle. \tag{4.10}$$

Mathematically speaking, the $a^\dagger_{\text{in}}$'s and $a^\dagger_{\text{out}}$'s generate two different bases of the Fock space. Since the physics that we want to describe must be independent of the choice of basis, expectation values expressed in terms of "in" and "out" operators and states must satisfy

$$\langle \text{in}|\,\phi_{\text{in}}(x)\,|\text{in}\rangle = \langle \text{out}|\,\phi_{\text{out}}(x)\,|\text{out}\rangle\,. \tag{4.11}$$

Here $|\text{in}\rangle$ and $|\text{out}\rangle$ denote generic "in" and "out" states. We can relate the two bases by introducing a unitary operator S such that

$$\phi_{\text{in}}(x) = S\,\phi_{\text{out}}(x)\,S^\dagger \tag{4.12}$$

$$|\text{in}\rangle = S\;|\text{out}\rangle\,, \quad |\text{out}\rangle = S^\dagger\;|\text{in}\rangle\,, \quad S^\dagger S = 1. \tag{4.13}$$

S is called the <u>S-matrix</u> or S-operator. Note that the plane wave solutions of ϕ_{in} and ϕ_{out} also imply that

$$a^\dagger_{\text{in}} = S\,a^\dagger_{\text{out}}\,S^\dagger, \qquad \hat{a}_{\text{in}} = S\,\hat{a}_{\text{out}}\,S^\dagger. \tag{4.14}$$

By comparing "in" with "out" states one can extract information about the interaction – this is the very essence of detector experiments, where one tries to infer the nature of the interaction by studying the products of the scattering of particles that have been collided with known energies. As we will see below, this information is contained in the elements of the S-matrix.

By contrast, in the absence of any interaction, i.e. for $\mathcal{L}_{\text{int}} = 0$ the distinction between ϕ_{in} and ϕ_{out} is not necessary. They can thus be identified, and then the relation between different bases of the Fock space becomes trivial, $S = 1$, as one would expect.

What we are ultimately interested in are transition amplitudes between an initial state i of, say, two particles of momenta $\mathbf{p}_1, \mathbf{p}_2$, and a final state f, for instance n particles of unequal momenta. The transition amplitude is then given by

$$\langle f,\,\text{out}|\,i,\,\text{in}\rangle = \langle f,\,\text{out}|\,S\,|i,\,\text{out}\rangle = \langle f,\,\text{in}|\,S\,|i,\,\text{in}\rangle \equiv S_{\text{fi}}. \tag{4.15}$$

The S-matrix element S_{fi} therefore describes the transition amplitude for the scattering process in question. The scattering cross section, which is a measurable quantity, is then proportional to $|S_{\text{fi}}|^2$. All information about the scattering is thus encoded in the S-matrix, which must therefore be closely related to the interaction Hamiltonian density $\mathcal{H}_{\text{int}}$. However, before we try to derive the relation between S and $\mathcal{H}_{\text{int}}$ we have to take a slight detour.

4.2 More on time evolution: Dirac picture

The operators $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ which we have encountered are Heisenberg fields and thus time-dependent. The state vectors are time-independent in the sense that they do not satisfy a non-trivial equation of motion. Nevertheless, state vectors in the Heisenberg picture can carry a time label. For instance, the "in"-states of the previous subsection are defined at $t = -\infty$. The relation of the Heisenberg operator $\phi_H(x)$ with its counterpart ϕ_S in the Schrödinger picture is given by

$$\phi_H(\mathbf{x}, t) = \mathrm{e}^{iHt}\,\phi_S\,\mathrm{e}^{-iHt}, \qquad H = H_0 + H_{\text{int}}, \tag{4.16}$$

Note that this relation involves the *full* Hamiltonian $H = H_0 + H_{\text{int}}$ in the interacting theory. We have so far found solutions to the Klein-Gordon equation in the free theory, and so we know how to handle time evolution in this case. However, in the interacting case the Klein-Gordon equation has an extra term,

$$(\Box + m^2)\phi(x) + \frac{\delta V_{\text{int}}(\phi)}{\delta\phi} = 0, \tag{4.17}$$

due to the potential of the interactions. Apart from very special cases of this potential, the equation cannot be solved anymore in closed form, and thus we no longer know the time evolution. It is therefore useful to introduce a new quantum picture for the interacting theory, in which the time dependence is governed by H_0 only. This is the so-called Dirac or Interaction picture. The relation between fields in the Interaction picture, ϕ_I, and in the Schrödinger picture, ϕ_S, is given by

$$\phi_I(\mathbf{x}, t) = e^{iH_0t}\,\phi_S\,e^{-iH_0t}. \tag{4.18}$$

At $t = -\infty$ the interaction vanishes, i.e. $H_{\text{int}} = 0$, and hence the fields in the Interaction and Heisenberg pictures are identical, i.e. $\phi_H(\mathbf{x}, t) = \phi_I(\mathbf{x}, t)$ for $t \to -\infty$. The relation between ϕ_H and ϕ_I can be worked out easily:

$$\begin{aligned}
\phi_H(\mathbf{x}, t) &= e^{iHt}\,\phi_S\,e^{-iHt} \\
&= e^{iHt}\,e^{-iH_0t}\,\underbrace{e^{iH_0t}\phi_S\,e^{-iH_0t}}_{\phi_I(\mathbf{x},t)}\,e^{iH_0t}\,e^{-iHt} \\
&= U^{-1}(t)\,\phi_I(\mathbf{x}, t)\,U(t),
\end{aligned} \tag{4.19}$$

where we have introduced the unitary operator $U(t)$

$$U(t) = e^{iH_0t}\,e^{-iHt}, \qquad U^\dagger U = 1. \tag{4.20}$$

The field $\phi_H(\mathbf{x}, t)$ contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the "in" and "out" field operators, we can now make the following identifications:

$$t \to -\infty \quad : \quad \phi_{\text{in}}(\mathbf{x}, t) = \phi_I(\mathbf{x}, t) = \phi_H(\mathbf{x}, t), \tag{4.21}$$

$$t \to +\infty \quad : \quad \phi_{\text{out}}(\mathbf{x}, t) = \phi_H(\mathbf{x}, t). \tag{4.22}$$

Furthermore, since the fields ϕ_I evolve over time with the free Hamiltonian H_0, they always act in the basis of "in" vectors, such that

$$\phi_{\text{in}}(\mathbf{x}, t) = \phi_I(\mathbf{x}, t), \qquad -\infty < t < \infty. \tag{4.23}$$

The relation between ϕ_I and ϕ_H at any time t is given by

$$\phi_I(\mathbf{x}, t) = U(t)\,\phi_H(\mathbf{x}, t)\,U^{-1}(t). \tag{4.24}$$

As $t \to \infty$ the identifications of eqs. (4.22) and (4.23) yield

$$\phi_{\text{in}} = U(\infty)\,\phi_{\text{out}}\,U^\dagger(\infty). \tag{4.25}$$

From the definition of the S-matrix, Eq. (4.12) we then read off that

$$\lim_{t\to\infty} U(t) = S. \tag{4.26}$$

We have thus derived a formal expression for the S-matrix in terms of the operator $U(t)$, which tells us how operators and state vectors deviate from the free theory at time t, measured relative to $t_0 = -\infty$, i.e. long before the interaction process.

An important boundary condition for $U(t)$ is

$$\lim_{t\to-\infty} U(t) = 1. \tag{4.27}$$

What we mean here is the following: the operator U actually describes the evolution relative to some initial time t_0, which we will normally suppress, i.e. we write $U(t)$ instead of $U(t, t_0)$. We regard t_0 merely as a time label and fix it at $-\infty$, where the interaction vanishes. Equation (4.27) then simply states that U becomes unity as $t \to t_0$, which means that in this limit there is no distinction between Heisenberg and Dirac fields.

Using the definition of $U(t)$, Eq. (4.20), it is an easy exercise to derive the equation of motion for $U(t)$:

$$i\frac{d}{dt}U(t) = H_{\rm int}(t)\,U(t), \qquad H_{\rm int}(t) = \mathrm{e}^{iH_0t}\,H_{\rm int}\,\mathrm{e}^{-iH_0t}. \tag{4.28}$$

The time-dependent operator $H_{\rm int}(t)$ is defined in the interaction picture, and depends on the fields $\phi_{\rm in}$, $\pi_{\rm in}$ in the "in" basis. Let us now solve the equation of motion for $U(t)$ with the boundary condition $\lim_{t\to-\infty} U(t) = 1$. Integrating Eq. (4.28) gives

$$\begin{aligned}
\int_{-\infty}^{t} \frac{d}{dt_1}U(t_1)\,dt_1 &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
U(t) - U(-\infty) &= -i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1 \\
\Rightarrow\ U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\,U(t_1)\,dt_1.
\end{aligned} \tag{4.29}$$

The rhs. still depends on U, but we can substitute our new expression for $U(t)$ into the integrand, which gives

$$\begin{aligned}
U(t) &= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)\left\{1 - i\int_{-\infty}^{t_1} H_{\rm int}(t_2)\,U(t_2)\,dt_2\right\}dt_1 \\
&= 1 - i\int_{-\infty}^{t} H_{\rm int}(t_1)dt_1 - \int_{-\infty}^{t} dt_1\,H_{\rm int}(t_1)\int_{-\infty}^{t_1} dt_2\,H_{\rm int}(t_2)\,U(t_2),
\end{aligned} \tag{4.30}$$

where $t_2 < t_1 < t$. This procedure can be iterated further, so that the nth term in the sum is

$$(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\,H_{\rm int}(t_1)\,H_{\rm int}(t_2)\cdots H_{\rm int}(t_n). \tag{4.31}$$

This iterative solution could be written in much more compact form, were it not for the fact that the upper integration bounds were all different, and that the ordering $t_n <$

$t_{n-1} < \ldots < t_1 < t$ had to be obeyed. Time ordering is an important issue, since one has to ensure that the interaction Hamiltonians act at the proper time, thereby ensuring the causality of the theory. By introducing the time-ordered product of operators, one can use a compact notation, such that the resulting expressions still obey causality. The time-ordered product of two fields $\phi(t_1)$ and $\phi(t_2)$ is defined as

$$\begin{aligned} T\{\phi(t_1)\,\phi(t_2)\} &= \begin{cases} \phi(t_1)\phi(t_2) & t_1 > t_2 \\ \phi(t_2)\phi(t_1) & t_1 < t_2 \end{cases} \\ &\equiv \theta(t_1 - t_2)\,\phi(t_1)\phi(t_2) + \theta(t_2 - t_1)\,\phi(t_2)\phi(t_1), \end{aligned} \tag{4.32}$$

where θ denotes the step function. The generalisation to products of n operators is obvious. Using time ordering for the nth term of Eq. (4.31) we obtain

$$\frac{(-i)^n}{n!} \prod_{i=1}^{n} \int_{-\infty}^{t} dt_i \; T\{H_{\text{int}}(t_1)\, H_{\text{int}}(t_2) \cdots H_{\text{int}}(t_n)\}\,, \tag{4.33}$$

and since this looks like the nth term in the series expansion of an exponential, we can finally rewrite the solution for $U(t)$ in compact form as

$$U(t) = T\,\exp\left\{-i \int_{-\infty}^{t} H_{\text{int}}(t')\,dt'\right\}, \tag{4.34}$$

where the "T" in front ensures the correct time ordering.

4.3 S-matrix and Green's functions

The S-matrix, which relates the "in" and "out" fields before and after the scattering process, can be written as

$$S = 1 + iT, \tag{4.35}$$

where T is commonly called the T-matrix. The fact that S contains the unit operator means that also the case where none of the particles scatter is encoded in S. On the other hand, the non-trivial case is described by the T-matrix, and this is what we are interested in. However, the S-matrix is not easily usable for practical calculations. As it stands now, it is a rather abstract concept, and we still have to relate it to the field operators appearing in our Lagrangian. This is achieved by establishing a general relation between S-matrix elements and n-point Green's functions,

$$G^n(x_1, \ldots x_n) = \langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle. \tag{4.36}$$

Once this step is completed, then for any given Lagrange density we may compute the Green's functions of the fields, which will in turn give us the S-matrix elements providing the link to experiment. In order to achieve this, we have to express the "in/out"-states in terms of creation operators $a^{\dagger}_{\text{in/out}}$ and the vacuum, then express the creation operators by the fields $\phi_{\text{in/out}}$, and finally use the time evolution to connect those with the fields ϕ in our Lagrangian.

Let us consider again the scattering process depicted in Fig. 4. The S-matrix element in this case is

$$\begin{aligned} S_{\text{fi}} &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n; \text{out}\Big|\mathbf{p}_1, \mathbf{p}_2; \text{in}\Big\rangle \\ &= \Big\langle \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n; \text{out}\Big|a^\dagger_{\text{in}}(\mathbf{p}_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle, \end{aligned} \tag{4.37}$$

where $a^\dagger_{\text{in}}$ is the creation operator pertaining to the "in" field ϕ_{in}. Our task is now to express $a^\dagger_{\text{in}}$ in terms of ϕ_{in}, and repeat this procedure for all other momenta labelling our Fock states.

The following identities will prove useful

$$\begin{aligned} a^\dagger(\mathbf{p}) &= i\int d^3x \left\{\left(\partial_0\, \mathrm{e}^{-iq\cdot x}\right)\phi(x) - \mathrm{e}^{-iq\cdot x}\left(\partial_0\phi(x)\right)\right\} \\ &\equiv -i\int d^3x\, \mathrm{e}^{-iq\cdot x}\, \overleftrightarrow{\partial_0}\, \phi(x), \end{aligned} \tag{4.38}$$

$$\begin{aligned} \hat{a}(\mathbf{p}) &\doteq -i\int d^3x \left\{\left(\partial_0\, \mathrm{e}^{iq\cdot x}\right)\phi(x) - \mathrm{e}^{iq\cdot x}\left(\partial_0\phi(x)\right)\right\} \\ &\equiv i\int d^3x\, \mathrm{e}^{iq\cdot x}\, \overleftrightarrow{\partial_0}\, \phi(x). \end{aligned} \tag{4.39}$$

The S-matrix element can then be rewritten as

$$\begin{aligned} S_{\text{fi}} &= -i\int d^3x_1\, \mathrm{e}^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial_0}\, \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|\phi_{\text{in}}(x_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle \\ &= -i\lim_{t_1\to-\infty}\int d^3x_1\, \mathrm{e}^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial_0}\, \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|\phi(x_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle, \end{aligned} \tag{4.40}$$

where in the last line we have used Eq. (4.4) to replace ϕ_{in} by ϕ. We can now rewrite $\lim_{t_1\to-\infty}$ using the following identity, which holds for an arbitrary, differentiable function $f(t)$, whose limit $t\to\pm\infty$ exists:

$$\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) - \int_{-\infty}^{+\infty} \frac{df}{dt}\, dt. \tag{4.41}$$

The S-matrix element then reads

$$\begin{aligned} S_{\text{fi}} &= -i\lim_{t_1\to+\infty}\int d^3x_1\, \mathrm{e}^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial_0}\, \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|\phi(x_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle \\ &\quad +i\int_{-\infty}^{+\infty} dt_1\, \frac{\partial}{\partial t_1}\left\{\int d^3x_1\, \mathrm{e}^{-ip_1\cdot x_1}\, \overleftrightarrow{\partial_0}\, \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|\phi(x_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle\right\}. \end{aligned} \tag{4.42}$$

The first term in this expression involves $\lim_{t_1\to+\infty}\phi = \phi_{\text{out}}$, which gives rise to a contribution

$$\propto \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|a^\dagger_{\text{out}}(\mathbf{p}_1)\Big|\mathbf{p}_2; \text{in}\Big\rangle. \tag{4.43}$$

This is non-zero only if $\mathbf{p}_1$ is equal to one of $\mathbf{k}_1, \ldots, \mathbf{k}_n$. This, however, means that the particle with momentum $\mathbf{p}_1$ does not scatter, and hence the first term does not contribute to the T-matrix of Eq. (4.35). We are then left with the following expression for S_{fi}:

$$S_{\text{fi}} = -i\int d^4x_1 \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out}\Big|\partial_0\left\{\left(\partial_0 \mathrm{e}^{-ip_1\cdot x_1}\right)\phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1}\left(\partial_0\phi(x_1)\right)\right\}\Big|\mathbf{p}_2; \text{in}\Big\rangle. \tag{4.44}$$

The time derivatives in the integrand can be worked out:

$$\begin{aligned}
&\partial_0 \left\{ \left(\partial_0 \mathrm{e}^{-ip_1\cdot x_1}\right) \phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1} \left(\partial_0 \phi(x_1)\right) \right\} \\
&= - \left[E(\mathbf{p}_1)\right]^2 \mathrm{e}^{-ip_1\cdot x_1}\, \phi(x_1) - \mathrm{e}^{-ip_1\cdot x_1}\, \partial_0^2 \phi(x_1) \\
&= - \left\{ \left(\left(-\nabla^2 + m^2\right) \mathrm{e}^{-ip_1\cdot x_1} \right) \phi(x_1) + \mathrm{e}^{-ip_1\cdot x_1}\, \partial_0^2\, \phi(x_1) \right\},
\end{aligned} \tag{4.45}$$

where we have used that $-\nabla^2 \mathrm{e}^{-ip_1\cdot x_1} = \mathbf{p}_1^2\, \mathrm{e}^{-ip_1\cdot x_1}$. For the S-matrix element one obtains

$$\begin{aligned}
S_{\mathrm{fi}} &= i \int d^4x_1\, \mathrm{e}^{-ip_1\cdot x_1} \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \mathrm{out} \Big| \left(\partial_0^2 - \nabla^2 + m^2\right) \phi(x_1) \Big| \mathbf{p}_2; \mathrm{in} \Big\rangle \\
&= i \int d^4x_1\, \mathrm{e}^{-ip_1\cdot x_1} \left(\Box_{x_1} + m^2\right) \Big\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \mathrm{out} \Big| \phi(x_1) \Big| \mathbf{p}_2; \mathrm{in} \Big\rangle.
\end{aligned} \tag{4.46}$$

What we have obtained after this rather lengthy step of algebra is an expression in which the field operator is sandwiched between Fock states, one of which has been reduced to a one-particle state. We can now successively eliminate all momentum variables from the Fock states, by repeating the procedure for the momentum $\mathbf{p}_2$, as well as the n momenta of the "out" state. The final expression for S_{fi} is

$$\begin{aligned}
S_{\mathrm{fi}} = {} & (i)^{n+2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \cdots \int d^4y_n\, \mathrm{e}^{(-ip_1\cdot x_1 - ip_2\cdot x_2 + ik_1\cdot y_1 + \cdots + k_n\cdot y_n)} \\
& \times \left(\Box_{x_1} + m^2\right) \left(\Box_{x_2} + m^2\right) \left(\Box_{y_1} + m^2\right) \cdots \left(\Box_{y_n} + m^2\right) \\
& \times \Big\langle 0; \mathrm{out} \Big| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \Big| 0; \mathrm{in} \Big\rangle,
\end{aligned} \tag{4.47}$$

where the time-ordering inside the vacuum expectation value (VEV) ensures that causality is obeyed. The above expression is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. It relates the formal definition of the scattering amplitude to a vacuum expectation value of time-ordered fields. Since the vacuum is uniquely the same for "in/out", the VEV in the LSZ formula for the scattering of two initial particles into n particles in the final state is recognised as the $(n+2)$-point Green's function:

$$G_{n+2}(y_1, y_2, \ldots, y_n, x_1, x_2) = \Big\langle 0 \Big| T\{\phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2)\} \Big| 0 \Big\rangle. \tag{4.48}$$

You will note that we still have not calculated or evaluated anything, but merely rewritten the expression for the scattering matrix elements. Nevertheless, the LSZ formula is of tremendous importance and a central piece of QFT. It provides the link between fields in the Lagrangian and the scattering amplitude S_{fi}^2, which yields the cross section, measurable in an experiment. Up to here no assumptions or approximations have been made, so this connection between physics and formalism is rather tight. It also illustrates a profound phenomenon of QFT and particle physics: the scattering properties of particles, in other words their interactions, are encoded in the vacuum structure, i.e. the vacuum is non-trivial!

4.4 How to compute Green's functions

Of course, in order to calculate cross sections, we need to compute the Green's functions. Alas, for any physically interesting and interacting theory this cannot be done exactly, contrary to the free theory discussed earlier. Instead, approximation methods have to be used in order to simplify the calculation, while hopefully still giving reliable results. Or one reformulates the entire QFT as a lattice field theory, which in principle allows to compute Green's functions without any approximations (in practice this still turns out to be a difficult task for physically relevant systems). This is what many theorists do for a living. But the formalism stands, and if there are discrepancies between theory and experiments, one "only" needs to check the accuracy with which the Green's functions have been calculated or measured, before approving or discarding a particular Lagrangian.

In the next section we shall discuss how to compute the Green's function of scalar field theory in perturbation theory. Before we can tackle the actual computation, we must take a further step. Let us consider the n-point Green's function

$$G_n(x_1,\ldots,x_n) = \langle 0\,|T\{\phi(x_1)\cdots\phi(x_n)\}|\,0\rangle\,. \tag{4.49}$$

The fields ϕ which appear in this expression are Heisenberg fields, whose time evolution is governed by the full Hamiltonian $H_0 + H_{\text{int}}$. In particular, the ϕ's are *not* the ϕ_{in}'s. We know how to handle the latter, because they correspond to a free field theory, but not the former, whose time evolution is governed by the interacting theory, whose solutions we do not know. Let us thus start to isolate the dependence of the fields on the interaction Hamiltonian. Recall the relation between the Heisenberg fields $\phi(t)$ and the "in"-fields[2]

$$\phi(t) = U^{-1}(t)\,\phi_{\text{in}}(t)\,U(t). \tag{4.50}$$

We now assume that the fields are properly time-ordered, i.e. $t_1 > t_2 > \ldots > t_n$, so that we can forget about writing $T(\cdots)$ everywhere. After inserting Eq. (4.50) into the definition of G_n one obtains

$$\begin{aligned} G_n = \langle 0|U^{-1}(t_1)\phi_{\text{in}}(t_1)U(t_1)\,U^{-1}(t_2)\phi_{\text{in}}(t_2)U(t_2)\cdots \\ \times\,U^{-1}(t_n)\phi_{\text{in}}(t_n)U(t_n)|0\rangle. \end{aligned} \tag{4.51}$$

Now we introduce another time label t such that $t \gg t_1$ and $-t \ll t_1$. For the n-point function we now obtain

$$\begin{aligned} G_n = \Big\langle 0\Big|U^{-1}(t)\Big\{U(t)U^{-1}(t_1)\phi_{\text{in}}(t_1)U(t_1)\,U^{-1}(t_2)\phi_{\text{in}}(t_2)U(t_2)\cdots \\ \times\,U^{-1}(t_n)\phi_{\text{in}}(t_n)U(t_n)U^{-1}(-t)\Big\}U(-t)\Big|0\Big\rangle. \end{aligned} \tag{4.52}$$

The expression in curly braces is now time-ordered by construction. An important observation at this point is that it involves pairs of U and its inverse, for instance

$$U(t)U^{-1}(t_1) \equiv U(t,t_1). \tag{4.53}$$

One can easily convince oneself that $U(t,t_1)$ provides the net time evolution from t_1 to t. We can now write G_n as

$$G_n = \Big\langle 0\Big|U^{-1}(t)\,T\Big\{\phi_{\text{in}}(t_1)\cdots\phi_{\text{in}}(t_n)\,\underbrace{U(t,t_1)\,U(t_1,t_2)\cdots U(t_n,-t)}_{U(t,-t)}\Big\}U(-t)\Big|0\Big\rangle. \tag{4.54}$$

[2]Here and in the following we suppress the spatial argument of the fields for the sake of brevity.

Let us now take $t \to \infty$. The relation between $U(t)$ and the S-matrix Eq. (4.26), as well as the boundary condition Eq. (4.27) tell us that

$$\lim_{t\to\infty} U(-t) = 1, \qquad \lim_{t\to\infty} U(t,-t) = S, \tag{4.55}$$

which can be inserted into the above expression. We still have to work out the meaning of $\langle 0|U^{-1}(\infty)$ in the expression for G_n. In a paper by Gell-Mann and Low it was argued that the time evolution operator must leave the vacuum invariant (up to a phase), which justifies the ansatz

$$\langle 0|U^{-1}(\infty) = K\langle 0|, \tag{4.56}$$

with K being the phase. Multiplying this relation with $|0\rangle$ from the right gives

$$\langle 0|U^{-1}(\infty)|0\rangle = K\langle 0|0\rangle = K. \tag{4.57}$$

Furthermore, Gell-Mann and Low showed that

$$\langle 0|U^{-1}(\infty)|0\rangle = \frac{1}{\langle 0|U(\infty)|0\rangle}, \tag{4.58}$$

which implies

$$K = \frac{1}{\langle 0|S|0\rangle}. \tag{4.59}$$

After inserting all these relations into the expression for G_n we obtain

$$G_n(x_1,\ldots,x_n) = \frac{\langle 0|\,T\{\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\,S\}\,|0\rangle}{\langle 0|S|0\rangle}. \tag{4.60}$$

The S-matrix is given by

$$S = T\,\exp\left\{-i\int_{-\infty}^{+\infty} H_{\rm int}(t)\,dt\right\}, \qquad H_{\rm int} = H_{\rm int}(\phi_{\rm in},\pi_{\rm in}), \tag{4.61}$$

and thus we have finally succeeded in expressing the n-point Green's function exclusively in terms of the "in"-fields. This completes the derivation of a relation between the general definition of the scattering amplitude $S_{\rm fi}$ and the VEV of time-ordered "in"-fields. The link between the scattering amplitude and the underlying field theory is provided by the n-point Green's function.

Problems

4.1 Using the definition $U(t) = e^{iH_0t}\,e^{-iHt}$, derive the evolution equation for $U(t)$:

$$i\frac{d}{dt}U(t) = H_{\rm int}(t)\,U(t),$$

where

$$H_{\rm int}(t) = e^{iH_0t}\,H_{\rm int}\,e^{-iH_0t}.$$

4.2 Given that $\phi_{\rm in}$ is a free field, obeying the Heisenberg equation of motion

$$i\dot{\phi}_{\rm in} = [H_0(\phi_{\rm in}, \pi_{\rm in}),\, \phi_{\rm in}]\,,$$

show that $\phi_{\rm out}$ is also a free field, which obeys

$$i\dot{\phi}_{\rm out} = [H_0(\phi_{\rm out}, \pi_{\rm out}),\, \phi_{\rm out}]\,.$$

[**Hint:** use $\phi_{\rm out} = S^\dagger \phi_{\rm in} S$ and $\pi_{\rm out} = S^\dagger \pi_{\rm in} S$. Keep in mind that the S-matrix has no explicit time dependence.]

5 Perturbation Theory

In this section we are going to calculate the Green's functions of scalar quantum field theory explicitly. We will specify the interaction Lagrangian in detail and use an approximation known as perturbation theory. At the end we will derive a set of rules, which represent a systematic prescription for the calculation of Green's functions, and can be easily generalised to apply to other, more complicated field theories. These are the famous Feynman rules.

We start by making a definite choice for the interaction Lagrangian $\mathcal{L}_{\rm int}$. Although one may think of many different expressions for $\mathcal{L}_{\rm int}$, one has to obey some basic principles: firstly, $\mathcal{L}_{\rm int}$ must be chosen such that the potential it generates is bounded from below – otherwise the system has no ground state. Secondly, our interacting theory should be renormalisable. Despite being of great importance, the second issue will not be addressed in these lectures. The requirement of renormalisability arises because the non-trivial vacuum, much like a medium, interacts with particles to modify their properties. Moreover, if one computes quantities like the energy or charge of a particle, one typically obtains a divergent result[3]. There are classes of quantum field theories, called renormalisable, in which these divergences can be removed by suitable redefinitions of the fields and the parameters (masses and coupling constants).

For our theory of a real scalar field in four space-time dimensions, it turns out that the only interaction term which leads to a renormalisable theory must be quartic in the fields. Thus we choose

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4!}\phi^4(x), \tag{5.1}$$

where the coupling constant λ describes the strength of the interaction between the scalar fields, much like, say, the electric charge describing the strength of the interaction between photons and electrons. The full Lagrangian of the theory then reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{5.2}$$

[3]This is despite the subtraction of the vacuum energy discussed earlier.

and the explicit expressions for the interaction Hamiltonian and the S-matrix are

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}, \quad H_{\text{int}} = \frac{\lambda}{4!}\int d^3x\, \phi_{\text{in}}^4(\mathbf{x},t)$$
$$S = T\,\exp\left\{-i\frac{\lambda}{4!}\int d^4x\, \phi_{\text{in}}^4(x)\right\}. \tag{5.3}$$

The n-point Green's function is

$$G_n(x_1,\ldots,x_n) = \frac{\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r \frac{1}{r!}\left\langle 0\left|T\left\{\phi_{\text{in}}(x_1)\cdots\phi_{\text{in}}(x_n)\left(\int d^4y\,\phi_{\text{in}}^4(y)\right)^r\right\}\right|0\right\rangle}{\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r \frac{1}{r!}\left\langle 0\left|T\left(\int d^4y\,\phi_{\text{in}}^4(y)\right)^r\right|0\right\rangle}. \tag{5.4}$$

This expression cannot be dealt with as it stands. In order to evaluate it we must expand G_n in powers of the coupling λ and truncate the series after a finite number of terms. This only makes sense if λ is sufficiently small. In other words, the interaction Lagrangian must act as a small perturbation on the system. As a consequence, the procedure of expanding Green's functions in powers of the coupling is referred to as perturbation theory.

5.1 Wick's Theorem

The n-point Green's function in Eq. (5.4) involves the time-ordered product over at least n fields. There is a method to express VEV's of n fields, i.e. $\langle 0|T\{\phi_{\text{in}}(x_1)\cdots\phi_{\text{in}}(x_n)\}|0\rangle$ in terms of VEV's involving two fields only. This is known as Wick's theorem.

Let us for the moment ignore the subscript "in" and return to the definition of normal-ordered fields. The normal-ordered product $:\phi(x_1)\phi(x_2):$ differs from $\phi(x_1)\phi(x_2)$ by the vacuum expectation value, i.e.

$$\phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2): + \langle 0|\phi(x_1)\phi(x_2)|0\rangle. \tag{5.5}$$

We are now going to combine normal-ordered products with time ordering. The time-ordered product $T\{\phi(x_1)\phi(x_2)\}$ is given by

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\} &= \phi(x_1)\phi(x_2)\theta(t_1-t_2) + \phi(x_2)\phi(x_1)\theta(t_2-t_1) \\ &= :\phi(x_1)\phi(x_2):\,\Big(\theta(t_1-t_2)+\theta(t_2-t_1)\Big) \\ &\quad +\langle 0|\phi(x_1)\phi(x_2)\theta(t_1-t_2) + \phi(x_2)\phi(x_1)\theta(t_2-t_1)|0\rangle. \end{aligned} \tag{5.6}$$

Here we have used the important observation that

$$:\phi(x_1)\phi(x_2): = :\phi(x_2)\phi(x_1):, \tag{5.7}$$

which means that normal-ordered products of fields are automatically time-ordered.[4] Equation (5.6) is Wick's theorem for the case of two fields:

$$T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2): + \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle. \tag{5.8}$$

[4] The reverse is, however, not true!

For the case of three fields, Wick's theorem yields

$$
\begin{aligned}
T\{\phi(x_1)\phi(x_2)\phi(x_3)\} &= \;:\phi(x_1)\phi(x_2)\phi(x_3): + :\phi(x_1):\langle 0|T\{\phi(x_2)\phi(x_3)\}|0\rangle \\
&\quad + :\phi(x_2):\langle 0|T\{\phi(x_1)\phi(x_3)\}|0\rangle + :\phi(x_3):\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \qquad (5.9)
\end{aligned}
$$

At this point the general pattern becomes clear: any time-ordered product of fields is equal to its normal-ordered version plus terms in which pairs of fields are removed from the normal-ordered product and sandwiched between the vacuum to form 2-point functions. Then one sums over all permutations. Without proof we give the expression for the general case of n fields (n even):

$$
\begin{aligned}
&T\{\phi(x_1)\cdots\phi(x_n)\} = \\
&\qquad :\phi(x_1)\cdots\phi(x_n): \\
&\qquad + :\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\phi(x_n): \langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle + \text{perms.} \\
&\qquad + :\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\widehat{\phi(x_j)}\cdots\widehat{\phi(x_k)}\cdots\widehat{\phi(x_l)}\cdots\phi(x_n): \\
&\qquad\quad \times\ \langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle\langle 0|T\{\phi(x_k)\phi(x_l)\}|0\rangle + \text{perms.} \\
&\qquad + \ldots + \\
&\qquad + \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle\langle 0|T\{\phi(x_3)\phi(x_4)\}|0\rangle\cdots\langle 0|T\{\phi(x_{n-1})\phi(x_n)\}|0\rangle \\
&\qquad\quad + \text{perms..} \qquad (5.10)
\end{aligned}
$$

The symbol $\widehat{\phi(x_i)}$ indicates that $\phi(x_i)$ has been removed from the normal-ordered product.

Let us now go back to $\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$. If we insert Wick's theorem, then we find that only the contribution in the last line of Eq. (5.10) survives: by definition the VEV of a normal-ordered product of fields vanishes, and it is precisely the last line of Wick's theorem in which no normal-ordered products are left. The only surviving contribution is that in which all fields have been paired or "contracted". Sometimes a contraction is represented by the notation:

$$
\underbrace{\phi(x_i)\phi(x_j)} \equiv \langle 0|T\{\phi(x_i)\phi(x_j)\}|0\rangle, \qquad (5.11)
$$

i.e. the pair of fields which is contracted is joined by the braces. Wick's theorem can now be rephrased as

$$
\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle \;=\; \text{sum of all possible contractions of } n \text{ fields.} \qquad (5.12)
$$

Let us look at a few examples. The first is the 4-point function

$$
\begin{aligned}
\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle &= \underbrace{\phi(x_1)\phi(x_2)}\underbrace{\phi(x_3)\phi(x_4)} \\
&+ \underbrace{\phi(x_1)\overbrace{\phi(x_2)\phi(x_3)}}\phi(x_4) + \phi(x_1)\underbrace{\phi(x_2)\overbrace{\phi(x_3)\phi(x_4)}} \qquad (5.13)
\end{aligned}
$$

The second example is again a 4-point function, where two of the fields are also normal-ordered:

$$
\begin{aligned}
\langle 0|T\{\phi(x_1)\phi(x_2):\phi(x_3)\phi(x_4):\}|0\rangle &= \underbrace{\phi(x_1)\phi(x_2)}:\underbrace{\phi(x_3)\phi(x_4)}: \\
&+ \underbrace{\phi(x_1)\overbrace{\phi(x_2):\phi(x_3)}}\phi(x_4): + \phi(x_1)\underbrace{\phi(x_2)\overbrace{:\phi(x_3)\phi(x_4)}}: \qquad (5.14)
\end{aligned}
$$

In this example, though, the contraction of $:\phi(x_3)\phi(x_4):$ vanishes by construction, so only the last two terms survive! As a general rule, contractions which only involve fields inside a normal-ordered product vanish. Such contractions contribute only to the vacuum. Normal ordering can therefore simplify the calculation of Green's functions quite considerably, as we shall see explicitly below.

5.2 The Feynman propagator

Using Wick's Theorem one can relate any n-point Green's functions to an expression involving only 2-point functions. Let us have a closer look at

$$G_2(x,y) = \langle 0|T\{\phi_{\rm in}(x)\phi_{\rm in}(y)\}|0\rangle. \tag{5.15}$$

We can now insert the solution for ϕ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we assume $t_x > t_y$ then $G_2(x,y)$ can be written as

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})} \\ &\quad \times \left\langle 0 \left| \left(\hat{a}^\dagger(\mathbf{p})\, e^{ip\cdot x} + \hat{a}(\mathbf{p})\, e^{-ip\cdot x}\right) \left(\hat{a}^\dagger(\mathbf{q})\, e^{iq\cdot y} + \hat{a}(\mathbf{q})\, e^{-iq\cdot y}\right) \right| 0 \right\rangle \\ &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, e^{-ip\cdot x + iq\cdot y} \left\langle 0 \left| \hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q}) \right| 0 \right\rangle. \end{aligned} \tag{5.16}$$

This shows that G_2 can be interpreted as the amplitude for a meson which is created at y and destroyed again at point x. We can now replace $\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})$ by its commutator:

$$\begin{aligned} G_2(x,y) &= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 4E(\mathbf{p})E(\mathbf{q})}\, e^{-ip\cdot x + iq\cdot y} \left\langle 0 \left| \left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})\right] \right| 0 \right\rangle \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, e^{-ip\cdot(x-y)}, \end{aligned} \tag{5.17}$$

and the general result, after restoring time-ordering, reads

$$G_2(x,y) = \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})} \left(e^{-ip\cdot(x-y)}\theta(t_x - t_y) + e^{ip\cdot(x-y)}\theta(t_y - t_x)\right). \tag{5.18}$$

Furthermore, using contour integration one can show that this expression can be rewritten as a 4-dimensional integral

$$G_2(x,y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{5.19}$$

where ϵ is a small parameter which ensures that G_2 does not develop a pole. This calculation has established that $G_2(x,y)$ actually depends only on the difference $(x-y)$. Equation (5.19) is called the Feynman propagator $G_F(x-y)$:

$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}. \tag{5.20}$$

The Feynman propagator is a Green's function of the Klein-Gordon equation, i.e. it satisfies

$$\left(\Box_x + m^2\right) G_F(x-y) = -i\delta^4(x-y), \tag{5.21}$$

and describes the propagation of a meson between the space-time points x and y.

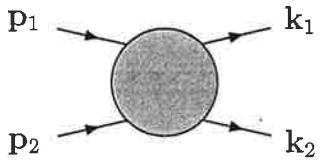


Figure 5: Scattering of two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ into 2 particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

5.3 Two-particle scattering to $O(\lambda)$

Let us now consider a scattering process in which two incoming particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter into two outgoing ones with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, as shown in Fig. 5. The S-matrix element in this case is

$$\begin{aligned} S_{\rm fi} &= \langle \mathbf{k}_1, \mathbf{k}_2; \text{out}|\mathbf{p}_1, \mathbf{p}_2; \text{in}\rangle \\ &= \langle \mathbf{k}_1, \mathbf{k}_2; \text{in}|S|\mathbf{p}_1, \mathbf{p}_2; \text{in}\rangle, \end{aligned} \tag{5.22}$$

and $S = 1 + iT$. The LSZ formula Eq. (4.47) tells us that we must compute G_4 in order to obtain $S_{\rm fi}$. Let us work out G_4 in powers of λ using Wick's theorem. To make life simpler, we shall introduce normal ordering into the definition of S, i.e.

$$S = T\,\exp\left\{-i\frac{\lambda}{4!}\int d^4x\, :\phi_{\rm in}^4(x):\right\} \tag{5.23}$$

Suppressing the subscripts "in" from now on, the expression we have to evaluate order by order in λ is

$$G_n(x_1,\ldots,x_n) \tag{5.24}$$
$$= \frac{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\left(\int d^4y\, :\phi^4(y):\right)^r\right\}\right|0\right\rangle}{\displaystyle\sum_{r=0}^{\infty}\left(-\frac{i\lambda}{4!}\right)^r\frac{1}{r!}\left\langle 0\left|T\left(\int d^4y\, :\phi^4(y):\right)^r\right|0\right\rangle}.$$

Starting with the denominator, we note that for $r = 0$ one finds

$$r = 0: \qquad \text{denominator} = 1. \tag{5.25}$$

If $r = 1$, then the expression in the denominator only involves fields which are normal-ordered. Following the discussion at the end of section 5.1 we conclude that these contributions must vanish, hence

$$r = 1: \qquad \text{denominator} = 0. \tag{5.26}$$

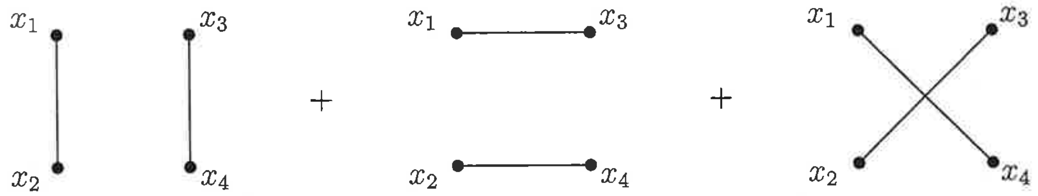
The contribution for $r = 2$, however, is non-zero. But then the case of $r = 2$ corresponds already to $O(\lambda^2)$, which is higher than the order which we are working to. Therefore

$$\text{denominator} = 1 \text{ to order } \lambda. \tag{5.27}$$

Turning now to the numerator, we start with $r = 0$ and apply Wick's theorem, which gives

$$
\begin{aligned}
r = 0: \quad & \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle \\
&= G_F(x_1 - x_2)\, G_F(x_3 - x_4) + G_F(x_1 - x_3)\, G_F(x_2 - x_4) \\
&\quad + G_F(x_1 - x_4)\, G_F(x_2 - x_3), \qquad (5.28)
\end{aligned}
$$

which can be graphically represented as



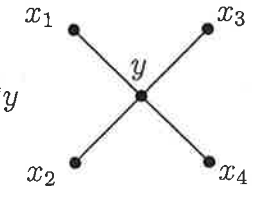
But this is the same answer as if we had set $\lambda = 0$, so $r = 0$ in the numerator does not describe scattering and is hence not a contribution to the T-matrix.

For $r = 1$ in the numerator we have to evaluate

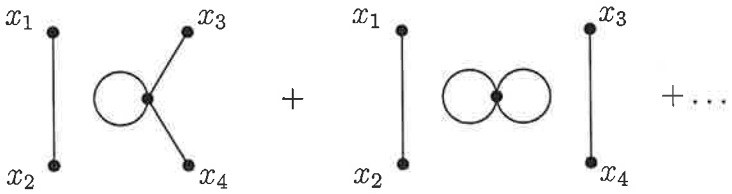
$$
\begin{aligned}
r = 1: \quad & -\frac{i\lambda}{4!}\left\langle 0 \left| T\left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \int d^4y\, \phi^4(y) : \right\} \right| 0 \right\rangle \\
&= -\frac{i\lambda}{4!} \int d^4y\; 4!\; G_F(x_1 - y) G_F(x_2 - y) G_F(x_3 - y) G_F(x_4 - y), \qquad (5.29)
\end{aligned}
$$

where we have taken into account that contractions involving two fields inside $: \cdots :$ vanish. The factor 4! inside the integrand is a combinatorial factor: it is equal to the number of permutations which must be summed over according to Wick's theorem and cancels the 4! in the denominator of the interaction Lagrangian. Graphically this contribution is represented by

$$-i\lambda \int d^4y$$



where the integration over y denotes the sum over all possible locations of the interaction point y. Without normal ordering we would have encountered the following contributions for $r = 1$:



Such contributions are corrections to the vacuum and are *cancelled* by the denominator. This demonstrates how normal ordering simplifies the calculation by automatically subtracting terms which do not contribute to the actual scattering process.

To summarise, the final answer for the scattering amplitude to $O(\lambda)$ is given by Eq. (5.29).

5.4 Graphical representation of the Wick expansion: Feynman rules

We have already encountered the graphical representation of the expansion of Green's functions in perturbation theory after applying Wick's theorem. It is possible to formulate a simple set of rules which allow to draw the graphs directly without using Wick's theorem and to write down the corresponding algebraic expressions.

We again consider a neutral scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5.30}$$

Suppose now that we want to compute the $O(\lambda^m)$ contribution to the n-point Green's function $G_n(x_1, \ldots, x_n)$. This is achieved by going through the following steps:

(1) Draw all distinct diagrams with n external lines and m 4-fold vertices:

- Draw n dots and label them $x_1, \ldots, x_n$ (external points)
- Draw m dots and label them $y_1, \ldots, y_m$ (vertices)
- Join the dots according to the following rules:
  - only one line emanates from each x_i
  - exactly four lines run into each y_j
  - the resulting diagram must be connected, i.e. there must be a continuous path between any two points.

(2) Assign a factor $-\frac{i\lambda}{4!}\int d^4y_i$ to the vertex at y_i

(3) Assign a factor $G_F(x_i - y_j)$ to the line joining x_i and y_j

(4) Multiply by the number of contractions $\mathcal{C}$ from the Wick expansion which lead to the same diagram.

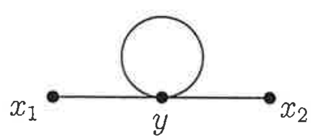
These are the Feynman rules for scalar field theory in position space.

Let us look at an example, namely the 2-point function. According to the Feynman rules the contributions up to order λ^2 are as follows:

$O(1)$:

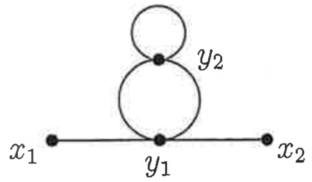
$= G_F(x_1 - x_2)$

$O(\lambda)$:



"tadpole diagram";
(cancelled by normal ordering)

$O(\lambda^2)$:



"tadpole diagram";
(cancelled by normal ordering)

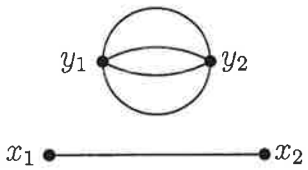
$O(\lambda^2)$:

$$= C\left(-\frac{i\lambda}{4!}\right)^2 \int d^4y_1 d^4y_2\; G_F(x_1 - y_1)\,[G_F(y_1 - y_2)]^3\, G_F(y_2 - x_2)$$

The combinatorial factor for this contribution is worked out as $C = 4 \cdot 4!$. Note that the same graph, but with the positions of y_1 and y_2 interchanged is topologically distinct. Numerically it has the same value as the above graph, and so the corresponding expression has to be multiplied by a factor 2.

Another contribution at order λ^2 is

$O(\lambda^2)$:



vacuum contribution;
not connected

This contribution must be discarded, since not all of the points are connected via a continuous line.

Let us end this discussion with a small remark on the tadpole diagrams encountered above. These contributions to the 2-point function are cancelled if the interaction term is normal-ordered. However, unlike the case of the 4-point function, the corresponding diagrams satisfy the Feynman rules listed above. In particular, the diagrams are connected and are not simply vacuum contributions. They must hence be included in the expression for the 2-point function.

5.5 Feynman rules in momentum space

It is often simpler to work in momentum space, and hence we will discuss the derivation of Feynman rules in this case. If one works in momentum space, the Green's functions are related to those in position space by a Fourier transform

$$G_n(x_1, \ldots, x_n) = \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_n}{(2\pi)^4}\, \mathrm{e}^{ip_1\cdot x_1 + \ldots + ip_n\cdot x_n}\, \widetilde{G}_n(p_1, \ldots, p_n). \tag{5.31}$$

The Feynman rules then serve to compute the Green's function $\widetilde{G}_n(p_1, \ldots, p_n)$ order by order in the coupling.

In every scattering process the overall momentum must be conserved, and hence

$$\sum_{i=1}^{n} p_i = 0. \tag{5.32}$$

This can be incorporated into the definition of the momentum space Green's function one is trying to compute:

$$\widetilde{G}_n(p_1,\ldots,p_n) = (2\pi)^4\delta^4\left(\sum_{i=1}^{n} p_i\right)\,\mathcal{G}_n(p_1,\ldots,p_n). \tag{5.33}$$

Here we won't be concerned with the exact derivation of the momentum space Feynman rules, but only list them as a recipe.

Feynman rules (momentum space)

(1) Draw all <u>distinct</u> diagrams with n external lines and m 4-fold vertices:

- Assign momenta $p_1,\ldots,p_n$ to the external lines
- Assign momenta k_j to the internal lines

(2) Assign to each external line a factor

$$\frac{i}{p_k^2-m^2+i\epsilon}$$

(3) Assign to each internal line a factor

$$\int\frac{d^4k_j}{(2\pi)^4}\,\frac{i}{k_j^2-m^2+i\epsilon}$$

(4) Each vertex contributes a factor

$$-\frac{i\lambda}{4!}(2\pi)^4\delta^4\left(\sum\text{momenta}\right),$$

(the delta function ensures that momentum is conserved at each vertex).

(5) Multiply by the combinatorial factor $\mathcal{C}$, which is the number of contractions leading to the same momentum space diagram (note that $\mathcal{C}$ may be different from the combinatorial factor for the same diagram considered in position space!)

5.6 S-matrix and truncated Green's functions

The final topic in these lectures is the derivation of a simple relation between the S-matrix element and a particular momentum space Green's function, which has its external legs amputated: the so-called truncated Green's function. This further simplifies the calculation of scattering amplitudes using Feynman rules.

Let us return to the LSZ formalism and consider the scattering of m initial particles (momenta $\mathbf{p}_1,\ldots,\mathbf{p}_m$) into n final particles with momenta $\mathbf{k}_1,\ldots,\mathbf{k}_n$. The LSZ formula tells us that the S-matrix element is given by

$$\begin{aligned}
&\Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \mathbf{p}_1, \dots, \mathbf{p}_m; \text{in} \Big\rangle \\
&= (i)^{n+m} \int \prod_{i=1}^{m} d^4x_i \int \prod_{j=1}^{n} d^4y_j \exp\left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\} \\
&\quad \times \prod_{i=1}^{m} \left(\Box_{x_i} + m^2\right) \prod_{j=1}^{n} \left(\Box_{y_j} + m^2\right) G_{n+m}(x_1, \dots, x_m, y_1, \dots, y_n). \qquad (5.34)
\end{aligned}$$

Let us have a closer look at $G_{n+m}(x_1, \dots, x_m, y_1, \dots, y_n)$. As shown in Fig. 6 it can be split into Feynman propagators, which connect the external points to the vertices at $z_1, \dots, z_{n+m}$, and a remaining Green's function $\overline{G}_{n+m}$, according to

$$G_{n+m} = \int d^4z_1 \cdots d^4z_{n+m}\, G_F(x_1 - z_1) \cdots G_F(y_n - z_{n+m})\, \overline{G}_{n+m}(z_1, \dots, z_{n+m}), \qquad (5.35)$$

where, perhaps for obvious reasons, $\overline{G}_{n+m}$ is called the truncated Green's function.

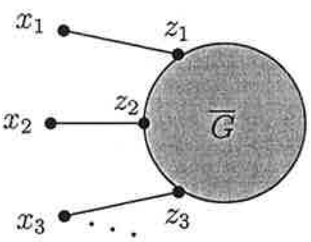


Figure 6: The construction of the truncated Green's function in position space.

Putting Eq. (5.35) back into the LSZ expression for the S-matrix element, and using that

$$\left(\Box_{x_i} + m^2\right) G_F(x_i - z_i) = -i\delta^4(x_i - z_i) \qquad (5.36)$$

one obtains

$$\begin{aligned}
&\Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \mathbf{p}_1, \dots, \mathbf{p}_m; \text{in} \Big\rangle \\
&= (i)^{n+m} \int \prod_{i=1}^{m} d^4x_i \int \prod_{j=1}^{n} d^4y_j \exp\left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\} \qquad (5.37) \\
&\quad \times (-i)^{n+m} \int d^4z_1 \cdots d^4z_{n+m}\, \delta^4(x_1 - z_1) \cdots \delta^4(y_n - z_{n+m})\, \overline{G}_{n+m}(z_1, \dots, z_{n+m}).
\end{aligned}$$

After performing all the integrations over the z_k's, the final relation becomes

$$\begin{aligned}
&\Big\langle \mathbf{k}_1, \dots, \mathbf{k}_n; \text{out} \Big| \mathbf{p}_1, \dots, \mathbf{p}_m; \text{in} \Big\rangle \\
&= \int \prod_{i=1}^{m} d^4x_i \prod_{j=1}^{n} d^4y_j \exp\left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\} \\
&\quad \times \overline{G}_{n+m}(x_1, \dots, x_m, y_1, \dots, y_n) \\
&\equiv \overline{\mathcal{G}}_{n+m}(p_1, \dots, p_m, k_1, \dots, k_n), \qquad (5.38)
\end{aligned}$$

where $\overline{\mathcal{G}}_{n+m}$ is the truncated $n+m$-point function in momentum space. This result shows that the scattering matrix element is directly given by the truncated Green's function in momentum space. The latter can be obtained using the Feynman rules without the expression for the external legs.

Problems

5.1 Verify that

$$:\phi(x_1)\phi(x_2):=:\phi(x_2)\phi(x_1):$$

Hint: write $\phi = \phi^+ + \phi^-$, where ϕ^+ and ϕ^- are creation and annihilation components of ϕ.

5.2 Verify that

$$G_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

is a Green's function of $(\partial^\mu\partial_\mu + m^2)$ as $\epsilon \to 0$ (where $\partial_\mu \equiv \partial/\partial x^\mu$).

5.3 Find the expressions corresponding to the following *momentum space* Feynman diagrams

Integrate out all the δ-functions but do not perform the remaining integrals.

6 Concluding remarks

Although we have missed out on many important topics in Quantum Field Theory, we got to the point where we established contact between the underlying formalism of Quantum Field Theory and the Feynman rules, which are widely used in perturbative calculations. The main concepts of the formulation were discussed: we introduced field operators, multi-particle states that live in Fock spaces, creation and annihilation operators, the connections between particles and fields as well as that between n-point Green's functions and scattering matrix elements. Besides slight complications in accounting for the additional degrees of freedom, the same basic ingredients can be used to formulate a quantum theory for electrons, photons or any other fields describing particles in the Standard Model and beyond. Starting from relativistic wave equations, this is discussed in the lectures by Nick Evans at this school. Renormalisation is a topic which is not so easily discussed in a relatively short period of time, and hence I refer the reader to standard textbooks on Quantum Field Theory, which are listed below. The same applies to the method of quantisation via path integrals.

Acknowledgements

I am indebted to Hartmut Wittig, on whose 2003 account of these lectures the current version is based. In particular Sections 4 and 5 have been taken over with little change. I would like to thank Tim Greenshaw for running the school so successfully, as well as Margaret Evans for her friendly and efficient organisation. Many thanks go to my fellow lecturers and the tutors for the pleasant and entertaining collaboration, and to all the students for their interest and questions, which made for a lively and inspiring atmosphere.

References

[1] F. Mandl and G. Shaw, *Quantum Field Theory*, Wiley 1984.

[2] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw-Hill 1987.

[3] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison Wesley 1995

[4] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1, CUP 1995

A Notation and conventions

4-vectors:

$$x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x})$$
$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\mathbf{x}) = (t, -\mathbf{x})$$
$$\text{Metric tensor:} \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Scalar product:

$$\begin{aligned} x^\mu x_\mu &= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 \\ &= t^2 - \mathbf{x}^2 \end{aligned}$$

Gradient operators:

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right)$$
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right)$$
$$\text{d'Alembertian:} \quad \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$$

Momentum operator:

$$\hat{p}^\mu = i\hbar\partial^\mu = \left(i\hbar\frac{\partial}{\partial t}, -i\hbar\nabla\right) = \left(\hat{E}, \hat{\mathbf{p}}\right) \quad \text{(as it should be)}$$

δ-functions:

$$\int d^3p\, f(\mathbf{p})\, \delta^3(\mathbf{p} - \mathbf{q}) = f(\mathbf{q})$$
$$\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3 \delta^3(\mathbf{p})$$
$$\int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x})$$

(similarly in four dimensions)

Note:

$$\begin{aligned} \delta(x^2 - x_0^2) &= \delta\{(x - x_0)(x + x_0)\} \\ &= \frac{1}{2x}\{\delta(x - x_0) + \delta(x + x_0)\} \end{aligned}$$

AN INTRODUCTION TO QED &QCD

Dr D J Miller
University of Glasgow

Lecture delivered at the School for Experimental High Energy Physics Students
Rutherford Appleton Laboratory, September 2006

An Introduction to QED & QCD

D.J. Miller

Department of Physics and Astronomy
University of Glasgow
Glasgow G12 8QQ

Lectures presented at the School for Young High Energy Physicists,
Rutherford Appleton Laboratory, September 2006.

Contents

1 Introduction

The aim of this course is to teach you how to calculate transition amplitudes, cross sections and decay rates, for elementary particles in the highly successful theories of Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD). Most of our work will be in understanding how to compute in QED. By the end of the course you should be able to go from a Feynman diagram, such as the one for $e^-e^- \to \mu^-\mu^-$ in figure 9, to a number for the cross section. To do this we will have to learn how to cope with *relativistic, quantum*, particles and *anti*-particles that carry *spin*. In fact all these properties of particles will emerge rather neatly from thinking about relativistic quantum mechanics. The rules for calculating in QCD are slightly more complicated that in QED, as we will briefly review, however, the basic techniques for the calculation are very similar.

We have a lot to cover so will necessarily have to take some short cuts. Our main fudge will be to work in relativistic quantum mechanics rather than the full Quantum Field Theory (QFT) (sometimes referred to as 'second quantization'). We will be in good company though since we will largely follow methods from Feynman's papers and text books such as Halzen and Martin. In quantum mechanics a *classical* wave is used to describe a particle whose motion is subject to the Uncertainty Principle. In a full QFT the wave's motion itself is subject to the Uncertainty Principle too - the quanta of that field are what we then refer to as particles. Luckily at lowest order in a perturbation theory calculation one neglects the quantum nature of the field and the two theories give the same answer. At higher orders the quantum nature of the field gives rise to virtual pair creation of particles - in the quantum mechanics version of the story these are included in a more ad hoc fashion as we will see. Luckily the simultaneous QFT course will give you a good grounding in more precise methodologies.

Thus our starting point will be ordinary Quantum Mechanics and our first goal (section 2) will be to write down a 'relativistic version' of Quantum Mechanics. This will lead us to look at relativistic wave equations, in particular the Dirac equation, which describes particles with spin 1/2. We will also develop a wave equation for photons and look at how they couple to our fermions (section 3) - this is the core of QED. A perturbation theory analysis will result in quantum mechanical probability amplitudes for particular processes. After this, we will work out how to go from the probability amplitudes to cross sections and decay rates (section 4). We will look at some examples of tree level QED processes. Here you will get hands-on experience of calculating transition amplitudes and getting from them to cross sections (section 5). We will restrict ourselves to calculations at *tree level* but, at the end of the course (section 6), we will also take a first look at higher order *loop* effects, which, amongst other things, are responsible for the running of the couplings. For QCD, this running means that the coupling appears weaker when measured at higher energy scales and is the reason why we can sometimes do perturbative QCD calculations. However, in higher order calculations divergences appear and we have to understand — at least in principle — how these divergences can be removed.

In reference [1] you will find a list of textbooks that may be useful.

1.1 Relativity Review

An event in a reference frame S is described by the four coordinates of a four-vector (in units where $c = 1$)

$$x^\mu = (t, \vec{x}), \tag{1.1}$$

where the Greek index $\mu \in \{0, 1, 2, 3\}$. These coordinates are *reference frame dependent.* The coordinates in another frame S' are given by x'^μ, related to those in S by a Lorentz Transformation (LT)

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.2}$$

where summation over repeated indices is understood. This transformation identifies x^μ as a *contravariant* 4-vector (often referred to simply as a *vector*). A familiar example of a LT is a boost along the z-axis, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{1.3}$$

with, as usual, $\beta = v$ and $\gamma = (1 - \beta^2)^{-1/2}$. LT's can be thought of as generalized rotations.

The "length" of the 4-vector $(t^2 - |\vec{x}|^2)$ is invariant to LTs. In general we define the Minkowski scalar product of two 4-vectors x and y as

$$x \cdot y \equiv x^\mu y^\nu g_{\mu\nu} \equiv x^\mu y_\mu, \tag{1.4}$$

where the metric

$$g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1), \qquad g^{\mu\lambda} g_{\lambda\nu} = g^\mu{}_\nu = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}, \tag{1.5}$$

has been introduced. The last step in eq. (1.4) is the definition of a *covariant* 4-vector (sometimes referred to as a *co-vector*),

$$x_\mu \equiv g_{\mu\nu} x^\nu. \tag{1.6}$$

This transforms under a LT according to

$$x_\mu \to x'_\mu = \Lambda_\mu{}^\nu x_\nu. \tag{1.7}$$

Note that the invariance of the scalar product implies

$$\Lambda^T g \Lambda = g \Rightarrow g \Lambda^T g = \Lambda^{-1}, \tag{1.8}$$

i.e. a generalization of the orthogonality property of the rotation matrix $R^T = R^{-1}$.

▷ **Exercise 1.1**
Show eq. (1.8), starting from the invariance of the scalar product.

To formulate a coherent relativistic theory of dynamics we define kinematic variables that are also 4-vectors (i.e. transform according to eq. (1.2)). For example, we define a 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{1.9}$$

where τ is the *proper time* measured by a clock moving with the particle. Everyone will agree what the clock says at a particular event so this measure of time is Lorentz invariant and u^μ transforms as x^μ. Note

$$u^\mu = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(1, \vec{v}) \tag{1.10}$$

and has invariant length

$$u^\mu u_\mu = \gamma^2(1^2 - |\vec{v}|^2) = 1. \tag{1.11}$$

Similarly 4-momentum provides a relativistic definition of energy and momentum

$$p^\mu = m u^\mu \equiv (E, \vec{p}). \tag{1.12}$$

The invariant length provides the crucial relation

$$p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2. \tag{1.13}$$

▷ **Exercise 1.2**
Check that $dt/d\tau = \gamma$ and that our relativistic definitions of E and $\vec{p}$ make sense in the non-relativistic limit.

The differentiation operator,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \quad \partial_\mu x^\nu = \delta^\nu_\mu, \tag{1.14}$$

is a covariant 4-vector (i.e. according to eq. (1.7)). This means that the contravariant equivalent 4-vector will have an extra minus sign in its space-like components,

$$\partial^\mu = (\frac{\partial}{\partial t}, -\vec{\nabla}). \tag{1.15}$$

The convention for the totally antisymmetric Levi-Civita tensor is

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\lambda,\sigma\} \text{ an even permutation of } \{0,1,2,3\} \\ -1 & \text{if an odd permutation} \\ 0 & \text{otherwise} \end{cases}. \tag{1.16}$$

Note that $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon_{\mu\nu\lambda\sigma}$, and $\epsilon^{\mu\nu\lambda\sigma}p_\mu q_\nu r_\lambda s_\sigma$ changes sign under a parity transformation since it contains an odd number of spatial components.

▷ **Exercise 1.3**
Verify the above two properties of $\epsilon^{\mu\nu\lambda\sigma}$.

I will use natural units, $c = 1$, $\hbar = 1$, so mass, energy, inverse length and inverse time all have the same dimensions. Generally think of energy as the basic unit, e.g. mass has units of GeV and distance has units of GeV^{-1}.

▷ **Exercise 1.4**
Noting that E has SI unit kg.m^2.s^{-2}, c has SI unit m.s^{-1} and $\hbar$ has SI unit kg.m^2.s^{-1}, what is a mass of 1 GeV in kg and what is a cross-section of 1 GeV^{-2} in microbarns?

2 Relativistic Wave Equations

Let's review how wave equations describe non-relativistic quantum particles. Experimentally we know that a particle with definite momentum $\vec{p}$ and energy E can be associated with a plane wave

$$\psi = e^{i(\vec{k}.\vec{x}-wt)}, \qquad \text{with} \qquad \vec{k} = \frac{\vec{p}}{\hbar}, \quad w = \frac{E}{\hbar}. \tag{2.1}$$

To extract E and $\vec{p}$ from the wave we use *operators*

$$E\psi = i\hbar\frac{d}{dt}\psi, \qquad \vec{p}\psi = -i\hbar\vec{\nabla}\psi. \tag{2.2}$$

In quantum mechanics, it is more usual to refer to the energy operator as the *Hamiltonian* H, and write (with $\hbar = 1$)

$$H\psi = i\frac{\partial\psi}{\partial t}. \tag{2.3}$$

I shall usually reserve the Greek symbol ψ for spin 1/2 fermions and ϕ for spin 0 bosons. So for pions and the like I shall write

$$H\phi = i\frac{\partial\phi}{\partial t}. \tag{2.4}$$

In non-relativistic systems, conservation of energy can be written

$$H = T + V, \tag{2.5}$$

where T is the kinetic energy and V is the potential energy. A particle of mass m and momentum $\vec{p}$ has non-relativistic kinetic energy,

$$T = \frac{\vec{p}^2}{2m}. \tag{2.6}$$

Replacing the energy and momentum operators with the forms seen in eq. (2.2), we arrive at the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{2.7}$$

In this equation ψ is the wave function describing the single particle probability amplitude. For a slow moving particle $v \ll c$ (e.g. an electron in a Hydrogen atom) this is adequate, but for relativistic systems ($v \sim c$) the Hamiltonian above is incorrect.

For a free relativistic particle the total energy E is given by the Einstein equation

$$E^2 = \vec{p}^2 + m^2. \tag{2.8}$$

Thus the square of the relativistic Hamiltonian H^2 is simply given by promoting the momentum to operator status:

$$H^2 = \vec{p}^2 + m^2. \tag{2.9}$$

So far, so good, but how should this be implemented into the wave equation of eq. (2.3), which is expressed in terms of H rather than H^2? Naively the relativistic wave equation looks like

$$\sqrt{\vec{p}^2 + m^2}\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{2.10}$$

but this is difficult to interpret because of the square root. There are two ways forward:

1. Work with H^2. By iterating the wave equation we have

$$H^2\phi(t) = -\frac{\partial^2\phi(t)}{\partial t^2} \quad \left[\text{or} \quad \left(\frac{i\partial}{\partial t} - V\right)^2 \phi(t)\right] \tag{2.11}$$

   This is known as the Klein-Gordon (KG) equation. In this case the wave function describes spinless bosons.

2. Invent a new Hamiltonian H_D that is linear in momentum, and whose square is equal to H^2 given above, $H_D^2 = \vec{p}^2 + m^2$. In this case we have

$$H_D\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{2.12}$$

   which is known as the Dirac equation, with H_D being the Dirac Hamiltonian. In this case the wave function describes spin 1/2 fermions, as we shall see.

2.1 The Klein-Gordon Equation

Let us now take a more detailed look at the KG equation (2.11). In position space we write the energy-momentum operator as

$$p^\mu \to i\partial^\mu, \tag{2.13}$$

so that the KG equation (for zero potential V) becomes

$$(\partial^2 + m^2)\,\phi(x) = 0 \tag{2.14}$$

where we recall the notation,

$$\partial^2 = \partial_\mu\partial^\mu = \partial^2/\partial t^2 - \nabla^2 \tag{2.15}$$

and x is the 4-vector $(t, \vec{x})$.

The operator ∂^2 is Lorentz invariant, so the Klein-Gordon equation is relativistically covariant (that is, transforms into an equation of the same form) if ϕ is a scalar function. That is to say, under a Lorentz transformation $(t, \vec{x}) \to (t', \vec{x}')$,

$$\phi(t, \vec{x}) \to \phi'(t', \vec{x}') = \phi(t, \vec{x}) \tag{2.16}$$

so ϕ is invariant. In particular ϕ is then invariant under spatial rotations so it represents a spin-zero particle (more on spin when we come to the Dirac equation); there being no preferred direction which could carry information on a spin orientation.

The Klein-Gordon equation has plane wave solutions:

$$\phi(x) = Ne^{-i(Et - \vec{p}\cdot\vec{x})} \tag{2.17}$$

where N is a normalization constant and $E = \pm\sqrt{\vec{p}^2 + m^2}$. Thus, there are both positive and negative energy solutions. The negative energy solutions pose a severe problem if we try to interpret ϕ as a wave function (as indeed we are trying to do). The spectrum is no longer bounded from below, and we can extract arbitrarily large amounts of energy from the system by driving it to ever more negative energy states. Any external perturbation capable of pushing a particle across the energy gap of $2m$ between the positive and negative energy continuum of states can uncover this difficulty. Furthermore, we cannot just throw away these solutions as unphysical since they appear as Fourier modes in any realistic solution of (2.14). Note that if one interprets ϕ as a quantum field there is no problem, as you will see in the field theory course. The positive and negative energy modes are just associated with operators which create or destroy particles.

A second problem with the wave function interpretation arises when trying to find a probability density. Since ϕ is Lorentz invariant, $|\phi|^2$ does not transform like a density (i.e. as the time component of a 4-vector) so we will not have a Lorentz covariant continuity equation $\partial\rho + \vec{\nabla} \cdot \vec{J} = 0$. To search for a candidate we derive such a continuity equation. Defining ρ and $\vec{J}$ by

$$\rho \equiv i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right), \quad \left[\text{or} \quad \phi^*\left(i\frac{\partial}{\partial t} - V\right) + \phi\left(-i\frac{\partial}{\partial t} - V\right)\phi^*\right], \tag{2.18}$$

$$\vec{J} \equiv -i\left(\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right), \tag{2.19}$$

we obtain (see problem) a covariant conservation equation

$$\partial_\mu J^\mu = 0, \tag{2.20}$$

where J is the 4-vector $(\rho, \vec{J})$. It is thus natural to interpret ρ as a probability density and $\vec{J}$ as a probability current. However, for a plane wave solution (2.17), $\rho = 2|N|^2 E$, so the negative energy solutions also have a negative probability!

▷ **Exercise 2.5**

Derive the continuity equation (2.20). Start with the Klein-Gordon equation multiplied by ϕ^* and subtract the complex conjugate of the KG equation multiplied by ϕ.

Thus, ρ may well be considered as the density of a conserved quantity (such as electric charge), but we cannot use it for a probability density. To Dirac, this and the existence of negative energy solutions seemed so overwhelming that he was led to introduce another equation, first order in time derivatives but still Lorentz covariant, hoping that the similarity to Schrödinger's equation would allow a probability interpretation. Dirac's original hopes were unfounded because his new equation turned out to admit negative energy solutions too! Even so, he did find the equation for spin-1/2 particles and predicted the existence of antiparticles.

Before turning to discuss what Dirac did, let us put things in context. We have found that the Klein-Gordon equation, a candidate for describing the quantum mechanics of spinless particles, admits unacceptable negative energy states when ϕ is interpreted as the single particle wave function. We could solve all our problems here and now, and restore our faith in the Klein-Gordon equation, by simply re-interpreting ϕ as a quantum field. However we will not do that. There is another way forward (this is the way followed in the textbook of Halzen & Martin) due to Feynman and Stückelberg. Causality forces us to ensure that positive energy states propagate forwards in time, but if we force the negative energy states to propagate only backwards in time then we find a theory that is consistent with the requirements of causality and that has none of the aforementioned problems. In fact, the negative energy states cause us problems only so long as we think of them as real physical states propagating forwards in time. Therefore, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$.

In order to become more familiar with this picture, consider a process with a π^+ and a photon in the initial state and final state. In figure 1(a) the π^+ starts from the point A and at a later time t_1 emits a photon at the point $\vec{x}_1$. If the energy of the π^+ is still positive, it travels on forwards in time and eventually will absorb the initial state photon at t_2 at the point $\vec{x}_2$. The final state is then again a photon and a (positive energy) π^+.

There is another process however, with the same initial and final state, shown in figure 1(b). Again, the π^+ starts from the point A and at a later time t_2 emits a photon at the point $\vec{x}_1$. But this time, the energy of the photon emitted is bigger than the energy of the initial π^+. Thus, the energy of the π^+ becomes negative and it is forced to travel backwards in time. Then at an *earlier* time t_1 it absorbs the initial state photon at the point $\vec{x}_2$, thereby rendering its energy positive again. From there, it travels forward in time and the final state is the same as in figure 1(a), namely a photon and a (positive energy) π^+.

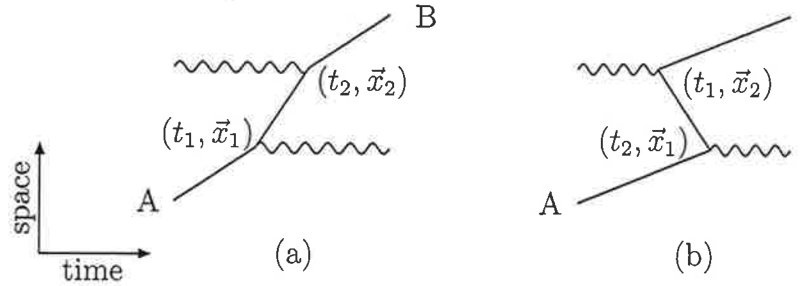


Figure 1: Interpretation of negative energy states

In todays language, the process in figure 1(b) would be described as follows: in the initial state we have an π^+ and a photon. At time t_1 and at the point $\vec{x}_2$ the photon creates an $\pi^+\pi^-$ pair. Both propagate forwards in time. The π^+ ends up in the final state, whereas the π^- is annihilated at (a later) time t_2 at the point $\vec{x}_1$ by the initial state π^+, thereby producing the final state photon. To someone observing in real time, the negative energy state moving backwards in time looks to all intents and purposes like a negatively charged pion with positive energy moving forwards in time.

▷ **Exercise 2.6**

Consider a wave incident on the potential step shown in figure 2. Show that if the step size $V > m + E_p$, where $E_p = \sqrt{\vec{p}^2 + m^2}$ then one cannot avoid using the negative square root $\vec{k} = -\sqrt{(E_p - V)^2 + m^2}$, resulting in negative currents and densities. Hint: use the continuity of $\phi(x)$ and $\partial\phi(x)/\partial x$ at $x = 0$, and ensure that the group velocity $v_g = \partial E/\partial k$ is positive for $x > 0$. Interpret the solution.

2.2 The Dirac Equation

Dirac wanted an equation first order in time derivatives and Lorentz covariant, so it had to be first order in spatial derivatives too. His starting point was to assume a

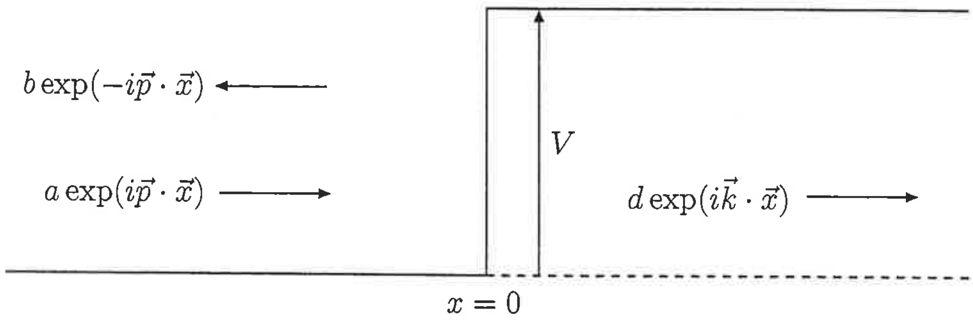


Figure 2: A potential step

Hamiltonian of the form,

$$H_D = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m, \tag{2.21}$$

where p_i are the three components of the momentum operator $\vec{p}$, and α_i and β are some unknown quantities, which we will show must be interpreted as 4×4 matrices. Substituting the expressions for the operators eq. (2.13) into the Dirac Hamiltonian of eq. (2.21) results in the equation

$$i\frac{\partial\psi}{\partial t} = (-i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \tag{2.22}$$

which is the position space Dirac equation.

If ψ is to describe a free particle it must satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among $\alpha_1, \alpha_2, \alpha_3$ and β. To see this, apply the Hamiltonian operator to ψ *twice*, to give

$$-\frac{\partial^2\psi}{\partial t^2} = [-\alpha^i\alpha^j\nabla^i\nabla^j - i\,(\beta\alpha^i + \alpha^i\beta)m\nabla^i + \beta^2 m^2]\psi, \tag{2.23}$$

with an implicit sum of i and j over 1 to 3. The Klein-Gordon equation by comparison is

$$-\frac{\partial^2\psi}{\partial t^2} = [-\nabla^i\nabla^i + m^2]\psi. \tag{2.24}$$

It is clear that we cannot recover the KG equation from the Dirac equation if the α^i and β are normal numbers. Insisting that the terms linear in ∇^i vanish independently would require either β to vanish or *all* the α^i to vanish. This would remove either $\nabla^i\nabla^j$ term or the m^2 term, both of which are unacceptable. Instead we must insist that the terms linear in ∇^i vanish in their sum *without* any of α^i or β vanishing, i.e. we must assume that α^i and β *anti-commute.* We recover the KG equation only if

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij} \\ \beta\alpha_i + \alpha_i\beta &= 0 \\ \beta^2 &= 1 \end{aligned} \tag{2.25}$$

for $i, j = 1, 2, 3$. In principle, these equations *define* α^i and β, and any objects which obey these relations are good representations of them. However, in practice, we will represent them by matrices. In this case, ψ is a multi-component *spinor* on which these matrices act.

▷ **Exercise 2.7**
Prove that any matrices $\vec{\alpha}$ and β satisfying eq. (2.25) are traceless with eigenvalues ± 1. Hence argue that they must be even dimensional.

In two dimensions a natural set of matrices for the $\vec{\alpha}$ would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.26}$$

However, there is no other independent 2×2 matrix with the right properties for β, so we must use a higher dimensional form. The smallest number of dimensions for which the Dirac matrices can be realized is four. One choice is the *Dirac representation*:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.27}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix. The spinor ψ therefore has four components.

There is a theorem due to Pauli that states that all sets of matrices obeying the relations in eq. (2.25) are equivalent. Since the hermitian conjugates $\vec{\alpha}^\dagger$ and $\beta^\dagger$ clearly obey the relations, you can, by a change of basis if necessary, assume that $\vec{\alpha}$ and β are hermitian. All the common choices of basis have this property. Furthermore, we would like α_i and β to be hermitian so that the Dirac Hamiltonian (2.42) is hermitian.

If we define

$$\rho = J^0 = \psi^\dagger \psi, \qquad \vec{J} = \psi^\dagger \vec{\alpha} \psi, \tag{2.28}$$

then it is a simple exercise using the Dirac equation to show that this satisfies the continuity equation $\partial_\mu J^\mu = 0$. We will see in section 2.8 that $(\rho, \vec{J})$ transforms, as it must, as a 4-vector. Note that ρ is now also positive definite.

2.3 Solutions to the Dirac Equation

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \chi(\vec{p}) \\ \phi(\vec{p}) \end{pmatrix} e^{-i(Et - \vec{p}\cdot\vec{x})} \tag{2.29}$$

where $\phi(\vec{p})$ and $\chi(\vec{p})$ are two-component spinors that depend on momentum $\vec{p}$ but are independent of $\vec{x}$. Using the Dirac representation of the matrices, and inserting the trial solution into the Dirac equation gives the pair of simultaneous equations

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}. \tag{2.30}$$

There are two simple cases for which eq. (2.30) can readily be solved, namely

1. $\vec{p} = 0$, $m \neq 0$, which might represent an electron in its rest frame.

2. $m = 0$, $\vec{p} \neq 0$, which describes a massless particle or a particle in the ultra-relativistic limit ($E \gg m$).

For case (1), an electron in its rest frame, the equations (2.30) decouple and become simply,

$$E\chi = m\chi, \qquad E\phi = -m\phi. \tag{2.31}$$

So, in this case, we see that χ corresponds to solutions with $E = m$, while ϕ corresponds to solutions with $E = -m$. In light of our earlier discussions, we no longer need to recoil in horror at the appearance of these negative energy states.

The negative energy solutions persist for an electron with $\vec{p} \neq 0$ for which the solutions to equation (2.30) are

$$\phi = \frac{\vec{\sigma} \cdot \vec{p}}{E+m}\chi, \qquad \chi = \frac{\vec{\sigma} \cdot \vec{p}}{E-m}\phi. \tag{2.32}$$

▷ **Exercise 2.8**
Show that $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$.

Using $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$ we see that $E = \pm|\sqrt{\vec{p}^2 + m^2}|$. We write the positive energy solutions with $E = +|\sqrt{\vec{p}^2 + m^2}|$ as

$$\psi(x) = \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix} e^{-i(Et-\vec{p}\cdot\vec{x})}, \tag{2.33}$$

while the general negative energy solutions with $E = -|\sqrt{\vec{p}^2 + m^2}|$ are

$$\psi(x) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\phi \\ \phi \end{pmatrix} e^{-i(Et-\vec{p}\cdot\vec{x})}, \tag{2.34}$$

for arbitrary constant ϕ and χ. Clearly when $\vec{p} = 0$ these solutions reduce to the positive and negative energy solutions discussed previously.

It is interesting to see how Dirac coped with the negative energy states. Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. On promoting one of these negative energy states to a positive energy one, by supplying energy, an electron-hole pair is created, i.e. a positive energy electron and a hole in the negative energy sea. The hole is seen in nature as a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics. The problem with Dirac's hole theory is that it does not work for bosons. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy.

It is convenient to rewrite the solutions, eqs. (2.33) and (2.34), introducing the spinors $u_\alpha^{(s)}(\vec{p})$ and $v_\alpha^{(s)}(\vec{p})$. The label $\alpha \in \{1,2,3,4\}$ is a spinor index that often will be suppressed, while $s \in \{1,2\}$ denotes the spin state of the fermion, as we shall see later. We take the positive energy solution eq. (2.33) and define

$$\sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} e^{-ip\cdot x} \equiv u^{(s)}(p)e^{-ip\cdot x}. \tag{2.35}$$

For the negative energy solution of eq. (2.34), change the sign of the energy, $E \rightarrow -E$, and the three-momentum, $\vec{p} \rightarrow -\vec{p}$, to obtain,

$$\sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix} e^{ip\cdot x} \equiv v^{(s)}(p)e^{ip\cdot x}. \tag{2.36}$$

In these two solutions E is now (and for the rest of the course) always positive and given by $E = (\vec{p}^2 + m^2)^{1/2}$. The χ_s for $s = 1, 2$ are

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.37}$$

For the simple case $\vec{p} = 0$ we may interpret χ_1 as the spin-up state and χ_2 as the spin-down state. Thus for $\vec{p} = 0$ the 4-component wave function has a very simple interpretation: the first two components describe electrons with spin-up and spin-down, while the second two components describe positrons with spin-up and spin-down. Thus we understand on physical grounds why the wave function had to have four components. The general case $\vec{p} \neq 0$ is slightly more involved and is considered in the next section.

The u-spinor solutions will correspond to particles and the v-spinor solutions to antiparticles. The role of the two χ's will become clear in the following section, where it will be shown that the two choices of s are spin labels. Note that each spinor solution depends on the three-momentum $\vec{p}$, so it is implicit that $p^0 = E$.

2.4 Orthogonality and Completeness

Our solutions to the Dirac equation take the form

$$\psi = Nu^{(s)}e^{-ip.x}, \quad \psi = Nv^{(r)}e^{ip.x}, \quad \text{with} \quad r, s = 1, 2, \tag{2.38}$$

where N is a normalization factor. We have already included a factor $\sqrt{E+m}$ in our spinors (see eqs. (2.35) and (2.36)), which results in

$$u^{(r)\dagger}(p)u^{(s)}(p) = v^{(r)\dagger}(p)v^{(s)}(p) = 2E\delta^{rs}. \tag{2.39}$$

This convention allows $u^\dagger u$ to transform as the time component of a 4-vector under Lorentz transformations, which is essential to its interpretation as a probability density (see eq. (2.28) and section 2.8). Also note that the spinors are *orthogonal.*

▷ **Exercise 2.9**
Check the normalization condition for the spinors in eq. (2.39).

We must further normalize the wave functions. A plane wave is not normalizable in an infinite space so we will work in a large box of volume V. The number of particles in the box will be

$$\int \psi^\dagger\psi\, d^3x = 2E\; N^2\; V, \tag{2.40}$$

so setting $N = 1/\sqrt{V}$ allows us to adopt the standard relativistic normalization convention of $2E$ particles per box of volume V. This is the convention that I will use throughout this course. I will usually set V to be a unit volume, but I will occasionally keep V explicit for clarity.

Remember that the solutions to the wave equation form a complete set of states meaning that we can expand (like a Fourier expansion) an arbitrary function $\chi(x)$ in terms of them

$$\chi(x) = \sum_n a_n \psi_n(x) \tag{2.41}$$

The a_n are the equivalent of Fourier coefficients and if χ is a wave function in some quantum mixed state then $|a_n|^2$ is the probability of being in the state ψ_n.

2.5 Spin

Now it is time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. The Dirac Hamiltonian in momentum space is given in eq. (2.21) as

$$H_D = \vec{\alpha} \cdot \vec{p} + \beta m, \tag{2.42}$$

and the orbital angular momentum operator is

$$\vec{L} = \vec{R} \times \vec{p}. \tag{2.43}$$

Evaluating the commutator of $\vec{L}$ with H_D,

$$\begin{aligned} [\vec{L}, H_D] &= [\vec{R} \times \vec{p}, \vec{\alpha} \cdot \vec{p}] \\ &= [\vec{R}, \vec{\alpha} \cdot \vec{p}] \times \vec{p} \\ &= i\vec{\alpha} \times \vec{p}, \end{aligned} \tag{2.44}$$

we see that the orbital angular momentum is *not* conserved (otherwise the commutator would be zero). We would like to find a *total* angular momentum $\vec{J}$ that *is* conserved, by adding an additional operator $\vec{S}$ to $\vec{L}$,

$$\vec{J} = \vec{L} + \vec{S}, \quad [\vec{J}, H_D] = 0. \tag{2.45}$$

To this end, consider the three matrices,

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha}, \tag{2.46}$$

where the first equivalence is merely a definition of $\vec{\Sigma}$ and the last equality can be verified by an explicit calculation. The $\vec{\Sigma}/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with $\vec{\alpha}$ and β are,

$$[\vec{\Sigma}, \beta] = 0, \qquad [\Sigma_i, \alpha_j] = 2i\varepsilon_{ijk}\alpha_k. \tag{2.47}$$

From the relations in (2.47) we find that

$$[\vec{\Sigma}, H_D] = -2i\vec{\alpha} \times \vec{p}. \tag{2.48}$$

▷ **Exercise 2.10**
Using $\alpha_1\alpha_2\alpha_3 \equiv \frac{1}{3}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k$ verify the commutation relations in eqs. (2.47) and (2.48).

Comparing eq. (2.48) with the commutator of $\vec{L}$ with H_D in eq. (2.44), you see that

$$[\vec{L} + \frac{1}{2}\vec{\Sigma}, H_D] = 0, \tag{2.49}$$

and we can identify

$$\vec{S} = \frac{1}{2}\vec{\Sigma} \tag{2.50}$$

as the additional quantity that, when added to $\vec{L}$ in equation (2.45), yields a conserved total angular momentum $\vec{J}$. We interpret $\vec{S}$ as an angular momentum *intrinsic* to the particle. Now

$$\vec{S}^2 = \frac{1}{4}\begin{pmatrix} \vec{\sigma}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\sigma} \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.51}$$

and, recalling that the eigenvalue of $\vec{J}^2$ for spin j is $j(j+1)$, we conclude that $\vec{S}$ represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised. We worked in the Dirac representation of the matrices for convenience, but the result is necessarily independent of the representation.

Now consider the u-spinor solutions $u^{(s)}(p)$ of eq. (2.35). Choose $\vec{p} = (0, 0, p_z)$ and write

$$u_\uparrow \equiv u^{(1)}(p) = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \qquad u_\downarrow \equiv u^{(2)}(p) = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \tag{2.52}$$

With these definitions, we get

$$S_z u_\uparrow = \frac{1}{2}u_\uparrow, \qquad S_z u_\downarrow = -\frac{1}{2}u_\downarrow. \tag{2.53}$$

So, these two spinors represent spin up and spin down along the z-axis respectively. For the v-spinors, with the same choice for $\vec{p}$, write,

$$v_\downarrow = v^{(1)}(p) = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \qquad v_\uparrow = v^{(2)}(p) = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \tag{2.54}$$

where now,

$$S_z v_\downarrow = \frac{1}{2}v_\downarrow, \qquad S_z v_\uparrow = -\frac{1}{2}v_\uparrow. \tag{2.55}$$

This apparently perverse choice of up and down for the v's is actually quite sensible when one realizes that a negative energy electron carrying spin $+1/2$ backwards in time looks just like a positive energy positron carrying spin $-1/2$ forwards in time.

2.6 Lorentz Covariance

There is a much more compact way of writing the Dirac equation, which requires that we get to grips with some more notation. Define the γ-*matrices*,

$$\gamma^0 = \beta, \qquad \vec{\gamma} = \beta\vec{\alpha}. \tag{2.56}$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{2.57}$$

In terms of these, the relations between the $\vec{\alpha}$ and β in eq. (2.25) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{2.58}$$

▷ **Exercise 2.11**
Prove that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Combinations like $a_\mu\gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu\gamma^\mu = a^\mu\gamma_\mu,$$

pronounced "a slash." Note that γ^μ is not, despite appearances, a 4-vector. It just denotes a set of four matrices. However, the notation is deliberately suggestive, for when combined with Dirac fields you can construct quantities that transform like vectors and other Lorentz tensors (see the next section).

Observe that using the γ-matrices the Dirac equation (2.22) becomes

$$(i\not{\partial} - m)\psi = 0, \tag{2.59}$$

or, in momentum space,

$$(\not{p} - m)\psi = 0. \tag{2.60}$$

The spinors u and v satisfy

$$(\not{p} - m)u^{(s)}(p) \;=\; 0, \tag{2.61}$$

$$(\not{p} + m)v^{(s)}(p) \;=\; 0, \tag{2.62}$$

since for $v^{(s)}(p)$, $E \to -E$ and $\vec{p} \to -\vec{p}$.

We want the Dirac equation (2.59) to preserve its form under Lorentz transformations eq. (1.2). We've just naively written the matrices in the Dirac equation as γ_μ however this does not make them a 4-vector! They are just a set of numbers in four matrices and there's no reason they should change when we do a boost. Since ∂^μ does transform, for the equation to be Lorentz covariant we are led to propose that ψ transforms too. We know that 4-vectors get their components mixed up by LT's, so we expect that the components of ψ might get mixed up too:

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \tag{2.63}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the spinor index of ψ. Note that the argument $\Lambda^{-1}x'$ is just a fancy way of writing x, i.e. each component of $\psi(x)$ is transformed into a linear combination of components of $\psi(x)$.

In order to appreciate the above it is useful to consider a vector field, where the corresponding transformation is

$$A^\mu(x) \to A'^\mu(x')$$

where $x' = \Lambda x$. This makes sense physically if one thinks of space rotations of a vector field. For example the wind arrows on a weather map are an example of a vector field: with each point on the map there is associated an arrow. Consider the wind direction at a particular point on the map, say Abingdon. If the map is rotated, then one would expect on physical grounds that the wind vector at Abingdon always point in the same physical direction and have the same length. In order to achieve this, both the vector itself must rotate, and the point to which it is attached (Abingdon) must be correctly identified after the rotation. Thus the vector at the point x' (corresponding to Abingdon in the rotated frame) is equal to the vector at the point x (corresponding to Abingdon in the unrotated frame), but rotated so as to keep the physical sense of the vector the same in the rotated frame (so that the wind always blows towards Oxford, say, in the two frames). Thus having correctly identified the same point in the two frames all we need to do is rotate the vector:

$$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x). \tag{2.64}$$

A similar thing also happens in the case of the 4-component spinor field above, except that we do not (yet) know how the components of the wave function themselves must transform, i.e. we do not know S.

We now need to figure out what S is. The requirement is that the Dirac equation has the same form in any inertial frame. Thus, if we make a LT from our original frame into another ('primed') frame and write down the Dirac equation in this frame, it has to have the same form.

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \quad \longrightarrow \quad (i\gamma^{\mu}\partial'_{\mu} - m)\psi'(x') = 0, \tag{2.65}$$

where we used the fact that m is a scalar, i.e. $m' = m$.

The derivative transforms as a covector, eq. (1.7), so using the orthogonality condition of eq. (1.8), we can write $\partial_{\mu} = \Lambda^{\sigma}{}_{\mu}\partial'_{\sigma}$ and multiplying the Dirac equation in the *original* frame by S it becomes

$$S(i\gamma^{\mu}\Lambda^{\sigma}{}_{\mu}\partial'_{\sigma} - m)\psi(x) = 0. \tag{2.66}$$

On the other hand, we can use the definition of S in eq. (2.63) to rewrite the equation in the primed frame as

$$(i\gamma^{\mu}\partial'_{\mu} - m)S\psi(x) = 0. \tag{2.67}$$

We can see that the second term (containing m) of eqs. (2.66) and (2.67) are now identical. To make the first term identical we need $S\Lambda^{\sigma}{}_{\mu}\gamma^{\mu} = \gamma^{\sigma}S$. Thus, in order for the Dirac equation to be Lorentz invariant, $S(\Lambda)$ has to satisfy

$$\Lambda^{\sigma}{}_{\mu}\gamma^{\mu} = S^{-1}\gamma^{\sigma}S \tag{2.68}$$

We still haven't solved for S explicitly. We need to find an S that satisfies eq. (2.68). Since S depends on the LT, we first have to find a convenient parameterization of a LT and then express $S(\Lambda)$ in terms of these parameters. For an infinitesimal LT, it can be shown that,

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{2.69}$$

where $\omega_{\mu\nu}$ is an antisymmetric set of infinitesimal parameters. For example, a boost along the z-axis corresponds to $\omega_{03} = -\omega_{30} = -\beta$ (remember that $\omega_{0i} = \omega^0{}_i = -\omega_0{}^i$ etc) with all other entries of $\omega_{\mu\nu}$ zero,

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix}. \tag{2.70}$$

This corresponds to eq. (1.3) when one makes an expansion in small β, i.e. $\gamma = 1 + \mathcal{O}(\beta^2)$. Non-zero ω_{01} or ω_{02} correspond to boosts along the x and y axes respectively. The remaining combinations, non-zero ω_{23}, ω_{31} or ω_{12}, correspond to infinitesimal anti-clockwise rotations through an angle ω_{ij} about the x, y and z axes respectively. It's a nice exercise to check this out.

For an infinitesimal LT we are at liberty to write

$$S(\Lambda) = 1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}, \tag{2.71}$$

which is nothing but a definition of the set of matrices $\sigma^{\mu\nu}$. Our task is to determine these matrices. To do this, substitute the expression for S, eq. (2.71), into eq. (2.68) (and remember that $S^{-1}(\Lambda) = 1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}$). After some algebra, we can convince ourselves that the solution is

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{2.72}$$

Thus S can be written explicitly in terms of γ-matrices for a general LT by building the finite transformation out of lots of infinitesimal ones.

▷ Exercise 2.12
Verify that eq. (2.72) is true.

Now that we now how ψ transforms we can find quantities that are Lorentz invariant, or transform as vectors or tensors under LT's. To this end, we will find it useful to introduce the Dirac *adjoint*. The Dirac adjoint $\bar{\psi}$ of a spinor ψ is defined by

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 \tag{2.73}$$

With the help of

$$S^\dagger(\Lambda)\gamma^0 = \gamma^0 S^{-1}(\Lambda) \tag{2.74}$$

we see that $\bar{\psi}$ transforms under LT's as

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}S^{-1}(\Lambda). \tag{2.75}$$

▷ Exercise 2.13

1. Verify that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.
2. Prove eq. (2.74)
3. Show that $\bar{\psi}$ satisfies the equation

$$\bar{\psi}\,(-i\overleftarrow{\partial\!\!\!/} - m) = 0$$

where the arrow over $\partial\!\!\!/$ implies the derivative acts to the left.

4. Hence prove that $\bar{\psi}$ transforms as in eq. (2.75).

Combining the transformation properties of ψ and $\bar{\psi}$ in eqs. (2.63) and (2.75) we see that the bilinear $\bar{\psi}\psi$ is Lorentz invariant. In section 2.8 we will consider the transformation properties of general bilinears.

Let's close this section by recasting the spinor normalization eq. (2.39) in terms of Dirac inner products. The conditions become

$$\begin{aligned}
\bar{u}^{(r)}(p)u^{(s)}(p) &= 2m\delta^{rs} \\
\bar{u}^{(r)}(p)v^{(s)}(p) &= \bar{v}^{(r)}(p)u^{(s)}(p) = 0 \\
\bar{v}^{(r)}(p)v^{(s)}(p) &= -2m\delta^{rs}
\end{aligned} \tag{2.76}$$

where, in analogy to eq. (2.73), we defined $\bar{u} \equiv u^\dagger\gamma^0$ and $\bar{v} \equiv v^\dagger\gamma^0$.

▷ **Exercise 2.14**
Verify the normalization properties in the above equations (2.76).

2.7 Parity, charge conjugation and time reversal

2.7.1 Parity

We usually use LT's which are in the connected Lorentz Group, $SO(3,1)$, meaning they can be obtained by a continuous deformation of the identity transformation (i.e. by lots of little transformations)[1]. This class of LT is often referred to as proper LT. However, the full Lorentz group consists not only of the proper transformations but also includes the discrete operations of *parity* (space inversion), P, and *time reversal*, T:

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.77}$$

LT's satisfy $\Lambda^T g \Lambda = g$, so taking determinants shows that $\det\Lambda = \pm 1$. Proper LT's are continuously connected to the identity so must have determinant 1, but both P and T operations have determinant -1.

Let us now find the action of parity on the Dirac wave function and determine the wave function ψ_P in the parity-reversed system. According to the discussion of the previous section, we need to find a matrix P satisfying

$$P^{-1}\gamma^0 P = \gamma^0, \qquad P^{-1}\gamma^i P = -\gamma^i. \tag{2.78}$$

Using some properties of the γ-matrices we see that $P = P^{-1} = \gamma^0$ is an acceptable solution (Clearly one could multiply γ^0 by a phase and still have an acceptable definition for the parity transformation.), from which it follows that the transformation is

$$\psi(t,\vec{x}) \to \psi_P(t,-\vec{x}) = P\psi(t,\vec{x}) = \gamma^0\psi(t,\vec{x}). \tag{2.79}$$

[1] Indeed in the last section we considered LT's very close to the identity in equation (2.69)

Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.80}$$

the u-spinors and v-spinors at rest have opposite eigenvalues, corresponding to particle and antiparticle having opposite *intrinsic* parities.

2.7.2 Charge Conjugation

Another discrete invariance of the Dirac equation is *charge conjugation*, which takes you from particle to antiparticle and vice versa. For scalar fields the symmetry is just complex conjugation, but in order for the charge conjugate Dirac field to remain a solution of the Dirac equation, you have to mix its components as well. The transformation on the fermion wavefunction is

$$\psi \to \psi_C = C\bar{\psi}^T, \tag{2.81}$$

where $\bar{\psi}^T = \left(\psi^\dagger\gamma^0\right)^T = \gamma^{0T}\psi^{\dagger T} = \gamma^0\psi^*$. To find the form of C, let's take the complex conjugate of the Dirac Equation,

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)^*\psi^* &= \left(i\left(\gamma^{\mu\dagger}\right)^T\partial_\mu - m\right)\left(\psi^\dagger\right)^T \\ &= \gamma^{0T}\left(-i\gamma^{\mu T}\partial_\mu - m\right)\bar{\psi}^T,\end{aligned} \tag{2.82}$$

where we have additionally used $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Premultiply by C and the Dirac equation becomes

$$\left(-iC\gamma^{\mu T}C^{-1}\partial_\mu - m\right)\psi_c = 0. \tag{2.83}$$

In order for ψ_C to satisfy the Dirac equation we require C to be a matrix satisfying the condition

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu \quad (C^{-1} = C^\dagger). \tag{2.84}$$

In the Dirac representation,a suitable choice for this operator is

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \tag{2.85}$$

The charge-conjugation transformation is then

$$\psi(t,\vec{x}) \to \psi_C(t,\vec{x}) = C\bar{\psi}^T(t,\vec{x}) = i\gamma^2\gamma^0\bar{\psi}^T(t,\vec{x}). \tag{2.86}$$

When Dirac wrote down his equation everybody thought parity and charge conjugation were exact symmetries of nature, so invariance under these transformations was essential. Now we know that neither of them, nor the combination *CP*, is respected by the standard electroweak model.

2.7.3 Time reversal

As already noted, time reversal is an improper LT, given by Λ_T in eq. (2.77). Naively one would expect to derive a time reversal operation in the same way as for parity. However, there is a subtlety that the momentum of a particle is a *rate of change*, so if we reverse the direction of time, the momentum must change direction. When we reverse the momentum $\vec{p}$ in a plane wave we find

$$e^{-i(Et-\vec{p}\cdot\vec{x})} \longrightarrow e^{-i(Et-(-\vec{p})\cdot\vec{x})} = e^{i(E(-t)-\vec{p}\cdot\vec{x})} = \left(e^{-i(E(-t)-\vec{p}\cdot\vec{x})}\right)^*. \tag{2.87}$$

In this example, taking the complex conjugate is the equivalent of reversing the time coordinate and reversing the momentum. So once again, we must take the *complex conjugate* of the field, transforming it according to

$$\psi(t,\vec{x}) \to \psi_T(-t,\vec{x}) = T\psi^*(t,\vec{x}). \tag{2.88}$$

To find the form of T, let's take the complex conjugate of the Dirac equation, premultiply by T and interchange $t \to -t$,

$$\begin{aligned}\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi(t,\vec{x}) &\longrightarrow S_T\left(-i\gamma^{0*}\frac{\partial}{\partial(-t)} - i\vec{\gamma}^*\cdot\vec{\nabla} - m\right)T^{-1}T\psi^*(-t,\vec{x})\\ &= \left(i\left[T\gamma^{0*}T^{-1}\right]\frac{\partial}{\partial t} + i\left[-T\vec{\gamma}^*T^{-1}\right]\cdot\vec{\nabla} - m\right)\psi_T(t,\vec{x}).\end{aligned} \tag{2.89}$$

For ψ_T to satisfy the Dirac equation we need

$$i\left[T\gamma^{0*}T^{-1}\right] = \gamma^0, \qquad \left[-T\vec{\gamma}^*T^{-1}\right] = -\vec{\gamma}. \tag{2.90}$$

A suitable choice is

$$T = i\gamma^1\gamma^3 = \begin{pmatrix} 0 & -i\sigma_1\sigma_3 \\ -i\sigma_1\sigma_3 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.91}$$

and the time reversal transformation on a fermion field is

$$\psi(t,\vec{x}) \to \psi_T(-t,\vec{x}) = T\psi^*(t,\vec{x}) = i\gamma^1\gamma^3\psi^*(t,\vec{x}) \tag{2.92}$$

2.7.4 CPT

We are now in the position to ask what is the effect of performing charge conjugation, parity and time-reversal all together on a Dirac field. The combined transformation is known as *CPT*. Using eqs. (2.79), (2.86) and (2.92), the CPT transformation is,

$$\begin{aligned}\psi(t,\vec{x}) \to \psi_{CPT}(-t,-\vec{x}) &= i\gamma^2\gamma^0\gamma^{0T}\left[\gamma^0 i\gamma^1\gamma^3\psi^*(t,\vec{x})\right]^*\\ &= i\gamma^2\gamma^0\gamma^0\gamma^0(-i)\gamma^1\gamma^3\psi(t,\vec{x})\\ &= \gamma^0\gamma^1\gamma^2\gamma^3\psi(t,\vec{x})\\ &= -i\gamma^5\psi(t,\vec{x})\end{aligned} \tag{2.93}$$

Thus, apart from the factor of γ^5, a particle moving forward in time is equivalent to an anti-particle moving backwards in time and in the opposite direction. In fact, the extra γ^5 makes no difference to observable quantities (see the next section) so this justifies the Feynman-Stückelberg interpretation of negative energy states we used earlier.

2.8 Bilinear Covariants

Now, as promised, we will construct and classify the bilinears. These are useful for defining quantities with particular properties under Lorentz transformations, and appearing in Lagrangians for fermion field theories.

To begin, note that by forming products of the γ-matrices it is possible to construct 16 linearly independent 4×4 matrices. Any constant 4×4 matrix can then be decomposed into a sum over these basis matrices. In equation (2.72) we have defined

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

and now it is convenient to define

$$\gamma^5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.94}$$

where the last equality is valid in the Dirac representation. This new matrix satisfies

$$\gamma^{5\dagger} = \gamma^5, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0, \qquad (\gamma^5)^2 = 1. \tag{2.95}$$

▷ **Exercise 2.15**
Prove the three results in eq. (2.95) independently of the γ-matrix representation.

Now, the set of 16 matrices

$$\left\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\right\}$$

form a basis for γ-matrix products. There are 16 matrices since there is 1 unit matrix, 1 γ^5 matrix, 4 γ^μ matrices and 4 $\gamma^\mu\gamma^5$ matrices, and 6 $\sigma^{\mu\nu}$ matrices (see equation (2.72) for the definition of $\sigma^{\mu\nu}$).

Using the transformations of ψ and $\bar{\psi}$ from eqs. (2.63) and (2.75), together with the transformation of γ^μ in eq. (2.74), the 16 fermion bilinears and their transformation properties can be written as follows:

$$\begin{array}{rcll}
\bar{\psi}\psi & \rightarrow & \bar{\psi}\psi & \text{S scalar} \\
\bar{\psi}\gamma^5\psi & \rightarrow & \det(\Lambda)\,\bar{\psi}\gamma^5\psi & \text{P pseudoscalar} \\
\bar{\psi}\gamma^\mu\psi & \rightarrow & \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi & \text{V vector} \\
\bar{\psi}\gamma^\mu\gamma^5\psi & \rightarrow & \det(\Lambda)\,\Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma^5\psi & \text{A axial vector} \\
\bar{\psi}\sigma^{\mu\nu}\psi & \rightarrow & \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\bar{\psi}\sigma^{\lambda\sigma}\psi & \text{T tensor}
\end{array} \tag{2.96}$$

In particular we note that

$$\bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = (\psi^\dagger\psi, \psi^\dagger\vec{\alpha}\psi) \tag{2.97}$$

which is our previous definition eq. (2.28) of the current 4–vector J^μ, i.e. we now see that it is really a 4–vector.

▷ **Exercise 2.16**
Derive the transformation properties of the bilinears in equation (2.96) under C, P, T and CPT transformations.

2.9 Massless (Ultra-relativistic) Fermions

At very high energies we may neglect the masses of particles ($E^2 \simeq |\vec{p}|^2$). Therefore, let us look at solutions of the Dirac equation with $m = 0$, on the basis that this will be an extremely good approximation for many situations.

From equation (2.30) we have in this case

$$E\phi = \vec{\sigma} \cdot \vec{p}\chi, \qquad E\chi = \vec{\sigma} \cdot \vec{p}\phi. \tag{2.98}$$

These equations can easily be decoupled by taking linear combinations and defining the two component spinors Ψ_L and Ψ_R,

$$\Psi_{R/L} \equiv \frac{\chi \pm \phi}{2}, \tag{2.99}$$

which leads to

$$E\Psi_R = \vec{\sigma} \cdot \vec{p}\Psi_R, \qquad E\Psi_L = -\vec{\sigma} \cdot \vec{p}\Psi_L. \tag{2.100}$$

The system is in fact described by two entirely separated two component spinors. If we take them to be moving in the z direction, and noting that $\sigma_3 = \text{diag}(1, -1)$, we see that there is one positive and one negative energy solution in each.

Further since $E = |\vec{p}|$ for massless particles, these equations may be written

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}\Psi_L = -\Psi_L, \qquad \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}\Psi_R = \Psi_R \tag{2.101}$$

Now, $\frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ is known as the *helicity* operator (i.e. it is the spin operator projected in the direction of motion of the momentum of the particle). We see that the Ψ_L corresponds to solutions with negative helicity, while Ψ_R corresponds to solutions with positive helicity. In other words Ψ_L describes a left-handed particle while Ψ_R describes a right-handed particle, and each type is described by a two-component spinor.

The two-component spinors transform very simply under LT's,

$$\Psi_L \longrightarrow e^{\frac{i}{2}\vec{\sigma}.(\vec{\theta}-i\vec{\phi})}\Psi_L \tag{2.102}$$

$$\Psi_R \longrightarrow e^{\frac{i}{2}\vec{\sigma}.(\vec{\theta}+i\vec{\phi})}\Psi_R \tag{2.103}$$

where $\vec{\theta} = \vec{n}\theta$ corresponds to space rotations through an angle θ about the unit $\vec{n}$ axis, and $\vec{\phi} = \vec{v}\phi$ corresponds to Lorentz boosts along the unit vector $\vec{v}$ with a speed $v = \tanh\phi$. Note that these transformations are consistent with the fact that it is not possible to boost past a massless particle (i.e. its helicity cannot be reversed).

However, under parity transformations $\vec{\sigma} \longrightarrow \vec{\sigma}$ (like $\vec{R} \times \vec{p}$), $\vec{p} \longrightarrow -\vec{p}$, therefore $\vec{\sigma} \cdot \vec{p} \longrightarrow -\vec{\sigma} \cdot \vec{p}$, i.e. the spinors transform into each other:

$$\Psi_L \leftrightarrow \Psi_R. \tag{2.104}$$

So a theory in which Ψ_L has different interactions to Ψ_R (such as the standard model in which the weak force only acts on left handed particles) manifestly violates parity.

Although massless particles can be described very simply using two component spinors as above, they may also be incorporated into the four-component formalism by using the γ^5 we defined earlier. Let's define *projection operators*

$$P_{R/L} \equiv \frac{1}{2}\left(1 \pm \gamma^5\right). \tag{2.105}$$

In the Dirac representation, these are,

$$P_{R/L} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \tag{2.106}$$

where 1 denotes the 2×2 identity matrix. Acting these projection operators on a general Dirac field of the form eq. (2.29) projects onto right- or left-handed eigenstates. To see this, first note that

$$P_{R/L}\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} \Psi_{R/L} \\ \Psi_{R/L} \end{pmatrix}. \tag{2.107}$$

The helicity operator in four-component Dirac space is given by $\vec{S} \cdot \vec{p}/|\vec{p}|$, with $\vec{S} = \frac{1}{2}\vec{\Sigma}$, where $\vec{\Sigma}$ is defined in equation (2.46). Acting this operator on the projected state gives

$$\frac{1}{2}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix}\begin{pmatrix} \Psi_{R/L} \\ \Psi_{R/L} \end{pmatrix} = \pm\frac{1}{2}\begin{pmatrix} \Psi_{R/L} \\ \Psi_{R/L} \end{pmatrix}, \tag{2.108}$$

indicating that the projected states are indeed right- or left-handed eigenstates with helicity $\pm\frac{1}{2}$.

This can be made more explicit by using a different representation for the γ-matrices. In the *chiral representation* (sometimes called the *Weyl representation*) we define the γ-matrices to be

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \vec{\gamma} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \tag{2.109}$$

so that, with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as before, the projection operators eq. (2.105) become

$$P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.110}$$

Now, the left-handed Weyl spinor sits in the upper two components of the Dirac spinor, while the right-handed Weyl spinor sits in the lower two components of the Dirac spinor. The projection operators pick out only the upper or lower component, e.g.

$$P_R\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}, \tag{2.111}$$

so the projected states are once again helicity eigenstates.

3 Quantum Electrodynamics

3.1 Classical Electromagnetism

So far, we have only considered relativistic wave equations for free particles. Now we want to include electromagnetic interactions, so let's start by reviewing Maxwell's Equations in differential form:

$$\vec{\nabla}.\vec{E} = \rho, \qquad \vec{\nabla}.\vec{B} = 0,$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}. \tag{3.1}$$

We can rewrite the Maxwell equations in terms of a scalar potential ϕ, and a vector potential $\vec{A}$. Writing

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi,$$
$$\vec{B} = \vec{\nabla}\times\vec{A}, \tag{3.2}$$

we automatically have solutions of two of the Maxwell equations,

$$\vec{\nabla}.\vec{B} = \vec{\nabla}.(\vec{\nabla}\times\vec{A}) \equiv 0 \tag{3.3}$$

and

$$\begin{aligned}\vec{\nabla}\times\vec{E} &= \vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right)\\ &= -\frac{\partial(\vec{\nabla}\times\vec{A})}{\partial t} - \vec{\nabla}\times(\vec{\nabla}\phi)\\ &= -\frac{\partial\vec{B}}{\partial t}.\end{aligned} \tag{3.4}$$

This simplifies things greatly since now there are only two Maxwell equations to solve. Let's write them out in terms of the potentials,

$$\vec{\nabla}.\vec{E} = -\nabla^2\phi - \frac{d(\vec{\nabla}.\vec{A})}{dt} = \rho, \tag{3.5}$$

and (since $\vec{\nabla}\times\vec{\nabla}\times\vec{A} \equiv -\nabla^2\vec{A} + \vec{\nabla}.(\vec{\nabla}.\vec{A})$),

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \vec{J} + \frac{\partial}{\partial t}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right). \tag{3.6}$$

or rearranging,

$$-\nabla^2\vec{A} + \frac{\partial^2\vec{A}}{\partial t^2} = \vec{J} - \vec{\nabla}(\vec{\nabla}.\vec{A} + \frac{\partial\phi}{\partial t}). \tag{3.7}$$

Unfortunately these two equations we are left with are quite complicated. To simplify them up we note that we can redefine our potentials,

$$\begin{aligned} \vec{A} &\rightarrow \vec{A}+\vec{\nabla}\psi, \\ \phi &\rightarrow \phi-\frac{\partial\psi}{\partial t}, \end{aligned} \qquad (3.8)$$

without changing $\vec{E}$ and $\vec{B}$. This redefinition of the potentials is known as a *gauge transformation.*

▷ **Exercise 3.17**
Check that $\vec{E}$ and $\vec{B}$ are invariant under the gauge transformation in eq. (3.8).

We can choose a gauge transformation such that

$$\vec{\nabla}.\vec{A} = -\frac{\partial\phi}{\partial t}. \qquad (3.9)$$

In this gauge (the Lorentz gauge) Maxwell's equations simplify to

$$-\nabla^2\phi + \frac{\partial^2\phi}{\partial t^2} = \rho, \qquad (3.10)$$

$$-\nabla^2\vec{A} + \frac{\partial^2\vec{A}}{\partial t^2} = \vec{J}. \qquad (3.11)$$

As well as being prettier, these equations also have a very suggestive form. They suggest we should define the 4-vectors,

$$J^\mu = (\rho, \vec{J}), \qquad A^\mu = (\phi, \vec{A}), \qquad (3.12)$$

so the Maxwell equations may be written in a manifestly covariant form,

$$\partial^2 A^\mu = J^\mu. \qquad (3.13)$$

The $\mu = 0$ equation is the ϕ eq. (3.10) and the $\mu = 1, 2, 3$ equations give the components of the eq. (3.11) for $\vec{A}$. The gauge condition, eq. (3.9), becomes

$$\partial^\mu A_\mu = 0. \qquad (3.14)$$

Eq. (3.13) is the classical wave equation for the electromagnetic field. In free space we have eq. (3.13) with no source, i.e.

$$\partial^2 A^\mu = 0, \qquad (3.15)$$

which has plane wave solutions,

$$A^\mu = \epsilon^\mu e^{iq.x}, \qquad (3.16)$$

where ϵ^μ is the polarization tensor and $q^2 = 0$.

The Lorentz condition, eq. (3.14), enforces

$$q^\mu \epsilon_\mu = 0, \tag{3.17}$$

which means the component of ϵ^μ in the direction of motion vanishes. Even after enforcing this condition, there is still room to make more gauge transformations,

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad \text{where} \quad \partial^2 \chi = 0. \tag{3.18}$$

This can be used to remove one extra degree of freedom from ϵ^μ. For example, in the *Coulomb gauge* we set

$$A^0 = 0. \tag{3.19}$$

3.2 The Dirac Equation in an Electromagnetic Field

We will now treat A^μ as a quantum mechanical wave function for photons. In the limit of a large number of photons the wave function is interpreted as a number density and produces the classical wave theory. But so far we have no interactions; to allow electrons to interact with electromagnetism we have to include the photon field into our Dirac equation.

The 'obvious' thing to do is to just be led by Lorentz invariance. The field A^μ is a vector field so we need to 'soak up' its free index with a γ-matrix. We therefore include it into the Dirac equation as

$$(i\gamma^\mu \partial_\mu - e\gamma_\mu A^\mu - m)\,\psi = 0, \tag{3.20}$$

where the factor of e is a free constant which quantifies how strongly the electron couples to the photon (the charge of the electron is $-e$).

It is convenient to incorporate this extra term into a new definition of a *covariant derivative*[2],

$$D^\mu \equiv \partial^\mu + ieA^\mu. \tag{3.21}$$

Our interacting Dirac equation was therefore obtained from the free Dirac equation by the *minimal substitution* $\partial^\mu \rightarrow D^\mu$, and the Dirac equation becomes

$$(i\not{D} - m)\psi = 0. \tag{3.22}$$

There is a much nicer and theoretically much more appealing way to get the interaction term. That is if we require the QED Lagrangian to be invariant under a *local gauge symmetry* consisting of the transformations

$$\psi \rightarrow e^{-ie\Lambda(x)}\psi, \qquad A^\mu \rightarrow A^\mu - \partial^\mu \Lambda(x). \tag{3.23}$$

then we are forced to the wave equation in eq. (3.22). For more details, I refer you to the Standard Model course.

We must also allow the electrons to enter into the photon wave equation but here the classical theory already tells us how a current density enters. We expect

$$\partial^2 A^\mu = J^\mu \tag{3.24}$$

where J^μ is just given by the charge times the Dirac equation number density $(-e\bar{\psi}\gamma^\mu\psi)$.

[2]Conventions for the covariant derivative vary. *Halzen and Martin*, and *Mandl and Shaw* both use $D^\mu \equiv \partial^\mu - ieA^\mu$ whereas *Peskin and Schroeder* both use eq. (3.21). Both conventions define the electron charge to be $-e$ but differ by a sign in the definition of the photon field, A^μ.

3.3 $g-2$ of the Electron

We now have a wave equation which describes how an electron behaves in an electromagnetic field, i.e. eq. (3.20). We will immediately put this to use by investigating the interaction between the spin of a non-relativistic electron and a magnetic field.

Writing the electron field in the form of eq. (2.29), we see that eq. (3.20) gives

$$\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot(-i\vec{\nabla}-e\vec{A}) \\ \vec{\sigma}\cdot(-i\vec{\nabla}-e\vec{A}) & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \tag{3.25}$$

Substituting the equation from the second row into the that from the first leads to,

$$\left(E - m + \frac{\left[\vec{\sigma}\cdot(-i\vec{\nabla}-e\vec{A})\right]^2}{E+m} \right) \chi = 0. \tag{3.26}$$

We can simplify this somewhat by using to relation

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{3.27}$$

to show

$$\left[\vec{\sigma}\cdot\left(-i\vec{\nabla}-e\vec{A}\right)\right]^2 = |-i\vec{\nabla}-e\vec{A}|^2 - e\left(\vec{\nabla}\times\vec{A}+\vec{A}\times\vec{\nabla}\right)\cdot\vec{\sigma}, \tag{3.28}$$

and note

$$\vec{\nabla}\times\vec{A}\psi + \vec{A}\times\vec{\nabla}\psi = \left(\vec{\nabla}\times\vec{A}\right)\psi = \vec{B}\psi. \tag{3.29}$$

Putting all this together we find,

$$\left(E - m + \frac{|\vec{p}-e\vec{A}|^2 - e\vec{B}\cdot\vec{\sigma}}{E+m} \right) \chi = 0. \tag{3.30}$$

In the non-relativistic limit we can write $E \approx m$ and observe that the lower 2-component spinor is

$$\phi \approx \frac{\vec{\sigma}\cdot(\vec{p}-e\vec{A})}{2m}\chi \ll \chi. \tag{3.31}$$

This allows us to write, for the 4-component spinor ψ,

$$\frac{1}{2m}|\vec{p}-e\vec{A}|^2\psi - \frac{e\vec{B}\cdot\vec{\Sigma}}{2m}\psi = 0. \tag{3.32}$$

Notice that we have a coupling between the magnetic field $\vec{B}$ and the spin of the electron $\vec{S} = \frac{1}{2}\vec{\Sigma}$. This is known as a *magnetic moment interaction* and takes the form

$$-\vec{\mu}\cdot\vec{B}. \tag{3.33}$$

Our Dirac equation in an electromagnetic field has predicted

$$\vec{\mu} = -\frac{e}{2m}\vec{\Sigma}. \tag{3.34}$$

In classical physics the *magnetic moment* of an orbiting charge is written

$$\vec{\mu}_{\rm orb} = -\frac{e}{2mc}\vec{L}. \tag{3.35}$$

This is the magnetic moment associated with orbital angular momentum. By analogy we define the magnetic moment due to intrinsic angular momentum (i.e. spin) as

$$\vec{\mu}_{\rm spin} = -g\frac{e}{2m}\vec{S} = -\frac{g}{2}\frac{e}{2m}\vec{\Sigma} \tag{3.36}$$

where g is the *gyromagnetic ratio* of the particle. The Dirac equation predicts

$$g = 2. \tag{3.37}$$

Experimentally one finds for the electron that

$$g = 2.0023193043738 \pm 0.0000000000082, \tag{3.38}$$

so the Dirac equations prediction is pretty close. It is not exactly correct, as we can see from the incredible precision with which this quantity has been measured. The discrepancy is due to us not yet including quantum corrections to our prediction. The interaction of an electron with a photon (and thus the gyromagnetic ratio) will be changed by processes of the form seen in fig. 3, and processes involving yet more particle loops. When

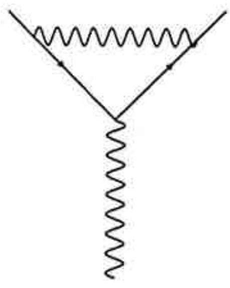


Figure 3: Quantum corrections to the electron-photon interaction.

one performs a more careful analysis, including these quantum effects, one predicts the deviation from 2 to be

$$\frac{g-2}{2} = 1 + \frac{\alpha}{2\pi} - 0.328\left(\frac{\alpha}{\pi}\right)^2 + 1.181\left(\frac{\alpha}{\pi}\right)^3 - 1.510\left(\frac{\alpha}{\pi}\right)^4 + \ldots + 4.393\times 10^{-12}, \tag{3.39}$$

and comparing this prediction with experiment:

$$\begin{aligned} \text{Theory}: \quad \frac{g-2}{2} &= 1159652140(28)\times 10^{-12}, \\ \text{Experiment}: \quad \frac{g-2}{2} &= 1159652186.9(4.1)\times 10^{-12}. \end{aligned} \tag{3.40}$$

The figure in brackets denotes the error on the last significant figure. We can see that the experimental measurement matches the theoretical prediction to 8 significant figures, making this prediction of QED the most precisely tested prediction in physics.

3.4 Interactions in Perturbation Theory

The principle technique for the computation of particle scattering cross-sections at high energies is perturbation theory. We assume that the coupling is small, i.e. $e \ll 1$ and expand our expressions around $e = 0$. We will be interested in processes such as that of fig. 4. Outside the shaded interaction region we assume the particles are free.

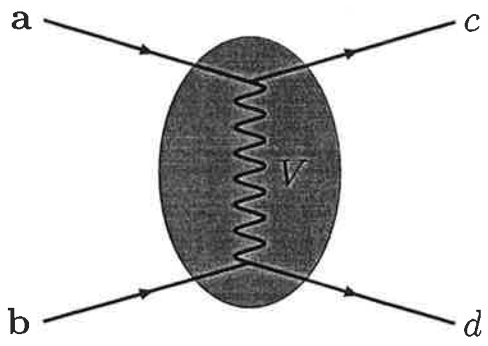


Figure 4: A $2 \to 2$ scattering.

The Dirac equation can be written,

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^i \partial_i \psi - m\psi - e\gamma^0\, V\, \psi = 0. \tag{3.41}$$

where the electromagnetic interaction is contained in V,

$$V = \gamma^0 \gamma^\mu A_\mu. \tag{3.42}$$

Note that $(\gamma^0)^2 = 1$ so the γ^0 have been included simply for notational convenience. It will be convenient to write this in terms of the *free* Dirac Hamiltonian (i.e. the Dirac Hamiltonian with *no* interaction term),

$$H_0 = -i\gamma^i \partial_i \psi + m\psi. \tag{3.43}$$

Then eq. (3.41) becomes

$$(H_0 + e\,V)\,\psi = i\frac{\partial \psi}{\partial t}. \tag{3.44}$$

We will assume that the scattering particles begin in a pure 4-momentum eigenstate and the interaction then scatters them to another 4-momentum eigenstate with some (small) probability. Let's denote this 4-momentum eigenstate, with 3-momentum $\vec{p}$, by $\Psi_{\vec{p}}(x)$ (we need only specify $\vec{p}$ since the energy is fixed by the on-shell condition $E^2 = m^2c^4 + |\vec{p}|^2c^2$). It is a solution of the free Dirac equation (since it is outside the interaction area), so,

$$H_0 \Psi_{\vec{p}}(x) = i\frac{\partial \Psi_{\vec{p}}(x)}{\partial t}. \tag{3.45}$$

These eigenstates form a complete set, so, in general, they may be used as a basis for writing any electron field, even inside the interaction area. We write

$$\psi(x) = \int d^3p\, \frac{1}{2E_{\vec{p}}}\, \kappa_{\vec{p}}(t) \Psi_{\vec{p}}(x). \tag{3.46}$$

The $\kappa_{\vec{p}}(t)$ are at this stage just coefficients which change with time, but we will later interpret their modulus-squared as a probability density for a final state of 3-momentum $\vec{p}$. The factor of $2E_{\vec{p}}$ in the denominator (where $E_{\vec{p}}$ is the energy associated with $\vec{p}$ by the on-shell condition) is to ensure that $\kappa_{\vec{p}}$ will have this interpretation. We substitute this expression into eq. (3.44), to give

$$\int d^3p \, \frac{1}{2E_{\vec{p}}} \left(i\gamma^0 \frac{\partial \kappa_{\vec{p}}}{\partial t} \Psi_{\vec{p}} + \kappa_{\vec{p}} i\gamma^0 \frac{\partial \Psi_{\vec{p}}}{\partial t} - \kappa_{\vec{p}} \gamma^0 H_0 \Psi_{\vec{p}} - e\gamma^0 V \kappa_{\vec{p}} \Psi_{\vec{p}} \right) = 0, \tag{3.47}$$

and note that the two middle terms cancel by eq. (3.45).

At time $t = -\infty$ the electron will be in a pure momentum eigenstate, so, if $\vec{p}_i$ is the initial momentum of the electron, then eq. (3.46) tells us that,

$$\kappa_{\vec{p}}(-\infty) = 2E_{\vec{p}} \, \delta^{(3)}(\vec{p} - \vec{p}_i). \tag{3.48}$$

If there were no interaction then these coefficients would not change, and if the interaction is small, they will only change very slightly. We can therefore perform an expansion around their unperturbed values in terms of the coupling, i.e. we write,

$$\kappa_{\vec{p}}(t) = \sum_{n=0}^{\infty} \tilde{\kappa}_{\vec{p}}^{(n)}(t) \, e^n. \tag{3.49}$$

We may now insert this into eq. (3.47) (with the two middle terms removed) and, since at this stage e is a free parameter, we will have an infinite tower of equations, one for each power of e. To leading order, i.e. e^0, the equation contains no dependence on V and is

$$\int d^3p \, \frac{1}{2E_{\vec{p}}} \left(i\gamma^0 \frac{\partial \tilde{\kappa}_{\vec{p}}^{(0)}}{\partial t} \Psi_{\vec{p}} \right) = 0, \tag{3.50}$$

which is simply a statement that the $\tilde{\kappa}_{\vec{p}}^{(0)}$ don't change. At order e^1 we have our first non-trivial equation,

$$\int d^3p \, \frac{1}{2E_{\vec{p}}} \left(i\gamma^0 \frac{\partial \tilde{\kappa}_{\vec{p}}^{(1)}}{\partial t} \Psi_{\vec{p}} - \gamma^0 V \tilde{\kappa}_{\vec{p}}^{(0)} \Psi_{\vec{p}} \right) = 0. \tag{3.51}$$

Now we will make use of the orthogonality of the momentum eigenstates $\Psi_{\vec{p}}$ to extract the coefficient $\tilde{\kappa}_{\vec{p}}^{(1)}$. We multiply through by $\Psi_{\vec{q}}^{\dagger}\gamma_0$, integrate over space, and use

$$\int d^3x \Psi_{\vec{q}}^{\dagger} \Psi_{\vec{p}} = 2E_{\vec{q}} \, \delta^{(3)}(\vec{p} - \vec{q}), \tag{3.52}$$

where $E_{\vec{q}}$ is the energy associated with $\vec{q}$, and we have used the usual normalisation of $2E$ particles in a unit box. Eq. (3.51) becomes,

$$i \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}}{\partial t} = \int d^3p \, \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(0)} \int d^3x \, \Psi_{\vec{q}} V \, \Psi_{\vec{p}}. \tag{3.53}$$

But we know that $\tilde{\kappa}_{\vec{p}}^{(0)}$ doesn't change with time, so to leading order, it must be given by eq. (3.48), allowing us to remove the integration over momentum states,

$$\frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}}{\partial t} = -i \int d^3x \, \Psi_{\vec{q}} V \, \Psi_{\vec{p}_i}. \tag{3.54}$$

We may now integrate this expression over time,

$$\int_{-\infty}^{\infty} dt \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}}{\partial t} = -i \int_{-\infty}^{\infty} dt \int d^3x \, \Psi_{\vec{q}} V \, \Psi_{\vec{p}_i} \tag{3.55}$$

$$\Rightarrow \tilde{\kappa}_{\vec{q}}^{(1)}(\infty) - \tilde{\kappa}_{\vec{q}}^{(1)}(-\infty) = -i \int d^4x \, \Psi_{\vec{q}} V \, \Psi_{\vec{p}_i} \tag{3.56}$$

Finally, we set $\vec{q}$ to the *final* electron momentum $\vec{p}_f$ and note that to order e^1, $\kappa_{\vec{p}_f} = e\tilde{\kappa}_{\vec{p}_f}^{(1)}$. Then $\tilde{\kappa}_{\vec{q}}^{(1)}(\infty) = \frac{1}{e}\kappa_{\vec{p}_f}$ and $\tilde{\kappa}_{\vec{q}}^{(1)}(-\infty) = 0$ if $\vec{p}_f \neq \vec{p}_i$. So, the transition amplitude κ_{fi} from an initial state ψ_i to a final state ψ_f is given by,

$$\kappa_{fi} = -i\, e \int d^4x \, \psi_f \, V \, \psi_i. \tag{3.57}$$

Now let's use our explicit form for V in QED and concentrate on the scattering of a particle $a \to c$ by a photon A^μ, as see in fig. 5. The transition amplitude is,

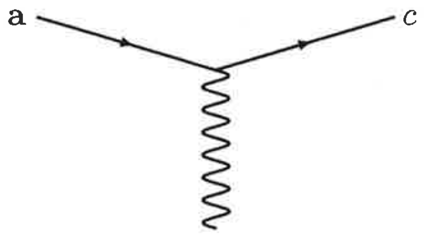


Figure 5: The scattering of a particle $a \to c$ by a photon A^μ.

$$\begin{aligned} \kappa_{ca} &= -i\, e \int \bar{\psi}_c \gamma_\mu A^\mu \psi_a \; d^4x \\ &= i \int J_\mu^{ca} A^\mu \; d^4x, \end{aligned} \tag{3.58}$$

where the current is,

$$J_\mu^{ca} = -e\, \bar{\psi}_c \gamma_\mu \psi_a = -e\, \bar{u}^c \gamma_\mu u^a \; e^{i(p_c - p_a)\cdot x}. \tag{3.59}$$

However, we are really interested in two particles scattering off each other so we'd better compute the A^μ field produced when another particle scatters from state $b \to d$, as seen in fig. 6. The photon field is given by,

$$\partial^2 A^\mu = J_{db}^\mu = -e\, \bar{u}_d \gamma^\mu u_b \, e^{i(p_d - p_b)\cdot x}, \tag{3.60}$$

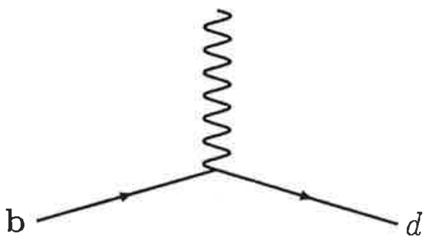


Figure 6: The photon field A^μ created by the current $a \to c$.

with solution

$$A^\mu = -\frac{1}{q^2} J^\mu_{db}, \quad \text{where } q = p_d - p_b. \tag{3.61}$$

Substituting this back into our expression for κ_{ca} we find

$$\kappa_{fi} = -i \int J^{ca}_\mu \frac{1}{q^2} J^\mu_{db}\, d^4x, \tag{3.62}$$

$$= -\bar{u}^c(ie\gamma^\mu)u^a \left(-i\frac{1}{q^2} g_{\mu\nu}\right) \bar{u}^d(ie\gamma^\nu)u^b \int e^{i(p_b+p_d-p_a-p_c)\cdot x} d^4x \tag{3.63}$$

The integral is just a delta function that ensures 4-momentum conservation in the interaction.

In order to make this result more memorable Feynman developed his famous rules that associate different parts of the expression with elements of a diagram of the scattering. For this precess, these factors are assigned as shown in fig. 7, with momentum

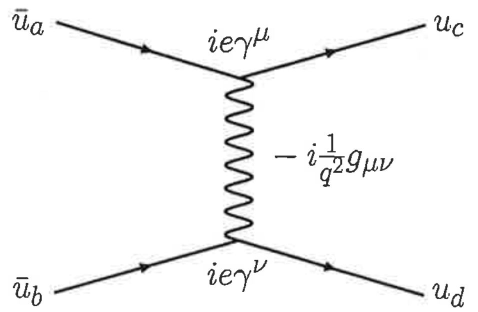


Figure 7: The scattering $ab \to cd$ and its associated Feynman terms.

conserved at the vertices. Applying these rules gives us $i\mathcal{M}_{fi}$ where

$$\kappa_{fi} = -i(2\pi)^4\delta^4(p_c + p_d - p_a - p_b)\mathcal{M}_{fi}. \tag{3.64}$$

▷ **Exercise 3.18**

Derive the Feynman rules for the scattering of two particles described by the Klein Gordon equation to leading order in e. You may assume the form of the result in eq. (3.57).

3.5 Internal Fermions and External Photons

We concentrated above on a scattering with external fermions interacting by the exchange of a photon. We can also imagine processes where there are external photon fields or internal virtual fermions. What are the Feynman rules for these cases? Given time constraints, rather than derive them, I'll present some simple arguments to motivate the rules.

If we have an external photon interacting with a fermion in some way, then the vertex rule is still $-ie\gamma^\mu$. Since the amplitude we wish to calculate is Lorentz invariant we can not allow a stray μ index to survive but must soak it up with a 4-vector. The obvious 4-vector associated with the external photon is its polarization vector ϵ^μ and indeed this is the appropriate factor for an external photon. Compare this to the way an external fermion closes the gamma matrix space indices, to give a number, with the external spinor.

We have seen that an internal photon (satisfying $\partial^2 A^\mu = 0$) generates a Feynman rule (or *propagator*),

$$\partial^2 A^\mu = 0 \qquad \rightarrow \qquad \frac{-ig_{\mu\nu}}{p^2} \tag{3.65}$$

Since a photon is just a collection of four scalar fields we can deduce that a massless, scalar field (which satisfies the KG equation $\partial\phi = 0$) will have a Feynman rule

$$\partial^2 \phi = 0 \qquad \rightarrow \qquad \frac{i}{p^2} \tag{3.66}$$

The sign is that of a space-like photon degree of freedom.

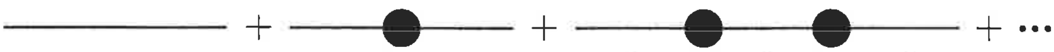
To find the propagator of a massive scalar field we can treat the mass as a perturbing interaction of the free particle. Writing the KG equation as

$$\partial^2 \phi = V\phi = -m^2\phi \tag{3.67}$$

will generate a Feynman rule for the scalar self interaction

$-im$

Now we can consider the set of perturbation theory diagrams that contribute to the full scalar propagator



$$\frac{i}{p^2} \quad + \quad \frac{i}{p^2}(-im)\frac{i}{p^2} \quad + \quad \frac{i}{p^2}(-im)\frac{i}{p^2}(-im)\frac{i}{p^2} \quad + \cdots$$

We can resum this series,

$$\frac{i}{p^2}\left(1 + \frac{m^2}{p^2} + \frac{m^4}{p^4} + ...\right) = \frac{i}{p^2}\left(\frac{1}{1 - \frac{m^2}{p^2}}\right) = \frac{i}{p^2 - m^2}, \tag{3.68}$$

and this is indeed the full propagator in the massive case.

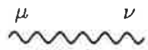
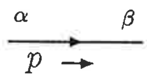
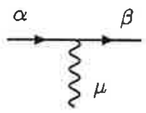
We can see that the propagator is just $-i$ times the inverse of the free field equation operator in momentum space. A sensible guess for the fermionic field is

$$(i\partial\!\!\!/ - m)\psi = 0 \qquad \longrightarrow \qquad \frac{i}{p\!\!\!/ - m} = \frac{i}{p\!\!\!/ - m}\frac{p\!\!\!/ + m}{p\!\!\!/ + m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}. \tag{3.69}$$

This is in fact the correct answer. You will gain more insight into these results from the Field Theory course.

3.6 Summary of Feynman Rules of QED

The Feynman rules for computing the amplitude $\mathcal{M}_{fi}$ for an arbitrary process in QED are summarized in Table 1.

For every ...	draw ...	write ...
Internal photon line		$\dfrac{-ig^{\mu\nu}}{p^2 + i\epsilon}$
Internal fermion line		$\dfrac{i(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$
Vertex		$ie\gamma^{\mu}_{\alpha\beta}$
Outgoing electron		$\bar{u}_\alpha(s,p)$
Incoming electron		$u_\alpha(s,p)$
Outgoing positron		$v_\alpha(s,p)$
Incoming positron		$\bar{v}_\alpha(s,p)$
Outgoing photon		$\varepsilon^{*\mu}(\lambda,p)$
Incoming photon		$\varepsilon^{\mu}(\lambda,p)$

- Attach a directed momentum to every internal line
- Conserve momentum at every vertex, i.e. include $\delta^{(4)}(\sum p_i)$
- Integrate over all internal momenta

Table 1: Feynman rules for QED. μ, ν are Lorentz indices, α, β are spinor indices and s and λ fix the polarization of the electron and photon respectively.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 1. This will become clear in the examples which follow. We have already met the Dirac spinors u and v. I will say more about the photon polarization vector ε

when we need to use it. Lastly, the $i\epsilon$ in the propagators is to avoid the singularity when $p^2 \rightarrow m^2$.

To summarize, the procedure for calculating the amplitude for any process in QED is:

1. Draw all possible distinct diagrams
2. Associate a directed 4-momentum with all lines
3. Apply the Feynman rules for the propagators, vertices and external legs
4. Ensure 4-momentum conservation at each vertex by adding $(2\pi)^4\delta^4(k_i - k_f)$,where k_i and k_f are the total incoming and outgoing 4-momenta of the vertex respectively
5. Perform the integration over all internal momenta with the measure $\int d^4k/(2\pi)^4$

It is also part of the Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is a reflection of Pauli's exclusion principle or equivalently of the anticommutation of the fermion operators discussed in the appendix. The Fermi statistics also dictate that we must multiply every closed fermion loop by -1. Note, however, that you don't need to get the absolute sign of any contribution to an amplitude right, just its sign relative to the other contributions, since it is the modulus of the amplitude squared that we ultimately need.

This sounds rather complicated. In particular there seem to be an awful lot of integrations to be done. However, at tree-level, i.e. if there are no loop diagrams, the delta functions attached to the vertices together with the integration over the internal momenta simply result in an overall 4-momentum conservation, i.e. a factor $(2\pi)^4\delta^4(P_i - P_f)$, where P_i and P_f are the total incoming and outgoing 4-momenta of the process. Thus at tree-level, no 'real' integration has to be done. At one loop, however, there is one non-trivial integration to be done. Generally, the calculation of an n-loop diagram involves n non-trivial integrations. Even worse, these integrals very often are divergent. Still, we can get perfectly reasonable theoretical predictions at any order in QED. The procedure to get these results is called renormalization and will be the topic of section 6.

At this point, some remarks concerning step 1, i.e. drawing all possible distinct Feynman diagrams, might be useful. In order to establish whether two diagrams are distinct, we have to try to convert one into the other. If this is possible without cutting lines or joining lines – that is solely by twisting and stretching the lines and rotating the whole figure – then the two diagrams are identical. It should be noted that the external lines are labeled in this process. Therefore, the two diagrams shown in fig, 10 are different. Finally, let me mention that the diagrams shown in fig. 1 are not Feynman diagrams. When drawing Feynman diagrams we are only interested in which particles are incoming and which are outgoing and there is no time direction involved.

4 Cross Sections and Decay Rates

Before explicitly calculating transition amplitudes we will examine how to connect those amplitudes to physical observables such as cross sections and particle widths.

4.1 The Transition Rate

Consider an arbitrary scattering process with an initial state i with total 4-momentum p_i and a final state f with total 4-momentum p_f. Let's assume we computed the scattering amplitude for this process in quantum field theory, i.e. we know the matrix element

$$-i\mathcal{M}_{fi}(2\pi)^4\delta^4(p_f - p_i). \tag{4.1}$$

Our task in this section is to convert this into a scattering cross section (relevant if there is more than 1 particle in the initial state) or a decay rate (relevant if there is just 1 particle in the initial state), see fig. 8.

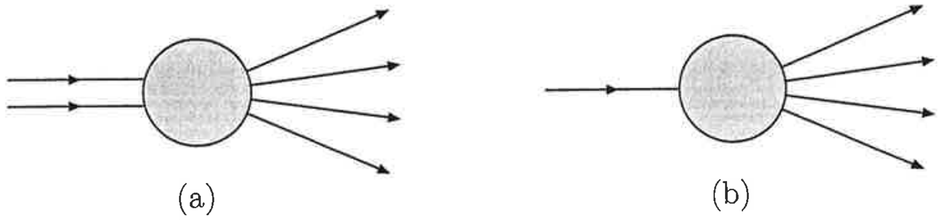


Figure 8: Scattering (a) and decay (b) processes.

The probability for the transition to occur is the square of the matrix element, i.e.

$$\text{Probability } = |-i\mathcal{M}_{fi}(2\pi)^4\delta^4(p_f - p_i)|^2. \tag{4.2}$$

Attempting to take the squared modulus of the amplitude produces a meaningless square of a delta function. This is a technical problem because our amplitude is expressed between plane wave states. These states are states of definite momentum and so extend throughout all of space-time. In a real experiment the incoming and outgoing states are localized (e.g. they might leave tracks in a detector). To deal with this properly we would have to construct normalized wave packet states which do become well separated in the far past and the far future. Instead of doing this we will do a much simpler but rather sloppy derivation. First of all, we will put our system in a box of volume $V = L^3$. We also imagine that the interaction is restricted to act only over a time of order T. The final answer is independent of V and T, reproducing the ones we would get if we worked with localized wave packets. Using

$$(2\pi)^4\delta^{(4)}(p_f - p_i) = \int e^{i(p_f - p_i)x}\, d^4x \tag{4.3}$$

we get in our space-time box the result

$$|(2\pi)^4\delta^{(4)}(p_f - p_i)|^2 \simeq (2\pi)^4\delta^{(4)}(p_f - p_i)\int e^{i(p_f - p_i)x}\, d^4x \simeq VT\,(2\pi)^4\delta^{(4)}(p_f - p_i). \tag{4.4}$$

The transition rate per unit volume per unit time W is then

$$W = \frac{|\kappa_{fi}|^2}{VT} = |\mathcal{M}_{fi}|^2(2\pi)^4\delta^{(4)}(p_f - p_i). \tag{4.5}$$

As expected, the dependence on V and T is canceled. Usually we are interested in much more detailed information than just the total transition rate. We want to know the differential transition rate dW, i.e. the transition rate into a particular element of the final state phase space per particle. To get dW we have to multiply by the number of states available to a particle in the (small) part of phase space under consideration.

For a particle in a box of side L, recall that boundary conditions will constrain the particle's momentum, $\vec{p}$, such that its components can only take discrete values $p_i = 2\pi n_i/L$, $n_i \in \mathbf{Z}$. Therefore the number of states available to each particle in the box is

$$dn = dn_x dn_y dn_z = \left(\frac{L}{2\pi}dp_x\right)\left(\frac{L}{2\pi}dp_y\right)\left(\frac{L}{2\pi}dp_z\right) = V\frac{d^3p}{(2\pi)^3} \tag{4.6}$$

where $V = L^3$ is the volume of the box, which we now set to the unit volume. However, this is the number states available to a single particle. Recall our normalisation is that we have $2E$ particles per unit volume. Therefore, the number of states available *per particle* is,

$$dn = \frac{1}{2E}\frac{d^3p}{(2\pi)^3}. \tag{4.7}$$

Notice that this is a covariant measure since,

$$\int \frac{1}{2E}\frac{d^3p}{(2\pi)^3} = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta\left(p^2 - m^2\right). \tag{4.8}$$

For an N particle final state, we have N momenta, $\vec{p}_k$, $1 < k < N$, with each state normalised to $2E_k$ particles per unit volume. The obvious generalization is,

$$dn = \prod_{k=1}^{N} \frac{1}{2E_k}\frac{d^3p_k}{(2\pi)^3}. \tag{4.9}$$

The transition rate for transitions into a particular element of final state phase space is thus given by combining eqs. (4.5) and (4.9),

$$\begin{aligned} dW &= |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{k=1}^{N} \frac{1}{2E_k}\frac{d^3p_k}{(2\pi)^3} \\ &= |\mathcal{M}_{fi}|^2 \, \mathrm{dLips}_N, \end{aligned} \tag{4.10}$$

where we have defined the *Lorentz invariant phase space* with N particles in the final state,

$$\begin{aligned} d\mathrm{Lips}_N &\equiv (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{k=1}^{N} \frac{d^3p_k}{(2\pi)^3\, 2E_k} \\ &= (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{k=1}^{N} 2\pi\delta\left(p_k^2 - m_k^2\right) \frac{d^4p_k}{(2\pi)^4}. \end{aligned} \tag{4.11}$$

4.2 Decay Rates

We turn now to the special case where we have only one particle with mass m in the initial state, i, and consider the decay of this particle into some final state, f. In this case, the transition rate is called the partial decay rate and denoted by $\Gamma_{i\to f}$. While we have included a correction for our normalization of the final state to ensure we are accounting for the correct number of particles, we have not yet done so for our initial state. Since we have $2E$ particles per unit volume in the initial state, we must divide dW by $2E_i$ to give,

$$\Gamma_{i\to f} = \frac{1}{2E_i}\int |\mathcal{M}_{fi}|^2 \mathrm{dLips} \tag{4.12}$$

For the important special case of two particles in the final state, there exists a particularly simple result. Consider the partial decay rate for a particle a of mass m_a and momentum p_a, into two particles b and c, with momenta p_a and p_b respectively. The Lorentz-invariant phase space is

$$\mathrm{dLips} = (2\pi)^4\delta^4(p_a - p_b - p_c)\frac{d^3\vec{p}_b}{(2\pi)^3\,2E_b}\frac{d^3\vec{p}_c}{(2\pi)^3\,2E_c}. \tag{4.13}$$

In the rest frame of a the four-vectors of the particles are,

$$p_a = (m, 0), \qquad p_b = (E_b, \vec{p}), \qquad p_c = (E_c, -\vec{p}). \tag{4.14}$$

We can eliminate one three-momentum in the phase space using three components of the δ-function,

$$\mathrm{dLips} = \frac{1}{(2\pi)^2}\delta(m_a - E_b - E_c)\frac{d^3p}{4E_bE_c}, \tag{4.15}$$

and hence the partial decay rate becomes,

$$\Gamma_{a\to bc} = \frac{1}{8m(2\pi)^2}\int |\mathcal{M}_{a\to bc}|^2\delta(m_a - E_b - E_c)\frac{|\vec{p}|^2}{E_bE_c}\,d|\vec{p}|\,d\Omega \tag{4.16}$$

where $d\Omega$ is the solid angle element for the angle of one of the outgoing particles with respect to some fixed direction. From the on-shell condition $E_b^2 = |\vec{p}|^2 + m_b^2$, we know $dE_b = |\vec{p}|d|\vec{p}|/E_b$ and similarly for c, so,

$$\begin{aligned} d(E_b + E_c) &= |\vec{p}|d|\vec{p}|\frac{E_b + E_c}{E_bE_c} \\ \Rightarrow \frac{|\vec{p}|^2}{E_bE_c}d|\vec{p}| &= \frac{|\vec{p}|}{E_b + E_c}d(E_b + E_c). \end{aligned} \tag{4.17}$$

Using this in eq. (4.16) and integrating over $(E_1 + E_2)$ we obtain the final result

$$\Gamma_{a\to bc} = \frac{1}{32\pi^2m_a^2}\int |\mathcal{M}_{a\to bc}|^2|\vec{p}|d\Omega. \tag{4.18}$$

The total decay rate of particle i is obtained by summation of the partial decay rates into all possible final states

$$\Gamma_{\mathrm{tot}} = \sum_f \Gamma_{if} \tag{4.19}$$

The total decay rate is related to the *mean life time* τ via $(\Gamma_{\text{tot}})^{-1} = \tau$. The *branching ratio* for the decay into a specific final state f is

$$B_f \equiv \frac{\Gamma_{if}}{\Gamma_{\text{tot}}}. \tag{4.20}$$

4.3 Cross Sections

The total cross section for a static target and a beam of incoming particles of momentum is defined as the total transition rate for a single target particle and a unit beam flux. The differential cross section is similarly related to the differential transition rate. However, we have calculated the differential transition rate with a choice of normalization corresponding to $2E$ 'target' particles in the box of unit volume, and a 'beam' corresponding also to $2E$ particles in the box. So our beam has N_a particles crossing unit area per unit time, where

$$N_a = 2E_a\,|\vec{v}|, \tag{4.21}$$

and $\vec{v}$ is the velocity of particles in the beam in the direction of the target. We must also take into account the stationary target which has

$$N_b = 2E_b \tag{4.22}$$

particles per unit volume. Thus, the *flux of the initial state* is,

$$F = N_a\,N_b = 4E_aE_b|\vec{v}| \tag{4.23}$$

and the differential cross section is related to the differential transition rate in eq. (4.10) by

$$d\sigma = \frac{dW}{F} = \frac{dW}{4E_aE_b|\vec{v}|}. \tag{4.24}$$

Now let us generalize to the case where we have two colliding beams. The first beam has a velocity $\vec{v}_a$ but now the 'target' particles are also moving with a velocity $\vec{v}_b$. In a colliding beam experiment $\vec{v}_a$ and $\vec{v}_b$ will point in opposite directions in the laboratory. In this case the definition of the cross section is retained as above, but now the beam flux of particles is effectively increased by the fact that the target particles are moving towards it. The flux is the total number of particles per unit area which run past each other per unit time,

$$F = 4E_aE_b|\vec{v}_a - \vec{v}_b|. \tag{4.25}$$

Notice that

$$F = 4\left(E_b|\vec{p}_a| - E_a|\vec{p}_b|\right) = 4\sqrt{\left[(p_a\cdot p_b)^2 - m_a^2m_b^2\right]}\,, \tag{4.26}$$

so this flux is covariant.

▷ **Exercise 4.19**
Prove eq. (4.26) in a frame where the momenta are collinear.

Finally, the differential cross-section for $2 \to N$ scattering becomes

$$\begin{aligned} d\sigma &= \frac{1}{F}|\mathcal{M}_{fi}|^2 d\mathrm{Lips}_N \\ &= \frac{1}{\sqrt{\left[(p_a\cdot p_b)^2 - m_a^2 m_b^2\right]}}|\mathcal{M}_{fi}|^2(2\pi)^4\delta^{(4)}(\sum_{k=1}^{N} p_k - p_a - p_b)\prod_{k=1}^{N}\frac{1}{2E_k}\frac{d^3p_k}{(2\pi)^3}, \quad (4.27)\end{aligned}$$

where p_a and p_b are the momenta of the initial state particles and p_k, $k = 1, ..., N$, are the momenta of the final state particles. The total cross-section can be obtained by integrating this expression over the final state momenta $\{p_k\}$.

Beware double counting when you have identical particles in the final state. If particles i and j are identical then a final state with $p_i = k$ and $p_j = q$ is the same as a final state with $p_i = q$ and $p_j = k$, so they should not be counted twice.

▷ **Exercise 4.20**
Show that in the centre-of-mass frame the differential cross section for the scattering $a(p_a) + b(p_b) \to c(p_c) + d(p_d)$ is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_c|}{|\vec{p}_a|}|\mathcal{M}_{fi}|^2. \quad (4.28)$$

4.4 Mandelstam Variables

Invariant $2 \to 2$ scattering amplitudes are frequently expressed in terms of the *Mandelstam variables.* These are defined by

$$\begin{aligned} s &\equiv (p_a + p_b)^2 = (p_c + p_d)^2, \\ t &\equiv (p_a - p_c)^2 = (p_b - p_d)^2, \\ u &\equiv (p_a - p_d)^2 = (p_b - p_c)^2. \quad (4.29)\end{aligned}$$

In fact there are only two independent Lorentz invariant combinations of the available momenta in this case, so there must be some relation between s, t and u.

▷ **Exercise 4.21**
Show that

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2. \quad (4.30)$$

▷ **Exercise 4.22**
Show that, for two body scattering of particles of equal mass m,

$$s \geq 4m^2, \qquad t \leq 0, \qquad u \leq 0.$$

(Hint: since all variables are invariant work in the centre of mass frame.)

5 Processes in QED and QCD

5.1 Electron–Muon Scattering

This is as simple a process as one can find since at lowest order in the electromagnetic coupling, just one diagram contributes. It is shown in figure 9. The amplitude obtained by applying the Feynman rules to this diagram is

$$i\mathcal{M}_{fi} = ie\,\bar{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)ie\,\bar{u}(p_d)\gamma^\nu u(p_b), \tag{5.1}$$

where $q^2 = (p_a - p_c)^2$. Note that, for clarity, I have dropped the spin label on the spinors. I will restore it when I need to. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication (which makes sense if you think of an unbarred spinor as a column vector and a barred spinor as a row vector and remember that the amplitude carries no spinor indices).

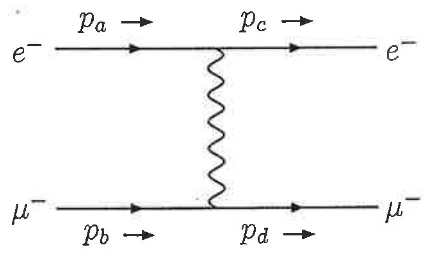


Figure 9: Lowest order Feynman diagram for $e^-\mu^- \to e^-\mu^-$ scattering.

The cross section involves the squared modulus of the amplitude, $|\mathcal{M}_{fi}|^2$. Let us see how we obtain a neat form for this. The hermitian conjugate of a 'spinor sandwich' is the same as its hermitian conjugate,

$$(\bar{u}(p_c)\gamma^\mu u(p_a))^* = (\bar{u}(p_c)\gamma^\mu u(p_a))^\dagger$$

since it is just a number. Using rules of matrix algebra we see that this is

$$\begin{aligned}(u(p_c)^\dagger\gamma^0\gamma^\mu u(p_a))^\dagger &= (u(p_a)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_c)) \\ &= (u(p_a)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_c)).\end{aligned} \tag{5.2}$$

But in section 2.6 we saw that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$, and so this becomes

$$(\bar{u}(p_c)\gamma^\mu u(p_a))^* = \bar{u}(p_a)\gamma^\mu u(p_c). \tag{5.3}$$

▷ **Exercise 5.23**

If Γ represents a string of γ-matrices (not including γ^5) and Γ_R is its reverse (i.e. the same γ-matrices in reverse order), show that,

$$[\bar{u}(k')\Gamma u(k)]^* = \bar{u}(k)\Gamma_R u(k').$$

Using this result in the expression for $|\mathcal{M}_{fi}|^2$ we obtain

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= \frac{e^4}{q^4}\bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_d)\gamma_\mu u(p_b)\bar{u}(p_a)\gamma^\nu u(p_c)\bar{u}(p_b)\gamma_\nu u(p_d) \\ &= \frac{e^4}{q^4} L^{\mu\nu}_{(e)} L_{(\mu)\,\mu\nu}, \end{aligned} \tag{5.4}$$

where the subscripts e and μ refer to the electron and muon respectively and

$$L^{\mu\nu}_{(e)} = \bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

Usually we have an unpolarized beam and target and do not measure the polarization of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different spin configurations are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the results

$$\begin{aligned} \sum_s u^{(s)}(p)\,\bar{u}^{(s)}(p) &= \not{p} + m, \\ \sum_s v^{(s)}(p)\,\bar{v}^{(s)}(p) &= \not{p} - m. \end{aligned} \tag{5.5}$$

Do not forget that by m, we really mean m times the unit 4×4 matrix.

▷ **Exercise 5.24**

Prove eq. (5.5).

Using the spin sums we find that

$$\begin{aligned} \sum_{\text{spins}} L^{\mu\nu}_{(e)} &= \sum_{s_a,s_c} \bar{u}^{(s_c)}_\alpha(p_c)\gamma^\mu_{\alpha\beta}u^{(s_a)}_\beta(p_a)\bar{u}^{(s_a)}_\rho(p_a)\gamma^\nu_{\rho\sigma}u^{(s_c)}_\sigma(p_c) \\ &= \gamma^\mu_{\alpha\beta}[\not{p}_a + m_e]_{\beta\rho}\gamma^\nu_{\rho\sigma}[\not{p}_c + m_e]_{\sigma\alpha} \\ &= \mathrm{Tr}\left(\gamma^\mu(\not{p}_a + m_e)\gamma^\nu(\not{p}_c + m_e)\right). \end{aligned} \tag{5.6}$$

where in the first line, we have make explicit the spinor indices in order to show how the trace emerges. Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of γ-matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook [1]. All these theorems can be derived from the fundamental anti-commutation relations of the γ-matrices in eq. (2.58) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use

$$\begin{aligned} \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd} \\ \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= g^{\mu_1\mu_2}\,\mathrm{Tr}(\gamma^{\mu_3}\ldots\gamma^{\mu_n}) - g^{\mu_1\mu_3}\,\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_n}) + \cdots \\ &\quad + g^{\mu_1\mu_n}\,\mathrm{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}}) \\ \mathrm{Tr}(\not{a}\not{b}) &= 4\,a\cdot b, \\ \mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a\cdot b\; c\cdot d - a\cdot c\; b\cdot d + a\cdot d\; b\cdot c). \end{aligned} \tag{5.7}$$

▷ **Exercise 5.25**
Derive the trace results in equation (5.7). (Hint: for the first one use $(\gamma^5)^2 = 1$.)

Using these trace theorems,

$$\sum_{\text{spins}} L_{(e)}^{\mu\nu} = 4(p_a^\mu p_c^\nu - g^{\mu\nu} p_a \cdot p_c + p_a^\nu p_c^\mu) + 4g^{\mu\nu} m_e^2, \tag{5.8}$$

with a similar result for $L_{(\mu)}^{\mu\nu}$. Putting this altogether, the spin averaged/summed amplitude squared is

$$\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \\
&= \frac{e^4}{q^4} 4\left(p_a^\mu p_c^\nu + p_a^\nu p_c^\mu - \left(p_a \cdot p_c - m_e^2\right) g^{\mu\nu}\right)\left(p_{b\,\mu} p_{d\,\nu} + p_{b\,\nu} p_{d\,\mu} - \left(p_b \cdot p_d - m_\mu^2\right) g_{\mu\nu}\right) \\
&= 8\frac{e^4}{q^4}\left((p_c \cdot p_d)(p_a \cdot p_b) + (p_c \cdot p_b)(p_a \cdot p_d) - m_e^2(p_b \cdot p_d) - m_\mu^2(p_a \cdot p_c) + 2m_e^2 m_\mu^2\right).
\end{aligned} \tag{5.9}$$

(Notice that we have divided by 4 since we are *averaging* over initial states, and there are 4 possible initial spin configurations.)

This takes on a more compact form if expressed in terms of the Mandelstam variables of eq. (4.29),

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{2e^4}{t^2}(s^2 + u^2 - 4(m_e^2 + m_\mu^2)(s+u) + 6(m_e^2 + m_\mu^2)^2). \tag{5.10}$$

Finally, we can derive the differential cross section for this process in the centre-of-mass frame using eq. (4.28). In the high energy limit where $s, |u| \gg m_e^2, m_\mu^2$, i.e. setting the masses to zero,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 s}\frac{s^2 + u^2}{t^2}. \tag{5.11}$$

Other calculations of cross sections or decay rates will follow the same steps we have used above. Draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more complications to be aware of, which we will illustrate below.

5.2 Electron–Electron Scattering

For the scattering $e^-e^- \to e^-e^-$ we now have identical particles in the final state which may only be distinguished by their momenta. Therefore we cannot just replace m_μ by m_e in the calculation we performed above. Labeling the momenta in the process according to $e^-(p_a) + e^-(p_b) \to e^-(p_c) + e^-(p_d)$ in analogy to $e^-\mu^-$ scattering, we realize that when particle a emits a photon we do not know whether it 'becomes' particle c (as it did in the $e^-\mu^-$ scattering) or 'becomes' particle d. Since either is possible, we need to include both cases, resulting in the two diagrams of fig. 10. Applying the Feynman rules, the

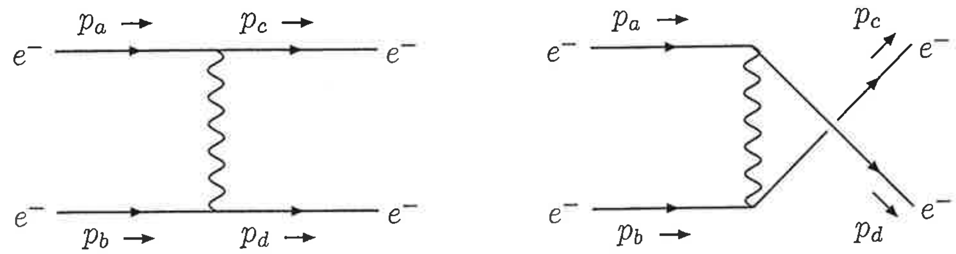


Figure 10: Lowest order Feynman diagrams for electron–electron scattering.

two diagrams give the amplitudes,

$$i\mathcal{M}_1 = \frac{ie^2}{t}\bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_d)\gamma_\mu u(p_b), \tag{5.12}$$

$$i\mathcal{M}_2 = -\frac{ie^2}{u}\bar{u}(p_d)\gamma^\mu u(p_a)\bar{u}(p_c)\gamma_\mu u(p_b). \tag{5.13}$$

Notice the additional minus sign in the second amplitude, which comes from the anti-commuting nature of fermion fields. Remember that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= |\mathcal{M}_1 + \mathcal{M}_2|^2 \\ &= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\mathrm{Re}\mathcal{M}_1^*\mathcal{M}_2, \end{aligned} \tag{5.14}$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams. $|\mathcal{M}_1|^2$ and $|\mathcal{M}_2|^2$ are very similar to the previous calculation. The interference term is a little more complicated due to a different trace structure.

Performing the calculation explicitly yields (in the limit of negligible fermion masses),

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu}\right). \tag{5.15}$$

▷ **Exercise 5.26**

Prove the result in eq. (5.15). It will be helpful first to prove

$$\begin{aligned} \gamma^\alpha\gamma^\mu\gamma_\alpha &= -2\gamma^\mu \\ \gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha &= 4g^{\mu\nu} \\ \gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha &= -2\gamma^\rho\gamma^\nu\gamma^\mu. \end{aligned} \tag{5.16}$$

5.3 Electron–Positron Annihilation

The two diagrams e^+e^- scattering are shown in fig. 11, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in fig. 10 if you twist round the fermion lines. The fact that the diagrams are related in this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing*. For our particular example, the squared amplitude for $e^+e^- \to e^+e^-$ is related to that for $e^-e^- \to e^-e^-$ by performing the interchange $s \leftrightarrow u$. Hence, squaring the amplitude and doing the traces yields (again neglecting fermion mass terms)

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{u^2+t^2}{s^2} + \frac{2u^2}{ts}\right). \tag{5.17}$$

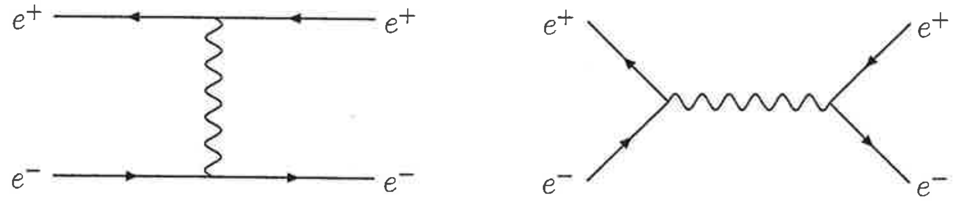


Figure 11: Lowest order Feynman diagrams for electron-positron scattering in QED.

If electrons and positrons collide and produce muon–antimuon or quark–antiquark pairs, then the annihilation diagram is the only one that contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{5.18}$$

where the sum is over quark flavours f and Q_f is the quark's charge in units of e. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of R as your e^+e^- collider's CM energy rose through a threshold for producing a new quark flavour. If you did not know about colour, the height of the step would seem too large. At the energies used at LEP you have to remember to include the diagram with a Z replacing the photon.

Finally, we compute the total cross section for $e^+e^- \to \mu^+\mu^-$, neglecting the lepton masses. Here we only have the annihilation diagram, and for the amplitude, we get

$$\begin{aligned}\mathcal{M}_{fi} &= (-ie)^2\bar{u}(p_d)\gamma^\mu v(p_c)\frac{-ig_{\mu\nu}}{s}\bar{v}(p_a)\gamma^\nu u(p_b) \\ &= \frac{ie^2}{s}\bar{u}_d\gamma^\mu v_c\bar{v}_a\gamma_\mu u_b. \end{aligned} \tag{5.19}$$

Summing over final state spins and averaging over initial spins gives,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4s^2}\mathrm{Tr}(\gamma^\mu \not{p}_c \gamma^\nu \not{p}_d)\mathrm{Tr}(\gamma_\mu \not{p}_b \gamma_\nu \not{p}_a),$$

where we have neglected m_e and m_μ. Using the results in equation (5.7) to evaluate the traces gives,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{8e^4}{s^2}(p_a\cdot p_d\, p_b\cdot p_c + p_a\cdot p_c\, p_b\cdot p_d).$$

Neglecting masses we have,

$$p_a\cdot p_c = p_b\cdot p_d = -t/2, \quad (5.20)$$

$$p_a\cdot p_d = p_b\cdot p_c = -u/2. \quad (5.21)$$

Hence

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = 2e^4\frac{t^2+u^2}{s^2}, \quad (5.22)$$

which incidentally is what you get by applying crossing to the electron–muon amplitude of section 5.1. We can use this in eq. (4.28) to find the differential cross section in the CM frame,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 s}\frac{t^2+u^2}{s^2}. \quad (5.23)$$

You could get straight to this point by noting that the appearance of v spinors instead of u spinors in $\mathcal{M}_{fi}$ does not change the answer since only quadratic terms in m_μ survive the Dirac algebra and we go on to neglect masses anyway. Hence you can use the result of eq. (5.11) with appropriate changes.

Neglecting masses, the CM momenta are

$$p_a = \frac{1}{2}\sqrt{s}(1,\vec{e}) \qquad p_c = \tfrac{1}{2}\sqrt{s}(1,\vec{e}\,') \quad (5.24)$$

$$p_b = \frac{1}{2}\sqrt{s}(1,\vec{e}) \qquad p_d = \tfrac{1}{2}\sqrt{s}(1,\vec{e}\,') \quad (5.25)$$

which gives $t = -s(1-\cos\theta)/2$ and $u = -s(1+\cos\theta)/2$, where $\cos\theta = \vec{e}\cdot\vec{e}\,'$. Hence, finally, the total cross section is,

$$\sigma = \int_{-1}^{1}\frac{d\sigma}{d\Omega}2\pi d(\cos\theta) = \frac{4\pi\alpha^2}{3s}. \quad (5.26)$$

5.4 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in fig. 12. For unpolarized initial and/or final states, the cross section calculation involves terms of the form

$$\sum_{\lambda}\varepsilon^{*\mu}(\lambda,p)\,\varepsilon^{\nu}(\lambda,p), \quad (5.27)$$

where λ represents the polarization of the photon of momentum p. Since the photon is massless, the sum is over the two transverse polarization states, and must vanish when contracted with p_μ or p_ν. In principle eq. (5.27) is a complicated object. However, there is a simplification as far as the amplitude calculation is concerned. The photon is coupled to the electromagnetic current $J^\mu = \bar{\psi}\gamma^\mu\psi$ of eq. (2.28). This is a conserved current, i.e. $\partial_\mu J^\mu = 0$, and in momentum space $p_\mu J^\mu = 0$. Hence, any term in the polarization sum, eq. (5.27), proportional to p^μ or p^ν does not contribute to the cross section. This means that in calculations one can make the replacement

$$\sum_\lambda \varepsilon^{*\mu}(\lambda, p)\, \varepsilon^\nu(\lambda, p) \to -g^{\mu\nu}, \tag{5.28}$$

and we have a simple, Lorentz-covariant prescription.

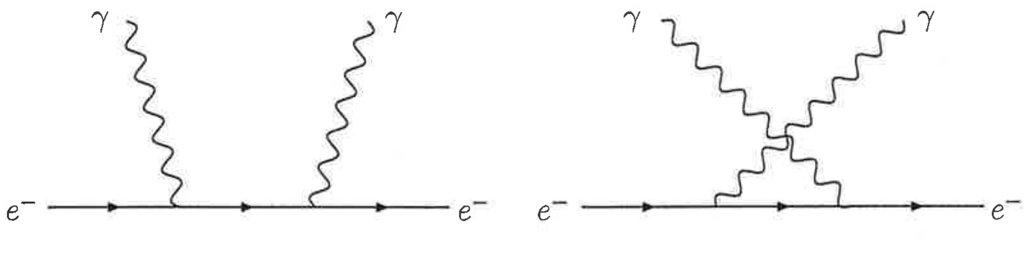


Figure 12: Lowest order Feynman diagrams for Compton scattering.

▷ **Exercise 5.27**
Show that the spin summed/averaged squared matrix element for Compton scattering in the massless limit is given by

$$|\mathcal{M}_{fi}|^2 = 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right) \tag{5.29}$$

Evaluate the total cross section using the expressions in the centre-of-mass frame at the end of the last sub-section. Why does this create a problem?

5.5 QCD Processes

The theory of quarks and gluons, QCD, is in many ways very similar to QED. We have done most of the hard work to calculate tree level amplitudes already. The main difference between the theories is that QCD has three types of charges (called 'colours', e.g. red, green and blue). We can write a quark as a vector with the three colour states shown

$$u = \begin{pmatrix} u^R \\ u^G \\ u^B \end{pmatrix} \tag{5.30}$$

There are more possible interactions than in QED which are mediated by eight photon-like gauge fields called "gluons". We encode the couplings of the gluons to the quarks by matrices which act on the above colour vector. For example there are two gluons with matrix "generators"

$$T^1 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.31}$$

These are just photon-like interactions with each of the two photons having different couplings to the different colours.

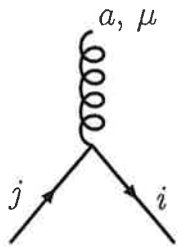
▷ **Exercise 5.28**
Check that the strength of a colour anti-colour quark pair scattering to itself at tree level is the same no matter which colour you pick. Show that the strength of a scattering of a colour anti-colour quark pair to a different colour pair is also the same no matter what colours you pick.

The remaining six gluons change the colour of the quark and are associated with generators of the form

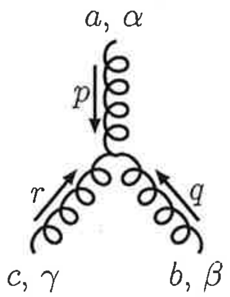
$$T^3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^4 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.32}$$

The remaining four generators are of the same form but interchange the other two colour combinations. Note these matrices are traceless and normalized so that $TrT^aT^b = \frac{1}{2}\delta^{ab}$.

You will learn more about the origin of these fields and their couplings in the Standard Model course. From the point of view of calculating cross sections though the Feynman Rules are all we need to proceed, and these are very similar to those of QED. The generator T^a is included in the Feynman rule for the gluon–quark–anti-quark vertex as shown in fig. 13 (upper), where g is the QCD coupling constant. Also, since a gluon



$$-igT^a\gamma^\mu$$



$$-gf^{abc}\left[(p-q)^\gamma g^{\alpha\beta} + (q-r)^\alpha g^{\beta\gamma} + (r-p)^\beta g^{\gamma\alpha}\right]$$

Figure 13: An example of some QCD Feynamn Rules.

associated with, for example, T^3 can pair produce a red quark and an anti-green quark we see that the gluons themselves are charged. Therefore gluons can interact with other gluons, and there are multi-gluon vertices that do not occur in QED where the photon is chargeless. The Feynman rule for these vertices are given in fig. 13 (lower), where f^{abc}, $a, b, c = 1, ..., 8$ are the QCD structure constants defined by

$$[T^a, T^b] = f^{abc}T^c, \tag{5.33}$$

The QCD Feynman rules will be discussed at greater length in the Standard Model course.

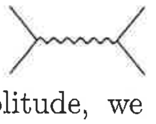
6 Introduction to Renormalization

6.1 Ultraviolet (UV) Singularities

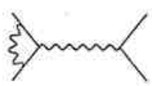
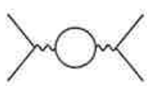
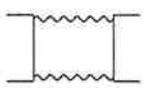
So far, everything was computed at tree-level, that is, at the lowest nontrivial order in perturbation theory. Very often, a more precise determination of a cross section is desirable and we are thus led to consider loop diagrams. In order to illustrate this, consider the example $e^+e^- \longrightarrow \mu^+\mu^-$. The perturbative expansion of the corresponding amplitude is written as

$$\mathcal{M} = \alpha\mathcal{M}_0 + \alpha^2\mathcal{M}_1 + \alpha^3\mathcal{M}_2 + \mathcal{O}(\alpha^4), \tag{6.1}$$

where $\alpha = \frac{e^2}{4\pi} \approx 1/137$. When we computed the corresponding amplitude in section 5.3 we only computed the leading order term

$\alpha\mathcal{M}_0 =$  $\propto e^2 \propto \alpha$ (6.2)

Using this expression for the amplitude, we will get the leading-order cross section $\sigma_0 \propto \alpha^2|\mathcal{M}_0|^2$. If we want to compute corrections of order α^3 to this result, we will have to compute the amplitude to an accuracy of order α^2.

$\mathcal{M} =$  $+$  $+$  $+$  $+ \dots$ (6.3)

In fact this set of diagrams is one place where the distinction between relativistic quantum mechanics and true field theory raises its head. The diagram with an internal quark loop is naturally generated in quantum field theory but not in a perturbative expansion in quantum mechanics. In principle, a quark must also be included in this loop, but in QM you have to treat the quark as an external particle that is put there by hand. While the Feynman rules we derived are correct, you will see a much more rigorous derivation of the (scalar theory) Feynman rules in your QFT course.

The one-loop correction to the cross section is related to the interference term of $\mathcal{M}_0$ and $\mathcal{M}_1$,

$$\sigma_1 \propto |\alpha\mathcal{M}_0 + \alpha^2\mathcal{M}_1 + \mathcal{O}(\alpha^3)|^2 = \alpha^2|\mathcal{M}_0|^2 + 2\alpha^3\mathrm{Re}(\mathcal{M}_0\,\mathcal{M}_1^*) + \mathcal{O}(\alpha^4). \tag{6.4}$$

The whole procedure looks pretty straightforward. However, if we try to compute a loop diagram, we run into trouble.

Consider as an example the vertex correction $\mathcal{V}$, depicted in fig. 14. Using the Feynman rules listed in section 3.6 we end up with an expression of the form

$$\mathcal{V} \propto \int \frac{d^4k}{(2\pi)^4} \frac{k\,k}{k^2((p_b+k)^2 - m^2)((p_a-k)^2 - m^2)} \tag{6.5}$$

where we did not bother to write down the full algebraic expression resulting from the spinor and Lorentz algebra but only the terms involving k. The two factors of k in the numerator stem from the two fermion propagators. The important point is that this integral diverges. Indeed, considering the limit $k \longrightarrow \infty$ we can neglect p_a, p_b and m and find

$$\mathcal{V} \sim \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^4} \sim \int \frac{dk}{(2\pi)}\frac{1}{k} = \infty \tag{6.6}$$

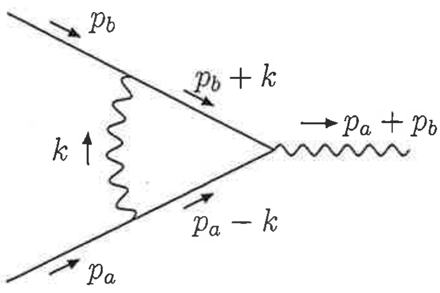


Figure 14: Vertex correction for $e^+e^- \longrightarrow \mu^+\mu^-$ scattering.

where we used $d^4k \sim k^3dk$. These singularities are called ultraviolet (UV) singularities because they come from the region $k \longrightarrow \infty$.

Similar problems are encountered if we try to compute the other one-loop diagrams and our final answer for the cross section at next-to-leading order seems to be infinity.

6.2 Infrared (IR) Singularities

There is another class of singularities that shows up in QED and QCD. As we saw in section 6.1 that UV singularities are related to the region of large k. However, there is also a potential danger of singularities from the region $k \sim 0$ or more generally, from zeros in the denominators of the integrand. These singularities are called infrared (IR) singularities. These occur if some (massless) particle becomes very soft or two become very collinear. These singularities have nothing to do with the UV singularities. The solution to the problem is completely different in the two cases. In fact, you already should have encountered an IR singularity. When you tried to compute the total cross section for Compton scattering in section 5.4 you should have found that the total cross section diverges. This is due to an IR singularity. Indeed, the final state photon can become arbitrarily soft, in which case the electron-photon pair becomes indistinguishable from a single electron. One possibility to get a well defined finite answer is to require that the final state photon has some minimal energy but the general solution will be discussed in the phenomenology course.

I will not discuss the IR singularities any further and will simply ignore them, safe in the knowledge that they can be dealt with in a manner totally different to that for the UV singularities. Thus in what follows I will call a cross section finite if it has no UV singularities, but it might well have IR singularities. Strictly speaking, we should replace every 'finite' below by 'UV-finite'.

6.3 Renormalization

It is important to realize that renormalization is not really about the removal of divergences, but simply an expression of the fact that in quantum field theories the value of certain parameters, e.g. the coupling constants, change with the energy scale used in a process. The infinities we encounter are then just a consequence of our ignorance of what is happening as $E \longrightarrow \infty$ although we integrate up to this limit in any loop diagrams. We will demonstrate this below, and show how results do turn out to be finite after all.

To to obtain a prediction for any measurable quantity $\mathcal{S}$, say a cross section, we started with wave equations from which we deduced the Feynman rules, which in turn were used to compute $\mathcal{S}$. The wave equations of QED, eqs. (3.22) and (3.24), have some parameters. So far, we denoted them by e, m and referred to them as mass and charge of the electron. Therefore, our result $\mathcal{S}$ will depend on these parameters. However, the parameter m in the Lagrangian is *not* the real mass of the electron, nor is e its charge. The identification of the parameter in the Lagrangian and the measurable quantity is only justified at tree level, because beyond this level the parameters themselves receive corrections, i.e. the propagator and vertex diagram which define the mass and coupling strength are themselves corrected. Therefore, from now on we will be more precise and denote the parameters in $\mathcal{L}$ by m_0 and e_0 and call them the *bare* mass and *bare* charge respectively. Note that the bare parameters are not measurable. The (measurable) physical mass and charge of the electron will be denoted (as always) by m and e. $\mathcal{L}$ also depends on the fields, which we denoted so far by ψ and A. From now on, we denote them by ψ_0 and A_0 and call them the bare fields.

We are now ready to reformulate the problem we encountered in section 6.1. If we try to compute a measurable quantity in terms of the unmeasurable bare quantities as a perturbative expansion in the coupling constant we generally encounter divergences. That is, if we compute

$$\mathcal{S}(e_0, m_0, \psi_0, A_0) = \mathcal{S}_0(e_0, m_0, \psi_0, A_0) + e_0^2\, \mathcal{S}_1(e_0, m_0, \psi_0, A_0) + \mathcal{O}(e_0^4) \tag{6.7}$$

then we may find that $\mathcal{S}_1(e_0, m_0, \psi_0, A_0) = \infty$. In particular, this is true for two special physical quantities, namely the mass and the charge of the electron,

$$\begin{aligned} m &= m_0 + e_0^2\, m_1(e_0, m_0, \psi_0, A_0) + \mathcal{O}(e_0^4) \\ e &= e_0 + e_0^2\, e_1(e_0, m_0, \psi_0, A_0) + \mathcal{O}(e_0^4). \end{aligned} \tag{6.8}$$

But this is an expression for two measurable quantities in terms of unknown parameters. If the unknowns m_0 and e_0 are finite then we would get divergences in m_1 and e_1 and hence in m and e. Since m and e are finite quantities we conclude that the bare quantities are infinite. This is the root of the problem. UV divergences in our perturbative calculations show up if we try to express our results in terms of the unmeasurable, unphysical bare parameters, i.e. the parameters of the original Lagrangian.

In order to save the situation, we have to find new parameters such that the result of any physical quantity expressed in these new parameters — at any order in perturbation theory — is finite. Is this possible? Generally, the answer is no. However, for some special theories (and luckily QED is one of them) it is possible. Such theories are called renormalizable theories. The new parameters are called the *renormalized* quantities and are denoted by e_R, m_R and ψ_R, A_R. They are related to the bare quantities as follows:

$$\begin{aligned} \psi_0 &= Z_2^{1/2}\psi_R \\ A_0 &= Z_3^{1/2}A_R \\ m_0 &= Z_m^{1/2}m_R \\ e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2} e_R \end{aligned} \tag{6.9}$$

This is simply a definition of the renormalization factors Z_1, Z_2, Z_3 and Z_m. Since the renormalization factors relate finite and divergent quantities, they have to be divergent themselves. More precisely, they can be written as a perturbative series with divergent coefficients.

To summarize, if we express the perturbative series for our physical quantity in terms of the renormalized quantities

$$\mathcal{S}(e_R, m_R, \psi_R, A_R) = \mathcal{S}_0(e_R, m_R, \psi_R, A_R) + e_R^2\, \mathcal{S}_1(e_R, m_R, \psi_R, A_R) + \mathcal{O}(e_R^4) \qquad (6.10)$$

there will be no UV-divergences at any order in perturbation theory. Some people refer to this as 'hiding the infinities'. What is meant by this statement is that if we have a small number of input values $(m_R, e_R \ldots)$ and express all results in terms of these input values we get finite answers for all measurable quantities. Thus, renormalizing QED enables us to relate any measurable quantity to a small number of measurable input values.

It is a highly non-trivial exercise to show that QED is indeed a renormalizable theory. But once we know that we can find a set of renormalized parameters e_R, m_R, ψ_R, A_R such that eq. (6.10) has finite coefficients at each order, it is clear that we can find as many other sets as we like. Indeed, if we chose $e'_R, m'_R, \psi'_R, A'_R$ such that m_R and m'_R (and all other parameters) are related by a finite series, then

$$\mathcal{S}'(e'_R, m'_R, \psi'_R, A'_R) = \mathcal{S}'_0(e'_R, m'_R, \psi'_R, A'_R) + (e'_R)^2\, \mathcal{S}'_1(e'_R, m'_R, \psi'_R, A'_R) + \mathcal{O}((e'_R)^4) \quad (6.11)$$

is also finite at each order in perturbation theory. In other words, the divergent pieces of the renormalization factors in eq. (6.9) are uniquely determined by requiring that the divergences cancel. However, we are completely free to fix the finite pieces to whatever we want. Choosing a particular set of renormalized quantities, that is, giving some prescription on how to fix the finite pieces of the renormalization factors, is called choosing the *renormalization scheme.* It is possible in QED that $m_R = m$ and $e_R = e$, i.e. the renormalized coupling is determined by real electron photon scattering. The renormalization scheme that satisfies these constraints is called the on-shell scheme. Alternatively, the renormalized coupling may be determined by scattering with, for example, a virtual photon. In this case the value of e_R will depend on the scale of the scattering, i.e. the coupling will "run" with the renormalization scale. To be precise let me also mention that one more constraint is needed to fix the scheme completely. Naively you would expect that four constraints are needed, since we have four renormalization factors to fix. However, two of them are related, $Z_1 = Z_2$. This identity is due to gauge invariance and is called the Ward identity. As a result, we only need three constraints to fix the renormalization scheme completely.

▷ **Exercise 6.29**

Why is it not possible in QCD to use the on-shell scheme?

Of course, the result of our calculation has to be independent of the renormalization scheme. This remark is not quite as innocuous as it looks. In fact, it is only true up to the order to which we decided to compute. If we decide to include the $\mathcal{O}(e_R^2)$ but not the higher order terms in our calculation, we have

$$\mathcal{S}(e_R, m_R, \psi_R, A_R) - \mathcal{S}'(e'_R, m'_R, \psi'_R, A'_R) = \mathcal{O}(e_R^4) \qquad (6.12)$$

The numerical result for our prediction will depend on the renormalization scheme! Even though the difference is formally of higher order it still can be numerically significant, in particular in QCD.

It's worth stressing again that this ability to hide UV divergences in the couplings is not as conspiratorial as it at first seems. In the IR a theory involves long wavelength modes that are insensitive to UV physics - indeed they (like us!) don't even know what the full UV theory of nature is. The incomplete IR theory will break down (generate infinities) if extended into the UV but since we know (presumably!) that the IR theory is part of a consistent UV theory there must be a way to hide the infinities. This is fundamentally why renormalization works.

6.4 Regularization

What we have learned so far is that we have to express the result of our calculation in terms of renormalized quantities rather than the bare ones. But since the starting point of any calculation is the Lagrangian, the first step in any calculation is to get the results in terms of bare quantities. Only then, we replace the bare quantities by the renormalized quantities, using eq. (6.9) and get a finite result. In intermediate steps we will have to deal with divergent expressions.

In order to give a mathematical meaning to these intermediate expressions, we will have to regularize the integrals. That is, we have to change them in a systematic way, such that they become finite. By doing so, we change the value of the integrals. However, at the end of our calculation, we are able to undo this change. Since the final result is finite, this step will not introduce a singularity.

There are — at least in principle — many different possibilities for regularizing the integrals. To illustrate the idea of regularization I will discuss first the method of introducing a cutoff, even though in practice this method is not really used. Consider again the vertex correction in eq. (6.5). As we saw, we got the UV singularity from the region $k \to \infty$. To regularize this expression, we introduce a cutoff Λ

$$\mathcal{V} \to \mathcal{V}_{\text{reg}} \sim \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{k\ k}{k^2((p_b+k)^2-m^2)((p_a-k)^2-m^2)} \tag{6.13}$$

Of course, by doing so we changed the value of the integral. At the end of our calculation we will have to let $\Lambda \to \infty$. Introducing this cutoff, however, gives us the possibility to deal with such intermediate expressions.

Let me illustrate the interplay between renormalization and regularization with an oversimplified example. Assume that with the cutoff regularization we get as a result of our calculation of some physical quantity, say a cross section

$$\mathcal{S} = e_0^4 A + e_0^6 \left(B \ln \frac{\Lambda}{m} + F_S \right) + \mathcal{O}(e_0^8) \tag{6.14}$$

where A, B and F_S are some *finite* terms. The originally divergent expression for $\mathcal{S}$ has been rendered finite by regularization. At this point we cannot let $\Lambda \to \infty$ since we would get $\mathcal{S} \to \infty$. However, we learned that we have to express our results in terms of e_R and not e_0 (For simplicity, I ignore the mass renormalization). This step is renormalization (not regularization). Computing the relation between e_0 and e_R, using the same regularization, we would find

$$e_R = e_0 - e_0^3\left(C\ln\frac{\Lambda}{m} + F_e\right) + \mathcal{O}(e_0^5) \tag{6.15}$$

and reversing this

$$e_0 = e_R + e_R^3\left(C\ln\frac{\Lambda}{m} + F_e\right) + \mathcal{O}(e_R^5) \tag{6.16}$$

where C and F_e are also finite. Plugging in eq. (6.16) into eq. (6.14) we get

$$\mathcal{S} = e_R^4 A + e_R^6\left((B+4AC)\ln\frac{\Lambda}{m} + F_S + 4AF_e\right) + \mathcal{O}(e_R^8) \tag{6.17}$$

and we would find $(B+4AC) = 0$. Since QED is a renormalizable theory this 'miracle' would happen for any measurable quantity. Finally, in the expression

$$\mathcal{S} = e_R^4 A + e_R^6 (F_S + 4AF_e) + \mathcal{O}(e_R^8) \tag{6.18}$$

we can let $\Lambda \to \infty$ and 'undo' the regularization.

To summarize, regularization enables us to work with divergent intermediate expressions. In the example above, instead of writing ∞ we write $\log\Lambda$ and have in mind $\Lambda \to \infty$. Renormalization, on the other hand removes the (would be) singularities, i.e. it removes the $\log\Lambda$ terms. Therefore, after renormalization we can (and have to) undo the regularization.

Note that we could have defined a different renormalized coupling

$$\tilde{e}_R = e_0 - e_0^3\left(C\ln\frac{\Lambda}{m} + G_e\right) + \mathcal{O}(e_0^5) \tag{6.19}$$

and this would have lead to

$$\mathcal{S} = \tilde{e}_R^4 A + \tilde{e}_R^6 (F_S + 4AG_e) + \mathcal{O}(\tilde{e}_R^8) \tag{6.20}$$

and we would have a different expression in terms of a different coupling - both equally valid, and identical up to the $\mathcal{O}(\tilde{e}_R^8)$ corrections.

As mentioned above, the method of introducing a cutoff for regularization is hardly ever used in actual calculations. The by far most popular method is to use dimensional regularization. The basic idea is to do the calculation not in 4 space-time dimensions but rather in D dimensions. Why does this help?

Consider once more our initial example of the vertex correction in eq. (6.5), which has an UV singularity in $D = 4$ space-time dimensions (see eq. (6.6)). For arbitrary D, using $d^Dk \sim k^{D-1}dk$ we get

$$\mathcal{V} \sim \int \frac{d^Dk}{(2\pi)^4}\frac{1}{k^4} \sim \int \frac{dk}{(2\pi)} k^{D-5} \tag{6.21}$$

and the integral is UV-finite for say $D \le 3$. Thus changing the dimension can regulate integrals. It is important to note that this is only a technicality. There is no Physics associated with $D \neq 4$ and at the end of the calculation we have to let $D \to 4$. If we did renormalize our theory properly this last step will not lead to UV divergences.

The reason why dimensional regularization is so popular is that it preserves gauge invariance and is technically relatively simple. Another very important issue is that this regularization not only regulates UV singularities, but also IR singularities. As mentioned in section 6.2, theories like QED or QCD are very often plagued by IR singularities. It is therefore very convenient if we do not have to introduce another regularization for IR singularities. Only after all UV and IR singularities have been removed, we can let $D \to 4$ and finally obtain a finite result.

7 QED as a Field Theory

7.1 Quantizing the Dirac Field

In this section we return to the Dirac equation and use it as the basis for a a field theory, which allows the creation and annihilation of particles naturally. Quantizing a field (or second quantization) basically means that the wave function becomes an operator. The space in which this operator acts is called the Fock space. The Fock space contains states with an arbitrary number of particles and therefore we will be able to describe processes where the number of states changes.

Dirac field theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{7.1}$$

Then the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{7.2}$$

leads to the Dirac equation. The canonical momentum is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = i\psi^\dagger(x) \tag{7.3}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial\psi}{\partial t}. \tag{7.4}$$

Now we want to regard ψ as a quantum field rather than as a wave function. In order to quantize this field, naively we would try to impose the usual equal time commutation relations, i.e.

$$\begin{aligned}
[\psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)] &= i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}), \\
[\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)] &= 0, \\
[\pi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)] &= 0,
\end{aligned} \tag{7.5}$$

where α and β label the spinor components of ψ and π. Without proving it for the moment we note that this would lead to a disaster. In particular, the Hamiltonian is unbounded from below - there is no ground state. The only way to cure the problem is to impose anti-commutation relations (we will soon see that this leads to the desired properties for spin-1/2):

$$\begin{aligned} \{\psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\} &= i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \\ \{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} &= 0, \\ \{\pi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\} &= 0. \end{aligned} \tag{7.6}$$

There is a very nice discussion in Peskin & Schroeder on this (Chapter 3). In particular, they show how anti-commutation relations really are the only solution.

The Heisenberg equations of motion for the field operators have the solution

$$\psi_\alpha(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b(s,\vec{k})u_\alpha(s,\vec{k})e^{-ik\cdot x} + d^\dagger(s,\vec{k})v_\alpha(s,\vec{k})e^{ik\cdot x}] \tag{7.7}$$

$$\bar{\psi}_\alpha(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b^\dagger(s,\vec{k})\bar{u}_\alpha(s,\vec{k})e^{ik\cdot x} + d(s,\vec{k})\bar{v}_\alpha(s,\vec{k})e^{-ik\cdot x}] \tag{7.8}$$

Since ψ is now an operator, so are the expansion coefficients $b^\dagger, d^\dagger, b$ and d. They are interpreted as creation and annihilation operators for electrons and positrons respectively. The anti-commutation relations for the fields, eq. (7.6), imply that

$$\begin{aligned} \left\{b(r,\vec{k}), b^\dagger(s,\vec{k}')\right\} &= (2\pi)^3\, 2E\, \delta^3(\vec{k}-\vec{k}')\delta_{sr} \\ \left\{d(r,\vec{k}), d^\dagger(s,\vec{k}')\right\} &= (2\pi)^3\, 2E\, \delta^3(\vec{k}-\vec{k}')\delta_{sr} \\ \left\{b(r,\vec{k}), b(s,\vec{k}')\right\} &= \left\{b^\dagger(r,\vec{k}), b^\dagger(s,\vec{k}')\right\} = 0 \\ \left\{d(r,\vec{k}), d(s,\vec{k}')\right\} &= \left\{d^\dagger(r,\vec{k}), d^\dagger(s,\vec{k}')\right\} = 0 \end{aligned} \tag{7.9}$$

▷ **Exercise 7.30**

Show that the anticommutation relations above lead to the correct anticommutation relations for the fields $\psi_\alpha(\vec{x},t)$ and $\pi_\beta(\vec{x},t)$. You will need the spinor sum relations in eq. (5.5).

The total Hamiltonian is

$$H = \int d^3\vec{x}\ :\mathcal{H}: \tag{7.10}$$

The symbols : : denote normal ordering of the operator inside, i.e. we put all creation operators to the left of all annihilation operators so that $H|0\rangle = 0$ by definition, and is the way we remove the ambiguity associated with the order of operators. Note that if we move an anti-commuting (fermion) operator through another such operator then we pick up a minus sign. Using eq. (7.4) after some algebra we get

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E}E\sum_{s=1,2}[b^\dagger(s,\vec{k})b(s,\vec{k}) + d^\dagger(s,\vec{k})d(s,\vec{k})]. \tag{7.11}$$

▷ **Exercise 7.31**
Verify the above form of the Hamiltonian. Can you see from the derivation why commutation relations for ψ and π and therefore for b and d would have led to a disaster?

The formula in eq. (7.11) has a very nice interpretation. The operator $b^\dagger b$ is nothing but the number operator for electrons and $d^\dagger d$ that for positrons. Thus, to get the total Hamiltonian, we have to count all electrons and positrons for all spin states s and momenta $\vec{k}$ and multiply this number by the corresponding energy E.

If we had tried to impose commutation relations, the $d^\dagger d$ term would have entered with a minus sign in front, which would signal that something has gone wrong. In particular, it would mean that $d^\dagger$ creates particles of negative energy. This is not supposed to happen in the quantized field theory. (We could try to fix the problem by simply re-labeling $d \leftrightarrow d^\dagger$ but it may be shown that this leads to acausal propagation.)

So, in order to quantize the Dirac field we are necessarily led to the introduction of anti-commutation relations. Remarkably we find that we have automatically taken into account the Pauli exclusion principle! For example,

$$\left\{b^\dagger(r,\vec{k}), b^\dagger(s,\vec{k}')\right\} = 0$$

implies that it is not possible to create two quanta in the same state, i.e.

$$b^\dagger(s,\vec{k})b^\dagger(s,\vec{k})|0\rangle = 0.$$

This intimate connection between spin and statistics is a direct consequence of desiring our theory to be consistent with the laws of relativity and quantum mechanics.

Finally consider the charge operator

$$Q = \int d^3\vec{x}\ :j_0(x): \ = \int d^3\vec{x}\ :\psi^\dagger\psi:$$

which, in terms of the creation and annihilation operators, is

$$Q = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b^\dagger(s,\vec{k})b(s,\vec{k}) - d^\dagger(s,\vec{k})d(s,\vec{k})] \tag{7.12}$$

This shows again that $b^\dagger$ creates fermions while $d^\dagger$ creates the associated antifermions of opposite charge.

7.2 Quantizing the Electromagnetic Field

The Maxwell equations can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\mu A^\mu \tag{7.13}$$

where the field strength tensor is

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{7.14}$$

and j_μ is a source for the field. Maxwell's equations do not change under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{7.15}$$

where $\Lambda(x)$ is some scalar field. This shows that there is some redundancy, and the 4 components of $A_\mu(x)$ are more than is required to describe the electromagnetic field (there are two transverse polarizations of e.m. radiation). This leads to a problem in quantization. To see this note that the canonically conjugate field to A_μ is

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} \tag{7.16}$$

and from this it follows that $\Pi^0 = 0$. This means there is no possibility of imposing a non-zero commutation relation between Π^0 and A^0, which we would need if we are to quantize the field.

To get around this problem we recognize that gauge invariance allows us to impose an extra condition, which we use to *fix* the gauge invariance, and effectively lower the degrees of freedom. For example, we can impose the Lorentz gauge condition, i.e.

$$\partial_\mu A^\mu = 0. \tag{7.17}$$

Note that, even after fixing the Lorentz gauge, we can perform another gauge transformation on A_μ, i.e. $A_\mu(x) \to A_\mu(x) + \partial_\mu \chi(x)$ where $\chi(x)$ must satisfy the wave equation, $\partial_\mu \partial^\mu \chi = 0$, i.e. we have two unphysical degrees of freedom and the two physical fields.

We impose the constraint by noting that since $\partial_\mu A^\mu = 0$, there is no harm in adding it to the Lagrangian density as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{7.18}$$

Indeed what we are doing here is following the Lagrange multiplier method of imposing constraints ($1/2\xi$ being the Lagrange multiplier), and recognizing that we should find the stationary points of $S = \int d^4x \mathcal{L}$ subject to the constraint $\int d^4x (\partial_\mu A^\mu)^2 = 0$, i.e. this comes from the "equation of motion" $\partial \mathcal{L}/\partial(1/2\xi) = 0$.

Using the gauge-fixed Lagrangian, the equations of motion are now

$$\partial^\mu F_{\mu\nu} - j_\nu + \frac{1}{\xi}\partial_\nu(\partial^\mu A_\mu) = 0.$$

If we require that these equations are satisfied *and then* also $\partial_\mu A^\mu = 0$, we have the original equations of motion but in a fixed gauge.

In the Feynman gauge $\xi = 1$, the Lagrangian is particularly simple (after some integration by parts under $\int d^4x$):

$$\mathcal{L} = \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - j_\mu A^\mu,$$

and quantization can now proceed: $\Pi^\mu = \partial_0 A^\mu$ and thus

$$[A^\mu(\vec{x}, t), \partial_0 A^\nu(\vec{y}, t)] = -i g^{\mu\nu} \delta^3(\vec{x} - \vec{y}) \tag{7.19}$$

with all other commutators vanishing. The Heisenberg operator corresponding to the photon field is

$$A_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E} \sum_{\lambda=0}^{3} \left[\varepsilon_\mu(\lambda, \vec{k}) a(\lambda, \vec{k}) e^{-ik\cdot x} + \varepsilon^*_\mu(\lambda, \vec{k}) a^\dagger(\lambda, \vec{k}) e^{ik\cdot x}\right] \tag{7.20}$$

where $\varepsilon_\mu(\lambda, \vec{k})$ are a set of four linearly independent basis 4-vectors for polarization ($\lambda = 0, 1, 2, 3$). For example, if $k = (k_0, \vec{k})$, we might choose $\varepsilon^\mu(0) = (1, 0, 0, 0)$, $\varepsilon^\mu(3) = (0, \vec{k})/k_0$, $\varepsilon^\mu(1) = (0, \vec{n}_1)$ and $\varepsilon^\mu(2) = (0, \vec{n}_2)$, where $k_0^2 = \vec{k}^2$, $\vec{n}_1 \cdot \vec{k} = 0$, $\vec{n}_2 \cdot \vec{k} = 0$ and $\vec{n}_1 \cdot \vec{n}_2 = 0$. $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ are therefore polarization vectors for transverse polarizations whilst $\varepsilon^\mu(0)$ is referred to as the timelike polarization vector and $\varepsilon^\mu(3)$ is referred to as the longitudinal polarization vector. For example, if $k = (k_0, 0, 0, k_0)$, $\varepsilon^\mu(0) = (1, 0, 0, 0)$, $\varepsilon^\mu(3) = (0, 0, 0, 1)$, $\varepsilon^\mu(1) = (0, 1, 0, 0)$ and $\varepsilon^\mu(2) = (0, 0, 1, 0)$.

The commutation relation (7.19) implies that

$$\left[a(\lambda, \vec{k}), a^\dagger(\lambda', \vec{k}')\right] = -g_{\lambda\lambda'}\, 2E\, (2\pi)^3 \delta^3(\vec{k} - \vec{k}'). \tag{7.21}$$

At a glance this looks fine, i.e. we interpret $a^\dagger(\lambda, \vec{k})$ as an operator that creates quanta of the electromagnetic field (photons) with polarization λ and momentum $\vec{k}$. However, for $\lambda = 0$ we have a problem since the sign on the RHS of (7.21) is opposite to that of the other 3 polarizations. This shows up in the fact that these timelike photons make a negative contribution to the energy:

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E} E \left(-a^\dagger(0, \vec{k}) a(0, \vec{k}) + \sum_{i=1,3} a^\dagger(i, \vec{k}) a(i, \vec{k}) \right). \tag{7.22}$$

Fortunately, although we might not realize it yet, we have already solved the problem. Recall that we still have to impose $\partial_\mu A^\mu = 0$. It turns out that it is impossible to do this at the operator level, but we can do it for all physical expectation values, i.e. we can impose the correct physics. It then turns out that contributions from the timelike and longitudinal photons always cancel. More explicitly, by demanding for any state $|\chi\rangle$ that

$$\langle \chi | \partial_\mu A^\mu | \chi \rangle = 0 \tag{7.23}$$

it follows that

$$\langle \chi | a^\dagger(3, \vec{k}) a(3, \vec{k}) - a^\dagger(0, \vec{k}) a(0, \vec{k}) | \chi \rangle = 0. \tag{7.24}$$

and therefore $\langle \chi | H | \chi \rangle \geq 0$. This is nice because it is in accord with our knowledge that free photons are transversely polarized.

▷ **Exercise 7.32**

Show that eq. (7.24) follows from eq. (7.23).

Acknowledgements

In all honesty, I cannot claim authorship of these notes. This lecture course has a long history; it was taught before me by Nick Evans, and I have only modified the LaTeX files with which he was kind enough to provide me. In turn, I understand that Nick built upon the foundation laid by Robert Thorne, Tim Morris, Jeff Forshaw, Adrian Signer and all the previous lecturers of this course who have been responsible for its development over the years.

Having said that, I have taken the liberty of introducing a few changes here and there. I have altered many of the conventions used (e.g. the normalization of spinors), included a discussion of time reversal symmetry and CPT, presented an alternate derivation of the Dirac equation's gyromagnetic ratio (and moved its position in the text), and reworked the discussion on time dependent perturbation theory. Any errors which I have managed to add while doing this (as well as any errors I have missed for sections left unchanged) are, of course, entirely my responsibility.

Many thanks also to Margaret Evans for the impeccable organization and to Tim Greenshaw for keeping us all under control. Thanks also to all the other lecturers and tutors and the students for making this school such a pleasant experience.

References

[1] J D Bjorken and S D Drell *Relativistic Quantum Mechanics* McGraw-Hill 1964
F Halzen and A D Martin *Quarks and Leptons* Wiley 1984
F Mandl and G Shaw *Quantum Field Theory* Wiley 1984
I J R Aitchison and A J G Hey *Gauge theories in particle physics* 2nd edition Adam Hilger 1989
M E Peskin and D V Schroeder *An Introduction to Quantum Field Theory* Addison Wesley 1995
B Hatfield *Quantum Field Theory of Point Particles and Strings* Addison Wesley 1992
L H Ryder *Quantum Field Theory* Cambridge University Press 1985
C Itzykson and J-B Zuber *Quantum Field Theory* McGraw-Hill 1987

Pre School Problems

Rotations, Angular Momentum and the Pauli Matrices

Show that a 3-dimensional rotation can be represented by a 3×3 orthogonal matrix R with determinant $+1$ (Start with $\vec{x}' = R\vec{x}$, and impose $\vec{x}' \cdot \vec{x}' = \vec{x} \cdot \vec{x}$). Such rotations form the special orthogonal group, $SO(3)$.

For an *infinitesimal* rotation, write $R = \mathbb{1} + iA$ where $\mathbb{1}$ is the identity matrix and A is a matrix with infinitesimal entries. Show that A is antisymmetric (the i is there to make A hermitian).

Parameterise A as

$$A = \begin{pmatrix} 0 & -ia_3 & ia_2 \\ ia_3 & 0 & -ia_1 \\ -ia_2 & ia_1 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} a_i L_i$$

where the a_i are infinitesimal and verify that the L_i satisfy the angular momentum commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k$$

Note that the Einstein summation convention is used here. Compute $L^2 \equiv L_1^2 + L_2^2 + L_3^2$. What is the interpretation of L^2 ?

The Pauli matrices σ_i are,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Verify that $\frac{1}{2}\sigma_i$ satisfy the same commutation relations as L_i.

Four Vectors

A Lorentz transformation on the coordinates $x^\mu = (ct, \vec{x})$ can be represented by a 4×4 matrix Λ as follows:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

For a boost along the x-axis to velocity v, show that

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (.25)$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$ as usual.

By imposing the condition

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu \qquad (.26)$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

show that

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g \Lambda = g$$

This is the analogue of the orthogonality relation for rotations. Check that it works for the Λ given by equation (.25) above.

Now introduce

$$x_{\mu} = g_{\mu\nu}x^{\nu}$$

and show, by reconsidering equation (.26) using $x^{\mu}x_{\mu}$, or otherwise, that

$$x'_{\mu} = x_{\nu}(\Lambda^{-1})^{\nu}{}_{\mu}$$

Vectors A^{μ} and B_{μ} that transform like x^{μ} and x_{μ} are sometimes called *contravariant* and *covariant* respectively. A simpler pair of names is *vector* and *covector*. A particularly important covector is obtained by letting $\partial/\partial x^{\mu}$ act on a scalar ϕ:

$$\frac{\partial \phi}{\partial x^{\mu}} \equiv \partial_{\mu}\phi$$

Show that ∂_{μ} does transform like x_{μ} and not x^{μ}.

Probability Density and Current Density

Starting from the Schrödinger equation for the wave function $\psi(\vec{x}, t)$, show that the probability density $\rho = \psi^*\psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \vec{J} = 0$$

where

$$\vec{J} = \frac{\hbar}{2im}[\psi^*(\nabla\psi) - (\nabla\psi^*)\psi]$$

What is the interpretation of $\vec{J}$? Verify that the continuity equation can be written in manifestly covariant form.

$$\partial_{\mu}J^{\mu} = 0$$

where $J^{\mu} = (c\rho, \vec{J})$.

THE STANDARD MODEL

By Dr S Davidson
Université C.B. Lyon

Lecture presented at the School for Experimental High Energy Physics Students
Rutherford Appleton Laboratory, September 2006

THE STANDARD MODEL

Sacha Davidson

IPNL, Université C.B. Lyon-1, 69622 Villeurbanne, France

Lectures presented at the School for Young High Energy Physicists,
Cosener's House, September 2006.

Contents

Introduction

An important feature of the Standard Model is that "it works": it is consistent with, or verified by, all available data. Secondly, it is a unified description, in terms of "gauge theories" of all the interactions of known particles (except gravity). A gauge theory is one that possesses invariance under a set of "local transformations" i.e. transformations whose parameters are space-time dependent.

Electromagnetism is a well-known example of a gauge theory. In this case the gauge transformations are local complex phase transformations of the fields of charged particles, and gauge invariance necessitates the introduction of a massless vector (spin-1) particle, called the photon, whose exchange mediates the electromagnetic interactions.

In the 1950's Yang and Mills considered (as a purely mathematical exercise) extending gauge invariance to include local non-abelian (i.e. non-commuting) transformations such as $SU(2)$. In this case one needs a set of massless vector fields (three in the case of $SU(2)$), which were formally called "Yang-Mills" fields, but are now known as "gauge fields".

In order to apply such a gauge theory to weak interactions, one considers particles which transform into each other under the weak interactions, such as a u-quark and a d-quark, or an electron and a neutrino to be arranged in doublets of weak isospin. The three gauge bosons are interpreted as the $W^{\pm}$ and Z bosons, that mediate weak interactions in the same way that the photon mediates electromagnetic interactions.

The difficulty in the case of weak interactions was that the weak interactions are known to be short range, mediated by very massive vector bosons, whereas Yang-Mills fields are required to be massless in order to preserve the gauge invariance. The apparent paradox was solved by the application of the "Higgs mechanism". This is a prescription for breaking the gauge symmetry spontaneously. In this scenario one starts with a theory that possesses the required gauge invariance, but the ground state of the theory is *not* invariant under the gauge transformations. The breaking of the invariance arises in the quantization of the theory, whereas the Lagrangian only contains terms which *are* invariant. One of the consequences of this is that the gauge bosons acquire a mass and the theory can thus be applied to weak interactions.

Spontaneous symmetry breaking and the Higgs mechanism has another extremely important consequence. It leads to a renormalizable theory with massive vector bosons. This means that one can carry out a programme of renormalization in which the infinities that arise in higher-order calculations can be reabsorbed into the parameters of the Lagrangian (as in the case of QED). Had one simply broken the gauge invariance explicitly by adding mass terms for the gauge bosons the resulting theory would not have been renormalizable and could not therefore have been used to carry out perturbative calculations. A consequence of the Higgs mechanism is the existence of a scalar (spin-0) particle, the Higgs boson.

The remaining step was to apply the ideas of gauge theories to the strong interactions. The gauge theory of strong interactions is called "Quantum ChromoDynamics" (QCD). In this theory the quarks possess an internal property called "color" and the gauge transformations are local transformations between quarks of different colors. The gauge bosons of QCD are called "gluons" and they mediate the strong interactions.

The union of QCD and the electroweak gauge theory, which describes the weak and electromagnetic interactions is known as the Standard Model. It has a very simple structure and the different forces of nature are treated in the same fashion, i.e. as gauge theories. It has eighteen fundamental parameters, most of which are associated with the masses of the gauge bosons, the quarks and leptons, and the Higgs. Nevertheless these are not all independent and, for example, the ratio of the W and Z boson masses are (correctly) predicted by the model. Since the theory is renormalizable, perturbative calculations can be performed that predict cross sections and decay rates both for strongly and weakly interacting processes. These predictions have met with considerable success when confronted with experimental data, which is becoming more and more precise, therefore posing increasingly stringent tests on the Standard Model.

1 QED as an Abelian Gauge Theory

The aim of this lecture is to start from a symmetry of the fermion Lagrangian, and show that "gauging" this symmetry (= making it well behaved) implies classical electromagnetism with its gauge invariance, the $e\bar{e}\gamma$ interaction, and that the photon should be massless.

1.1 Preliminaries

In the Field Theory lectures at this school, the quantum theory of an interacting scalar field was introduced, and the voyage from the Lagrangian to the Feynman rules was made. Fermions can be quantised in a similar way, and the propagators one obtains are the Greens functions for (inverse of) the Dirac wave equation of the QED/QCD course. In this course, I will start from the Lagrangian (as opposed to wave equation) of a free Dirac fermion, and add interactions, to construct the Standard Model Lagrangian in classical field theory. That is, the fields are treated as functions, and I will not discuss creation and annihiliation operators. However, to extract Feynman rules from the Lagrangian, I will implicitly rely on the rules developed for scalar fields in the Field Theory course.

1.2 Gauge Transformations

Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{1.1}$$

This Lagrangian density is invariant under a phase transformation of the fermion field

$$\psi \rightarrow e^{iQ\omega}\psi, \qquad \overline{\psi} \rightarrow e^{iQ\omega}\overline{\psi}, \tag{1.2}$$

where Q is the charge operator ($Q\psi = +\psi, Q\bar{\psi} = -\bar{\psi}$), ω is a real constant (i.e. independent of x) and $\overline{\psi}$ is the conjugate field.

The set of all numbers $e^{-i\omega}$ form a group[1]. This particular group is "abelian" which is to say that any two elements of the group commute. This just means that

$$e^{-i\omega_1}e^{-i\omega_2} = e^{-i\omega_2}e^{-i\omega_1}. \tag{1.3}$$

[1]A group is a mathematical term for a set, where multiplication of elements is defined and results in another element of the set. Furthermore, there has to be a 1 element (s.t. $1 \times a = a$) and an inverse (s.t. $a \times a^{-1} = 1$) for each element a of the set

This particular group is called $U(1)$ which means the group of all unitary 1×1 matrices. A unitary matrix satisfies $U^+ = U^{-1}$ with U^+ being the adjoint matrix.

We can now state the invariance of the Lagrangian eq. (1.1) under phase transformations in a more fancy way by saying that the Lagrangian is invariant under global $U(1)$ transformations. By global we mean that ω does not depend on x.

For the purposes of these lectures it will usually be sufficient to consider infinitesimal group transformations, i.e. we assume that the parameter ω is sufficiently small that we can expand in ω and neglect all but the linear term. Thus we write

$$e^{-i\omega} = 1 \; - \; i\,\omega \; + \; \mathcal{O}(\omega^2). \tag{1.4}$$

Under such infinitesimal phase transformations the field ψ changes according to

$$\psi \to \psi + \delta\psi = \psi + \; i\,Q\omega\,\psi, \tag{1.5}$$

and the conjugate field $\overline{\psi}$ by

$$\overline{\psi} \to \overline{\psi} + \delta\overline{\psi} \; = \overline{\psi} + i\,Q\omega\,\overline{\psi} = \overline{\psi} - \; i\,\omega\,\overline{\psi}, \tag{1.6}$$

such that the Lagrangian density remains unchanged (to order ω).

At this point we should note that global transformations are not very attractive from a theoretical point of view. The reason is that making the same transformation at every space-time point requires that all these points 'know' about the transformation. But if I make a certain transformation at the top of the Mont Blanc, how can a point somewhere in England know about it? It would take some time for a signal to travel from the Alps to England (and anyway, might it not want to stop of in France on the way?).

Thus, we have two options at this point. Either, we simply note the invariance of eq. (1.1) under global $U(1)$ transformations and put this aside as a curiosity, or we insist that invariance under gauge transformations is a fundamental property of nature. If we take the latter option we have to require invariance under local transformations. Local means that the parameter of the transformation, ω, now depends on the space-time point x. Such local (i.e. space-time dependent) transformations are called "gauge transformations".

If the parameter ω depends on the space-time point then the field ψ transforms as follows under infinitesimal transformations

$$\delta\psi(x) = i\,\omega(x)\,\psi(x); \qquad \delta\overline{\psi}(x) = -i\,\omega(x)\,\overline{\psi}(x). \tag{1.7}$$

Note that the Lagrangian density eq. (1.1) now is *no longer* invariant under these transformations, because of the partial derivative that is interposed between $\overline{\psi}$ and ψ. This derivative will act on the space-time dependent parameter $\omega(x)$ such that the Lagrangian density changes by an amount $\delta\mathcal{L}$, where

$$\delta\mathcal{L} \;=\; -\overline{\psi}(x)\,\gamma^{\mu}\,[\partial_{\mu}Q\omega(x)]\,\psi(x). \tag{1.8}$$

The square brackets in $[\partial_{\mu}\,\omega(x)]$ are introduced to indicate that the derivative ∂_{μ} acts only inside the brackets. It turns out that we can restore gauge invariance if we assume that the fermion field interacts with a vector field A_{μ}, called a "gauge field", with an interaction term

$$-\,e\,\overline{\psi}\,\gamma^{\mu}A_{\mu}Q\psi \tag{1.9}$$

added to the Lagrangian density which now becomes

$$\mathcal{L} = \overline{\psi}\,\left(i\gamma^{\mu}\left(\partial_{\mu} + i\,e\,QA_{\mu}\right) - m\right)\psi. \tag{1.10}$$

In order for this to work we must also assume that apart from the fermion field transforming under a gauge transformation according to eq. (1.7) the gauge field, A_{μ}, also changes according to

$$-eQA_{\mu} \rightarrow -eQ(A_{\mu} + \delta A_{\mu}(x)) \;=\; -eQ\;A_{\mu} + Q\partial_{\mu}\,\omega(x). \tag{1.11}$$

So $\delta A_{\mu}(x)) = -Q\partial_{\mu}\,\omega(x)./e$.

> **Exercise 1.1**
> Using eqs. (1.7) and (1.11) show that under a gauge transformation $\delta(-e\,\overline{\psi}\,\gamma^{\mu}A_{\mu}\psi) \;=\; -\overline{\psi}(x)\,\gamma^{\mu}\,[\partial_{\mu}\omega(x)]\,\psi(x)$.

This change exactly cancels with eq. (1.8), so that once this interaction term has been added the gauge invariance is restored. We recognize eq. (1.10) as being the fermionic part of the Lagrangian density for QED, where e is the electric charge of the fermion and A_{μ} is the photon field.

In order to have a proper quantum field theory, in which we can expand the photon field, A_{μ}, in terms of creation and annihilation operators for photons, we need a kinetic term for the field, A_{μ}, i.e. a term which is quadratic in the derivative of the field. Without such a term the Euler-Lagrange equation for the gauge field would be an algebraic equation and we could use it to eliminate the gauge field altogether from the Lagrangian. We need to ensure that in introducing a kinetic term we do not spoil the invariance under gauge transformations. This is achieved by defining the field strength, $F_{\mu\nu}$, as

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{1.12}$$

where the derivative is understood to act on the A-field only[2] It is easy to see that under the gauge transformation eq. (1.11) each of the two terms on the right hand side of eq. (1.12) changes, but the changes cancel out. Thus we may add to the Lagrangian any term which depends on $F_{\mu\nu}$ (and which is Lorentz invariant, thus, with all Lorentz indices contracted). Such a term is $aF_{\mu\nu}F^{\mu\nu}$. This gives the desired term quadratic in the derivative of the field A_μ. If we choose the constant a to be $-1/4$ then the Lagrange equations of motion match exactly the (relativistic formulation) of Maxwell's equations[3].

We have thus arrived at the Lagrangian density for QED, but from the viewpoint of demanding invariance under $U(1)$ gauge transformations rather than starting with Maxwell's equations and formulating the equivalent quantum field theory.

The Lagrangian density for QED is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu\left(\partial_\mu + i\,e\,QA_\mu\right) - m\right)\psi. \tag{1.13}$$

Exercise 1.2
Starting with the Lagrangian density for QED write down the Euler-Lagrange equations for the gauge field A_μ and show that this results in Maxwell's equations

In the Field Theory lectures, we saw that a term $\lambda\phi^4$ in the Lagrangian gave $4!\lambda$ as the coupling of four ϕs in perturbation theory. Neglecting the combinatoric factors, it is believable that eqn (1.13) gives the $\gamma - \bar{e} - e$ Feynman Rule used in the QED course: $-ie\gamma^\mu$, for negatively charged particles.

Note that we are *not* allowed to add a mass term for the photon. A term such as $M^2A_\mu A^\mu$ added to the Lagrangian density is not invariant under gauge transformations, but would give us a transformation

$$\delta\mathcal{L} = \frac{2M^2}{e}A^\mu(x)\partial_\mu\omega(x) \neq 0. \tag{1.14}$$

Thus the masslessness of the photon can be understood in terms of the requirement that the Lagrangian be gauge invariant.

[2] thus strictly speaking we should write $F_{\mu\nu} = [\partial_\mu A_\nu] - [\partial_\nu A_\mu]$

[3] The determination of this constant a is the *only* place that a match to QED has been used. The rest of the Lagrangian density is obtained purely from the requirement of local $U(1)$ invariance. A different constant would simply mean a different normalization of the photon field.

1.3 Covariant Derivatives

Before leaving the abelian case, it is useful to introduce the concept of a "covariant derivative". This is not essential for abelian gauge theories, but will be an invaluable tool when we extend these ideas to non-abelian gauge theories.

The covariant derivative D_μ is defined to be

$$D_\mu \equiv \partial_\mu + i\, e\, A_\mu. \tag{1.15}$$

This has the property that given the transformations of the fermion field eq. (1.7) and the gauge field eq. (1.11) the quantity $D_\mu \psi$ transforms in the same way under gauge transformations as ψ.

Exercise 1.3
Show that under an infinitesimal gauge transformation $D_\mu \psi$ transforms as $D_\mu \psi \to D_\mu \psi + \delta(D_\mu \psi)$ with $\delta(D_\mu \psi) = -i\omega(x) D_\mu \psi$.

We may thus rewrite the QED Lagrangian density as

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi. \tag{1.16}$$

Furthermore the field strength $F_{\mu\nu}$ can be expressed in terms of the commutator of two covariant derivatives, i.e.

$$\begin{aligned} F_{\mu\nu} \;&=\; -\frac{i}{e}\,[D_\mu, D_\nu] = -\frac{i}{e}[\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + i\, e\,[A_\mu, A_\nu] \\ &=\; = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{1.17}$$

where in the last line we adopted the conventional notation again and left out the square brackets.Notice that in using eq. (1.17) the derivatives act only on the A-field.

1.4 Gauge Fixing

The guiding principle of this chapter has been to hold onto the U(1) symmetry. This forced the introduction of a new massless field A_μ which we could interpret as the photon. In this subsection we will try to quantise the photon field (*e.g.* calculate its propagator) by naively following the prescription used for scalars and fermions...which will not work. This should not be surprising, because A_μ is has four real components, some of which should be ensuring that the gauge symmetry is maintained. And the physical photon has two polarisation states. This calculational difficulty can be resolved by "fixing the gauge" (breaking our precious gauge symmetry) in the Lagrangian, in such a way as to maintain the gauge symmetry in observables [4].

In general, if the part of the action that is quadratic in some field, $\phi(x)$, is given, in terms of the Fourier transform $\tilde{\phi}(p)$ by

$$S_\phi = \int d^4p\, \tilde{\phi}(-p)\mathcal{O}(p)\tilde{\phi}(p), \tag{1.18}$$

then the propagator for the field ϕ may be written as

$$i\,\mathcal{O}^{-1}(p). \tag{1.19}$$

In the case of QED the part of the Lagrangian that is quadratic in the photon field is given by $-1/4\, F^{\mu\nu}F_{\mu\nu} = -1/2\, A^\mu \left(-g_{\mu\nu}\partial^\sigma\partial_\sigma + \partial_\mu\partial_\nu\right) A^\nu$, where we used partial integration to obtain the second expression. In momentum space, the quadratic part of the action is then given by

$$S_A = \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\,p^2 + p_\mu p_\nu\right)\tilde{A}^\nu(p) \tag{1.20}$$

Unfortunately the operator $(-g_{\mu\nu}\,p^2 + p_\mu p_\nu)$ does not have an inverse. This can be most easily seen by noting $(-g_{\mu\nu}\,p^2 + p_\mu p_\nu)\,p^\nu = 0$. This means that the operator $(-g_{\mu\nu}\,p^2 + p_\mu p_\nu)$ has an eigenvector (p^ν) with eigenvalue 0 and therefore is not invertible. Thus it seems we are not able to write down the propagator of the photon. We solve this problem by adding to the Lagrangian density a gauge fixing term

$$-\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu\right)^2. \tag{1.21}$$

With this term included (again in momentum space), S_A becomes

$$S_A = \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\,p^2 - \frac{\xi}{1-\xi}p_\mu p_\nu\right)\tilde{A}^\nu(p), \tag{1.22}$$

and, noting the relation

$$\left(g_{\mu\nu}\,p^2 + \frac{\xi}{1-\xi}p_\mu p_\nu\right)\left(g^{\nu\rho} - \xi\frac{p^\nu p^\rho}{p^2}\right) = p^2 g_\mu{}^\rho, \tag{1.23}$$

we see that the propagator for the photon may now be written as

$$-i\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}. \tag{1.24}$$

[4] The gauge symmetry is also preserved in the Path Integral, which is a sum over all field configurations weighted by $\exp\{i\int \mathcal{L}d^4x\}$. In path integral quantisation, an alternative to the canonical approach used in the Field Theory lectures, greens functions are calculated from the path integral. So it is unimportant that the gauge symmetry seems broken in the Lagrangian.

The special choice $\xi = 0$ is known as the Feynman gauge. In this gauge the propagator eq. (1.24) is particularly simple and we will use it most of the time.

This procedure of gauge fixing seems strange: first we worked hard to get a gauge invariant Lagrangian, and then we spoil gauge invariance by introducing a gauge fixing term.

The point is that we have to fix the gauge in order to be able to perform a calculation. Once we computed a physical quantity, the dependence on the gauge cancels. In other words, it does not matter how we fix the gauge and in particular, what value for ξ we take. The choice $\xi = 0$ is simply a matter of convenience. A more careful procedure would be to leave ξ arbitrary and check that all ξ-dependence in the final result cancels. This gives us a strong check on the calculation, however, at the price of making the computation much more tedious.

The procedure of fixing the gauge in order to be able to perform a calculation, even though the final result does not depend on how we fixed the gauge, can be understood by the following analogy. Assume we wanted to calculate some scalar quantity (say the time it takes for a point mass to get from one point to another) in our ordinary 3-dimensional Euclidean space. To do so, we choose a coordinate system, perform the calculation and get our final result. Of course, the result does not depend on how we chose the coordinate system, but in order to be able to perform the calculation we had to fix it somehow. Picking a coordinate system corresponds to fixing a gauge and the independence of the result on the coordinate system chosen corresponds to the gauge invariance of physical quantities. To take this one step further we remark that by far not all quantities are independent of the coordinate system. For example, the x-coordinate of the position of the point mass at a certain time depends on our choice. Similarly, there are important quantities that are gauge dependent. An example is the gauge boson propagator given in eq. (1.24). However, all measurable quantities are gauge invariant. This is where our analogy breaks down, in that in our Euclidean example there are measurable quantities that do depend on the choice of the coordinate system.

Finally we should mention that eq. (1.21) is by far not the only way to fix the gauge but it will be sufficient for these lectures to consider gauges defined through eq. (1.21). These gauges are called covariant gauges.

1.5 Summary

- It is possible for the Lagrangian for a (complex) Dirac field to be invariant under local U(1) transformations (phase rotations), in which the phase parameter depends on space-time. In order to accomplish this we include an interaction with a vector gauge boson which transforms under the local (gauge) transformation according to eq. (1.11).

- This interaction is encoded by replacing the derivative ∂_μ by the covariant derivative D_μ defined by eq. (1.15). $D_\mu \psi$ transforms under gauge transformations as $e^{-i\omega} D_\mu \psi$.
- The kinetic term for the gauge boson is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is proportional to the commutator $[D_\mu, D_\nu]$ and is invariant under gauge transformations.
- The gauge boson is massless, since a term proportional to $A_\mu A^\mu$ is *not* invariant under gauge transformations.
- The resulting Lagrangian is identical to that of QED.

2 Non-Abelian Gauge Theories

In this lecture, the "gauge" concept will be constructed so that the gauge bosons have self-interactions—as are observed among the gluons of QCD, and the $W^{\pm}$, Z and γ of the electroweak sector. Unfortunately, the gauge bosons will still be massless. (We will see how to give the $W^{\pm}$ and Z their observed masses in the Higgs chapter.)

2.1 Global Non-Abelian Transformations

We apply the ideas of the previous lecture to the case where the transformations do not commute with each other, ie the group is "non-abelian".

Consider n free fermion fields $\{\psi_i\}$, arranged in a multiplet ψ:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix} \tag{2.1}$$

for which the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi, \\ &\equiv \overline{\psi}^{i}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_i, \end{aligned} \tag{2.2}$$

where the index i is summed from 1 to n. Eq. (2.2) is therefore a shorthand for

$$\mathcal{L} = \overline{\psi}^{1}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_1 + \overline{\psi}^{2}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_2 + \dots. \tag{2.3}$$

The Lagangian density (2.2) is invariant under (space-time *independent*) complex rotations in ψ_i space:

$$\psi \to \mathbf{U}\psi, \qquad \bar{\psi} \to \bar{\psi}\mathbf{U}^{\dagger}, \tag{2.4}$$

where $\mathbf{U}$ is an $n \times n$ matrix such that

$$\mathbf{U}\mathbf{U}^{\dagger} = 1, \qquad det[\mathbf{U}] = 1. \tag{2.5}$$

The transformation (2.4) is called an internal symmetry, which rotates the fields (*e.g.* quarks of different colour) among themselves.

The group of matrices satisfying the conditions (2.5) is called $SU(n)$. This is the group of special, unitary $n \times n$ matrices. Special in this context means that the determinant is equal to 1. In order to specify an $SU(n)$ matrix completely we need n^2-1 real parameters. Indeed, we need $2n^2$ real parameters to determine an arbitrary complex $n \times n$ matrix. But there are n^2 constraints due to the unitary requirements and one additional constraint due to the requirement det $= 1$.

An arbitrary $SU(n)$ matrix can be written as

$$\mathbf{U} = e^{-i\sum_{a=1}^{n^2-1}\omega^a \mathbf{T}^a} \equiv e^{-i\omega^a \mathbf{T}^a} \tag{2.6}$$

where we adopted the summation convention. The ω^a, $a \in \{1 \dots n^2-1\}$ are the real parameters and the $\mathbf{T}^a$ are called the generators of the group.

> **Exercise 2.1**
> Show that the unitarity of the $SU(n)$ matrices entails hermiticity of the generators and that the requirement of det $= 1$ means that the generators have to be traceless.

In the case of $U(1)$ there was just one generator. Here we have n^2-1 generators $\mathbf{T}^a$. There is still some freedom left on how to normalize the generators. We will adopt the usual normalization convention

$$\text{tr}(\mathbf{T}^a\mathbf{T}^b) = \frac{1}{2}\delta_{ab} \tag{2.7}$$

The reason we can always enforce eq. (2.7) is that $\text{tr}(\mathbf{T}^a\mathbf{T}^b)$ is a real matrix symmetric in $a \leftrightarrow b$. Thus it can be diagonalized. If you have problems with getting on friendly terms with the concept of generators, for the moment you can think of them as traceless, hermitian $n \times n$ matrices (this is, however, not the complete picture).

The crucial new feature of the group $SU(n)$ is that two elements of $SU(n)$ generally do not commute, i.e.

$$e^{-i\omega_1^a \mathbf{T}^a} e^{-i\omega_2^b \mathbf{T}^b} \neq e^{-i\omega_2^b \mathbf{T}^b} e^{-i\omega_1^a \mathbf{T}^a} \tag{2.8}$$

(compare to eq. (1.3)). To put this in a different way, the group algebra is not trivial. For the commutator of two generators we have

$$[\mathbf{T}^a, \mathbf{T}^b] \equiv i f^{abc}\mathbf{T}^c \neq 0 \tag{2.9}$$

where we defined the structure constants of the group, f^{abc} and used the summation convention again. The structure constants are totally antisymmetric. This can be seen as follows: from eq. (2.9) it is obvious that $f^{abc} = -f^{bac}$. To convince us of the antisymmetry in the other indices as well, we note that multiplying eq. (2.9) by $\mathbf{T}^d$ and taking the trace, using eq. (2.7), we get $2if^{abd} = \text{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \text{tr}(\mathbf{T}^b\mathbf{T}^a\mathbf{T}^d) = \text{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \text{tr}(\mathbf{T}^a\mathbf{T}^d\mathbf{T}^b)$.

2.2 Non-Abelian Gauge Fields

Now suppose we allow the transformation U to depend on space-time. Then the Lagrangian density changes by $\delta\mathcal{L}$ under this "non abelian gauge transformation", where

$$\delta\mathcal{L} = \overline{\psi}\,\mathbf{U}^\dagger\gamma^\mu\left(\partial_\mu\mathbf{U}\right)\psi. \tag{2.10}$$

The local gauge symmetry can be restored by introducing a covariant derivative $\mathbf{D}_\mu$, giving interactions with gauge bosons, such that

$$\mathbf{D}_\mu\mathbf{U}(x)\psi(x) = \mathbf{U}(x)\mathbf{D}_\mu\psi(x) \tag{2.11}$$

This is like the electromagnetic case, except that $\mathbf{D}_\mu$ is now a matrix:

$$i\mathbf{D}_\mu = i\mathbf{I}\partial_\mu - g\mathbf{A}_\mu \tag{2.12}$$

where $\mathbf{A}_\mu = \mathbf{T}^a A^a_\mu$. It contains n^2-1 vector (spin one) gauge bosons, A^a_μ, one for each generator of $SU(n)$. Under a gauge transformation U, $\mathbf{A}_\mu$ should transform as

$$\mathbf{A}_\mu \to \mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}\left(\partial_\mu\mathbf{U}\right)\mathbf{U}^\dagger \tag{2.13}$$

This ensures that the Lagrangian density :

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\mathbf{D}_\mu - m\right)\psi, \tag{2.14}$$

is invariant under local $SU(n)$ gauge transformations. It can be checked that (eqn 2.13) reduces to the gauge transformation of electromagnetism in the abelian limit.

Exercise 2.2

(For algebraically ambitious people): perform an infinitesimal gauge transformation on $\psi, \bar{\psi}$ and $\mathbf{D}$, using (2.6), and show that to linear order in the ω_a, $\bar{\psi}\gamma_\mu\mathbf{D}^\mu\psi$ is invariant.

Exercise 2.3

Show that in the SU(2) case, the covariant derivative is

$$i\mathbf{D}_\mu = \begin{pmatrix} i\partial_\mu - \frac{g}{2}W^3_\mu & -\frac{g}{2}(W^1_\mu - iW^2_\mu) \\ -\frac{g}{2}(W^1_\mu + iW^2_\mu) & i\partial_\mu + \frac{g}{2}W^3_\mu \end{pmatrix},$$

and find the usual charged current interactions for the lepton doublet

$$\psi = \begin{pmatrix} \nu \\ e \end{pmatrix}$$

by defining $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$

Exercise 2.4
Include the U(1) hypercharge interaction in the previous question; show that the covariant derivative acting on the lepton doublet (of hypercharge $Y = -1/2$) is

$$i\mathbf{D}_\mu = \begin{pmatrix} i\partial_\mu - \frac{g}{2}W_\mu^3 - g'YB_\mu & -\frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ -\frac{g}{2}(W_\mu^1 + iW_\mu^2) & i\partial_\mu + \frac{g}{2}W_\mu^3 - g'YB_\mu \end{pmatrix}.$$

Define

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

and write the diagonal (neutral) interactions in terms of Z_μ and A_μ. Extract $\sin\theta_W$ in terms of g and g'. (Recall that the photon does not interact with the neutrino.)

The kinetic term for the gauge bosons is again constructed from the field strengths $F^a_{\mu\nu}$ which are defined from the commutator of two covariant derivatives:

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right]. \tag{2.15}$$

where the matrix $\mathbf{F}_{\mu\nu}$ is given by

$$\mathbf{F}_{\mu\nu} = \mathbf{T}^a F^a_{\mu\nu}, \tag{2.16}$$

This gives us

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu \tag{2.17}$$

Notice that $\mathbf{F}_{\mu\nu}$ is gauge *variant*, unlike the U(1) case. We know the transformation of $\mathbf{D}$ from (2.13), so

$$[\mathbf{D}_\mu, \mathbf{D}_\nu] \to \mathbf{U}\,[\mathbf{D}_\mu, \mathbf{D}_\nu]\,\mathbf{U}^\dagger \tag{2.18}$$

The gauge invariant kinetic term for the gauge bosons is therefore

$$-\frac{1}{2}\mathbf{Tr}\ \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}, \tag{2.19}$$

where the Trace is in $SU(n)$ space, and a summation over the index a is implied.

In sharp contrast with the abelian case, this term does not only contain the terms which are quadratic in the derivatives of the gauge boson fields, but also the terms

$$g\, f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu. \tag{2.20}$$

This means that there is a very important difference between abelian and non-abelian gauge theories. For non-abelian gauge theories the gauge bosons interact with each other via both three-point and four-point interaction terms. The three point interaction term contains a derivative, which means that the Feynman rule for the three-point vertex involves the momenta of the particles going into the vertex. We shall write down the Feynman rules in detail later.

Once again, a mass term for the gauge bosons is forbidden, since a term proportional to $A_\mu^a A^{a\,\mu}$ is *not* invariant under gauge transformations.

2.3 Gauge Fixing

As in the case of QED, we need to add a gauge fixing term in order to be able to derive a propagator for the gauge bosons. In the Feynman gauge this means adding the term $-\frac{1}{2}(\partial^\mu A_\mu^a)^2$ to the Lagrangian density and the propagator (in momentum space) becomes

$$-i\,\delta_{ab}\,\frac{g_{\mu\nu}}{p^2}.$$

There is one unfortunate complication, which is included briefly here for the sake of completeness, although one only needs to know about it for the purpose of performing higher loop calculations with non-abelian gauge theories.

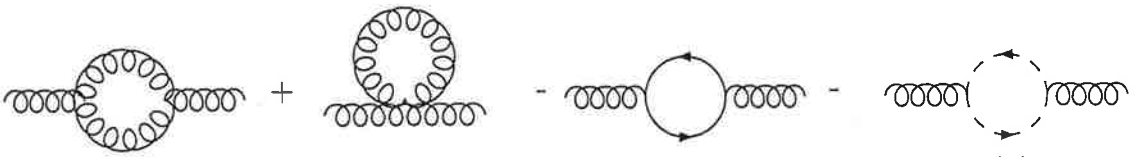
If one goes through the formalism of gauge fixing carefully, it turns out that at higher orders this leads to extra loop diagrams. These diagrams involve additional particles that are mathematically equivalent to interacting scalar particles and are known as a "Faddeev-Popov ghost". For each gauge field there is such a ghost field. These are *not* to be interpreted as physical scalar particles which could in principle be observed experimentally, but merely as part of the gauge-fixing programme. For this reason they are referred to as 'ghosts". Furthermore they have two peculiarities

1. They only occur inside loops. This is because they are not really particles and cannot occur in initial or final states, but are introduced to clean up a difficulty that arises in the gauge-fixing mechanism.

2. They behave like fermions even though they are scalars (spin zero). This means that we need to count a minus sign for each loop of Faddeev-Popov ghosts in any Feynman diagram.

We shall display the Feynman rules for these "ghosts" later.

Thus, for example, the Feynman diagrams which contribute to the one-loop corrections to the gauge boson propagator are



(a) (b) (c) (d)

Diagram (a) involves the three-point interaction between the gauge bosons, diagram (b) involves the four-point interaction between the gauge bosons, diagram (c) involves a loop of fermions, and diagram (d) is the extra diagram involving the Faddeev-Popov ghosts. Note that both diagrams (c) and (d) have a minus sign in front of them because both the fermions and Faddeev-Popov ghosts obey Fermi statistics.

2.4 The Lagrangian for a General Non-Abelian Gauge Theory

Let us summarize what we have found so far: Consider a group gauge group, $\mathcal{G}$ of "dimension" N (for $SU(n)$: $N \equiv n^2 - 1$), whose N generators, $\mathbf{T}^a$, obey the commutation relations $\left[\mathbf{T}^a, \mathbf{T}^b\right] = i f_{abc} \mathbf{T}^c$, where f_{abc} are called the "structure constants" of the group.

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i is given (in Feynman gauge) by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i\overline{\psi}\left(\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)\psi - \frac{1}{2}(\partial^\mu A^a_\mu)^2 + \mathcal{L}_{FP} \tag{2.21}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu, \tag{2.22}$$

$$\mathbf{D}_\mu = \partial_\mu \mathbf{I} + i\, g\, \mathbf{T}^a A^a_\mu \tag{2.23}$$

and

$$\mathcal{L}_{FP} = -\xi^a \partial^\mu \partial_\mu \eta^a + g\, f_{acb} \xi^a A^c_\mu (\partial^\mu \eta^b) \tag{2.24}$$

Under an infinitesimal gauge transformation, the N gauge bosons, A^a_μ change by an amount that contains a term which is not linear in A^a_μ:

$$\delta A^a_\mu(x) = -f^{abc} A^b_\mu(x) \omega^c(x) + \frac{1}{g}\partial_\mu \omega^a(x), \tag{2.25}$$

whereas the field strengths $F^a_{\mu\nu}$ transform by a change

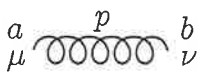
$$\delta F^a_{\mu\nu}(x) \;=\; -f^{abc}\, F^b_{\mu\nu}(x)\, \omega^c. \tag{2.26}$$

In other words they transform as the "adjoint" representation of the group (which has as many components as there are generators). This means that the quantity $F^a_{\mu\nu} F^{a\,\mu\nu}$ (summation over a implied) is invariant under gauge transformations.

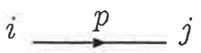
2.5 Feynman Rules

The Feynman rules for such a gauge theory can directly be read off from the Lagrangian. As mentioned previously, the propagators are obtained by taking all terms bilinear in the field and inverting the corresponding operator (and multiplying by i). The rules for the vertices are obtained by simply taking (i times) the factor multiplying the corresponding term in the Lagrangian. The explicit rules are given by:

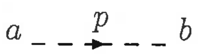
Propagators:



Gluon: $-i\, \delta_{ab}\, g_{\mu\nu}/p^2$

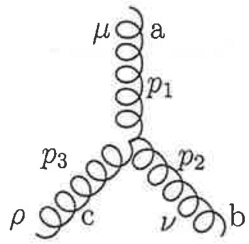


Fermion: $i\, \delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$

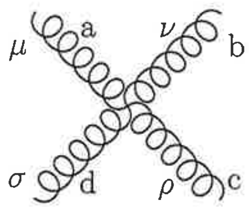


Faddeev-Popov ghost: $i\, \delta_{ab}/p^2$

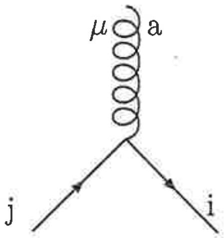
Vertices: (all momenta are flowing into the vertex).



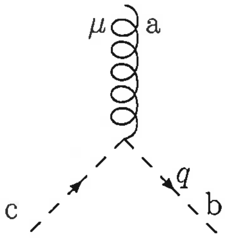
$$-g\, f^{abc}\left(g_{\mu\nu}\left(p_1-p_2\right)_\rho+g_{\nu\rho}\left(p_2-p_3\right)_\mu+g_{\rho\mu}\left(p_3-p_1\right)_\nu\right)$$



$$-i\, g^2 f^{eab} f^{ecd}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)$$
$$-i\, g^2 f^{eac} f^{ebd}\left(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)$$
$$-i\, g^2 f^{ead} f^{ebc}\left(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma}\right)$$



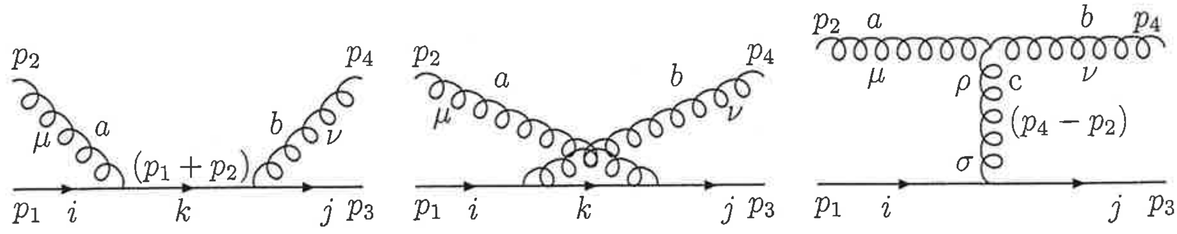
$$-i\, g\, \gamma^\mu\left(T^a\right)_{ij}$$



$$g\, f^{abc}\, q_\mu$$

2.6 An Example

As an example of the application of these Feynman rules, we consider the process of Compton scattering, but this time for the scattering of non-abelian gauge bosons and fermions, rather than photons. We need to calculate the amplitude for a gauge boson of momentum p_2 and gauge label a to scatter fermion of momentum p_1 and gauge label i producing a fermion of momentum p_3 and gauge label j and a gauge boson of momentum p_4 and gauge label b. Note that $i, j \in \{1 \ldots n\}$ whereas $a, b \in \{1 \ldots n^2-1\}$. In addition to the two Feynman diagrams one gets in the QED case there is a third diagram involving the self-interaction of the gauge bosons.



(a) (b) (c)

We will assume that the fermions are massless (i.e. that we are at sufficiently high energies that we may neglect their masses), and work in terms of the Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned}$$

The polarizations are accounted for by contracting the amplitude obtained for the above diagrams with the polarization vectors $\epsilon^\mu(\lambda_2)$ and $\epsilon^\nu(\lambda_4)$. Each diagram consists of two vertices and a propagator and so their contributions can be read off from the Feynman rules.

For diagram (a) we get

$$\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^k_j\right)\left(i\,\frac{\gamma\cdot(p_2+p_2)}{s}\right)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^i_k\right)u_i(p_1)$$

$$= -i\,\frac{g^3}{s}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1+p_2)\gamma^\mu\right)\left(\mathbf{T}^b\mathbf{T}^a\right)u(p_1).$$

For diagram (b) we get

$$\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^k_j\right)\left(i\,\frac{\gamma\cdot(p_1-p_4)}{u}\right)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^i_k\right)u_i(p_1)$$

$$= -i\,\frac{g^3}{u}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1-p_4)\gamma^\mu\right)\left(\mathbf{T}^a\mathbf{T}^b\right)u(p_1).$$

Note here that the order of the $\mathbf{T}$ matrices is the other way around from diagram (a).

Diagram (c) involves the three-point gauge-boson self-coupling. Since the Feynman rule for this vertex is given with incoming momenta, it is useful to replace the outgoing gauge-boson momentum p_4 by $-p_4$ and understand this to be an incoming momentum. Note that the internal gauge-boson line carries momentum $p_4 - p_2$ coming into the vertex. The three incoming momenta that are to be substituted into the Feynman rule for the vertex are therefore p_2, $-p_4$, $p_4 - p_2$. The vertex thus becomes

$$-g\,f_{abc}\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right)$$

and the diagram gives

$$\epsilon^{\mu}(\lambda_2)\epsilon^{\nu}(\lambda_4)\overline{u}^{j}(p_3)\left(-i\,g\,\gamma_{\sigma}(\mathbf{T}^c)^i_j\right)u_i(p_1)\left(-i\frac{g^{\rho\sigma}}{t}\right)$$
$$\times\ (-g\,f_{abc})\,(g_{\mu\nu}(p_2+p_4)_{\rho}+g_{\rho\nu}(p_2-2p_4)_{\mu}+g_{\mu\rho}(p_4-2p_2)_{\nu})$$
$$= -i\frac{g^3}{t}\epsilon^{\mu}(\lambda_2)\epsilon^{\nu}(\lambda_4)\overline{u}(p_3)\left[\mathbf{T}^a,\mathbf{T}^b\right]\gamma^{\rho}u(p_1)\left(g_{\mu\nu}(p_2+p_4)_{\rho}-2(p_4)_{\mu}g_{\nu\rho}-2(p_2)_{\nu}g_{\mu\rho}\right),$$

where in the last step we have used the fact that the polarization vectors are transverse so that $p_2\cdot\epsilon(\lambda_2)=0$ and $p_4\cdot\epsilon(\lambda_4)=0$ and the commutation relations eq. (2.9).

Exercise 2.4
Draw all the Feynman diagrams for the tree level amplitude for two gauge bosons with momenta p_1 and p_2 to scatter into two gauge bosons with momenta q_1 and q_2. Label the momenta of the external gauge boson lines.

2.7 Summary

- A non-abelian gauge theory is one in which the Lagrangian is invariant under local transformations of a non-abelian group.
- This invariance is achieved by introducing a gauge boson for each generator of the group. The partial derivative in the Lagrangian for the fermion field is replaced by a covariant derivative defined in eq. (2.23).
- The gauge bosons transform under infinitesimal gauge transformations in a non-linear way given by eq. (2.25).
- The field strengths, $F^a_{\mu\nu}$ are obtained from the commutator of two covariant derivatives and are given by eq. (2.22). They transform as the adjoint representation under gauge transformations such that the quantity $F^a_{\mu\nu}F^{a\,\mu\nu}$ is invariant.
- $F^a_{\mu\nu}F^{a\,\mu\nu}$ contains terms which are cubic and quartic in the gauge bosons, indicating that these gauge bosons interact with each other.
- The gauge fixing mechanism leads to the introduction of Faddeev-Popov ghosts which are scalar particles that occur only inside loops and obey Fermi statistics.

3 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is the theory of the strong interactions. It is nothing but a non-abelian gauge theory with the group $SU(3)$. Thus, the quarks are described by a field ψ_i where i runs from 1 to 3. The quantum number associated with the label i is called color. The eight gauge bosons which have to be introduced in order to preserve local gauge invariance are the eight 'gluons'. These are taken to be the carriers which mediate the strong interactions in exactly the same way that photons are the carriers which mediate the electromagnetic interactions.

The Feynman rules for QCD are therefore simply the Feynman rules listed in the previous lecture, with the gauge coupling constant, g, taken to be the strong coupling, g_s, (more about this later), the generators $\mathbf{T}^a$ taken to be the eight generators of $SU(3)$ in the triplet representation, and f^{abc}, a, b, c, $= 1 \ldots 8$ are the structure constants of $SU(3)$ (you look these up in a book whenever you need them).

Thus we now have a quantum field theory which can be used to describe the strong interactions.

3.1 Running Coupling

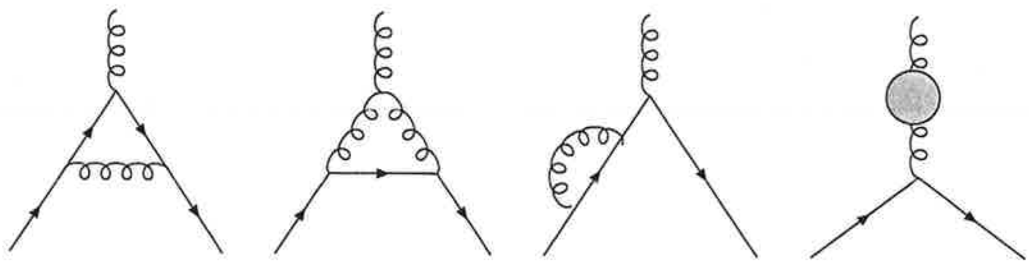
The coupling for the strong interactions is the QCD gauge coupling, g_s. We usually work in terms of α_s defined as

$$\alpha_s = \frac{g_s^2}{4\pi}. \tag{3.1}$$

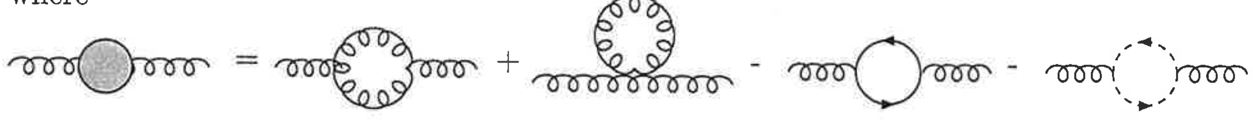
Since the interactions are strong, we would expect α_s to be too large to perform reliable calculations in perturbation theory. On the other hand the Feynman rules are only useful within the context of perturbation theory.

This difficulty is resolved when we understand that 'coupling constants' are not constant at all. The electromagnetic fine structure constant, α, only has the value 1/137 at energies which are not large compared with the electron mass. At higher energies it is larger than this. For example, at LEP energies it takes a value closer to 1/128. On the other hand, it turns out that for non-abelian gauge theories the coupling *decreases* as the energy increases.

To see how this works within the context of QCD, we note that when we perform higher order perturbative calculations there are loop diagrams which have the effect of dressing the couplings. For example, the one-loop diagrams which dress the coupling between a quark and a gluon are:



where



are the diagrams needed to calculate the one-loop corrections to the gluon propagator.

These diagrams contain UV divergences and need to be renormalized by subtracting at some renormalization scale μ. This scale then appears inside a logarithm for the renormalized quantities. This means that if the square-momenta of all the external particles coming into the vertex are of order Q^2, where $Q \gg \mu$, then the above diagrams give rise to a correction which contains a logarithm of the ratio Q^2/μ^2:

$$-\alpha_s^2 \beta_0 \ln\left(Q^2/\mu^2\right). \tag{3.2}$$

This correction is interpreted as the correction to the effective QCD coupling, $\alpha_s(Q^2)$, at momentum scale Q. i.e

$$\alpha_s(Q^2) = \alpha_s(\mu^2) - \alpha_s(\mu^2)^2 \beta_0 \ln\left(Q^2/\mu^2\right) + \dots \tag{3.3}$$

β_0 is calculated to be

$$\beta_0 = \frac{11 N_c - 2 n_f}{12 \pi}, \tag{3.4}$$

where N_c is the number of colors (=3), n_f is the number of active flavors, i.e. the number of flavors whose mass threshold is below the momentum scale, Q. Note that β_0 is *positive*, which means that the coefficient in front of the logarithm in eq. (3.3) is *negative*, so that the effective coupling *decreases* as the momentum scale is increased.

A more precise analysis shows that the effective coupling obeys the differential equation

$$\frac{\partial \alpha_s(Q^2)}{\partial \ln(Q^2)} = \beta\left(\alpha_s(Q^2)\right) \tag{3.5}$$

where β has a perturbative expansion

$$\beta(\alpha) = -\beta_0 \alpha^2 + \mathcal{O}(\alpha^3) + \cdots \tag{3.6}$$

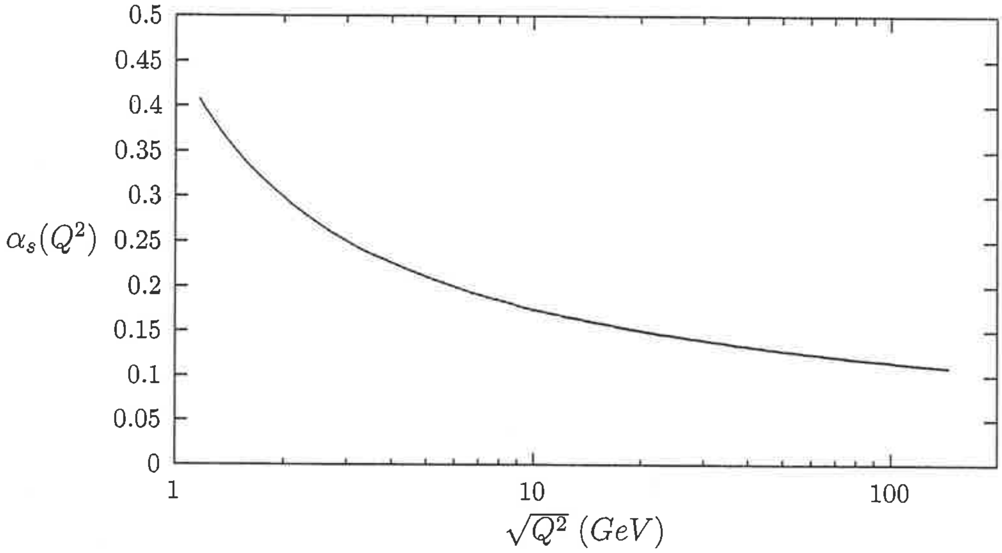


Figure 3.1: The running of $\alpha_s(Q^2)$ with β taken to two loops.

In order to solve this differential equation we need a boundary value. Nowadays this is taken to be the measured value of the coupling at the Z boson mass[5] (= 91 GeV), which is measured to be

$$\alpha_s(M_Z^2) \;=\; 0.118 \pm 0.002 \tag{3.7}$$

This is one of the free parameters of the Standard Model.

The running of $\alpha_s(Q^2)$ is shown in the figure 3.1. We can see that for momentum scales above about 2 GeV the coupling is less than 0.3 so that one can hope to carry out reliable perturbative calculations for QCD processes with energy scales larger than this.

Gauge invariance requires that the gauge coupling for the interaction between gluons must be exactly the same as the gauge coupling for the interaction between quarks and gluons. The β-function could therefore have been calculated from the higher order corrections to the three-gluon (or four-gluon) vertex and must yield the same result, despite the fact that it is calculated from a completely different set of diagrams.

[5]Previously the solution to eq. (3.5) (to leading order) was written as $\alpha_s(Q^2) = 4\pi/\beta_0 \ln(Q^2/\Lambda_{QCD}^2)$ and the scale Λ_{QCD} was used as the standard parameter which sets the scale for the magnitude of the strong coupling. This turns out to be rather inconvenient since it needs to be adjusted every time higher order corrections are taken into consideration and the number of active flavors has to be specified. This parametrization is now hardly ever used.

Exercise 3.1
Draw the Feynman diagrams needed for the calculation of the one-loop correction to the triple gluon coupling (don't forget the Faddeev-Popov ghost loops).

Exercise 3.2
Solve equation (3.5) using β to leading order only, and calculate the value of α_s at a momentum scale of 10 GeV. Use the value at M_z given by eq. (3.7). Calculate also the error in α_s at 10 GeV.

3.2 Quark (and Gluon) Confinement

This argument can be inverted to provide an answer to the question 'why have we never seen quarks or gluons in a laboratory ? '.

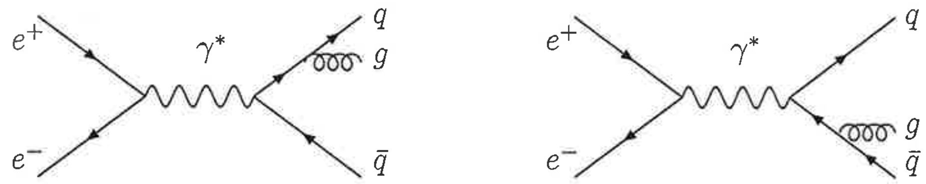
Asymptotic freedom, which tells us that the effective coupling between quarks becomes weaker as we go to short distances (this is equivalent to going to high energies) implies, conversely, that effective couplings grow as we go to large distances. Therefore, the complicated system of gluon exchanges, which leads to the binding of quarks (and antiquarks) inside hadrons, leads to a stronger and stronger binding as we attempt to pull the quarks apart. This means that we can never isolate a quark (or a gluon) at large distances since we require more and more energy to overcome the binding as the distance between the quarks grows.

The upshot of this is that the only free particles which can be observed at macroscopic distances from each other are color singlets. This mechanism is known as "quark confinement". The details of how it works are not fully understood. Nevertheless the argument presented here is suggestive of such confinement and at the level of non-perturbative field theory, lattice calculations have confirmed that for non-abelian gauge theories the binding energy does indeed grow as the distance between quarks increases.

Thus we have two different pictures of the world. At sufficiently short distances, which can be probed at sufficiently large energies, we can consider quarks and gluons interacting with each other. We can perform calculations of the scattering cross sections between quarks and gluons (called the "hard cross section") at short distances because the running coupling is sufficiently small that we can rely on perturbation theory.

On the other hand, before we can make a direct comparison with what is observed in accelerator experiments, we need to take into account the fact that these quarks and gluons bind into color singlet hadrons and it is only these color singlet states that are observed. The mechanism for such binding is beyond the scope of perturbation theory and is not understood in detail. Nevertheless several Monte Carlo programs have been developed which simulate this binding in such a way that the results of the short-distance perturbative calculations at the level of quarks and gluons can be confronted with experiment in a successful way.

Thus, for example, if we wish to calculate the cross section for electron-positron annihilation into three jets (at high energies), we first calculate, in perturbation theory, the process for electron plus positron to annihilate into a virtual photon which then decays into a quark, and antiquark and a gluon. The two Feynman diagrams for this process are:



However, before we can compare the results of this perturbative calculation with experimental data on three jets of observed hadrons, we need to perform a convolution of this calculated cross section with a Monte Carlo simulation that accounts for the way in which the final state partons (quarks and gluons) bind with other quarks and gluons to produce observed hadrons. It is only after such a convolution has been performed that one can get a reliable comparison of the calculated cross section with data.

Likewise, if we want to calculate cross sections for initial state hadrons we need to account for the probability of finding a particular quark or gluon inside an initial hadron with a given fraction of the initial hadron's momentum (these are called "parton distribution functions").

> **Exercise 3.3**
> Draw the (tree level) Feynman diagrams for the process $e^+ + e^- \rightarrow$ 4 jets. Consider only one photon exchange plus the QCD contributions only (do not include $Z-$boson exchange or $W-W$ production.)

3.3 $\theta-$ Parameter

There is one more gauge invariant term that can be written down:

$$\mathcal{L}_\theta = \theta \frac{\alpha_s}{8\pi} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}, \tag{3.8}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (in four dimensions) and the factor 8π is purely conventional. Since we should work with the most general gauge invariant Lagrangian there is no reason to omit this term. However, adding this term to the Lagrangian leads to a problem, the strong CP problem.

To understand the nature of the problem, we first convince ourselves that this term violates CP. In QED we would have

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \;=\; \mathbf{E}\cdot\mathbf{B}, \tag{3.9}$$

and for QCD we have a similar expression except that $\mathbf{E}^a$ and $\mathbf{B}^a$ carry a color index – they are known the chromoelectric and chromomagnetic fields. Under charge conjugation both the electric and magnetic field change sign. But under parity the electric field, which is a proper vector, changes sign, whereas the magnetic field, which is a polar vector, does not change sign. Thus we see that the term $\mathbf{E}\cdot\mathbf{B}$ is odd under CP.

For this reason, the parameter θ in front of this term must be exceedingly small in order not to give rise to strong interaction contributions to CP violating quantities such as the electric dipole moment of the neutron. The current experimental limits on this dipole moment tell us that $\theta \; < \; 10^{-9}$. Thus we are tempted to think that θ is zero. Nevertheless, strictly speaking θ is a free parameter of QCD, and is often considered to be the nineteenth free parameter of the Standard Model.

Of course we simply could set θ to zero (or a very small number) and be happy with it. However, whenever a free parameter is zero or extremely small, we would like to understand the reason. The fact that we do not know why this term is absent (or so small) is the strong CP problem.

There are several possible solutions to the strong CP problem that offer explanations why this term is absent (or small). But none of these solutions has been confirmed yet and the problem is still unresolved.

Another question is why this is not a problem in QED. In fact a term like eq. (3.8) can also be written down in QED. A thorough discussion of this point is beyond the scope of this lecture. Suffice to say that this term can be written (in QED and QCD) as a total divergence. So it seems that it can be eliminated from the Lagrangian altogether. However, in QCD (but not in QED) there are non-perturbative effects (related to so called instantons) which prevent us from neglecting the θ-term.

3.4 Summary

- Quarks transform as a triplet representation of color $SU(3)$ (each quark can have one of three colors.)
- The eight gauge bosons of QCD are the gluons which are the carriers that mediate the strong interactions.

- The coupling of quarks to gluons (and gluons to each other) decreases as the energy scale increases. Therefore, at high energies one can perform reliable perturbative calculations for strong interacting processes.

- As the distance between quarks increases the binding increases, such that it is impossible to isolate individual quarks or gluons. The only observable particles are color singlet hadrons. Perturbative calculations performed at the quark and gluon level must be modified by accounting for the recombination of final state quarks and gluons into observed hadrons as well as the probability of finding these quarks and gluons inside the initial state hadrons.

- QCD admits a gauge invariant strong CP violating term with a coefficient θ. This parameter is known to be very small from limits on CP violating phenomena such as the electric dipole moment of the neutron.

4 Spontaneous Symmetry Breaking

We have seen that in an unbroken gauge theory the gauge bosons must be massless. This is exactly what we want for QED (massless photon) and QCD (massless gluons). However, if we wish to extend the ideas of describing interactions by a gauge theory to the weak interactions, the symmetry must somehow be broken since the carriers of the weak interactions (W and Z bosons) are massive (weak interactions are very short range). We could simply break the symmetry by hand by adding a mass term for the gauge bosons, which we know violates the gauge symmetry. However, this would destroy renormalizability of our theory.

Renormalizable theories are prefered because they are more predictive. It is written in the Field Theory and QED lectures, that there are divergent results (infinities) in QED and QCD, and these are said to be renormalizable theories. So what could be worse about a non-renormalizable theory? The critical issue is the number of divergences: few in a renormalizable theory, and infinite in the non-renormalizable case. Associated to every divergence is a parameter that must be extracted from data, so renormalizable theories can make testable predictions once a few parameters are measured. For instance, in QCD, the coupling g has a divergence. But once α_s is measured in one process, the theory can be tested in other processes.

In this chapter, we will see a way to give masses to the W and Z, called "spontaneous symmetry breaking", which maintains the renormalizability of the theory. In this scenario, the Lagrangian maintains its symmetry under a set of local gauge transformations. On the other hand, the lowest energy state, which we interpret as the vacuum state, is *not* a singlet of the gauge symmetry. There is an infinite number of states each with the same ground-state energy and nature chooses one of these states as the 'true' vacuum.

4.1 Massive Gauge Bosons and Renormalizability

In this subsection we will convince ourselves that simply adding by hand a mass term for the gauge bosons will destroy the renormalizability of the theory. It will not be a rigorous argument, but it will illustrate the difference between introducing mass terms for the gauge bosons in a brute force way and introducing them via spontaneous symmetry breaking.

Higher order (loop) corrections generate ultraviolet divergences. In a renormalizable theory, these divergences can be absorbed into the parameters of the theory we started with, and in this way can be 'hidden'. As we go to higher orders we need to absorb more and more terms into these parameters, but there are only as many divergent quantities as there are parameters. So, for instance in QED, the Lagrangian we start with contains the fermion field, the gauge boson field, and interactions whose strength is controlled by e and m. Being a renormalisable theory, all divergences of diagrams can be absorbed into these quantities (irrespective of the number of loops or legs), and once e and m are measured, all other observables ($g-2$, etc) can be predicted.

In order to ensure that the programme can be carried out there have to be restrictions on the allowed interaction terms. Furthermore all the propagators have to decrease like $1/p^2$ as the momentum $p \longrightarrow \infty$. Note that this is how the massless gauge-boson propagator eq. (1.24) behaves. If these conditions are not fulfilled then the theory generates more and more divergent terms as one calculates to higher orders and it is not possible to absorb these divergences into the parameters of the theory. Such theories are said to be "non-renormalizable".

Now we can convince ourselves that simply adding a mass term $M^2 A_\mu A^\mu$ to the Lagrangian given in eq. (2.21) will lead to a non-renormalizable theory. To start with we note that such a term will modify the propagator. Collecting all terms bilinear in the gauge fields in momentum space and Feynman gauge we get

$$\frac{1}{2} A_\mu \left(-g^{\mu\nu}(p^2 - M^2) + p^\mu p^\nu \right) A_\nu \tag{4.1}$$

We have to invert this operator to get the propagator which now takes the form

$$\frac{i}{p^2 - M^2} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \right) \tag{4.2}$$

Note that this propagator, eq. (4.2), has a much worse ultraviolet behavior in that it goes to a constant for $p \longrightarrow \infty$. Thus, it is clear that the ultraviolet properties of a theory with a propagator as given in eq. (4.2) are worse than for a theory with a propagator as given in eq. (1.24). According to our discussion at the beginning of this subsection we conclude that without the explicit mass term $M^2 A_\mu A^\mu$ the theory is renormalizable whereas with this term it is not. In fact, it is precisely the gauge symmetry that ensures renormalizability. Breaking this symmetry results in the loss of renormalizability.

The aim of spontaneous symmetry breaking is to break the gauge symmetry in a more subtle way, such that we still can give the gauge bosons a mass but retain renormalizability.

4.2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a phenomenon that is by far not restricted to gauge symmetries. It is a subtle way to break a symmetry by still requiring that the Lagrangian remains invariant under the symmetry transformation. However, the ground state of the symmetry is *not* invariant, i.e. *not* a singlet under a symmetry transformation.

In order to illustrate the idea of spontaneous symmetry breaking consider a pen that is completely symmetric with respect to rotations around its axis. If we balance this pen on its tip on a table, and start to press on it with a force precisely along the axis we have a perfectly symmetric situation. This corresponds to a Lagrangian that is symmetric (under rotations around the axis of the pen in this case). However, if we increase the force, at some point the pen will bend (and eventually break). The question then is in which direction it will bend. Of course we do not know, since all directions are equal. But the pen will pick one and by doing so it will break the rotational symmetry. This is the spontaneous symmetry breaking.

A better example can be given by looking at point mass in a potential

$$V(\vec{r}) = \mu^2\vec{r}\cdot\vec{r} + \lambda(\vec{r}\cdot\vec{r})^2. \tag{4.3}$$

This potential is symmetric under rotations and we assume $\lambda > 0$. For $\mu^2 > 0$ the potential has a minimum at $\vec{r} = 0$, thus the point mass will simply fall to this point. The situation is more interesting, if $\mu^2 < 0$. For 2-dimensional $\vec{r}$ the potential is shown in Fig. 4.1. If the point mass sits at $\vec{r} = 0$ the system is not in the ground state but the situation is completely symmetric. In order to reach the ground state, the symmetry has to be broken, i.e. if the point mass wants to roll down, it has to decide in which direction. Any directions is equally good, but one has to be picked. This is exactly what spontaneous symmetry breaking means. The Lagrangian (here the potential) is symmetric (here under rotations around the z-axis), but the ground state (here the position of the point mass once it rolled down) is not.

Let us formulate this in a slightly more mathematical way. Since we will apply these ideas to gauge symmetries, we will now return to gauge symmetries. We denote the ground state by $|0\rangle$. A spontaneously broken gauge theory is a theory whose Lagrangian is invariant under gauge transformations, which is exactly what we have done in sections 1 and 2. The new feature in a spontaneously broken theory is that the ground state is not invariant under gauge transformations. This means

$$e^{-i\omega^a\mathbf{T}^a}|0\rangle \neq |0\rangle \tag{4.4}$$

which entails

$$\mathbf{T}^a|0\rangle \neq 0 \qquad \text{for some } a \tag{4.5}$$

Eq. (4.5) follows from eq. (4.4) upon expansion in ω^a. Thus, the theory is spontaneously broken if there exists at least one generator that does not annihilate the vacuum.

In the next section we will apply these ideas in more detail to gauge symmetries and we will see that indeed, this way of breaking the gauge symmetry has all the desired features.

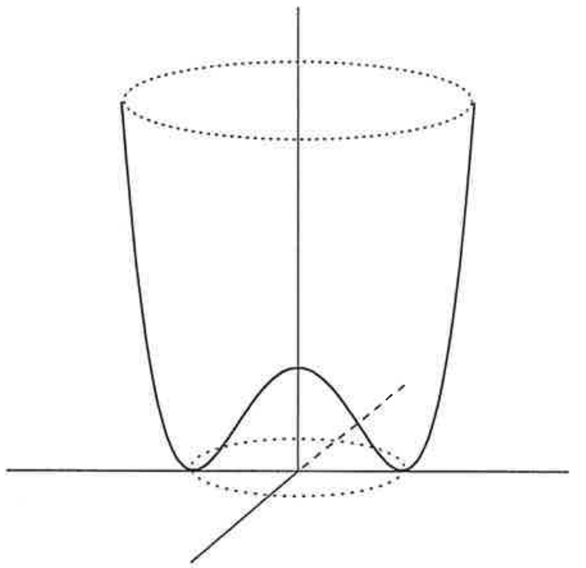


Figure 4.1: A potential that leads to spontaneous symmetry breaking

4.3 The Abelian Higgs Model

For simplicity, we will start by spontaneously breaking the $U(1)$ gauge symmetry in a theory of a complex scalar field. In the Standard Model, it will be a non-abelian gauge theory that is spontaneously broken but all the important ideas can simply be translated from the $U(1)$ case considered here.

The Lagrangian density for a gauged complex scalar field, with a mass term and a quartic self-interaction, may be written

$$\mathcal{L} = (D_\mu \Phi)^* D^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\Phi), \tag{4.6}$$

where the potential $V(\Phi)$, is given by

$$V(\Phi) = \mu^2 \Phi^* \Phi + \lambda \left| \Phi^* \Phi \right|^2 . \tag{4.7}$$

and the covariant derivative D_μ and the field-strength tensor $F_{\mu\nu}$ are given in eqs. (1.15) and (1.12) respectively. This Lagrangian is invariant under $U(1)$ gauge transformations

$$\Phi \longrightarrow e^{-i\omega(x)} \Phi. \tag{4.8}$$

Provided μ^2 is positive this potential has a minimum at $\Phi = 0$. We call the $\Phi = 0$ state the vacuum and expand Φ in terms of creation and annihilation operators that populate the higher energy states. In terms of a quantum field theory, where Φ is an operator, the precise statement is that the operator Φ has zero vacuum expectation value, i.e. $\langle 0|\Phi|0\rangle = 0$.

Now suppose we *reverse* the sign of μ^2, so that the potential becomes

$$V(\Phi) \; = \; -\mu^2\Phi^*\Phi + \lambda\left|\Phi^*\Phi\right|^2. \tag{4.9}$$

with $\mu^2 > 0$. We see that this potential no longer has a minimum at $\Phi = 0$, but a *maximum.* The minimum occurs at

$$\Phi = e^{i\theta}\sqrt{\frac{\mu^2}{2\lambda}} \equiv e^{i\theta}\frac{v}{\sqrt{2}} \tag{4.10}$$

where θ takes any value from 0 to 2π. There is an infinite number of states each with the same lowest energy, i.e. we have a degenerate vacuum. The symmetry breaking occurs in the choice made for the value of θ which represents the true vacuum. For convenience we shall choose $\theta = 0$ to be this vacuum. Such a choice constitutes a spontaneous breaking of the $U(1)$ invariance, since a $U(1)$ transformation takes us to a different lowest energy state. In other words the vacuum breaks $U(1)$ invariance. In quantum field theory we say that the field Φ has a non-zero vacuum expectation value

$$\langle\Phi\rangle \; = \; \frac{v}{\sqrt{2}}. \tag{4.11}$$

But this means that there are 'excitations' with zero energy, that take us from the vacuum to one of the other states with the same energy. The only particles which can have zero energy are massless particles (with zero momentum). We therefore expect a massless particle in such a theory.

To see that we do indeed get a massless particle, we expand Φ around its vacuum expectation value as

$$\Phi \; = \; \frac{e^{i\phi/v}}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H\right) \simeq \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi\right). \tag{4.12}$$

The fields H and ϕ have zero vacuum expectation value and it is these fields that are expanded in terms of creation and annihilation operators of the particles that populate the excited states. Of course, it is the H-field that corresponds to the Higgs field.

We now want to write the Lagrangian in terms of the H and ϕ fields. In order to get the potential we insert eq. (4.12) into eq. (4.9) and find

$$V \; = \; \mu^2H^2 + \mu\sqrt{\lambda}\left(H^3 + \phi^2H\right) + \frac{\lambda}{4}\left(H^4 + \phi^4 + 2H^2\phi^2\right) + \frac{\mu^4}{4\lambda}. \tag{4.13}$$

Note that in eq. (4.13) there is a mass term for the H-field, $\mu^2H^2 \equiv M_H/2H^2$ where we defined[6]

$$M_H = \sqrt{2}\mu \tag{4.14}$$

[6]Note that for a real field ϕ representing a particle of mass m the mass term is $\frac{1}{2}m^2\phi^2$ whereas for a complex field the mass term is $m^2\phi^\dagger\phi$.

However, there is *no* mass term for the field ϕ. Thus ϕ is a field for a massless particle called a "Goldstone boson". We will look at this issue in a more general way in section 4.4. Next we consider the kinetic term. We plug eq. (4.12) into $(D_\mu\Phi)^* D^\mu\Phi$ and get

$$\begin{aligned}(D_\mu\Phi)^* D^\mu\Phi &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g^2v^2A_\mu A^\mu + \frac{1}{2}g^2A_\mu A^\mu(H^2+\phi^2) \\ &- gA_\mu(\phi\partial_\mu H - H\partial_\mu\phi) + gvA_\mu\partial^\mu\phi + g^2vA_\mu A^\mu H \end{aligned} \qquad (4.15)$$

There are several important features in eq. (4.15). Firstly, the gauge boson acquired a mass term $1/2g^2v^2A_\mu A^\mu \equiv 1/2M_A^2A_\mu A^\mu$, where we defined

$$M_A = gv \qquad (4.16)$$

Secondly, there is a coupling of the gauge field to the H-field

$$g^2vA_\mu A^\mu H = gM_AA_\mu A^\mu H \qquad (4.17)$$

It is important to remember that this coupling is proportional to the mass of the gauge boson. Finally, there is also the bilinear term $g\,v\,A^\mu\partial_\mu\phi$, which after integrating by parts (for the action S) may be written as $-M_A\,\phi\,\partial_\mu A^\mu$. This mixes the Goldstone boson, ϕ, with the longitudinal component of the gauge boson, with strength M_A (when the gauge-boson field A_μ is separated into its transverse and longitudinal components, $A_\mu = A_\mu^L + A_\mu^T$, where $\partial^\mu A_\mu^T = 0$). Later on, we will use the gauge freedom to get rid of this mixing term.

4.4 Goldstone Bosons

In the previous subsection we have seen that there is a massless boson, a Goldstone boson, associated with the flat direction in the potential. Goldstone's Theorem describes the appearance of massless bosons when a global (not gauge) symmetry is spontaneously broken.

Suppose we have a theory whose Lagrangian is invariant under a symmetry group $\mathcal{G}$ with N generators $\mathbf{T}^a$ and the symmetry group of the vacuum forms a subgroup $\mathcal{H}$ of $\mathcal{G}$, with m generators. This means that the vacuum state is still invariant under transformations generated by the m generators of $\mathcal{H}$, but not the remaining $N-m$ generators of the original symmetry group $\mathcal{G}$. Thus we have

$$\begin{aligned}\mathbf{T}^a|0\rangle &= 0 \qquad a = 1\ldots m \\ \mathbf{T}^a|0\rangle &\neq 0 \qquad a = m+1\ldots N\end{aligned} \qquad (4.18)$$

Goldstone's theorem states that then there will be $N-m$ massless particles (one for each broken generator of the group). The case considered in this section is special case in that there is only one generator of the symmetry group (i.e. $N = 1$) which is broken by the vacuum. Thus, there is no generator that leaves the vacuum invariant (i.e. $m = 0$) and we get $N - m = 1$ Goldstone boson.

Like all good general theorems, Goldstone's theorem has a loophole, which arises when one considers a gauge theory, i.e. when one allows the original symmetry transformations to be local. In a spontaneously broken gauge theory, the choice of which vacuum is the true vacuum is equivalent to choosing a gauge, which is necessary in order to be able to quantize the theory. What this means is that the Goldstone bosons, which can, in principle, transform the vacuum into one of the states degenerate with the vacuum, now affect transitions into states which are not consistent with the original gauge choice. This means that the Goldstone bosons are "unphysical" and are often called "Goldstone ghosts".

On the other hand the quantum degrees of freedom associated with the Goldstone bosons are certainly there *ab initio* (before a choice of gauge is made). What happens to them?

A massless vector boson has only two degrees of freedom (the two directions of polarization of a photon), whereas a massive vector (spin-one) particle has three possible values for the helicity of the particle. In a spontaneously broken gauge theory, the Goldstone boson associated with each broken generator provides the third degree of freedom to the gauge bosons. This means that the gauge bosons become massive. The Goldstone boson is said to be "eaten" by the gauge boson. This is related to the mixing term between A_L^μ and ϕ of the previous subsection. Thus, in our abelian model, the two degrees of freedom of the complex field Φ turn out to be the Higgs field and the longitudinal component of the (now massive) gauge boson. There is no physical, massless particle associated with the degree of freedom ϕ present in Φ.

4.5 The Unitary Gauge

As mentioned above, we want to use the gauge freedom to choose a gauge such that there are no mixing terms between the longitudinal component of the gauge field and the Goldstone boson. Recall

$$\Phi = \frac{1}{\sqrt{2}} (v + H)\, e^{i\phi/v} = \frac{1}{\sqrt{2}} \left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi + \dots \right). \tag{4.19}$$

where the dots stand for nonlinear terms in ϕ. Next we make a gauge transformation (see eq. (1.2))

$$\Phi \to \Phi' = e^{-i\phi/v}\, \Phi \tag{4.20}$$

In other words, we fix the gauge such that the imaginary part of Φ vanishes. Under the gauge transformation eq. (4.20) the gauge field transforms according to (see eq. (1.11))

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{gv}[\partial_\mu \phi] \tag{4.21}$$

This is in fact the superposition of A_μ and ϕ that make up the physical field. Note that the change from A_μ to A'_μ made in eq. (4.21) affects only the longitudinal component. If we express now the Lagrangian in terms of Φ' and A'_μ there will be no mixing term. Even better, the ϕ field vanishes altogether! This can easily be seen by noting that under a gauge transformation the covariant derivative $D_\mu\Phi$ transforms in the same way as Φ, thus

$$D_\mu\Phi \to (D_\mu\Phi)' = e^{-i\phi/v} D_\mu\Phi = e^{-i\phi/v}\frac{1}{\sqrt{2}}\left(\partial_\mu H + igA'_\mu(v+H)\right) \tag{4.22}$$

and $(D_\mu\Phi)'^*(D^\mu\Phi)'$ is independent of ϕ. Performing the algebra (and dropping the ' for the A-field) we get for the Lagrangian in the unitary gauge

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{M_A^2}{2}A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{M_H^2}{2}H^2 \\ &+ gM_A A_\mu A^\mu H + \frac{g^2}{2}A_\mu A^\mu H^2 - \frac{\lambda}{4}H^4 - \sqrt{\frac{\lambda}{2}}M_H H^3 \end{aligned} \tag{4.23}$$

with M_A and M_H as defined in eqs. (4.16) and (4.14) respectively. All the terms quadratic in A_μ may be written (in momentum space) as

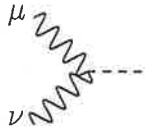
$$A_\mu(-p)\left(-g^{\mu\nu}p^2 + p^\mu p^\nu + g^{\mu\nu}M_A^2\right)A_\nu(p). \tag{4.24}$$

The gauge boson propagator is the inverse of the coefficient of $A_\mu(-p)A_\nu(p)$, which is

$$-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_A^2}\right)\frac{1}{(p^2 - M_A^2)}, \tag{4.25}$$

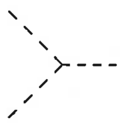
which is the usual expression for the propagator of a massive spin-one particle, eq. (4.2). The only other remaining particle is the scalar, H, with mass $m_H = \sqrt{2}\,\mu$ which is the Higgs boson. This is a physical particle, which interacts with the gauge boson and also has cubic and quartic self-interactions. The Lagrangian given in eq. (4.23) leads to the following vertices and Feynman rules.

$2\,i\,e^2 g_{\mu\nu}$



$2\,i\,eM_A g_{\mu\nu}$

$6\,i\,\lambda$



$6\,i\,m_H\sqrt{2\lambda}$

The advantage of the unitary gauge is that there appear no unphysical particles, i.e. the ϕ-field has completely disappeared. The disadvantage is that the propagator of the gauge field, eq. (4.25), behaves as p^0 for $p \to \infty$. As discussed in section 4.1 this seems to indicate that the theory is non-renormalizable. It seems that we have not gained anything at all by breaking the theory spontaneously rather than by simply adding a mass term by hand. Fortunately this is not true. In order to see that the theory is still renormalizable, in spite of eq. (4.25), it is very useful to consider a different type of gauges, namely the R_ξ gauges discussed in the next subsection.

4.6 R_ξ Gauges (Feynman Gauge)

The class of R_ξ gauges is a more conventional way to fix the gauge. Recall that in QED we fixed the gauge by adding a term, eq. (1.21) in the Lagrangian. This is exactly what we do here. The gauge fixing term we are adding to the Lagrangian density eq. (4.6) is

$$\begin{aligned}\mathcal{L}_R &\equiv -\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu - (1-\xi)M_A\phi\right)^2 \\ &= -\frac{1}{2(1-\xi)}\partial_\mu A^\mu \partial_\nu A^\nu + M_A\phi\partial_\mu A^\mu - \frac{1-\xi}{2}M_A^2\phi^2 \end{aligned} \tag{4.26}$$

The special value $\xi = 0$ again corresponds to the Feynman gauge. The second term in eq. (4.26) cancels precisely the mixing term in eq. (4.15). Thus, we achieved our goal. Note however, that in this case, contrary to the unitary gauge, the unphysical ϕ-field does not disappear. The first term in eq. (4.26) is bilinear in the gauge field, thus it contributes to the gauge-boson propagator. The terms bilinear in the A-field are

$$-\frac{1}{2}A^{\mu}(-p)\left(-g_{\mu\nu}(p^2 - M_a^2) + p_\mu p_\nu - \frac{p_\mu p_\nu}{1-\xi}\right)A^{\nu}(p) \tag{4.27}$$

which leads to the gauge boson propagator

$$\frac{-i}{(p^2 - M_A^2)}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2 - (1-\xi)M_A^2}\right) \tag{4.28}$$

Note that in Feynman gauge, the propagator becomes particularly simple. The crucial feature of eq. (4.28), however, is that this propagator behaves as p^{-2} for $p \to \infty$. Thus, this class of gauges is manifestly renormalizable. There is, however, a price to pay. The Goldstone boson is still present. It has acquired a mass, M_A, from the gauge fixing term, and it has interactions with the gauge boson, with the Higgs scalar and with itself. Furthermore, for the purposes of higher order corrections in non-Abelain theories, we need to introduce Faddeev-Popov ghosts which interact with the gauge bosons, the Higgs scalar and the Goldstone bosons.

Let us stress that there is no contradiction at all between the apparent non-renormalizability of the theory in the unitary gauge and the manifest renormalizability in the R_ξ gauge. Since physical quantities are gauge invariant, any physical quantity can be calculated in a gauge where renormalizability is manifest. As mentioned above, the price we pay for this is that there are more particles and many more interactions, leading to a plethora of Feynman diagrams. We therefore only work in such gauges if we want to compute higher order corrections. For the rest of these lectures we shall confine ourselves to tree-level calculations and work solely in unitary gauge.

Nevertheless, one cannot over-stress the fact that only when the gauge bosons acquire masses through the Higgs mechanism do we have a renormalizable theory. It is this mechanism that makes it possible to write down a consistent Quantum Field Theory which describes the weak interactions.

4.7 Summary

- In the case of a gauge theory the Goldstone bosons provide the longitudinal component of the gauge bosons, which therefore acquire a mass. The mass is proportional to the magnitude of the vacuum expectation value and the gauge coupling constant. The Goldstone bosons themselves are unphysical.

- It is possible to work in the unitary gauge where the Goldstone boson fields are set to zero.

- When gauge bosons acquire masses by this (Higgs) mechanism, renormalizability is maintained. This can be seen explicitly if one works in a R_ξ gauge, in which the gauge boson propagator decreases as $1/p^2$ as $p \to \infty$. This is a necessary condition for renormalizability. If one does work in such a gauge, however, one needs to work with Goldstone boson fields, even though the Goldstone bosons are unphysical. The number of interactions and the number of Feynman graphs required for the calculation of some process is greatly increased.

5 The Standard Model with one Family

To write down the Lagrangian of a theory, one first needs to choose the particle content and the symmetries (gauge and global)— and then write down every allowed renormalizable interaction. In this section we shall use this recipe to construct the Standard Model with one family. The Lagrangian should contain pieces

$$\mathcal{L}_{(SM,1)} = \mathcal{L}_{\text{gauge bosons}} + \mathcal{L}_{\text{fermion masses}} + \mathcal{L}_{\text{fermionKT}} + \mathcal{L}_{Higgs} \quad . \tag{5.1}$$

The terms are written out in eqns 5.15, 5.29, 5.30 and 5.55.

5.1 Left- and Right- Handed Fermions

The weak interactions are known to violate parity. Parity non-invariant interactions for fermions can be constructed by giving different interactions to the "left-handed" and "right-handed" components, where these are defined in eqn 5.4. Thus, in writing down the Standard Model, we will treat the left-handed and right-handed parts separately.

A Dirac field, ψ, representing a fermion, can be expressed as the sum of a left-handed part, ψ_L, and a right-handed part, ψ_R,

$$\psi \; = \; \psi_L \; + \; \psi_R, \tag{5.2}$$

where

$$\psi_L \; = \; P_L \psi \quad \text{with} \quad P_L = \frac{(1-\gamma_5)}{2} \tag{5.3}$$

$$\psi_R \; = \; P_R \psi \quad \text{with} \quad P_R = \frac{(1+\gamma_5)}{2} \tag{5.4}$$

P_L and P_R are projection operators in the sense that

$$P_L P_L = P_L, \quad P_R P_R = P_R, \quad \text{and } P_L P_R = 0. \tag{5.5}$$

They project out the left-handed (negative) and right-handed (positive) *chirality* states of the fermion respectively. This is the definition of chirality, which is a property of fermion fields, but not a physical observable.

The kinetic term of the Dirac Lagrangian and the interaction term of a fermion with a vector field can also be written as a sum of two terms each involving only one chirality

$$\overline{\psi}\gamma^\mu\partial_\mu\psi \; = \; \overline{\psi_L}\gamma^\mu\partial_\mu\psi_L \; + \overline{\psi_R}\gamma^\mu\partial_\mu\psi_R, \tag{5.6}$$

$$\overline{\psi}\gamma^\mu A_\mu\psi \; = \; \overline{\psi_L}\gamma^\mu A_\mu\psi_L \; + \overline{\psi_R}\gamma^\mu A_\mu\psi_R. \tag{5.7}$$

On the other hand, a mass term mixes the two chiralities

$$m\overline{\psi}\psi \;=\; m\overline{\psi_L}\psi_R \;+ m\overline{\psi_R}\psi_L. \tag{5.8}$$

Exercise 5.1
Use $(\gamma_5)^2 = 1$ to verify eq. (5.5) and $\overline{\psi} = \psi^\dagger\gamma^0$, $\gamma^{5\dagger} = \gamma^5$ as well as $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ to verify eq. (5.7).

In the limit where the fermions are massless (or sufficiently relativistic), chirality becomes *helicity*, which is the projection of spin on the direction of motion, and is a physical observable. Thus, if the fermions are massless, we can treat the left-handed and right-handed chiralities as separate particles of conserved helicity. We can understand this physically from the fact that if a fermion is massive and is moving along the *positive* z-axis along which its spin has a *positive* component, so that the helicity is *positive* in this frame, one can always boost into a frame in which the fermion is moving along the *negative* z-axis, but the component of spin is unchanged. In the new frame the helicity will be *negative*. On the other hand if the particle is massless and travels with the speed of light no such boost is possible and in that case helicity/chirality is a good quantum number.

Exercise 5.2
For a massless spinor

$$u(p) \;=\; \frac{1}{\sqrt{E}}\begin{pmatrix} E\chi \\ \underline{\sigma}\cdot\underline{p}\chi \end{pmatrix},$$

where χ is a two-component spinor, show that

$$(1 \pm \gamma^5)u(p)$$

are eigenstates of $\underline{\sigma}\cdot\underline{p}/E$ with eigenvalues ± 1 respectively. Take

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and in 4×4 matrix notation $\underline{\sigma}\cdot\underline{p}$ means

$$\begin{pmatrix} \underline{\sigma}\cdot\underline{p} & 0 \\ 0 & \underline{\sigma}\cdot\underline{p} \end{pmatrix}$$

5.2 Symmetries and Particle Content

We have made all the preparations to write down a gauge invariant Lagrangian. We now have only to pick the gauge group and the matter content of the theory. It should be noticed that there are no theoretical reasons to pick a certain group or certain matter content. It is to match experimental observations that we pick the gauge group for the Standard Model to be

$$U(1)_Y \times SU(2) \times SU(3) \tag{5.9}$$

To indicate that the abelian $U(1)$ group is *not* the gauge group of QED but of hypercharge a subscript Y has been added. The corresponding coupling and gauge boson is denoted by g' and B^μ respectively.

The $SU(2)$ group has three generators ($\mathbf{T}_a = \sigma_a/2$), the coupling is denoted by g and the three gauge bosons are denoted by $W^1_\mu, W^2_\mu, W^3_\mu$. None of these gauge bosons (and neither B_μ) are physical particles. As we will see, linear combinations of these gauge bosons will make up the photon as well as the W and Z boson.

Finally, the $SU(3)$ is the group of the strong interactions. The corresponding eight gauge bosons are the gluons. In this section we will concentrate on the other two groups, with one generation of fermions. The strong interactions are dealt with in section 3, and extra generations are in next section.

As matter content for the first family, we have

$$q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}; \; u_R; \; d_R; \; \ell_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}; \; e_R; \; \{\nu_R!!\} \tag{5.10}$$

Note that a right-handed neutrino ν_R has appeared. It is gauge singlet (no strong interactions, no weak interactions, no electric charge), so is unneccessary in a model with massless neutrinos. However, neutrinos are known to have small masses, which can be described by adding a ν_R. Neutrino masses will be further discussed in chapter 7.

Note also that the left- and right-handed fermion components have been given different weak interactions. The Standard Model is constructed this way, because the weak interactions are known to violate parity. The left-handed components form doublets under $SU(2)$ whereas the right-handed components are singlets. This means that under a $SU(2)$ gauge transformations we have

$$e_R \;\rightarrow\; e'_R = e_R \tag{5.11}$$

$$\ell_L \;\rightarrow\; \ell'_L = e^{-i\omega^a \mathbf{T}^a} \ell_L \tag{5.12}$$

Thus, the $SU(2)$ singlets e_R, ν_R, u_R and d_R are invariant under $SU(2)$ transformations and do not couple to the corresponding gauge bosons $W^1_\mu, W^2_\mu, W^3_\mu$.

Since this separation of the electron into its left- and right-handed helicity only makes sense for a massless electron we also need to assume that the electron *is* massless in the exact $SU(2)$ limit and that the mass for the electron arises as a result of spontaneous symmetry breaking in a similar way as the masses for the gauge bosons arise. We will come back to this later.

Under $U(1)_Y$ gauge transformations, the matter fields transform as follows

$$\psi \to \psi' = e^{-i\omega Y(\psi)}\psi \tag{5.13}$$

where Y is the hypercharge of the particle under consideration. It is chosen to give the observed electric charge of the particles. The explicit values for the hypercharges of the particles listed in eq. (5.10) are as follows:

$$Y(\ell_L) = -\frac{1}{2};\ Y(e_R) = -1;\ Y(\nu_R) = 0;\ Y(q_L) = \frac{1}{6};\ Y(u_R) = \frac{2}{3};\ Y(d_R) = -\frac{1}{3}; \tag{5.14}$$

Under $SU(3)$ the lepton fields ℓ_L, e_R, ν_R are singlets, i.e. they do not transform at all. This means that they do not couple to the gluons. The quarks on the other hand form triplets under $SU(3)$. The strong interaction does not distinguish between left- and right-handed particles.

We have now listed all fermions that belong to the first family, together with their transformation properties under the various gauge transformations. However, since we ultimately want massive weak gauge bosons, we will have to break the $U(1)_Y \times SU(2)$ gauge group spontaneously, by introducing some type of Higgs scalar. The transformation properties of this scalar will be deduced in the discussion of fermion masses.

5.3 Kinetic Terms for the Gauge Bosons

The gauge kinetic terms for abelian and non-abelian theories were presented in the first two lectures. From the general expression of eqn 2.21, we extract for the SM gauge bosons:

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{4}F^A_{\mu\nu}F^{A\,\mu\nu} + \mathcal{L}_{gauge-fix} + \mathcal{L}_{FP\ ghosts} \tag{5.15}$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the hypercharge field strength, the second term contains the SU(2) field strength so a runs from one to three (over the three vector bosons of SU(2)), and third is the gluon kinetic term so A : 1..8. To do an explicit perturbative calculation, additional gauge fixing terms, and Fadeev-Popov ghosts, must be included. The form of these terms depends on the choice of gauge.

5.4 Fermion Masses and Yukawa Couplings

We cannot have an explicit mass term for the quarks or electrons, since a mass term mixes left-handed and right-handed fermions and we have assigned these to different multiplets of weak SU(2). However, if an SU(2) doublet Higgs is introduced, there is a gauge invariant interaction that will look like a mass when the Higgs gets a vev. Such an interaction is called a 'Yukawa interaction' and is written

$$\mathcal{L}_{\text{Yukawa}} = -Y_e\, \overline{l_L}^i\, \Phi_i e_R + \text{h.c.} \tag{5.16}$$

Note that the Higgs doublet must have $Y = 1/2$, to ensure that this term has zero weak hypercharge.

Recalling eq. (5.19) we introduce a scalar "Higgs" field, which is a doublet under $SU(2)$, singlet under SU(3) (no colour), and has a scalar potential as given in eq. (4.9), i.e.

$$V(\Phi) = -\mu^2\Phi^*\Phi + \lambda\,|\Phi^*\Phi|^2\,. \tag{5.17}$$

This potential has a minimum at $\Phi^*\Phi = \frac{1}{2}\mu^2/\lambda$, so some component of the Higgs doublet should get a "vev" (vacuum expectation value). In unitary gauge, this vev can be written

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.18}$$

with $v = \mu/\sqrt{\lambda}$.

Recall from the previous chapter, that Φ can be written as its "radial" degree of freedom $\times$ an exponential containing the broken generators of the gauge symmetry:

$$\Phi = \frac{e^{i(\omega_a \mathbf{T}^a - \omega_3 \mathbf{Y})}}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H \end{pmatrix}. \tag{5.19}$$

The unitary gauge choice consists of absorbing this exponential with a gauge transformation, so that in the unitary gauge, eqn (5.16) is

$$\mathcal{L}_{\text{Yukawa}} = -\frac{Y_e}{\sqrt{2}}\begin{pmatrix} \overline{\nu_L} & \overline{e_L} \end{pmatrix}\begin{pmatrix} 0 \\ v + H \end{pmatrix} e_R + \text{h.c.} \tag{5.20}$$

The part proportional to the vev is simply

$$-\frac{Y_e v}{\sqrt{2}}\,(\overline{e_L}\, e_R + \overline{e_R}\, e_L) = \frac{Y_e v}{\sqrt{2}}\,\overline{e}\, e, \tag{5.21}$$

and we see that the electron has acquired a mass, which is proportional to the vev of the scalar field. This immediately gives us a relation for the Yukawa coupling in terms of the electron mass, m_e, and the W mass, M_W,

$$Y_e = g\frac{m_e}{\sqrt{2}\,M_W}. \tag{5.22}$$

Thus, as for the gauge bosons, the strength of the coupling of the Higgs to fermions is proportional to the mass of the fermions.

The quarks also acquire a mass through the spontaneous symmetry breaking mechanism, via their Yukawa coupling with the scalars. The interaction term

$$- Y_d \, \overline{q_L}^i \, \Phi_i \, d_R \; + \; \text{h.c.} \tag{5.23}$$

gives a mass, m_d to the d quark, when we replace Φ_i by its vev. This mass is given by

$$m_d \;=\; \frac{Y_d}{\sqrt{2}} v \;=\; \sqrt{2} \frac{Y_d \, M_W}{g} \tag{5.24}$$

Since the vev is in the lower component of the Higgs doublet, we must do a little more work to obtain a mass for the upper element u of the quark doublet. In the case of $SU(2)$ there is a second way in which we can construct an invariant for the Yukawa interaction:

$$- Y_u \; \epsilon_{ij} \, \overline{q_L}^i \, \Phi^{*j} u_R + \; \text{h.c.} \qquad (i, j \;=\; 1, 2), \tag{5.25}$$

where ϵ_{ij} is the two-dimensional antisymmetric tensor. Note that

$$\Phi^c = \epsilon_{ij} \Phi^{j*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_+^* \\ \Phi_0^* \end{pmatrix} \tag{5.26}$$

has $Y = -1/2$, as required by the $U(1)$ symmetry. This term does indeed give a mass m_u to the u quark, where

$$m_u \;=\; \frac{Y_u}{\sqrt{2}} v \;=\; \sqrt{2} \frac{Y_u \, M_W}{g} \tag{5.27}$$

So the SM Higgs scalar couples to both the u quark and the d quark, with interaction terms

$$- g \frac{m_u}{2 \, M_W} \, \overline{u} \, H \, u \;-\; g \frac{m_d}{2 \, M_W} \, \overline{d} \, H \, d. \tag{5.28}$$

The terms in the Lagrangian that give masses to the first generation quarks and charged leptons are

$$\mathcal{L}_{\text{fermion masses}} \;=\; - Y_e \, \overline{l_L}^i \, \Phi_i e_R \;-\; Y_d \, \overline{q_L}^i \, \Phi_i \, d_R \;-\; Y_u \; \epsilon_{ij} \, \overline{q_L}^i \, \Phi^{*j} u_R + \; \text{h.c.} \tag{5.29}$$

We could also included a Yukawa mass term for the neutrinos: $- Y_\nu \; \epsilon_{ij} \, \overline{\ell_L}^i \, \Phi^{*j} \nu_R +$ h.c. . However, neutrino masses do not neccessarily arise from a Yukawa interaction (this will be discussed in a later chapter).

5.5 Kinetic Terms for Fermions

The fermionic kinetic terms should be familiar from chapter 2:

$$\begin{aligned}\mathcal{L}_{\text{fermionKT}} &= i\,\overline{\ell_L}^T\,\gamma^\mu \mathbf{D}_\mu\,\ell_L + i\overline{e_R}\,\gamma^\mu D_\mu\,e_R + i\overline{\nu_R}\,\gamma^\mu \partial_\mu\,\nu_R \\ &\quad + i\,\overline{q_L}^T\,\gamma^\mu \mathbf{D}_\mu\,q_L + i\overline{d_R}\,\gamma^\mu D_\mu\,d_R + i\overline{u_R}\,\gamma^\mu \partial_\mu\,u_R \qquad (5.30)\end{aligned}$$

where the covariant derivatives include the hypercharge, SU(2) and SU(3) gauge bosons as required. For instance:

$$D_\mu = \partial_\mu + ig\mathbf{T}^a W_\mu^a + ig'Y(\ell_L)B_\mu \qquad \text{for } \ell_L \qquad (5.31)$$

$$D_\mu = \partial_\mu + ig'Y(e_R)B_\mu \qquad \text{for } e_R \qquad (5.32)$$

$$D_\mu = \partial_\mu + ig_s\mathbf{T_s}^a G_\mu^a + ig'Y(d_R)B_\mu \qquad \text{for } d_R \qquad (5.33)$$

where the strong coupling (g_s), the eight generators of $SU(3)$ ($\mathbf{T_s}^a$) and the corresponding gluon fields (G_μ^a) have been introduced, and $Y(f)$ is the hypercharge of fermion f.

This gives the following interaction terms between the leptons and the gauge bosons:

$$-\frac{g}{2}\begin{pmatrix}\overline{\nu_L}\\ \overline{e_L}\end{pmatrix}^T \gamma^\mu \left(\begin{pmatrix} W_\mu^0 & \sqrt{2}\,W_\mu^- \\ \sqrt{2}\,W_\mu^+ & -W_\mu^0\end{pmatrix} - \tan\theta_W B_\mu\right)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} - i\,g\,\tan\theta_W\,\overline{e_R}\,\gamma^\mu B_\mu e_R. \qquad (5.34)$$

where we have used

$$Z_\mu \equiv \cos\theta_W\,W_\mu^3 - \sin\theta_W B_\mu \qquad (5.35)$$

$$A_\mu \equiv \cos\theta_W\,B_\mu + \sin\theta_W W_\mu^3 \qquad (5.36)$$

to replace the B_μ and W_μ^3 by the physical particles Z_μ and A_μ. (In the exercises of chapter 2, these definitions followed from requiring that the photon not interact with the neutrino. In section 5.6, we will see that the photon is also massless).

Writing out the projection operators for left- and right-handed fermions, eqs. (5.3) and (5.4), we obtain the following interactions:

1. A coupling of the charged vector bosons $W^\pm$, which mediate transitions between neutrinos and electrons (or u and d quarks) with an interaction term

$$-\frac{g}{2\sqrt{2}}\,\overline{\nu}\,\gamma^\mu\left(1-\gamma^5\right)e\,W_\mu^- - \frac{g}{2\sqrt{2}}\,\overline{u}\,\gamma^\mu\left(1-\gamma^5\right)d\,W_\mu^- + \text{h.c.} \qquad (5.37)$$

(h.c. means 'hermitian conjugate' and gives the interaction involving an emitted W_μ^+ where the incoming particle is a neutrino (u) and the outgoing particle is an electron (d).

2. The usual coupling of the photon with the charged fermions is (using, for instance, the relation eq. (5.54))

$$g \sin\theta_W \, \overline{e}\, \gamma^\mu \, e \, A_\mu - \frac{2}{3} g \sin\theta_W \, \overline{u}\, \gamma^\mu \, u \, A_\mu + \frac{1}{3} g \sin\theta_W \, \overline{d}\, \gamma^\mu \, d \, A_\mu. \tag{5.38}$$

   Note that the left- and right-handed fermions have exactly the same coupling to the photon so that the electromagnetic coupling turns out to be purely vector (i.e. no γ^5 term).

3. The coupling of neutrinos to the neutral weak gauge boson, Z_μ

$$-\frac{g}{4\cos\theta_W} \overline{\nu}\, \gamma^\mu \left(1-\gamma^5\right) \nu \, Z_\mu \tag{5.39}$$

4. The coupling of both the left- and right-handed electron to the Z

$$\frac{g}{4\cos\theta_W} \overline{e} \left(\gamma^\mu \left(1-\gamma^5\right) - 4\sin^2\theta_W \gamma^\mu\right) e \, Z_\mu \tag{5.40}$$

5. The coupling of the quarks to the Z can be written in the general form

$$-\frac{g}{2\cos\theta_W} \overline{q_i} \left(T_i^3 \gamma^\mu \left(1-\gamma^5\right) - 2Q_i \sin^2\theta_W \gamma^\mu\right) q_i \, Z_\mu, \tag{5.41}$$

   where quark i has third component of weak isospin T_i^3 and electric charge Q_i.

From these terms in the Lagrangian we can read off directly the Feynman rules for the three-point vertices with two fermions and one weak gauge boson. Then we can use these vertices to calculate weak interactions of the quarks and leptons. This allows us, for example, to calculate the total decay width of the Z or W boson, by calculating the decay width into all possible quarks and leptons. However, quarks are not free particles, so for inclusive processes, in which we trigger on known initial or final state hadrons, information is needed about the probability to find a quark with given properties inside an initial hadron or the probability that a quark with given properties will decay ("fragment") into a final state hadron.

Exercise 5.3

The decay rate for the Z into a fermion-antifermion pair, $Z \to f\,\bar{f}$, is

$$\Gamma \;=\; \frac{1}{2M_Z}\int d^{LIPS}\,|\mathcal{M}|^2 \;=\; \frac{1}{64\,\pi^2\,M_Z}\int d\Omega\,|\mathcal{M}|^2,$$

where d^{LIPS} means integrate over the phase-space for the two final-state fermions, and $d\Omega$ means integrate over the solid angle of one final-state particle. Write the general interaction term for the coupling of the Z boson to a fermion as

$$-\frac{g}{2\cos\theta_W}\gamma^\mu\left(v_f - a_f\,\gamma^5\right).$$

Show that the square matrix element, summed over the fermion spins and averaged over the boson spins is

$$|\mathcal{M}|^2 \;=\; -\frac{1}{12}\,g_{\mu\nu}\,\frac{g^2}{\cos^2\theta_W}\,\left((v_f)^2+(a_f)^2\right)\,\mathrm{Tr}\left(\gamma^\mu\gamma\cdot k_1\gamma^\nu\gamma\cdot k_2\right),$$

where k_1 and k_2 are the momenta of the outgoing fermions and the gauge polarization sum is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\,\epsilon_\nu^{(\lambda)} \;=\; -g_{\mu\nu}+\frac{q_\mu\,q_\nu}{M_Z^2}$$

($q = k_1 + k_2$ is the initial momentum of the Z boson). Hence show that

$$\Gamma \;=\; \frac{1}{48\,\pi}\frac{g^2}{\cos^2\theta_W}\,\left((v_f)^2+(a_f)^2\right)\,M_Z.$$

Neglect the masses of the fermions in comparison with the Z mass.

Exercise 5.4

The Z boson can decay leptonically into a pair of neutrinos or charged leptons of all three generations and hadronically into u quarks, d quarks, c quarks, s quarks, or b quarks (c quarks couple like u quarks, whereas s quarks and b quarks couple like d quarks). Deduce the values of v_f and a_f for each of these cases and consequently estimate the decay width of the Z boson. (The experimental value is 2.49 GeV.)

[Take M_Z=91 GeV, $\sin^2\theta_W$=0.23, and the fine-structure constant, $\alpha = 1/129$ (why this value ?)].

5.6 The Higgs Part and Gauge Boson Masses

The doublet Higgs Lagrangian should contain a "spontaneous symmetry breaking" potential, which will give the Higgs a vev and self-interactions, and kinetic terms, which will generate gauge boson masses and interactions of the Higgs with the gauge bosons. We first consider the potential:

$$V(\Phi) = -\mu^2 \Phi_i^* \Phi^i + \lambda \left(\Phi_i^* \Phi^i\right)^2 . \tag{5.42}$$

This potential has a minimum at $\Phi_i^+ \Phi_i = \frac{1}{2}\mu^2/\lambda$. Writing Φ in the form of eqn 5.19, and replacing this in the potential eq. (5.42), we find that we get a mass term for the real Higgs field H, with value $m_H = \sqrt{2}\mu$. As expected, the ω_a do not appear in the potential. In an ungauged theory, they would be the massless goldstone bosons. In a gauge theory like the Standard Model, they will reappear as the longitudinal degree of freedom of the massive gauge bosons.

The remaining term of the Φ Lagrangian is the kinetic term $(D_\mu \Phi)^\dagger (D^\mu \Phi)$. Looking at this term more carefully will help us to understand where the "physical" gauge bosons (i.e. the $W^\pm$, Z and photon) come from, and how they are related to the $W_\mu^1, W_\mu^2, W_\mu^3, B_\mu$. To see the effect of the Higgs vev on the gauge boson masses, it is most simple to work in unitary gauge, that is, we absorb the exponential of eqn 5.19 with a gauge transformation. In this gauge, the covariant derivative acting on the Higgs doublet is

$$\mathbf{D}_\mu \Phi = \frac{1}{\sqrt{2}} \left(\partial_\mu + i \frac{g}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2} W_\mu^- \\ \sqrt{2} W_\mu^+ & -W_\mu^3 \end{pmatrix} + i \frac{g'}{2} B_\mu \right) \begin{pmatrix} 0 \\ v + H \end{pmatrix}, \tag{5.43}$$

so that

$$|\mathbf{D}_\mu \Phi|^2 = \frac{1}{2} (\partial_\mu H)^2 + \frac{g^2 v^2}{4} W^{+\mu} W_\mu^- + \frac{v^2}{8} \left(g W_\mu^3 - g' B_\mu \right)^2 + \text{interaction terms}. \tag{5.44}$$

where the 'interaction terms' are terms involving three fields (two gauge fields and the H-field). Eq. (5.44) tells us that the W_μ^3 and B_μ fields mix (as do W_μ^1 and W_μ^2) and the physical gauge bosons must be superpositions of these fields, such that there are no mixing terms. Thus we define

$$Z_\mu \equiv \cos\theta_W \, W_\mu^3 - \sin\theta_W B_\mu \tag{5.45}$$

$$A_\mu \equiv \cos\theta_W \, B_\mu + \sin\theta_W W_\mu^3 \tag{5.46}$$

with the weak mixing angle θ_W defined by

$$\tan\theta_W \equiv \frac{g'}{g} \tag{5.47}$$

Then eq. (5.44) is rewritten as

$$|\mathbf{D}_\mu \Phi|^2 \;=\; \frac{1}{2}(\partial_\mu H)^2 + \frac{g^2 v^2}{4} W^+_\mu W^{-\,\mu} + \frac{v^2 g^2}{8\cos^2\theta_W} Z_\mu Z^\mu + 0 A_\mu A^\mu \tag{5.48}$$

Here we see how the $SU(2)$ and $U(1)$ are unified (or at least 'entangled') in the sense that the neutral gauge boson that acquires a mass through the Higgs mechanism is the linear superposition of a gauge boson from the $SU(2)$ and the $U(1)_Y$ gauge boson.

From eq. (5.48) we can read off the masses of the gauge bosons. The last term tells us that the linear combination eq. (5.46) remains massless. This field is identified with the photon. For the other fields we have

$$M_W = \frac{1}{2} g v; \qquad M_Z = \frac{1}{2}\frac{g v}{\cos\theta_W} \tag{5.49}$$

The Z boson mediates the neutral current weak interactions. These were not observed until after the development of the model. From the magnitude of amplitudes involving weak neutral currents (exchange of a Z boson), one can infer the (tree level) magnitude of the weak mixing angle, θ_W. The ratio of the mass of the Z and W bosons is a prediction of the Standard Model. More precisely, we define a quantity known as the ρ-parameter by

$$M_W^2 = \rho\, M_Z^2 \cos^2\theta_W. \tag{5.50}$$

In the Standard Model $\rho = 1$ at tree level. In higher orders there is a small correction, which depends on the definition used for $\sin\theta_W$ (that is, which loop corrections are included in $\sin\theta_W$). Note that the ρ-parameter would be very different from one if the symmetry breaking were due to a scalar multiplet which was not a doublet of weak isospin.

Exercise 6.3

Using the experimental values, $M_W = 80.39 \pm 0.07$ Gev, $M_Z = 91.187 \pm 0.002$ GeV, and $\sin^2\theta_W = 0.2316 \pm 0.0002$, calculate the experimental value of ρ (together with its experimental error).

The spontaneous symmetry breaking mechanism breaks $SU(2) \times U(1)_Y$ down to $U(1)$. It is this surviving $U(1)$ that is identified as the $U(1)$ of electromagnetism. It is not the $U(1)_Y$ of the original gauge group but a set of transformations generated by a particular linear combination of the original $U(1)$ and rotations about the third axis of weak isospin. To see this we note that the explicit representation of the generator $\mathbf{Y}$ as a 2×2 matrix that can be combined with the explicit representation of $\mathbf{T}^1$, $\mathbf{T}^2$ and $\mathbf{T}^3$, is given by

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.51}$$

The factor 1/2 ensures the normalization[7] condition eq. (2.7). Using eq. (5.51) it can easily be seen that the symmetry associated with the generator

$$Q \equiv Y + T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.52}$$

is not broken, i.e. $Q|0\rangle = 0$ (see eq. (4.5)). Thus, starting with four generators, we get only three Goldstone bosons. These will become the longitudinal component of three gauge bosons, thereby giving them a mass, but the fourth is left massless.

The coupling of any particle to the photon is always proportional to

$$g \sin\theta_W \,(Y \;+\; T_3) \;=\; g \sin\theta_W \, Q \tag{5.53}$$

Thus we can identify $g \sin\theta_W$ with one unit of electric charge and we have a relationship between the weak coupling, g, and the electron charge, e,

$$e \;=\; g \sin\theta_W \tag{5.54}$$

We end this subsection by giving the remaining piece of the SM Lagrangian, from eqns 5.44 and 5.42

$$\begin{aligned} \mathcal{L}_{Higgs} &= |\mathbf{D}_\mu \Phi|^2 \; - \mu^2 \Phi_i^* \Phi^i + \lambda \left(\Phi_i^* \Phi^i\right)^2 \\ &= \frac{1}{2}(\partial_\mu H)^2 + \mu^2 H^2 + \frac{g^2 v^2}{4} W^{+\mu} W^-_\mu + \frac{v^2 g^2}{8\cos^2\theta_W} Z_\mu Z^\mu \\ &\quad + \text{interaction terms} \end{aligned} \tag{5.55}$$

5.7 Classifying the Free Parameters

The free parameters in the Standard Model for one generation, are

- The two gauge couplings for the $SU(2)$ and $U(1)$ gauge groups, g and g'.
- The two parameters, μ and λ in the scalar potential $V(\Phi)$
- The Yukawa coupling constants, Y_u, Y_d, Y_e, Y_ν.

It is convenient to replace these parameters by others which are more directly experimentally measurable, namely e, $\sin\theta_W, m_e$, and m_W, and m_H, m_u, m_d and m_ν. (Notice that the

[7] We warn the reader that in the literature sometimes a different normalization is used such that eq. (5.52) reads $Q = Y/2 + T^3$

gauge sector is well measured, but the quark masses are not very observable, we have yet to see the Higgs, and although we see neutrino mass differences, measuring the absolute mass scale is difficult — and the neutrino masses might not be directly proportional to Yukawa couplings anyway.) The relation between these physical parameters and the parameters of the initial Lagrangian are

$$\tan\theta_W = \frac{g'}{g}, \tag{5.56}$$

$$e = g\sin\theta_W, \tag{5.57}$$

$$m_H = \sqrt{2}\mu, \tag{5.58}$$

$$M_W = \frac{g\,\mu}{2\sqrt{\lambda}}, \tag{5.59}$$

$$m_e = Y_e\frac{\mu}{\sqrt{\lambda}}. \tag{5.60}$$

Note that when we add more generations of fermions, we will acquire more parameters: additional masses (4 parameters per generation), and also mixing angles, as we will see in the next chapter.

In terms of these measured quantities, the Z mass, M_Z, and the Fermi-coupling, G_F are *predictions* (although historically G_F was known for many years before the discovery of the W boson and its value was used to predict the W mass).

5.8 Summary

- Weak interactions are mediated by the SU(2) gauge bosons, which act only on the left-handed components of fermions.
- The (left-handed) neutrino and left-handed component of the electron form an SU(2) doublet, whereas the right-handed components of the electron and neutino are SU(2) singlets. Similarly for the first generation quarks.
- There is also a weak hypercharge $U(1)_Y$ gauge symmetry. Both left- and right-handed quarks transform under this $U(1)_Y$ with a hypercharge which is related to the electric charge of the by the relation eq. (5.54). The left-handed leptons, and the e_R also carry hypercharge, but the ν_R has no SM gauge interactions.
- In the symmetry limit the fermions with SU(2) gauge interactions[8] are massless. The spontaneous symmetry breaking mechanism that gives a vev to the scalar field also generates fermion masses.

[8] so this does not apply to ν_R, which *can* have an explicit mass term

- The scalar multiplet that is responsible for the spontaneous symmetry breaking also carries weak hypercharge. As a result, one neutral gauge boson (the Z) acquires a mass, whereas its orthogonal superposition is the massless photon. The magnitude of the electron charge, e, is then given by $e = g \sin\theta_W$.

- The weak interactions proceed via the exchange of massive charged or neutral gauge bosons. The old four-fermi weak Hamiltonian is an effective Hamiltonian which is valid for low energy processes in which all momenta are small compared with the W mass. The Fermi coupling is obtained in terms of e, M_W and $\sin\theta_W$ by the relation eq. (6.16).

For completeness, a full set of Feynman rules for the case of a single family of leptons is given in the appendix to this lecture.

Feynman Rules in the Unitary Gauge (for one Generation of Leptons)

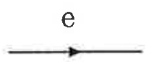
Propagators:

All propagators carry momentum p.

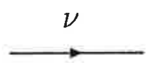
$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_W^2)/(p^2 - M_W^2)$

$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_Z^2)/(p^2 - M_Z^2)$

$-i\, g_{\mu\nu}/p^2$



$i\,(\gamma \cdot p + m_e)/(p^2 - m_e^2)$

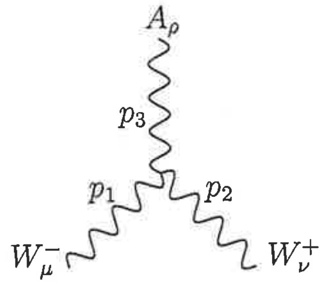


$i\,\gamma \cdot p/p^2$

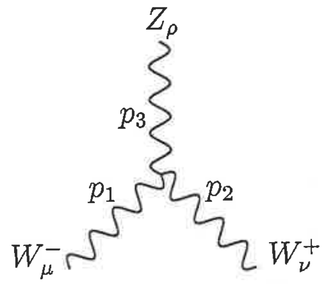
$i/(p^2 - m_H^2)$

Three-point gauge-boson couplings:

All momenta are incoming

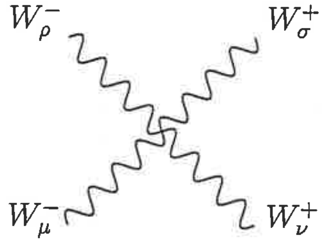


$$i\,g\,\sin\theta_W\,\left((p_1-p_2)_\rho\,g_{\mu\nu}+(p_2-p_3)_\mu\,g_{\nu\rho}+(p_3-p_1)_\nu\,g_{\rho\mu}\right)$$

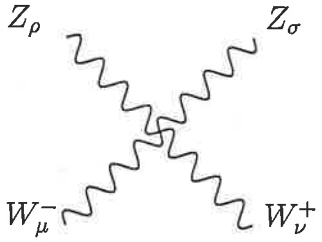


$$i\,g\,\cos\theta_W\,\left((p_1-p_2)_\rho\,g_{\mu\nu}+(p_2-p_3)_\mu\,g_{\nu\rho}+(p_3-p_1)_\nu\,g_{\rho\mu}\right)$$

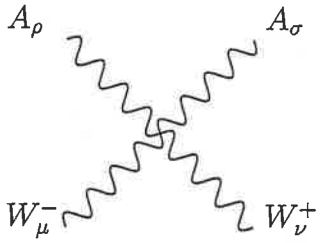
Four-point gauge-boson couplings:



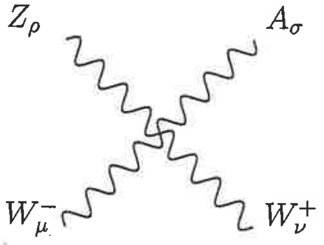
$$i\,g^2\,\left(2g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$



$$i\,g^2\cos^2\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

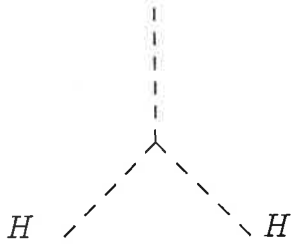


$$i\,g^2\sin^2\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

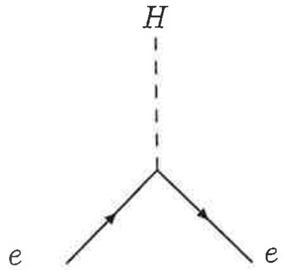


$$i\,g^2\cos\theta_W\,\sin\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

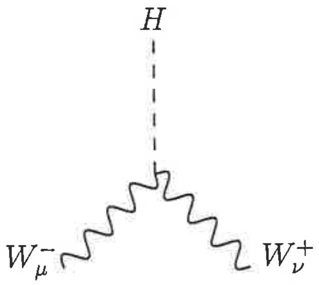
Three-point couplings with Higgs scalars:



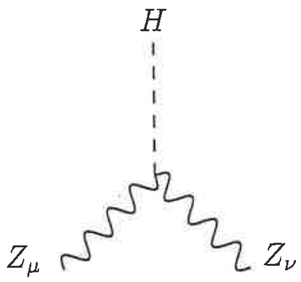
$-\frac{3}{2}\, i\, g\, m_H^2 / M_W$



$-\frac{1}{2}\, i\, g\, m_e / M_W$

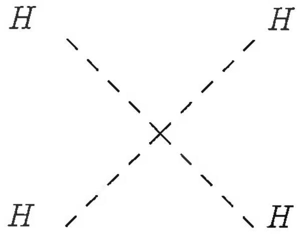


$i\, g\, M_W\, g_{\mu\nu}$

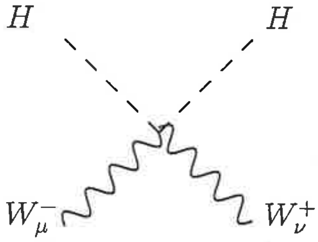


$i\ (g/\cos^2\theta_W)\ M_W\, g_{\mu\nu}$

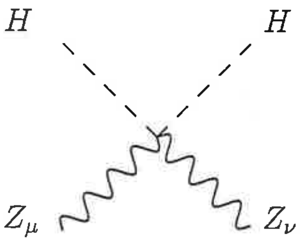
Four-point couplings with Higgs scalars:



$$-\tfrac{3}{4}\, i\, g^2\, (m_H^2/M_W^2)$$

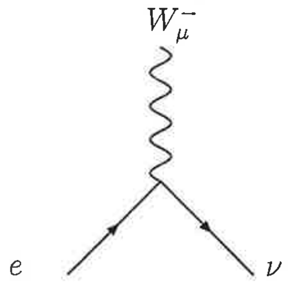


$$\tfrac{1}{2}\, i\, g^2\, g_{\mu\nu}$$

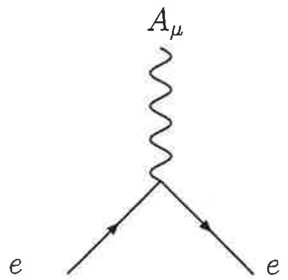


$$\tfrac{1}{2}\, i\, (g^2/\cos^2\theta_W)\, g_{\mu\nu}$$

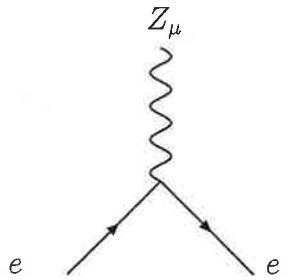
Fermion interactions with gauge bosons:



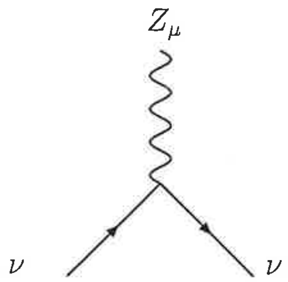
$$-i\,\left(g/2\sqrt{2}\right)\,\gamma_\mu\,(1-\gamma^5)$$



$$+\,i\,g\,\sin\theta_W\,\gamma_\mu$$



$$+\,\tfrac{1}{4}\,i\,(g/\cos\theta_W)\,\gamma_\mu\,\left(1-4\,\sin^2\theta_W\,-\gamma^5\right)$$



$$-\,\tfrac{1}{4}\,i\,(g/\cos\theta_W)\,\gamma_\mu\,(1-\gamma^5)$$

6 Additional Generations

In the previous section, the Lagrangian of the Standard Model with one family was given. Here we include additional "families", or "generations", and briefly outline the phenomenological consequences in the quark sector. Family-changing processes among the leptons will be discussed in the neutrino chapter.

6.1 A Second Quark Generation

The second generation of quarks consists of a c quark, which has electric charge $+\frac{2}{3}$ and an s quark, with electric charge $-\frac{1}{3}$. We can just add a copy of the left-handed isodoublet and copies of the right-handed singlets in order to include this generation.

The only difference would be in the Yukawa interaction terms where the coupling constants are chosen to reproduce the correct masses for the new quarks. But in this case there is a further complication. It is possible to write down Yukawa terms which mix quarks of different generations, e.g. the Yukawa couplings of the previous section become matrices in flavour space:

$$- [Y_d]_{ij}\overline{q_{Li}}\, \Phi\, d_{Rj} - [Y_u]_{ij}\overline{q_{Li}}\, \Phi^c\, u_{Rj} + h.c. \tag{6.1}$$

where i, j are generation indices. The off-diagonal element $[Y_d]_{12}$ seems to gives rise to a mass mixing between the d quark and s quark.

The Yukawa matrices are $n_f \times n_f$ matrices, where n_f is the number of flavours, and can be diagonalised by independant unitary transformations on the left and right (because $YY^\dagger$ and $Y^\dagger Y$ are hermitian). The physical particles are those that diagonalize the mass matrix. So it is convenient to rotate to the eigenbasis of the mass matrix, where there is *no* Yukawa mixing between quarks of different generations.

Notice that when we add a second generation, it has the same gauge interactions as the first. So if we make a unitary transformation in *generation* space, the fermion kinetic terms remain unchanged. Taking advantage of this freedom, we can rotate u_R, d_R and q_L respectively to the masss eigenstate bases of the u_R, d_R and u_L.

This means, however, that the quark doublets which couple to the gauge bosons *are*, in general, superposition of physical quarks, because we have written the d_{Li} in the u_{Li} mass eigenstate basis:

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L , \tag{6.2}$$

and

$$\begin{pmatrix} c \\ \tilde{s} \end{pmatrix}_L , \tag{6.3}$$

where $\tilde{d}$ and $\tilde{s}$ are related to the physical d quark and s quark by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \end{pmatrix} = \mathbf{V_C} \begin{pmatrix} d \\ s \end{pmatrix}, \tag{6.4}$$

where $\mathbf{V_C}$ is a unitary 2×2 matrix.

Terms which are diagonal in the quarks are unaffected by this unitary transformation of the quarks. Thus the coupling to photons or Z bosons is the same whether written in terms of $\tilde{d}$, $\tilde{s}$ or simply s, d. We return to this later.

On the other hand the coupling to the charged gauge bosons is

$$-\frac{g}{2\sqrt{2}}\overline{u}\gamma^\mu(1-\gamma^5)\,\tilde{d}\,W_\mu^- \;-\frac{g}{2\sqrt{2}}\overline{c}\gamma^\mu(1-\gamma^5)\,\tilde{s}\,W_\mu^- \quad + \text{h.c.} \tag{6.5}$$

which we may write as

$$-\frac{g}{2\sqrt{2}}\begin{pmatrix} \overline{u} \\ \overline{c} \end{pmatrix}^T \gamma^\mu(1-\gamma^5)\,\mathbf{V_C}\begin{pmatrix} d \\ s \end{pmatrix} W_\mu^- \quad + \text{h.c.} \tag{6.6}$$

The most general 2×2 unitary matrix may be written as

$$\begin{pmatrix} e^{-i\gamma} & \\ & 1 \end{pmatrix}\begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}\begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix}. \tag{6.7}$$

We have set one of the phases to unity since we can always absorb an overall phase by adjusting the remaining phases, α, β, and γ.

The phases, α, β, γ can be absorbed by performing a global phase transformation on the d, s and u quarks respectively. This again has no effect on the neutral terms. Thus the most general observable unitary matrix is given by

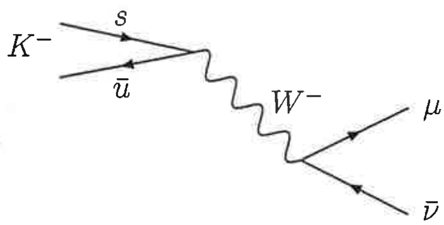
$$\mathbf{V_C} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}, \tag{6.8}$$

where θ_C is the Cabibbo angle.

In terms of the physical quarks, the charged gauge boson interaction terms are

$$\begin{aligned} -\frac{g}{2\sqrt{2}}\Big(&\cos\theta_C\,\overline{u}\gamma^\mu(1-\gamma^5)\,d \;+\; \sin\theta_C\,\overline{u}\gamma^\mu(1-\gamma^5)\,s \\ &+\;\cos\theta_C\,\overline{c}\gamma^\mu(1-\gamma^5)\,s \;-\; \sin\theta_C\,\overline{c}\gamma^\mu(1-\gamma^5)\,d\Big)\,W_\mu^- \quad + \text{h.c.} \end{aligned} \tag{6.9}$$

This means that the u quark can undergo weak interactions in which it is converted into an s quark, with an amplitude that is proportional to $\sin\theta_C$. It is this that gives rise to strangeness violating weak interaction processes, such as the leptonic decay of K^- into a muon and antineutrino. The Feynman diagram for this process is



6.2 Flavor Changing Neutral Currents

Although there are charged weak interactions that violate strangeness conservation, there are no known neutral weak interactions that violate strangeness. For example, the K^0 does not decay into a muon pair or two neutrinos (branching ratio $< 10^{-5}$). This means that the the Z boson only interacts with quarks of the same flavor. We can see this by noting that the Z-boson interaction terms are unaffected by a unitary transformation. This absence of flavor changing neutral currents (FCNC) in experimental data is rather important. As we will see, in the Standard Model there are no FCNC at tree level and the absence of FCNC is an important constraint for any extension of the Standard Model.

The Z-boson interactions with d and s quarks are proportional to

$$\bar{\tilde{d}}\,\tilde{d} + \bar{\tilde{s}}\,\tilde{s}. \tag{6.10}$$

(we have suppressed the γ-matrices which act between the fermion fields). Writing this out in terms of the physical quarks we get

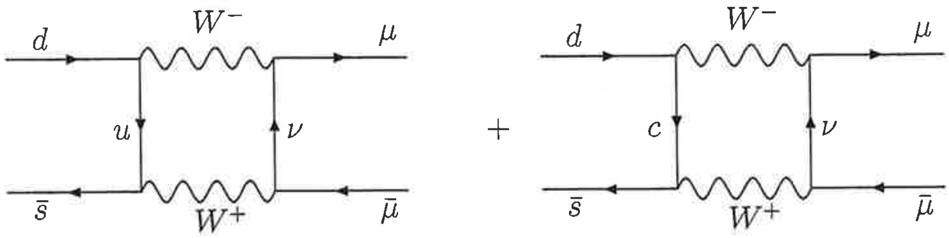
$$\begin{aligned} &\cos^2\theta_C\,\bar{d}\,d \;+\; \sin\theta_C\,\cos\theta_C\,\bar{s}\,d \;+\; \cos\theta_C\,\sin\theta_C\,\bar{d}\,s \;+\; \sin^2\theta_C\,\bar{s}\,s \\ &+\; \cos^2\theta_C\,\bar{s}\,s \;-\; \sin\theta_C\,\cos\theta_C\,\bar{d}\,s \;-\; \cos\theta_C\,\sin\theta_C\,\bar{s}\,d \;+\; \sin^2\theta_C\,\bar{d}\,d. \end{aligned} \tag{6.11}$$

We see that the cross terms cancel out and we are left with simply

$$\bar{d}\,d \;+\; \bar{s}\,s. \tag{6.12}$$

This cancellation is the reason for the absence of FCNC and is simply a consequence of the unitarity of the mixing matrix eq. (6.7). This effect is also known as the "GIM" (Glashow-Iliopoulous-Maiani) mechanism. It was used to predict the existence of the c quark.

There can be a small contribution to strangeness changing neutral processes from higher order corrections in which we do not exchange a Z boson, but two charged W bosons. The Feynman diagrams for such a contribution to the leptonic decay of a K^0 (which consists of a d quark and an s antiquark) are



These diagrams differ in the flavor of the internal quark which is exchanged, being a u quark in the first diagram and a c quark in the second. Both of these diagrams are allowed because of the Cabibbo mixing. The first of these diagrams gives a contribution proportional to

$$+\sin\theta_C\,\cos\theta_C,$$

which arises from the product of the two couplings involving the emission of the W bosons. The second diagram gives a term proportional to

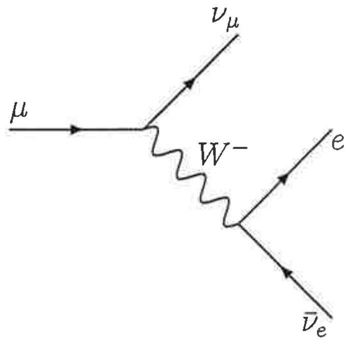
$$-\cos\theta_C\,\sin\theta_C.$$

If the c quark and u quark had identical mass then these two contributions would cancel precisely. However, because the c quark is much more massive than the u quark, there is some residual contribution. This was used to limit the mass of the c quark to < 5 GeV, before it was discovered.

6.3 Adding Another Lepton Generation

We first neglect the ν_R and neutrino masses. In this approximation, there will be no generation mixing in the lepton sector, so we can include other lepton families, the muon and its neutrino, and the tau-lepton with its neutrino, simply as copies of what we have for the electron and its neutrino For each family we have a weak isodoublet of left-handed leptons and a right-handed isosinglet for the charged lepton.

Thus, the mechanism which determines the decay of the muon (μ) is one in which the muon converts into its neutrino and emits a charged W^-, which then decays into an electron and (anti)neutrino. The Feynman diagram is



The amplitude for this process is given by the product of the vertex rules for the emission (or absorption) of a W^- with a propagator for the W boson between them. Up to corrections of order m_μ^2/M_W^2, we may neglect the effect of the term $q^\mu q^\nu/M_W^2$ in the W-boson propagator, so that we have

$$\left(-i\frac{g}{2\sqrt{2}}\overline{\nu_\mu}\gamma^\rho(1-\gamma^5)\mu\right)\left(\frac{-i\,g_{\rho\sigma}}{q^2-M_W^2}\right)\left(-i\frac{g}{2\sqrt{2}}\overline{e}\gamma^\sigma(1-\gamma^5)\,\nu_e\right), \tag{6.13}$$

where q is the momentum transferred from the muon to its neutrino. Since this is negligible in comparison with M_W we may neglect it and the expression for the amplitude simplifies to

$$i\,\frac{g^2}{8M_W^2}\,\overline{\nu_\mu}\gamma^\rho(1-\gamma^5)\mu\,\overline{e}\gamma_\rho(1-\gamma^5)\nu_e. \tag{6.14}$$

Before the development of this model, weak interactions were described by the "four-fermi model" with a weak interaction Hamiltonian given by

$$\mathcal{H}_{ijkl} \;=\; \frac{G_F}{\sqrt{2}}\,\overline{\psi}_i\,\gamma^\mu(1-\gamma^5)\psi_j\;\overline{\psi}_k\,\gamma_\mu(1-\gamma^5)\psi_l. \tag{6.15}$$

We now recognize this as an effective low-energy Hamiltonian which may be used when the energy scales involved in the weak process are negligible compared with the mass of the W boson. The Fermi coupling constant, G_F is related to the electric charge, e, the W mass and the weak mixing angle by

$$G_F \;=\; \frac{e^2}{4\sqrt{2}\,M_W^2\,\sin^2\theta_W}. \tag{6.16}$$

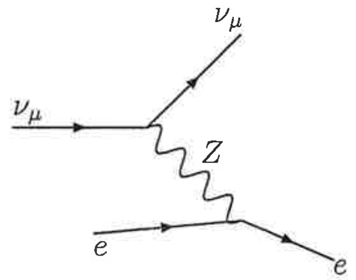
This gives us a value for G_F,

$$G_F \;= 1.12\times 10^{-5}\ \mathrm{GeV}^{-2}, \tag{6.17}$$

which is very close to the value of 1.17×10^{-5} GeV^{-2}, measured from the lifetime of the muon.

We see that the weak interactions are 'weak', not because the coupling is particularly small (the $SU(2)$ gauge coupling is about twice as large as the electromagnetic coupling), but because the exchanged boson is very massive, so that the Fermi coupling constant of the four-fermi theory is very small. The large mass of the W boson is also responsible for the fact that the weak interactions are short range (of order 10^{-18} m).

In the Standard Model, however, we also have neutral weak currents. Thus, for example, we can have elastic scattering of muon-type neutrinos against electrons via the exchange of the Z boson. The Feynman diagram for such a process is



Exercise 6.1
Let us write the four-fermi interaction for this process as

$$\mathcal{H} = \frac{G_F}{\sqrt{2}} \overline{\nu_e}\gamma^\rho(1-\gamma^5)\nu_e \, \overline{\mu}\gamma_\rho(v - a\gamma^5)\mu,$$

where v and a give us the vector and axial-vector coupling of the muon to the Z-boson (the muon couples in an identical way to the electron). Determine v and a in terms of θ_W.

6.4 Adding a Third Generation (of Quarks)

Adding a third generation is achieved in the same way. In this case the three weak isodoublets of left-handed fermions are

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}, \quad \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}, \quad \begin{pmatrix} t \\ \tilde{b} \end{pmatrix}, \qquad (6.18)$$

where $\tilde{d}$, $\tilde{s}$ and $\tilde{b}$ are related to the physical d, s and b quarks by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \\ \tilde{b} \end{pmatrix} = \mathbf{V_{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \qquad (6.19)$$

The 3×3 unitary matrix $\mathbf{V_{CKM}}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Once again it only affects the charged weak processes in which a W boson is exchanged. For this reason the elements are written as

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{6.20}$$

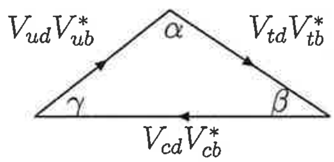
A 3×3 unitary matrix can have nine independent parameters (counting the real and imaginary parts of a complex element as two parameters). In this case there are six possible fermions involved in the charged weak processes and so we can have five relative phase transformations, thereby absorbing five of the nine parameters.

This means that whereas the Cabibbo matrix only has one parameter (the Cabibbo angle, θ_C) the CKM matrix has four independent parameters. If the CKM matrix were real it would only have three independent parameters. This means that in the case of the CKM matrix some of the elements may be complex. The four independent parameters can be thought of as three mixing angles between the three pairs of generations and a complex phase.

The requirement of unitarity puts various constraints on the elements of the CKM matrix. For example we have

$$V_{ud}\,V_{ub}^* \,+\, V_{cd}\,V_{cb}^* \,+\, V_{td}\,V_{tb}^* \;=\; 0$$

This can be represented as a triangle in the complex plane known as the "unitarity triangle"



The angles of the triangle are related to ratios of elements of the CKM matrix

$$\alpha \;=\; -\arg\left\{\frac{V_{td}\,V_{tb}^*}{V_{ud}\,V_{ub}^*}\right\} \tag{6.21}$$

$$\beta \;=\; -\arg\left\{\frac{V_{td}\,V_{tb}^*}{V_{cd}\,V_{cb}^*}\right\} \tag{6.22}$$

$$\gamma \;=\; -\arg\left\{\frac{V_{ud}\,V_{ub}^*}{V_{cd}\,V_{cb}^*}\right\} \tag{6.23}$$

A popular representation of the CKM matrix is due to Wolfenstein and uses parameters A, which is assumed to be of order unity, a complex number, $(\rho \,+\, i\,\eta)$ and a small number λ, which is approximately equal to $\sin\theta_C$. In terms of these parameters the CKM matrix is written

$$\mathbf{V_{CKM}} \;=\; \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\,\lambda^3\,(\rho - i\,\eta) \\ -\lambda & 1-\lambda^2/2 & A\,\lambda^2 \\ A\lambda^3\,(1-\rho-i\,\eta) & -A\,\lambda^2 & 1 \end{pmatrix} \;+\; \mathcal{O}(\lambda^4). \tag{6.24}$$

We see that whereas the W bosons can mediate a transition between a u quark and a b quark (V_{ub}) or between a t quark and a d quark (V_{td}), the amplitude for such transitions are suppressed as the cube of the small quantity which determines the amplitude for transitions between the first and second generations, λ. The $\mathcal{O}(\lambda^4)$ corrections are needed to ensure the unitarity of the CKM matrix and these corrections have several matrix elements which are complex.

6.5 *CP* Violation

The possibility that some of the elements of the CKM matrix may be complex provides a mechanism for the violation of CP conservation. Violation of CP conservation has been observed in the K^0, $\overline{K}^0$ system, and is currently being investigated for B-mesons.

Higher-order corrections to the masses of B^0 and $\overline{B^0}$, give rise to mixing between the two states. Thus the mass matrix can be written

$$\begin{pmatrix} M_{B^0} & \Delta M \\ (\Delta M)^* & M_{B^0} \end{pmatrix}. \tag{6.25}$$

The mass eigenstates are therefore

$$|B_L\rangle = p\,|B^0\rangle + q\,|\overline{B}^0\rangle, \tag{6.26}$$

whose mass is $M - \frac{1}{2}\Delta m$ and

$$|B_H\rangle = p\,|B^0\rangle - q\,|\overline{B}^0\rangle, \tag{6.27}$$

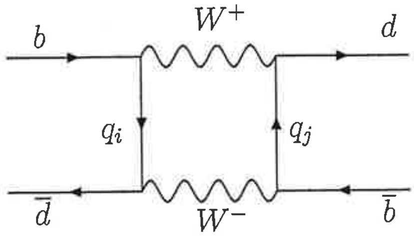
whose mass is $M + \frac{1}{2}\Delta m$, where we introduced the mass difference between the two mass eigenstates $\Delta m \equiv 2\sqrt{\Delta M(\Delta M)^*}$.

If ΔM were real then we would have $p = q = 1/\sqrt{2}$ and these mass eigenstates would be CP eigenstates, using the fact that

$$CP|B^0\rangle = -|\overline{B}^0\rangle.$$

However, the non-zero phases in the CKM matrix give rise to a complex phase for ΔM, so that the ratio of p and q is a complex phase, indicating that B_L and B_H are *not* CP eigenstates.

A typical weak interaction contribution to the mass-mixing term, ΔM is given by the Feynman diagram



Note that on the left we have a B^0, consisting of a b quark and an d antiquark, whereas on the right we have a $\overline{B}^0$ consisting of an d quark and a b antiquark. The internal quarks marked q_i and q_j can each be u, c or t quarks and each of the vertices carries some element of the CKM matrix. The total contribution, therefore, may be written

$$\sum_{i=u,c,t} \sum_{j=u,c,t} V_{ib} V_{id}^* V_{jb}^* V_{jd} a_{ij}.$$

Once again, if all the masses of the quarks were equal then the amplitudes a_{ij} would all be equal, and this would vanish by the unitarity constraints imposed on the elements V_{ik}. Since the quarks do not all have the same mass, there is some residual contribution. Indeed, the above diagram is dominated by the term in which a t quark is exchanged on both sides, since this quark is much more massive than the rest.

Restricting ourselves to the t quark exchange contribution, we can read off the phase of this contribution, without calculating the diagram itself. It is given by the phase of the products, of the CKM matrix elements entering in the diagram, namely

$$(V_{td}^* V_{tb})^2.$$

The phase of this quantity is the square of the ratio of p and q, so we have

$$\frac{p}{q} = \frac{V_{td}^* V_{tb}}{V_{td} V_{tb}^*}$$

Now suppose that at time $t = 0$ we prepare a state which is purely B^0. Accounting for the fact that the B^0 meson has a decay rate Γ, we can use eqs.(6.26, 6.27) to write the state at time t as

$$|B(t)\rangle = e^{-iMt} e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) |B^0\rangle + i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) |\overline{B}^0\rangle . \right) \tag{6.28}$$

Now suppose that the amplitude for a state $|B^0\rangle$ to decay into some CP eigenstate $|f\rangle$ is A_f, whereas the amplitude for a state $|\overline{B}^0\rangle$ to decay into the state $|f\rangle$ is $\overline{A}_f$. Once again, if CP were conserved, we would have

$$A_f = \pm \overline{A}_f,$$

but the CP violating phases give rise to a more general complex phase for the ratio of these two amplitudes.

This means that the amplitude to find the state $|f\rangle$ after time t is given by

$$\langle f|\mathcal{H}_{wk}|B(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2}\left(\cos\left(\frac{\Delta m}{2}t\right) A_f \;+\; i\frac{q}{p}\sin\left(\frac{\Delta m}{2}t\right)\overline{A}_f\right). \tag{6.29}$$

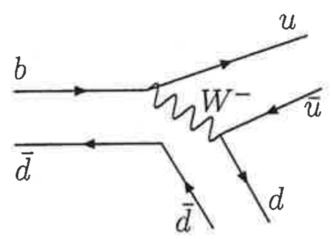
Similarly, if we had prepared a $\overline{B}^0$ at $t=0$ the amplitude to find the state $|f\rangle$ would be

$$\langle f|\mathcal{H}_{wk}|\overline{B}(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2}\left(\cos\left(\frac{\Delta m}{2}t\right) \overline{A}_f \;-\; i\frac{p}{q}\sin\left(\frac{\Delta m}{2}t\right) A_f\right). \tag{6.30}$$

Taking the square moduli, to find the decay rates we arrive at the result

$$\frac{\Gamma(B(t)\to f)-\Gamma(\overline{B}(t)\to f)}{\Gamma(B(t)\to f)+\Gamma(\overline{B}(t)\to f)} \;=\; -\;\sin\left(\Delta m\, t\right)\Im m\left(\frac{q}{p}\frac{\overline{A}_f}{A_f}\right). \tag{6.31}$$

For example, if the state $|f\rangle$ is the CP even two-pion state $|\pi^0\,\pi^0\rangle$, the Feynman diagram at the quark level for $A_{2\pi}$ is



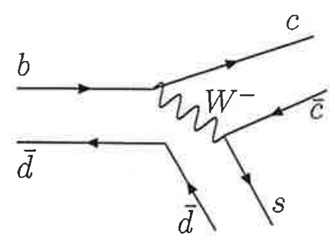
To calculate the decay amplitudes we would need to know the wave functions for the mesons in terms of the constituent quark-antiquark pairs, but for the ratio $\overline{A}_{2\pi}/A_{2\pi}$ we just need the ratios of the CKM matrix elements occurring in this diagram, namely

$$\frac{\overline{A}_{2\pi}}{A_{2\pi}} \;=\; \frac{V_{ub}\,V^*_{ud}}{V^*_{ub}\,V_{ud}},$$

so that (using eq. (6.21))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{2\pi}}{A_{2\pi}}\right) \;=\; \frac{V_{td}\,V^*_{tb}\,V_{ub}\,V^*_{ud}}{V^*_{td}\,V_{tb}\,V^*_{ub}\,V_{ud}} \;=\; -\sin\left(2\alpha\right) \tag{6.32}$$

As a further example, we consider the so-called "golden channel" where $|f\rangle$ is the state $|J/\psi\,K_S\rangle$. In this case the quark level Feynman diagram is



Here there is a further complication since the outgoing state $(s\,\bar{d})$ is actually $\overline{K}^0$ (and likewise for the $\overline{B}^0$ decay it would be a K^0). As in the B^0 system, the mass eigenstates are given by

$$\begin{aligned} |K_S\rangle &= p_K|K^0\rangle + q_K|\overline{K}^0\rangle \\ |K_L\rangle &= p_K|K^0\rangle - q_K|\overline{K}^0\rangle. \end{aligned} \tag{6.33}$$

Once again, if CP were conserved we would have $p_K = q_K = 1/\sqrt{2}$, and these mass eigenstates would be eigenstates of CP. The phases in the CKM matrix introduce a phase in the ratio of p_K and q_K, calculated from diagrams similar to the ones for the B^0 system (but with the b quark replaced by an s quark). In this case it is the diagram with internal c quark exchange that dominates (although the mass of the c quark is much less than the t quark the CKM matrix elements are much larger for c quark exchange than t quark exchange and this effect dominates), so we have a factor

$$\frac{q_K}{p_K} = \frac{V_{cd}^* V_{cs}}{V_{cd} V_{cs}^*}$$

which enters in the ratio of the decay amplitudes, giving

$$\frac{\overline{A}_{J/\psi K_S}}{A_{J/\psi K_S}} = -\frac{V_{cb} V_{cs}^* V_{cd}^* V_{cs}}{V_{cb}^* V_{cs} V_{cd} V_{cs}^*} = -\frac{V_{cb} V_{cd}^*}{V_{cb}^* V_{cd}},$$

(a minus occurs because the $J/\psi\, K_S$ state is CP odd) so that (using eq. (6.22))

$$\Im m\left(\frac{q\,\overline{A}_{J/\psi K_S}}{p\,A_{J/\psi K_S}}\right) = -\frac{V_{td} V_{tb}^* V_{cb} V_{cd}^*}{V_{td}^* V_{tb} V_{cb}^* V_{cd}} = \sin(2\beta)\,. \tag{6.34}$$

6.6 Summary

- Additional generations may be added, with gauge interactions copied from the first, but in this case one can have mass-mixing between quarks of different generations. In terms of the mass eigenstates, the charged W bosons mediate transitions between a $T^3 = +\frac{1}{2}$ quark (u, c or t) and a superposition of $T^3 = -\frac{1}{2}$ quarks (d, s and b).

In two generations, this mechanism allows weak interactions that violate strangeness conservation. The mixing matrix has only one independent parameter, the Cabibbo angle.

- The unitarity of the mixing matrix guarantees that there are no strangeness changing neutral processes. Weak interactions involving the exchange of a Z boson do not change flavor. There is a small violation of this in higher orders owing to the mass splitting between the quarks.

- Including a third generation, the mixing matrix for the $T^3 = -1/2$ quarks (d, s and b) is the CKM matrix. This matrix has four independent parameters, so that some of the matrix elements may be complex.

- The possibility that some of the elements of the CKM matrix may be complex leads to a weak interaction contribution to a mass mixing of B^0 and $\overline{B}^0$ which can be complex. This gives rise to CP violation, since the eigenstates of the B^0 mass matrix are then no longer eigenstates of CP. The CKM matrix also introduces phases in the ratios of the decay amplitudes for B^0 and $\overline{B}^0$ to a given CP eigenstate. Products of the phase of the mass mixing and the ratio of the decay amplitudes can be observed directly in tagged B-meson experiments and the angles α and β of the unitarity triangle can be directly measured.

7 Neutrinos

In its original formulation, the Standard Model had massless neutrinos—this presumably because neutrino masses were not measured at the time. We now know that neutrinos have (very little) mass, which can be accomodated in the SM in a straighforward way. We will discuss this in the second part of this chapter. There are two possible types of neutrino mass term ("Dirac" and "Majorana"), because the neutrino has zero electric charge. This makes neutrino mass terms a bit different those of the other fermions, and can explain why neutrinos are much lighter than SM fermions.

In the first part of this chapter, we focus on the currently observed consequence of small neutrino masses—oscillations.

7.1 Neutrino Oscillations

Recall that in the quark sector, there were flavour-changing charged current processes—that is, the W could interact with an up-type quark of one generation, and a down-type quark of another. If the neutrinos have mass, we should get exactly the same effect in the lepton sector, except that the mixing matrix U_{fm} is called the PMNS matrix (for Pontecorvo, Maki, Nakagawa and Sakata), rather than CKM. The index order "flavour-mass" indicates that U rotates a vector from the neutrino mass basis to the neutrino "flavour" basis, which is the charged lepton mass basis.

The physical consequences of mixing angles are quite different between the lepton sector and the quarks. This is because neutrinos are very light, and have only weak interactions. In the quark sector, one can differentiate $D \to K\bar{\mu}\nu$ from $D \to \pi\bar{\mu}\nu$, because the π and K have strong and electromagnetic interactions, which allows to track them in the detector, and they have sufficiently different masses that the tracks are distinguishable. This is not the case in trying to distinguish $\mu \to e\nu_3\bar{\nu}_2$ from $\mu \to e\nu_3\bar{\nu}_1$.

The small masses and weak interactions of neutrinos imply that the wave packets corresponding to different neutrino mass eigenstates remain superposed over long distances. The effects of flavour mixing can therefore be seen via oscillations.

For simplicity we will consider the case of two generations [9] which in the charged leptons we will take to be the electron and muon. We label the neutrino mass eigenstates as ν_1 and ν_2.

[9] Of course, in the Standard Model we have three families, but the important concepts can all be understood in the simpler case.

They are related by an equation very similar, to eq. (6.4)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{7.1}$$

Now we would like to compute the amplitude for an oscillation process. We suppose that we have an initial beam of muons which decays to relativisitic neutrinos of energy E and momentum k. The neutrinos travel a distance $L = \tau$ to a detector where they produce an e or a μ by CC scattering. The amplitude will be

$$\mathcal{A}_{\mu\alpha} \sim \sum_j U_{\mu j} \times e^{-i(E_j\tau - k_j L)} \times U^*_{\alpha j} \tag{7.2}$$

where the three pieces arise from production, propagation and detection. (From your field theory notes, you can check that the Feynman propagator in position space, $G(0, (\tau, L))$, is the exponential. Except its integrated over momentum; we suppose that the momentum integral was taken care of in the production process, when neutrinos of 4-momentum (E, k) were produced.)

First, suppose that we neglect the neutrino masses, so $(E_j, k_j) = (E_n, k_n)$ for any j, n. The propagation exponential can then be factored out, and (7.2) is the unitarity condition for U:

$$U_{\mu j} U^*_{\alpha j} = \delta_{\mu\alpha} \tag{7.3}$$

Recall that for quarks, in three generations, this relation gave the unitarity triangle.

Now allow the neutrinos to have small masses, $m \ll E, k$, so that $L \simeq \tau$ remains. Then the exponent can be written

$$-i(E_j\tau - k_j L) \simeq -i(E_j - k_j)L = -i\frac{E_j^2 - k_j^2}{E + k} L \simeq -i\frac{m_j^2}{2E} L \tag{7.4}$$

so

$$\mathcal{P}_{\mu\alpha} = |\mathcal{A}_{\mu\alpha}|^2 = |\sum_j U_{\mu j} e^{-i\Delta m_j^2 L/(2E)} U^*_{\alpha j}|^2 \tag{7.5}$$

Using the explicit form of U given in eqn (7.1), one obtains for the muon survival probability

$$\mathcal{P}_{\mu\mu} = 1 - \sin^2 2\theta \sin^2 \frac{(m_2^2 - m_1^2)L}{4E} \tag{7.6}$$

In reality, there are three generations of leptons in the SM, so the MNS matrix U is 3×3, and there are 3 mass eigenstates in the sum of eqn (7.5). As in the case of CKM, MNS can be written in terms of three angles and one phase:

$$\hat{U} = \begin{bmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{7.7}$$

$$\simeq \begin{bmatrix} c_{12} & s_{12} & s_{13}e^{-i\delta} \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} . \tag{7.8}$$

where the "solar" angle $\theta_{12} \simeq \pi/6$, and we have used the approximate measured value of the atmospheric angle $\theta_{23} \simeq \pi/4$. $s_{13} = \sin\theta_{13} \leq .2$ is experimentally bounded to be significantly smaller than its fellows. Note that, unlike the quark sector, some mixing angles are large. Combined with the small neutrino masses, this is puzzling and provoking to theorists, who expend much effort building models of this.

It is often said that MNS has three phases, so let's recall the phase choices that allow to write eqn (7.7), so as to understand where the other two phases could be.

- A 3×3 complex matrix has 18 real parameters
- the unitarity condition $UU^\dagger = 1$ reduces this to 9, which can be taken as 3 angles and 6 phases.
- five of those phases are relative phases between the fields $e, \mu, \tau, \nu_1, \nu_2$ and ν_3
- ...so if we are free to choose the phases of all the LH fermions, we are left with one phase in the mixing matrix. This was the case with the quarks, where any potential phase in the quark masses could be absorbed by the RH fermion fields.
- if the RH fields do not appear in our physical process (which means the masses appear as mm^*), then we are free to make the above phase choice, and our process is independant of any possible phase on the masses. This is the case for neutrino oscillations.
- we will see in a later section that the ν_L can have so-called "Majorana" masses, between themselves and their antiparticle. This means that it is the LH neutrino field which must absorb the phase off the Majorana mass. So in physical processes where the Majorana mass appears linearly (not as mm^*; eg $0\nu2\beta$), one can choose the phase such that the mass is real—in which case one can remove one less phase from MNS, or one can keep MNS with one phase, and allow complex masses.
- it is always possible to remove the phase from one majorana mass, by using the global overall phase of all the leptons (which corresponds to the global symmetry of lepton

number conservation in a theory without majorana masses. This is the sixth phase of $e, \mu, \tau, \nu_1, \nu_2$ and ν_3, which we could not use to remove phases from the lepton number conserving MNS matrix). So in three generations, there are possibly two complex majorana neutrino masses, so two "Majorana" phases in addition to the "Dirac" phase δ of MNS.

Although there are 3 generations, it is well known that for the oscillation probabilities we observe, with the mixing angles that are measured, two neutrino probabilities are a very good approximation. Why is this?

Lets return to the oscillation amplitude $\mathcal{A}_{\alpha\beta}(L)$, and imagine it as the sum of three vectors in the complex plane. If $\alpha = \beta$, the unitarity condition at $L = 0$ says they should sum to a vector of length one. If $\alpha \neq \beta$, then they should sum to zero and this is the unitarity triangle. At non-zero L, two of the vectors rotate in the complex plane, with frequencies $(m_j^2 - m_1^2)/2E$ — so neutrino oscillations correspond, in some sense, to time-dependent non-unitarity.

Consider the oscillation probabilities $\mathcal{P}_{\mu\alpha}$, measured for atmospheric neutrinos, on length scales corresponding to $m_3^2 - m_1^2$. The solar mass difference can be neglected, because $m_2^2 - m_1^2 \ll m_3^2 - m_1^2$, so there is only one relevant mass difference, and the survival probability behaves as for 2 generations. This is easy to visualise in the complex plane, where only the vector $U_{\mu 3} U_{\alpha 3}^*$ rotates with L. The stationary sum $U_{\mu 2} U_{\alpha 2}^* + U_{\mu 1} U_{\alpha 1}^*$ can be treated as a single vector, so this looks like a 2 generation system. So "atmospheric" oscillations can be approximated as two-neutrino oscillations because the atmospheric mass difference is that largest.

In the case of the solar mass difference, measured for instance at KamLAND, two neutrino approximations are good because θ_{13} is small. The observed survival probability is $\mathcal{P}_{ee}$, and since $U_{e3} \ll U_{ej}$, for $j = 1, 2$ the last term can be dropped in

$$\mathcal{A}_{ee} = \sum_j U_{ej} e^{-i\Delta m_j^2 L/(2E)} U_{ej}^* \tag{7.9}$$

7.2 Oscillations as quantum mechanics, in vaccum and matter

This subsection reviews a more conventional derivation of neutrino oscillations in 2 generations, and includes neutrino oscillations in matter. Electron neutrinos acquire an effective mass term from their interactions with dense matter— this is the MSW effect—which can have significant effects in the sun and supernovae, and over long baselines in the earth.

In the mass eigenbasis we have the Schrödinger equation

$$i \frac{d}{dt} \Psi = H \cdot \Psi \tag{7.10}$$

with a diagonal Hamiltonian

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \tag{7.11}$$

This Schrödinger equation can easily be solved. Defining our initial states at $t = 0$ as $|1\rangle \equiv |1(t=0)\rangle, |2\rangle \equiv |2(t=0)\rangle$ we get the time dependent states

$$\begin{aligned} |1(t)\rangle &= e^{-iE_1t}|1\rangle \\ |2(t)\rangle &= e^{-iE_1t}|2\rangle \end{aligned} \tag{7.12}$$

Let us repeat the last few steps in the interaction eigenbasis. Multiplying eq. (7.10) by V from the left we get the corresponding Schrödinger equation as

$$i\frac{d}{dt}\tilde{\Psi} = \tilde{H} \cdot \tilde{\Psi} \tag{7.13}$$

with

$$\tilde{H} \equiv V \cdot H \cdot V^{-1} = \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \tag{7.14}$$

where

$$a = \frac{1}{2}(E_1 + E_2) \tag{7.15}$$

$$b = \frac{1}{2}(E_1 - E_2)\cos(2\theta) \tag{7.16}$$

$$c = -\frac{1}{2}(E_1 - E_2)\sin(2\theta) \tag{7.17}$$

The crucial feature of the new Hamiltonian is that it is no longer diagonal. As a result, if we start at time $t = 0$ with an interaction eigenstate $|\alpha\rangle$, at a later time we get a superposition of $|\alpha\rangle$ and $|\beta\rangle$ interaction eigenstates. Indeed, using eq. (7.1) for the time dependent states we get

$$|\alpha(t)\rangle = e^{-iE_1t}\cos\theta\,|1\rangle + e^{-iE_2t}\sin\theta\,|2\rangle \tag{7.18}$$

$$|\beta(t)\rangle = -e^{-iE_1t}\sin\theta\,|1\rangle + e^{-iE_2t}\cos\theta\,|2\rangle \tag{7.19}$$

Let us now use this relation to compute the oscillation probability $\mathcal{P}_{\alpha\to\beta}(t)$. What we mean by this is the following: assume that at $t = 0$ we know that our state is a pure interaction eigenstate $|\alpha\rangle$. To be concrete we can assume this is an electron neutrino ν_e created in the sun. $\mathcal{P}_{\alpha\to\beta}(t)$ then gives us the probability that at a later time t this sate has evolved into an interaction eigenstate $|\beta\rangle$. Of course, this probability is simply the absolute value squared of the amplitude

$$\mathcal{P}_{\alpha\to\beta}(t) = |\langle\beta|\alpha(t)\rangle|^2$$

$$
\begin{aligned}
&= \left|-\sin\theta\,\cos\theta\left(e^{-iE_1t}-e^{-iE_2t}\right)\right|^2 \\
&= \frac{1}{2}\sin^2(2\theta)\,(1-\cos(E_2-E_1)t) \\
&= \sin^2(2\theta)\sin^2\left(\frac{E_2-E_1}{2}\,t\right)
\end{aligned}
\tag{7.20}
$$

In the first step we used eq. (7.18) and the orthogonality of the mass eigenstates $\langle i|j\rangle = \delta_{ij}$. The expression for $\mathcal{P}_{\alpha\to\beta}(t)$ can be brought into a more useful form by noting that

$$
E_i = \sqrt{p^2+m_i^2} = p + \frac{m_i^2}{2p} + \ldots \tag{7.21}
$$

and, therefore,

$$
\frac{1}{2}(E_2-E_1) = \frac{m_2^2-m_1^2}{4E} \equiv \frac{\Delta m^2}{4E} \tag{7.22}
$$

where E is the energy of the beam[10]. Furthermore, since the neutrinos travel at the speed of light, we have $L = vt = ct = t$, where L is the distance travelled by the neutrino. Thus, we arrive at the final expression for the oscillation probability

$$
\mathcal{P}_{\alpha\to\beta}(t) = \sin^2(2\theta)\sin^2\left(L\,\frac{\Delta m^2}{4E}\right) \tag{7.23}
$$

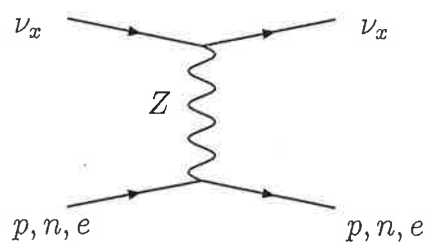
Eq. (7.23) has the expected properties in that the probability vanishes for $L \to 0$, $\theta \to 0$ and most notably for $\Delta m^2 \to 0$. This last limit tells us that there is no mixing if the two neutrino species have the same mass and, in particular, if they are massless.

So far we have considered oscillations in vacuum, i.e. we assumed that the neutrinos were travelling through the vacuum. While this is true for most of the time, the neutrinos produced in the sun first have to travel through the sun before they can reach us. The matter surrounding the neutrinos can have a crucial effect on the oscillation probability for the neutrinos. This effect is called the matter effect or the Mikheyev-Smirnov-Wolfenstein effect.

The question at the heart of the problem is: how does the Hamiltonian $\tilde{H}$, eq. (7.14), change through interactions of the neutrinos with surrounding matter? There are basically neutral and charged current interactions.

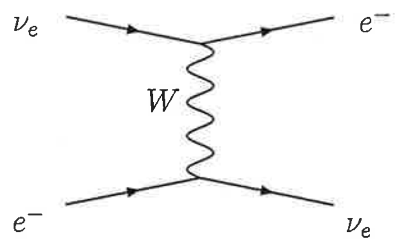
As we have learnt, neutral current interactions are mediated by the exchange of a Z boson. Taking into account that the surrounding matter is basically made of protons, neutrons and electrons a typical Feynman diagram is

[10]This argument can be made more rigorously using wave packets



The important point is that these interactions are independent on the flavor x of the neutrino. Thus they affect the two diagonal entries of the Hamiltonian in the same way. This means they change a, eq. (7.15), i.e. the Hamiltonian is modified by $a \longrightarrow \bar{a}$. As we will see later, this change is irrelevant.

The charged current interactions are mediated by a $W^{\pm}$. A typical Feynman diagram is



These interactions take place only for electron neutrinos since there are no μ (or τ) in the surrounding matter. In our convention where we identify the $|\alpha\rangle$ state with an electron neutrino, this means that only the top-left entry of the Hamiltonian, eq. (7.14) is modified. Thus, including the matter effects we arrive at the following Hamiltonian

$$\tilde{H}_{\mathrm{MSW}} = \begin{pmatrix} \bar{a} + b + w & c \\ c & \bar{a} - b \end{pmatrix} \tag{7.24}$$

where w comes from the charged current interactions. The explicit form of w is not important for us. What we want to know is how the w-term modifies the mixing angle. To find the modified mixing angle θ_{MSW} we have to diagonalize $\tilde{H}_{\mathrm{MSW}}$, i.e. we have to find

$$V_{\mathrm{MSW}} = \begin{pmatrix} \cos\theta_{\mathrm{MSW}} & \sin\theta_{\mathrm{MSW}} \\ -\sin\theta_{\mathrm{MSW}} & \cos\theta_{\mathrm{MSW}} \end{pmatrix} \tag{7.25}$$

such that

$$H_{\mathrm{MSW}} \equiv V_{\mathrm{MSW}}^{-1} \cdot \tilde{H}_{\mathrm{MSW}} \cdot V_{\mathrm{MSW}} \tag{7.26}$$

is diagonal. If we plug the explicit forms for V_{MSW}, eq. (7.25), and $\tilde{H}_{\mathrm{MSW}}$, eq. (7.24), into eq. (7.26) we find for the off-diagonal terms of H_{MSW}

$$c\cos(2\theta_{\mathrm{MSW}}) + \frac{2b+w}{2}\sin(2\theta_{\mathrm{MSW}}) \tag{7.27}$$

This vanishes for

$$\tan(2\theta_{\mathrm{MSW}}) = -\frac{2c}{2b+w} = \frac{-\Delta m^2 \sin(2\theta)}{4Ew - \Delta m^2\cos(2\theta)} \tag{7.28}$$

where we have used eqs. (7.16) and (7.17).

We note that θ_{MSW} does not depend on a, thus as mentioned above, the change $a \longrightarrow \bar{a}$ induced by the neutral current interactions does not matter at all. The important point is that for $4Ew \sim \Delta m^2\cos(2\theta)$ there can be a dramatic effect and the oscillation probability can increase substantially. In fact, this effect is very important in the explanation of experimental results.

7.3 The See-Saw Mechanism

In this section we are concerned with neutrino masses and offer a possible explanation why they might be so small compared to other fermion masses. We will restrict ourselves to the case of one family.

As mentioned previously, introducing a right-handed neutrino allows us to write down the same kind of Yukawa coupling as for the u-type quarks, eq. (5.25). This will result in a 'usual' Dirac mass term for the neutrinos of the form

$$m_D\,\overline{\nu}\nu = m_D(\overline{\nu}_L\nu_R + \overline{\nu}_R\nu_L) \tag{7.29}$$

(compare to eq. (5.27)). There is no doubt that such a term can be introduced in the Lagrangian, but it leads immediately to the question why the ν mass is so much smaller than the other fermion masses. In fact, we would expect that the Yukawa couplings of all fermions are roughly of the same order. This would lead to neutrino masses roughly of the same size as the masses of the other leptons, obviously in sharp contrast to observations.

However, the very special properties of the right-handed neutrinos allow us to write down yet another term in the Lagrangian. Recall that we have to write down the most general gauge invariant Lagrangian, given the gauge group and the matter content. In fact, since the ν_R is a singlet under all gauge transformations, we can (have to!!) add a term something like

$$M\nu_R\nu_R + \text{h.c.} \tag{7.30}$$

Note that for this term to be gauge invariant it is mandatory that $Y(\nu_R) = 0$ and that ν_R neither couples to $SU(2)$ nor $SU(3)$ gauge bosons.

This is a Majorana mass term, but its fermion index contraction is perhaps unclear. Lets consider this with care:

- The Dirac mass for a 4-component spinor is of the form

$$m_D\bar{\psi}\psi = m_D\psi^\dagger\gamma_0(P_L^2 + P_R^2)\psi = m_D(\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R) \tag{7.31}$$

  So to get a Lorentz scalar, a LH 2-component fermion must be contracted with a RH 2-component fermion.

- Recall that the antiparticle of a chiral fermion has opposite chirality from the particle:
  1) the negative energy solutions of momentum $\vec{p}$ became the positive energy solutions of $-\vec{p}$.
  2) For a massless (= chiral) particle, helicity = chirality, and helicity is $\vec{s}\cdot\vec{p}$
  ...so the antiparticle has opposite chirality from the particle.

By analogy with the Dirac mass term, one could try to write a mass term between the chiral ψ_L and its antiparticle as:

$$m\overline{(\psi_L)^c}\psi_L + h.c. \tag{7.32}$$

One should take care in the literature with such expressions, because the operations $\bar{}$, C and P_L do not commute, and different authors perform them in different order. Eqn (7.32) is a Lorentz scalar, can also be expressed as $m\psi_L^T i\sigma_2\psi_L$ and is often written $m\psi_L\psi_L$, with the index contraction understood. This is the notation of eqn (7.30).

Whereas m_D is expected to be of the same size as charged-lepton masses the most natural value for M is much larger. Ultimately we expect that at a high energy scale (maybe the GUT scale $M \sim 10^{15}$ GeV) there is a theory that explains all the fermion masses. Then, the natural value for the fermion masses is of the order M. However, all fermion masses except for the ν_R are protected by the chiral symmetry. This explains why $m_D \ll M$. To understand the consequences of $M \gg m_D$ consider the neutrino mass matrix

$$\begin{pmatrix} \overline{(\nu_L)^c} & \overline{\nu_R} \end{pmatrix} \begin{bmatrix} 0 & m_D \\ m_D & M \end{bmatrix} \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} \tag{7.33}$$

In order to get the masses of the physical particles, i.e. the eigenstates of the mass matrix, we have to diagonalize this matrix. The eigenvalues are approximately given by

$$\frac{m_D^2}{M} \quad \text{and} \quad M \tag{7.34}$$

where we used $m_D \ll M$. Thus we can see that the physical neutrinos are a (nearly) left-handed neutrino with mass m_D^2/M and a (nearly) right-handed neutrino with mass M. Taking $m_D \sim m_{top}$ and $M \sim 10^{15}$ GeV, we get $m \sim .03$ eV, not to far form the measured atmospheric mass difference. So this provides us with an explanation why the mass of the left-handed neutrino is so much smaller than the mass of the other leptons.

If this explanation is correct, there should also be very heavy (nearly sterile) right-handed neutrinos. They may not experimentally interesting, if they have GUT-scale masses, but they can be relevant in cosmology. If the ν_R are produced in the Universe after inflation, they could produce a lepton asymmetry in their decay. The Standard Model has non-pertubative B+L violating interactions, which are rapid at temperatures $T > m_W$, which would partially transform this lepton asymmetry into a baryon asymmetry. This scenario, called leptogenesis, appears to work and adds to the attraction of the seesaw model.

7.4 Summary

- When neutrino mass are included in the Lagrangian, mixing angles appear at the CC vertex, as in the quark case.
- The experimental signature of (small) neutrino masses is oscillations: a neutrino produced from one flavour of charged lepton, can be detected by the appearance of a different charged lepton. Thus, an electron neutrino produced in the sun can arrive as a neutrino of a different flavor on earth.
- If the neutrinos travel through matter rather than the vacuum the oscillation pattern can change dramatically.
- The see-saw mechanism provides us with an explanation why the neutrino masses are so much smaller than the other lepton masses.

8 Supersymmetry

This is the only section truly beyond the Standard Model. However, supersymmetry (SUSY) plays an important role in particle physics phenomenology, so in this section we will outline the basic ideas of this new symmetry, why so many theorists like it and sketch how to supersymmetrise the Standard Model.

Supersymmetry is a big topic, and this is a short lecture. There are books and review articles for readers of all tastes. In preparing this lecture, I used (among others) a phenomenological introduction by S Martin, hep-ph/9709356 ($\sim$ 100 pages) — which uses the space-time metric (-, +,+,+), and also a review of BSM physics by M Peskin, hep-ph/9705479.

8.1 Why supersymmetry?

We learned from LEP that loop calculations work. This is a shining success for the Standard Model: we calculate, as a function of few inputs, the one-loop gauge corrections to many observables, and what we measure agrees with the calculations. However, the Standard Model requires a Higgs boson of mass $\sim$ 100 GeV, and if we calculate the loop corrections to the Higgs boson mass, they are "quadratically divergent", that is, proportional to Λ_{NP}^2, where Λ_{NP} is the scale of New (BSM) Physics. There are various conclusions that one can draw: there is new physics close to the electroweak scale that does not appear in the precision observables of LEP, or the loop contributions cancel against each other, or the Higgs mass in the Lagrangian has just the right value to cancel the quadratic divergences (called "fine tuning"—unpopular among theorists). We will see that supersymmetry is a combination of the first and second solutions.

Supersymmetry is a transformation which turns bosons into fermions, and fermions into bosons. If it is a symmetry of the Lagrangian, then every fermion must have a bosonic partner and vice versa, and the interactions are restricted by the symmetry. When we supersymmetrise (exactly) the Standard Model, we will therefore (more than) double the number of particles—but the number of coupling constants stays (almost) the same.

Exercise 8.1

Consider the interaction Lagrangian

$$\mathcal{L} = y_f H(\overline{t_L} t_R + \overline{t_R} t_L) + \frac{y_s^2}{2} H^2 (T_1 T_1^* + T_2 T_2^*)$$

where t_L, t_R are chiral fermions (the top?), H is a real scalar and T_1 and T_2 are complex scalars.

• Draw the Feynman diagrams for the one-loop contributions to the Higgs mass from t, T_1 and T_2.

• Using Feynam rules from the lectures, calculate the leading (= most divergent) part of the diagrams at zero external momentum.

• Find a desirable relation between y_f and y_S, such that the divergences cancel

• Include soft scalar masses

$$\delta\mathcal{L} = m_T^2 (T_1 T_1^* + T_2 T_2^*)$$

take the supersymmetric relation that you found between y_f and y_s, and estimate the same one-loop diagrams.

8.2 A symmetry: boson $\leftrightarrow$ fermion?

Recall that a symmetry, be it local gauge, or global like Poincaré, is defined by operators which generate the transformations under which the Lagrangian transforms to itself + a total divergence. These operators are called generators.

We are looking for an operator Q, acting on bosons $|b\rangle$ and fermions $|f\rangle$ such that

$$Q|b\rangle = |f\rangle, \qquad Q|f\rangle = |b\rangle \tag{8.1}$$

Bosons have even spin and mass dimension (where I am counting the mass dimension of a field in 4-d), fermions have odd spin and mass dimension, so we conclude that the operator Q should have spin 1/2 and mass dimension 1/2. And since it transforms bosons to fermions, and fermions to bosons, our supersymmetric Lagrangian should have exactly the same number of fermionic and bosonic degrees of freedom. So there is a complex scalar for every chiral fermion, a chiral fermion for each massless vector, and fundamental real scalars are not allowed.

Since Q is a fermion, it should have a spinor index. By statistics and dimensional analysis, we can imagine it acting on fields (operators) as

$$\begin{aligned} &[Q^\alpha, \phi] \sim \psi^\alpha \\ &\{Q^\alpha, \psi\} \sim \partial_\mu \phi + m\phi + g\phi^2, ...A_\mu \end{aligned} \tag{8.2}$$

It is clear that Q^α changes spin, so mixes into the Poincaré group of translations and rotations. It can be shown that there is one way, and only one way, of extending the commutation relations of the Poincaré group (Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula theorem). And this extension is supersymmetry, with the properties we were looking for above. More precisely, one may introduce fermionic generators $\mathbf{Q}_\alpha$ satisfying the following algebra

$$\{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_\beta\} = 2\sigma^\mu_{\alpha\beta}\mathbf{P}_\mu \tag{8.3}$$

$$[\mathbf{Q}_\alpha, \mathbf{P}_\mu] = 0 \tag{8.4}$$

$$[\mathbf{Q}_\alpha, \mathbf{M}_{\mu\nu}] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q_\beta \tag{8.5}$$

The labels α and β are spinor indices taking the values 1 and 2, the bar denotes conjugation and the algebra involves anticommutators and commutators. Another important point to note is that in eqs. (8.3), (8.4) and (8.5) the new generators mix with the other Poincaré generators.

A theory is supersymmetric if it is invariant under the group of transformations generated by $\mathbf{P}_\mu$, $\mathbf{M}_{\mu\nu}$ and $\mathbf{Q}_\alpha$.

In such a theory, for every bosonic state there is a fermionic state with the same energy, and vice-versa. This follows directly from the fact that the Hamiltonian ($\mathbf{P}_0$) commutes with $\mathbf{Q}$.

Another interesting feature is that the cosmological constant vanishes: the Hamiltonian is bounded from below and the ground state has zero energy (if susy is not spontaneously broken). To understand this we simply have to note that since σ^0 is equal to the unit matrix and $\mathbf{P}$ is the Hamiltonian, eq. (8.3) entails

$$\{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_\beta\} = 2\delta_{\alpha\beta}\mathbf{H} \tag{8.6}$$

From this we conclude for an arbitrary state $|\psi\rangle$

$$\langle\psi|\mathbf{H}|\psi\rangle = \langle\psi|\mathbf{Q}\overline{\mathbf{Q}}|\psi\rangle = ||\overline{\mathbf{Q}}|\psi\rangle|| \geq 0 \tag{8.7}$$

At the same time we see that

$$\langle\psi|\mathbf{H}|\psi\rangle = 0 \quad \Leftrightarrow \quad \overline{\mathbf{Q}}|\psi\rangle = 0 \tag{8.8}$$

which is precisely the condition for susy not to be spontaneously broken (see eq. (4.18)).

8.3 The Supersymmetric Harmonic Oscillator

In this subsection we will consider the simplest supersymmetric model and convince ourselves that this model has indeed all the nice properties we expect.

Let us start with the usual (bosonic) harmonic oscillator. The Hamiltonian is given by

$$H_B = \frac{1}{2}\left(p^2 + \omega_B^2\, x^2\right) \tag{8.9}$$

If we define creation and annihilation operators

$$a \equiv \frac{1}{\sqrt{2\omega_B}}\,(p - i\omega_B x)\,; \qquad a^+ \equiv \frac{1}{\sqrt{2\omega_B}}\,(p + i\omega_B x)\,; \tag{8.10}$$

then the canonical commutation relation $[p, x] = -i$ entails the usual commutation relations for the creation and annihilation operators

$$[a, a^+] = 1; \quad [a, a] = [a^+, a^+] = 0 \tag{8.11}$$

If we write the Hamiltonian eq. (8.9) in terms of the creation and annihilation operators we get

$$H_B = \frac{\omega_B}{2}\left(a^+\, a + a\, a^+\right) = \omega_B\left(N_B + \frac{1}{2}\right) \tag{8.12}$$

where we defined the counting operator $N_B \equiv a^+\, a$. The energy spectrum of this Hamiltonian (i.e. its eigenvalues) is given by

$$E_{n_B} = \omega_B\left(n_B + \frac{1}{2}\right) \quad \text{with} \quad n_B = 0, 1, 2, 3 \ldots \tag{8.13}$$

A point to note is that the ground-state energy E_0 is 1/2 and not 0. In a quantum field theory this leads to the problem with the infinite ground-sate energy. This problem is solved by normal ordering.

Let us now repeat these steps for a fermionic harmonic oscillator. We introduce fermionic creation and annihilation operators b and b^+. They satisfy

$$\{b, b^+\} = 1; \quad \{b, b\} = \{b^+, b^+\} = 0 \tag{8.14}$$

These relations correspond to eq. (8.11). However, since we are dealing with fermionic operators now, the commutators are replaced by anticommutators. In analogy to eq. (8.12) we write the Hamiltonian of the fermionic harmonic oscillator as

$$H_F = \frac{\omega_F}{2}\left(b^+\, b - b\, b^+\right) = \omega_F\left(N_F - \frac{1}{2}\right) \tag{8.15}$$

where we introduced another counting operator $N_F \equiv b^+\, b$. Note that there is a relative minus sign between the $b^+\, b$ and $b\, b^+$ term. This sign is due to the fermionic nature of the creation and annihilation operator.

The energy spectrum of this Hamiltonian is given by

$$E_{n_F} = \omega_F\left(n_F - \frac{1}{2}\right) \quad \text{with} \quad n_F = 0, 1 \tag{8.16}$$

Note that contrary to eq. (8.13), n_F can only take the values 0 or 1. This is a reflection of Pauli's exclusion principle in that there cannot be two fermions in the same state.

If we wish we can find an explicit representation of the creation and annihilation operators in terms of Pauli matrices

$$\begin{aligned} b = \sigma_1 - i\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ b^+ = \sigma_1 + i\sigma_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{8.17}$$

In this representation the Hamiltonian eq. (8.15) is given by

$$H_F = \frac{\omega_F}{2}\,\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.18}$$

and we see that the eigenvalues of H_F are indeed $\pm\omega_F/2$ as given in eq. (8.16).

Now we are ready to combine the fermionic and the bosonic harmonic oscillator. If we just add the two, we do not increase the symmetry of the theory. In order to do this we also have to require $\omega_B = \omega_F \equiv \omega$. Only in this case do we end up with a supersymmetric model. The Hamiltonian then is

$$H \equiv H_B + H_F\Big|_{\omega_B=\omega_F\equiv\omega} = \frac{\omega}{2}\left(a^+\,a + a\,a^+ + b^+\,b - b\,b^+\right) = \omega\left(a^+\,a + b^+\,b\right) \tag{8.19}$$

First of all we see that naively H has an additional symmetry $a \leftrightarrow b$. A state is now determined by two quantum numbers n_B and n_F and the energy spectrum is

$$E_{n_B,n_F} = \omega\,(n_B + n_F) \quad \text{with} \quad n_B = 0,1,2,3\ldots;\ n_F = 0,1 \tag{8.20}$$

Note that the ground-state energy is $E_{0,0} = 0$. Thus as advertised in a previous section, the ground state has zero energy. This is simply because the bosonic ground state energy $+1/2$ and the fermionic ground state energy $-1/2$ cancel.

The other feature mentioned before, namely that the states appear in pairs (a fermionic and a bosonic state) with the same energy can be seen from eq. (8.20). Indeed, the states $|n_B, n_F = 0\rangle$ and $|n_B - 1, n_F = 1\rangle$ have the same energy. Furthermore $|n_B, n_F = 0\rangle$ is a bosonic state (integer spin), whereas $|n_B - 1, n_F = 1\rangle$ is a fermionic state (half-integer spin).

8.4 Supercharges

In this subsection we want to look at the symmetry found in an earlier subsection in a somewhat more formal way.

Through the Noether theorem, a symmetry is always related to a conserved current and a conserved charge. Thus, in a supersymmetric theory there is a conserved supercurrent and a conserved supercharge. It is the latter that generates the transformations and we denote it by $\mathbf{Q}$. Since it is conserved it has to commute with the Hamiltonian.

For the supersymmetric harmonic oscillator the supercharge is given by

$$\mathbf{Q}_1 = \sqrt{\omega}\left(a^+ b + a\, b^+\right); \qquad \mathbf{Q}_2 = i\sqrt{\omega}\left(a^+ b - a\, b^+\right); \tag{8.21}$$

where, as mentioned after eq. (8.3), $\mathbf{Q}$ has a spinor index. We now show that the supercharges as defined in eq. (8.21) have the desired properties. Using the (anti)commutation rules for the creation and annihilation operators, eqs. (8.11) and (8.14), we can compute

$$\begin{aligned}
\{\mathbf{Q}_1, \mathbf{Q}_1\} &= \omega\{a^+ b + a\, b^+, a^+ b + a\, b^+\} \\
&= \omega\{a^+ b, a\, b^+\} + \omega\{a\, b^+, a^+ b\} \\
&= 2\omega\left(a^+ a\,(1 - b^+ b) + (1 + a^+ a)\, b^+ b\right) \\
&= 2\omega(a^+ a + b^+ b) = 2H
\end{aligned} \tag{8.22}$$

In a similar way we can compute the remaining anticommutators of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and we get

$$\{\mathbf{Q}_1, \mathbf{Q}_1\} = \{\mathbf{Q}_2, \mathbf{Q}_2\} = 2\mathbf{H}; \qquad \{\mathbf{Q}_1, \mathbf{Q}_2\} = 0; \tag{8.23}$$

Note that this is in agreement with eq. (8.6). Now we can see that, as promised, the supercharge is conserved.

$$[\mathbf{Q}_1, \mathbf{H}] = \left[\mathbf{Q}_1, (\mathbf{Q}_1)^2\right] = 0 \tag{8.24}$$

$$[\mathbf{Q}_2, \mathbf{H}] = \left[\mathbf{Q}_2, (\mathbf{Q}_2)^2\right] = 0 \tag{8.25}$$

Eqs. (8.24) and (8.25) allow us to see that the states in this theory come in pairs. In fact, let $|\Psi\rangle$ be an eigenstate of $\mathbf{H}$, i.e. $\mathbf{H}|\Psi\rangle = E_\Psi|\Psi\rangle$. Then $\mathbf{Q}_1|\Psi\rangle$ is an eigenstate of $\mathbf{H}$ with the same energy.

$$\mathbf{H}\mathbf{Q}_1|\Psi\rangle = \mathbf{Q}_1\mathbf{H}|\Psi\rangle = \mathbf{Q}_1 E_\Psi|\Psi\rangle = E_\Psi \mathbf{Q}_1|\Psi\rangle \tag{8.26}$$

If $|\Psi\rangle$ is a bosonic state containing n_B bosons and no fermions, then

$$\mathbf{Q}_1|\Psi\rangle = \mathbf{Q}_1|n_B, 0\rangle = \sqrt{\omega}\left(a^+ b + a\, b^+\right)|n_B, 0\rangle = |n_B - 1, 1\rangle \tag{8.27}$$

is a fermionic state with the same energy. Similarly, if $|\Psi\rangle = |n_B, 1\rangle$ is a fermionic state, $\mathbf{Q}_1|\Psi\rangle = |n_B + 1, 0\rangle$ is a bosonic state with the same energy. Thus, the states come indeed in pairs with the same energy, one fermionic and one bosonic.

Of course, the same argument could have been made with $\mathbf{Q}_2$ rather than with $\mathbf{Q}_1$. However, $\mathbf{Q}_2$ acting on a state $|\Psi\rangle$ produces the same state as $\mathbf{Q}_1$ acting on a state $|\Psi\rangle$. Thus, there are not four but only two states with the same energy.

What we have seen is that if we start with the usual bosonic harmonic oscillator and want to make this theory supersymmetric, then we are led to introduce for every bosonic (fermionic) state a fermionic (bosonic) state with the same energy. This is exactly what happens if we want to make the Standard Model supersymmetric. For each boson (fermion) we have to introduce a fermionic (bosonic) partner, thereby doubling the particle spectrum.

8.5 Superfields

The superfield is a very convenient piece of SUSY notation, which rests on the abstract idea of supersymmetrising space-time. Suppose that for the four (bosonic) dimensions we know, that is x, y, z and t, we add a pair of fermionic dimension η and $\bar{\eta}$. The SUSY transformations Q and $\bar{Q}$ are translations in the fermionic directions of this "superspace". Being η and $\bar{\eta}$ fermions, they anticommute with themselves, so the taylor expansion in these fermionic dimensions ends quickly!

The superfield associated with, say, the Higgs, is a function of superspace:

$$H(x^\mu, \eta) = H(x^\mu) + \eta h(x^\mu) + \eta\eta F(x^\mu) \tag{8.28}$$

H is an example of a (left-handed) "chiral superfield", a simple sort of superfield that is independent of $\bar{\eta}$, suitable for decribing a matter multiplet made of a LH fermion and complex scalar. By a standard abuse of notation, the superfield has the same symbol as its scalar component. So on the RHS of the equality, H is the scalar Higgs, h is the higgsino, and F is a bosonic field of mass dimension 2, who therefore cannot have kinetic terms and can be removed from the Lagrangian by using its equations of motion (something like a Lagrange multiplier). We make no more mention of F, other than to note that it is the origin of calling part of the SUSY Lagrangian "F-terms".

The reason that superfields are convenient, is that one can compactly write all the SM Yukawa interactions, and their supersymmetric relatives (of which there are very many), as the "superpotential":

$$W = Y_e H_d L E^c + Y_\nu H_u L N^c + Y_d H_d L D^c + Y_u H_u L U^c \tag{8.29}$$

For simplicity, one can consider only one generation. Y_f is the Yukawa coupling for fermion f, and the RH fermions (*e.g.* $\overline{e_R}$) have been written as LH anti-particles (e^c). Notice that there are two physically distinct Higgs doublets H_u and H_d, where in the SM we used one doublet and its charge conjugate. We will return later to the reason for this extra field.

To obtain supersymmetric interactions of component fields, in ordinary 4-dimensional space, one should extract the "F-term" of W. That is, expand each field as in eqn (8.28), and pick out all the terms $\propto \eta^2$. It is clear that this will include the SM Yukawas, because each fermion comes with an η. It also gives scalar four point interactions. The full expression is

$$\begin{aligned}
\mathcal{L}_{SSM} &= \text{kinetic terms} + \sum_{ij} \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j - \sum_k |\frac{\partial W}{\partial \Phi_i}|^2 \\
&= ... + Y_e H_d \ell e^c + ... - |Y_e|^2 (H_d L)(H_d L)^* - ...
\end{aligned} \tag{8.30}$$

where i, j, k run over all the superfields in W, and on the second line are the parts coming from derivatives with respect to E^c. The fermion index contraction is in the same shorthand as eqn (7.30). The kinetic terms and gauge interactions come from another function of the superfields.

It is possible to draw diagrams and do calculations in superspace; this can be useful for obtaining exactly supersymmetric cancellations.

8.6 The MSSM particle content (partially)

The Lagrangian for the Minimal Supersymmetric SM can be motivated as follows:

1. add a boson for all SM fermions, and a fermion for all SM bosons
2. "supersymmetrise" SM Feynman diagrams
3. observe that step 2 gave superpartners with the same mass as their SM relatives. As we have not observed any superpartners, add "SUSY breaking" mass terms to make them heavier than current experimental sensitivities. (These masses are called "soft" because the quadratic diveregences still cancel— as you discovered in the problem.)

This heuristic recipe will give a Lagrangian with ~ 125 free parameters—compared to 19 in the SM. The vast majority of the additional parameters come in the SUSY breaking sector, and make the theory unwieldy to study. It is therefore common to work with simplified SUSY-breaking sectors (eg MSUGRA...) with fewer parameters.

In this subsection, we restrict outselves to the first step, of outlining the particle content of the MSSM. Feynman rules can be found elsewhere.

Superpartners are often written as capitalised, or "tilded" SM particles. The partners of one generation of SM leptons are a slepton doublet, a singlet selectron and a "right-handed" sneutrino:

$$\begin{array}{llll} \ell = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} & \rightarrow \tilde{\ell} = \begin{pmatrix} \tilde{e}_L \\ \tilde{\nu}_L \end{pmatrix} & \text{or } L = \begin{pmatrix} E_L \\ N_L \end{pmatrix} \\ e^c & \rightarrow \tilde{e^c} & \text{or } E^c \\ (\nu_R)^c & \rightarrow \widetilde{(\nu_R^c)} & \text{or } N^c \end{array} \tag{8.31}$$

and sometimes, abusively, the ^{c} is dropped from the singlets, although they remain "left-handed". Similarly, one introduces squark partners, of all colours and flavours, for the quarks.

The spartners of the SM bosons are the "-inos", who can be names according to whether they are added before (Bino and 3 Winos) or after (Photino, Zino and 2 Winos) spontaneous symmetry breaking:

$$\begin{array}{ll} \gamma & \rightarrow \tilde{\gamma} \\ Z & \rightarrow \tilde{z} \text{ or } z \\ W^{\pm} & \rightarrow \tilde{w}^{\pm}, \text{ or } w^{\pm} \\ H = \begin{pmatrix} H^+ \\ H_0 \end{pmatrix} & \rightarrow \tilde{h}_u = \begin{pmatrix} \tilde{h}_u^+ \\ \tilde{h}_u^0 \end{pmatrix} \end{array} \tag{8.32}$$

In supersymmetry, we need a second Higgs doublet. One can see this from the formal structure of the theory, or from considerations of anomaly cancellation, or by counting fermionic degrees of freedom. Lets do the last: suppose we break the electroweak gauge symmetry in an exactly supersymmetric SM. The spartners must therefore have the same masses as the SM particles, and notice in the SM after SSB, there are no massless charged bosons. However, among the inos in eqn (8.32), there are 3 *chiral* charged fermions, and it takes 2 chiral fermions to make a massive charged "Dirac" fermion (a Majorana mass would break charge conservation). The solution to this problem is to add a second Higgs:

$$H_d = \begin{pmatrix} H^0 \\ H^- \end{pmatrix} \rightarrow \tilde{h}_d = \begin{pmatrix} \tilde{h}_d^0 \\ \tilde{h}_d^- \end{pmatrix} \tag{8.33}$$

who gives mass to the d quarks and charged leptons.

Recall that we must add soft masses for all these new fermions, to ensure that we do not see them, so the physical mass eigenstates will be 4 neutralinos and 2 (4 component fermion) charginos, respectively linear combinations of $\tilde{\gamma}, \tilde{z}, h_u$ and h_d and $\tilde{w}^{\pm}, h_u^+, h_d^-$.

8.7 Summary

- Supersymmetry transforms bosons $\leftrightarrow$ fermions. It is an (the only possible) extension of the Poincaré algebra.

- since fermion loops come with a relative − sign, the Higgs mass would have no quadratic divergence in an exactly supersymmetric theory.
- to supersymmetrise the SM, one adds a boson (sfermion) for every fermion, and a fermion (-ino) for every boson. Then one adds a second Higgs doublet and its SUSY partners.
- No spartners have been observed, so one gives them masses in excess of current experimental bounds. This breaks the supersymmetry, and allows finite corrections to the Higgs mass of order m^2_{SUSY}.
- At the time of writing this sentence, supersymmetry has not been found.

Acknowledgements

Large parts of these notes (in particular, chapters 3 and 6), come directly from the original version by Douglas Ross, modified afterwards by Adrian Signer. I thank them for the invaluable LaTeX-files, and AS also for his lecture notes.

Chapters 1,2 4 and 5 aim to follow G Altarelli's Standard Model course from Les Houches, 1990. I fear my version does little justice to the clarity of the original, which I am grateful to have been able to attend.

I thank the lecturers, tutors and students for their questions and comments, and for the especially friendly atmosphere at this school. Finally , a special thanks to Tim and Margaret, for the seamless organisation that made it possible.

SELECTED TOPICS IN PHENOMENOLOGY

By Dr S Moretti
University of Southampton

Lectures presented at the School for Experimental High Energy Physics Students

Rutherford Appleton Laboratory, September 2006

Selected topics in

Phenomenology

Stefano Moretti

School of Physics and Astronomy
University of Southampton
Southampton SO17 1BJ, UK

and

Particle Physics Department
Rutherford Appleton Laboratory
Chilton, Didcot, Oxon OX11 0QX

Foreword

The tradition is that this course is given using transparencies, unlike the other courses in the school and that the transparencies are simply reproduced in the proceedings. This year I have used a mixture of slides and whiteboard. These notes attempt to combine all the material I used throughout the course and also contain some which I could not treat extensively in the lecture theater. In preparing my course I used material from Nigel Glover, Mike Seymour and Michael Krämer, who preceded me as lecturers of the Phenomenology course. I am greatly indebted to them for letting me using it. I have also ripped off some slides from Gavin Salam's talks and presentations for some of the QCD topics and from Laura Reina in the case of Higgs physics. A special thank goes to Dan Tovey for letting me use many slides from one of his talks for that very last lecture the day after the school dinner (and aftermath !). The references I have used to prepare the course are collected at the end and should ideally provide a good starting point for those who want to learn more about some of the topics. I would like to thank Tim Greenshaw for organising the school so well and for his support throughout. Lots of thanks also go to the other lecturers, to the tutors and primarily to the students. Finally, I am grateful to Margaret Evans for all the practical arrangements and for coping with my extreme lateness in preparing these notes: I was again the last one ...

Introduction

- Kant's Critique of Pure Reason (as interpreted by Wikipedia):

1. Phenomenon: Phenomena constitute the world as we experience it, as opposed to the world as it exists independently of our experiences (thing-in-themselves, 'das ding an sich'). Humans cannot, according to Kant, know things-in-themselves, only things as we experience them.

2. Noumenon: "Thing in itself (Ding an sich)" is an allegedly unknowable, undescribable reality that, in some way, lies "behind" observed phenomena. Noumena are sometimes spoken of, though the very notion of individuating items in "the noumenal world" is problematic, since the very notions of number and individuality are among the categories of the understanding, which are supposed to apply only to phenomena, not noumena.

- (The concept of 'Phenomena' led to a tradition of philosophy known as Phenomenology: Hegel, Heidegger, etc. – which we will ignore here !)
- Phenomenon in the general sense: stands for any observable event; phenomena make up the raw data of science.
- Famous quotes: "No phenomenon is a phenomenon until it is an observed phenomenon" (Niels Bohr).
- (I will nonetheless discuss Supersymmetry ...)

My definition of (high energy) phenomenology

- Branch of high-energy physics that seeks knowledge by:

1. Exploiting the hints and clues available in observable phenomena (aka experimental data), without any preconception on the theory governing the latter.

2. Parametrise theories into a set of observables (predictions) that can directly be tested by experiment, thus confirming or disproving the former.

- Phenomenology: bridge between theory and experiment !

Outline

- Introduction: The Standard Model & Beyond
- Tests of the Standard Model
  - QCD: running coupling; infrared safety; factorisation; parton distribution functions; jet production; searches for new physics
  - Electro-Weak (EW) Physics: weak interactions from unitarity; Z line-shape; precision tests; W boson production; indirect search for the Higgs boson
- Higgs Boson Hunting
  - The Higgs mechanism
  - The Higgs picture
  - The Higgs profile
  - Collider searches
- Supersymmetry (SUSY)
  - Why supersymmetry ?
  - The hierarchy problem and gauge coupling unification
  - The Minimal Supersymmetric Standard Model (MSSM)
  - Indirect searches: $g-2$
  - Collider searches
- Epilogue

Introduction

- Current theoretical framework of particle physics is

Standard Model (SM)

- SM is $SU(3) \times SU(2) \times U(1)$ gauge theory with

Matter fields:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} s \\ c \end{pmatrix}_L \begin{pmatrix} b \\ t \end{pmatrix}_L \quad d_R \; u_R \; s_R \; c_R \; b_R \; t_R \text{ (quarks)}$$

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}_L \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L \quad e_R \quad \mu_R \quad \tau_R \quad \text{(leptons)}$$

Force fields:

$$\gamma, W^{\pm}, Z, g \text{ (Vector bosons)}$$

and a

$$H \text{ (Higgs scalar)}$$

Q: Why do we believe in the Standard Model ?

A: Because confirmed by experiment !

Q: Why look Beyond the Standard Model (BSM) ?

A: Because SM lacks explanation of fundamental quantities !

SM Flaws

- SM does not explain quantum numbers:

→ EM charge, weak isospin, hypercharge and colour

- Contains (at least) 19 arbitrary parameters:

3 gauge couplings

1 CP-violating vacuum angle

6 quark masses

3 charged lepton masses

3 weak mixing angles

1 CP-violating CKM phase

1 W mass

1 Higgs mass

and (possibly) 9 more parameters in the neutrino sector:

3 neutrino masses

3 neutrino mixing angles

3 CP-violating phases

- More crucially: it does not incorporate gravity !

Beyond the Standard Model

- Three kind of problems:

1. Mass:
- What is the origin of particle masses ?
- Are the masses due to a Higgs boson ?
- What sets the scale of fermion masses ?

2. Unification:
- Is there a theory unifying all particle interactions ?

3. Flavour:
- Why are there so many types of quarks and leptons ?
- What is the origin of CP-violation ?

- Solutions should incorporate gravity (space-time origin/structure)
- String theory best (only ?) candidate, but not yet predictive !
- Supersymmetry (SUSY) to play a role in solving problems:

1. (Gauge) coupling unification best with light sparticles;
2. Mass hierarchy needs light sparticles for stabilisation;
3. SUSY seems essential for the consistency of string theory.

Jargon: a sparticle is a SUSY particle !

What New Physics (NP) ?

- Which way to go about ?

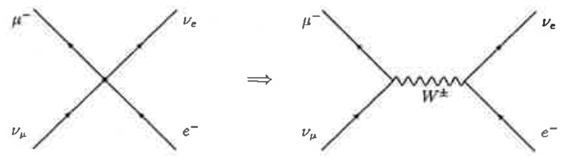
1. Come up with theory, devise model for it, get out predictions, compare with experiment !
2. Treat SM as effective theory below some high scale Λ: NP described by operators of dimension ≥ 6 suppressed by powers of E^2/Λ^2 ($E \to$ relevant energy).

Historic example: Fermi's theory of weak interactions,

- $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ decay described by effective Lagrangian:

$$\mathcal{L} = \frac{G_F}{\sqrt{2}} [\bar{\nu}_\mu \gamma_\lambda (1 - \gamma_5) \mu][\bar{e} \gamma^\lambda (1 - \gamma_5) \nu_e].$$

- From experiment $G_F \approx 1.17 \times 10^{-5}$ GeV^{-2} (Fermi coupling).
- As $\Lambda \approx M_W$, W appears as deviations from effective theory.



- Hence, precision tests of the SM can reveal NP !

- Crucial question for phenomenology is:

What is the scale of new physics ? $\Lambda \lesssim 1$ TeV ? Higher ?

- We do not know for sure, so we push up collider energies !

Test of the SM: QCD

Outline

- Importance of QCD
- The QCD coupling
- $e^+e^- \to$ hadrons
- Infrared safe quantities
- Jets
- Parton shower
- Hadronisation
- Deeply inelastic scattering
- Hadron-Hadron collisions
- New physics searches

Importance of QCD

QCD is the correct* theory of strong interactions

(*in the described sense of a low-energy effective theory)

→ Why QCD studies ?

1) A Quantum Field Theory (QFT) with unique features:

   – asymptotic freedom
   – infrared slavery (confinement)

2) We need to understand QCD also to search for NP:

   – for new particles hadro-production (Tevatron and LHC)
   – to predict the SM backgrounds to NP signals

- QCD degrees of freedom: quarks & gluons (aka partons).
- Will study their interactions in e^+e^-, $e^{\pm}p$, pp and $p\bar{p}$.
- (See Nick's course for Lagrangian & Feynman rules)

- The QCD Lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\,\mu\nu} + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{ghost}$$

where $F^A_{\alpha\beta}$ is the field strength tensor derived from the gluon field $\mathcal{A}^a_\alpha$,

$$F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu - g f^{ABC} \mathcal{A}^B_\mu \mathcal{A}^\mu_\nu$$

and the indices A, B, C run over the eight colour degrees of freedom of the gluon field. The quark fields q_a are in the triplet representation of the SU(3) colour group and D is the covariant derivative:

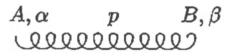
$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} + ig(t^c \mathcal{A}^c_\mu)_{ab}$$

The t are matrices in the fundamental representation of SU(3) and satisfy:

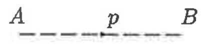
$$[t^A, t^B] = i f^{ABC} t^C$$

For a discussion of the gauge-fixing and ghost terms of the QCD Lagrangian see Nick's course.

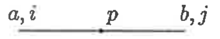
The **Feynman rules** can be derived from the QCD Lagrangian:



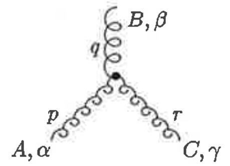
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\varepsilon}\right]\frac{i}{p^2+i\varepsilon}$$



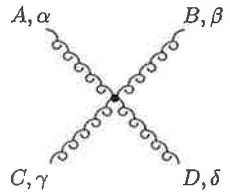
$$\delta^{AB}\frac{i}{p^2+i\varepsilon}$$



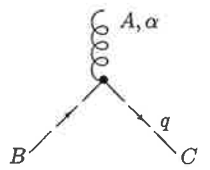
$$\delta^{ab}\frac{i}{(\not p-m+i\varepsilon)_{ji}}$$



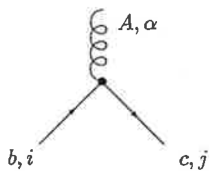
$$-gf^{ABC}\big[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\big]$$
(all momenta incoming)



$$\begin{aligned}&-ig^2 f^{XAC}f^{XBD}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\\ &-ig^2 f^{XAD}f^{XBC}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta}\right)\\ &-ig^2 f^{XAB}f^{XCD}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\end{aligned}$$

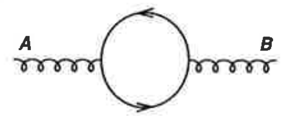


$$gf^{ABC}q^\alpha$$

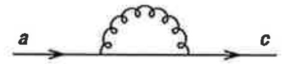


$$-ig\left(t^A\right)_{cb}(\gamma^\alpha)_{ji}$$

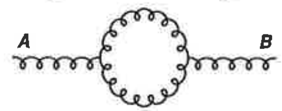
$$\mathrm{Tr}(t^At^B)=T_R\delta^{AB}\,,\quad T_R=\tfrac{1}{2}$$



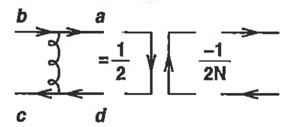
$$\textstyle\sum_A t^A_{ab}t^A_{bc}=C_F\delta_{ac}\,,\quad C_F=\frac{N_c^2-1}{2N_c}=\frac{4}{3}$$



$$\textstyle\sum_{C,D} f^{ACD}f^{BCD}=C_A\delta^{AB}\,,\quad C_A=N_c=3$$



$$t^A_{ab}t^A_{cd}=\frac{1}{2}\delta_{bc}\delta_{ad}-\frac{1}{2N_c}\delta_{ab}\delta_{cd}\ \text{(Fierz)}$$



The QCD coupling

... is running !

- Quantum corrections alter particle masses and couplings.
- Ultraviolet divergences removed by renormalisation.

Renormalisation introduces a mass scale μ – the subtraction point of UV divergences – and the <u>renormalised</u> coupling α_s depends on μ:

$$\alpha_s \to \alpha_s(\mu) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)},$$

$\beta_0 = (11N_C - 2n_f)/12\pi$, $N_C = 3$, $n_f = \#$ of active flavours.

- $\Lambda \equiv \Lambda_{QCD}$ (≈ 200 MeV) is an integration constant:

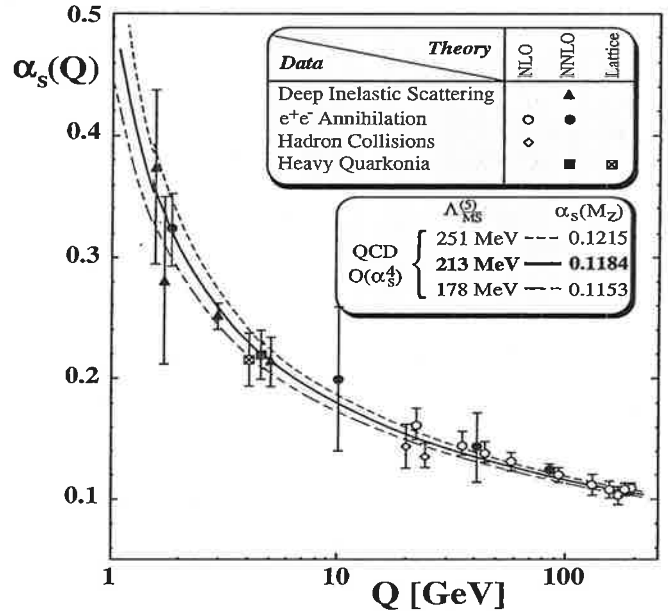
$$\mu^2 \frac{d\alpha_s}{d\mu^2} \equiv \beta(\alpha_s) = -\beta_0 \alpha_s^2 + \dots$$

- **<u>Asymptotic freedom</u>: $\alpha_s \to 0$ as $\mu \to \infty$**
  $\to$ we can use perturbation theory for processes involving large momentum scales (small distances).

[Sign of β is crucial: in QED, $\beta < 0$ and α increases as $\mu \to \infty$.]

- **<u>Infrared slavery</u>: $\alpha_s \to \infty$ as $\mu \to \Lambda$**
  $\to$ confinement: quarks & gluons are only found in colour-singlet bound states.
  $\to$ we have to use non-perturbative methods (e.g. lattice) at low momentum scales (large distances).

- Running of α_s has been established experimentally !



(Compilation of data by Siggi Bethke)

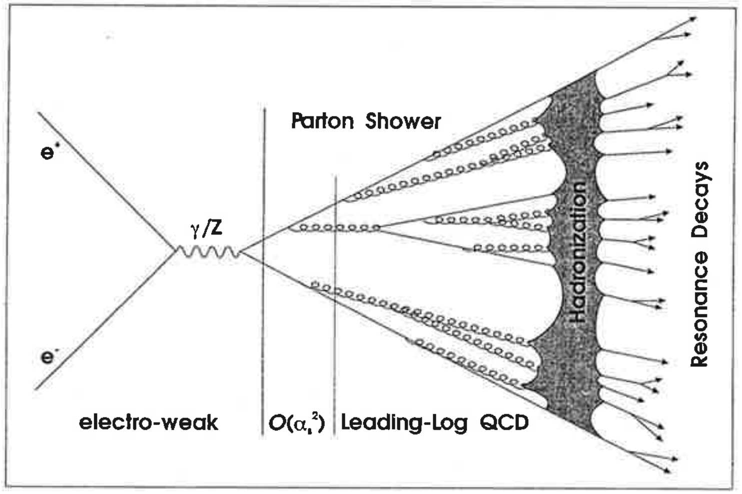
- But how do we <u>measure</u> α_s ?

$e^+e^- \to$ hadrons

- But QCD Feynman rules tell us only about partons !
- Hadron formation (long distance) is not perturbative !

  → how to calculate $e^+e^- \to$ hadrons ?

Plenty of physics between partons and hadrons !



- Each event has different hadronic final state !

  → how to sum over all of these ?

- Symmetries can help us !

→ Matrix Element (ME) to produce n hadrons $h_1...h_n$:

$$\mathcal{M} \sim \{\bar{v}(p_{e^+}) e \gamma_\mu u(p_{e^-})\} \; \frac{-g^{\mu\nu}}{q^2} \; T_\nu(n, q, \{p_{h_1} \dots p_{h_n}\}),$$

with T_ν parametrisation of the unknown part.

→ Gives total cross section:

$$\sigma = \frac{1}{2s}\,\frac{1}{4}\,\frac{e^2}{s^2}\mathrm{Tr}(\not{p}_{e^+}\gamma^\mu \; \not{p}_{e^-}\gamma^\nu)$$

$$\times \sum_n \int d\mathrm{PS}_n \; T_\mu(n, q, \{p_{h_1} \dots p_{h_n}\}) \; T^*_\nu(n, q, \{p_{h_1} \dots p_{h_n}\}).$$

→ Define: $H_{\mu\nu}(q) \equiv \sum_n \int dPS_n \; T_\mu \; T^*_\nu$.

→ Impose Lorentz covariance:

$H_{\mu\nu} = A g_{\mu\nu} + B q_\mu q_\nu$, $\quad$ (A, B functions only of q^2).

→ Impose gauge invariance:

$$q^\mu H_{\mu\nu} = q^\nu H_{\mu\nu} = 0 \;\Rightarrow\; A = -q^2 B.$$

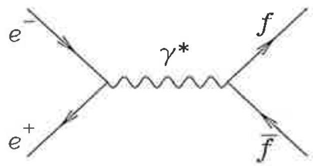
→ Hence, $\sigma = \frac{e^2}{2s} B(s)$ and $B(s)$ dimensionless.

→ Gives fundamental prediction:

$$R \equiv R(e^+e^-) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \text{constant},$$

without knowing anything about hadron interactions !

$e^+e^- \to$ hadrons at leading order

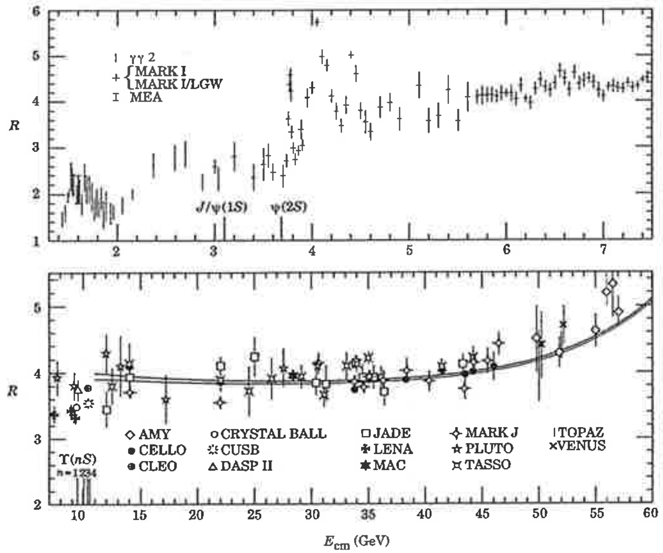


$$\sigma_0 = \frac{4\pi\alpha^2}{3s} N_C \sum_q e_q^2 \;(\text{if}\, f = q)$$

$$\Rightarrow R_0 \equiv \frac{\sigma_0(e^+e^- \to \text{hadrons})}{\sigma_0(e^+e^- \to \mu^+\mu^-)} = N_C \sum_q e_Q^2$$

$\to$ first evidence for colour ($N_C = 3$) !

- Kinematically allowed if $\sqrt{s} > 2m_q$, steps at $\sqrt{s} = 2m_q$.



(e.g. $\sqrt{s} = 34$ GeV, $R = \frac{11}{3}$, cf PETRA data: 3.88 ± 0.03)

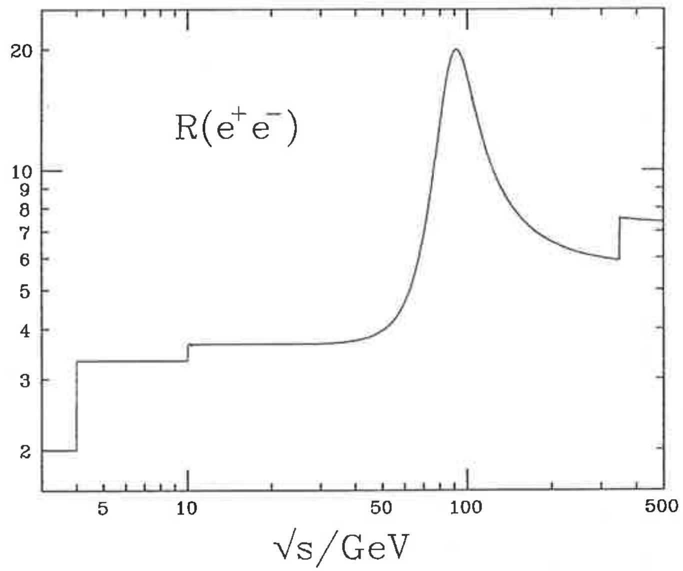
R at the Z peak

- Can also tell us about EW couplings:

$$R = N_C \frac{\sum_q \mathcal{A}_q}{\mathcal{A}_\mu} = 20.095, \qquad \mathcal{A}_f = v_f^2 + a_f^2.$$

(cf LEP average: 20.775 ± 0.027)

- In general, sensitive to $\gamma - Z$ interference:



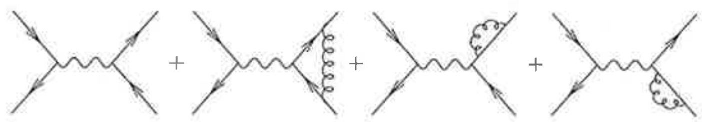
(in fact, $R = 19.984$ on Z peak)

$e^+e^- \to$ hadrons beyond leading order

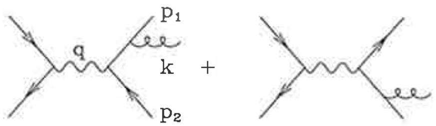
- α_s largest coupling: expect QCD corrections largest !

→ start with them !

- At $\mathcal{O}(\alpha_s)$:



(a)



(b)

- Virtual corrections: <u>interfere</u> tree-level diagrams with one-loop ones in (a)

→ can be negative

- Real corrections: <u>square</u> tree-level diagrams in (b)

→ positive definite

(b) real gluon emission

- 3-body phase space: $d\Phi_3 = [\ldots]\, d\alpha\, d\beta\, d\gamma\, dx_1\, dx_2$

where α, β, γ are Euler angles and $x_1 = 2E_q/\sqrt{s}$ and $x_2 = 2E_{\bar{q}}/\sqrt{s}$ are energy fractions of final-state quark and antiquark.

- Applying Feynman rules and integrating over Euler angles:

$$\sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

with integration region $0 \le x_1, x_2 \le 1$.

- Integral is divergent at $x_1, x_2 = 1$:

$$\begin{aligned} 1 - x_1 &= x_2 x_3 \,(1 - \cos\theta_{qg})/2, \\ 1 - x_2 &= x_1 x_3 \,(1 - \cos\theta_{\bar{q}g})/2, \end{aligned}$$

where $x_3 = 2E_g/\sqrt{s}$ (E_g gluon energy) and $\theta_{ig} (i = 1, 2)$ are angles between gluon and quarks.

→ collinear divergence: $\theta_{qg} \to 0$ or $\theta_{\bar{q}g} \to 0$

→ soft divergence: $E_g \to 0$

- Singularities indicate breakdown of perturbation theory when mass scales approach Λ.

- Fortunately, collinear/soft regions do not make important contributions to total cross section:

→ they cancel !

- Make integral finite using e.g. dimensional regularisation:

$D = 4 - 2\epsilon \Rightarrow$

$$\sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon) \int dx_1 dx_2 \frac{(1-\epsilon)(x_1^2 + x_2^2) + 2\epsilon(1-x_3)}{(1-x_3)^\epsilon[(1-x_1)(1-x_2)]^{1+\epsilon}}$$

where $H(\epsilon) = \frac{3(1-\epsilon)(4\pi)^{2\epsilon}}{(3-2\epsilon)\Gamma(2-2\epsilon)} = 1 + \mathcal{O}(\epsilon)$.

- Hence

$$\sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon)\right]$$

→ soft/collinear divergences are regulated, appearing as poles at $D = 4$ ($\epsilon = 0$).

(b) virtual gluon exchange

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{1 + C_F \frac{\alpha_s}{2\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\}$$

⇒ Adding real and virtual corrections, the infrared/collinear poles cancel and the result is finite as $\epsilon \to 0$:

$$R = R_0 \left\{1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right\}$$

- Other regularisation schemes available: e.g. finite g mass $m_g \equiv \epsilon s$ (non-gauge invariant !).

⇒ R is an infrared safe quantity !

⇒ R is finite and regularisation scheme-independent !

$e^+e^- \to$ hadrons cross section at NLO

1. First α_s measurement:

$$\begin{aligned} R(\text{LEP}) &= 20.775 \pm 0.027 \\ R_0(M_Z) &= 19.984 \\ \to \alpha_s(M_Z) &= 0.124 \pm 0.004 \end{aligned}$$

2. Second α_s measurement:

$$\begin{aligned} R(\text{PETRA}) &= 3.88 \pm 0.03 \\ R_0(34\ \text{GeV}) &= 3.69 \\ \to \alpha_s &= 0.162 \pm 0.026 \\ \to \alpha_s(M_Z) &= 0.134 \pm 0.018\ \text{(upon running)} \end{aligned}$$

- PETRA agrees with LEP:

⇒ test of QCD in intervening energy range !

Note: τ-decays

- Related measurement: $R \equiv R(\tau) = \frac{\text{BR}(\tau \to \text{hadrons})}{\text{BR}(\tau \to \text{electrons, muons})}$

→ one of best α_s measurements:

$$\begin{aligned} \alpha_s(m_\tau = 1.77\ \text{GeV}) &= 0.33 \pm 0.03 \\ \to \alpha_s(M_Z) &= 0.118 \pm 0.004\ \text{(upon running)} \end{aligned}$$

- Cross section now known through NNNLO → fig

- And theoretical errors ? Where are they ?

Dependence of total cross section on only *hard* gluons is reflected in 'good behaviour' of perturbation series:

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + 1.045 \frac{\alpha_s(Q)}{\pi} + 0.94 \left(\frac{\alpha_s(Q)}{\pi} \right)^2 - 15 \left(\frac{\alpha_s(Q)}{\pi} \right)^3 + \cdots \right)$$

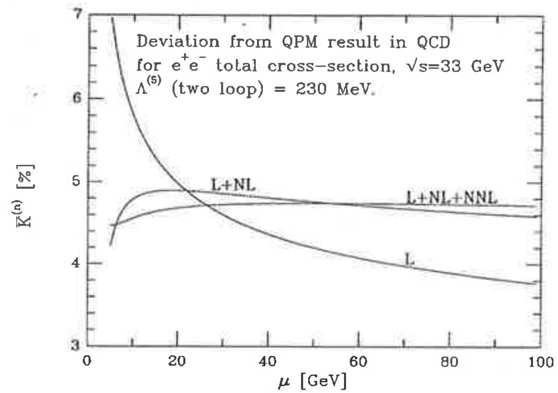
(Coefficients given for $Q = M_Z$)

Renormalisation scale dependence

- Recall $\alpha_s(\mu)$, μ arbitrary: would disappear to all orders ...

$\Rightarrow$ Use dependence as estimate of uncertainty due to truncating perturbative series $\rightarrow$ smaller at each order

- Vary μ (by some factor) to estimate theoretical uncertainty:



- What scale to use for <u>central</u> α_s value ?

1. Physical scale, $\mu = \sqrt{s}$
2. Principle of Minimal Sensitivity: where $d\sigma/d\mu = 0$
3. Fastest Apparent Convergence: where NLO=LO

$\rightarrow$ theoretical predictions rather subjective !

Infrared safe quantities

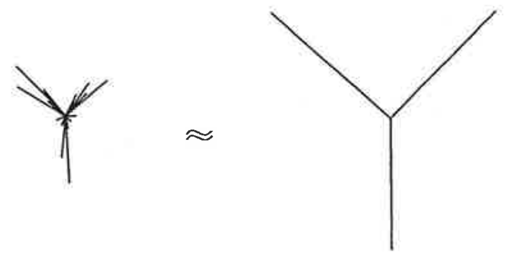
• $R(e^+e^-)$ and $R(\tau)$ are very <u>inclusive</u> quantities: total cross sections or decay rates !

• Infrared safety guaranteed by 'theorems', e.g. Bloch and Nordsieck (BN) plus Kinoshita, Lee and Nauenberg (KLN):

$\rightarrow$ <u>suitably defined quantities</u> are free of singularities.

• Physical meaning: events with hadrons give approximately the same measurement as parton ones.

• Computational meaning: infinities cancel when adding real gluon emission and virtual gluon exchange.

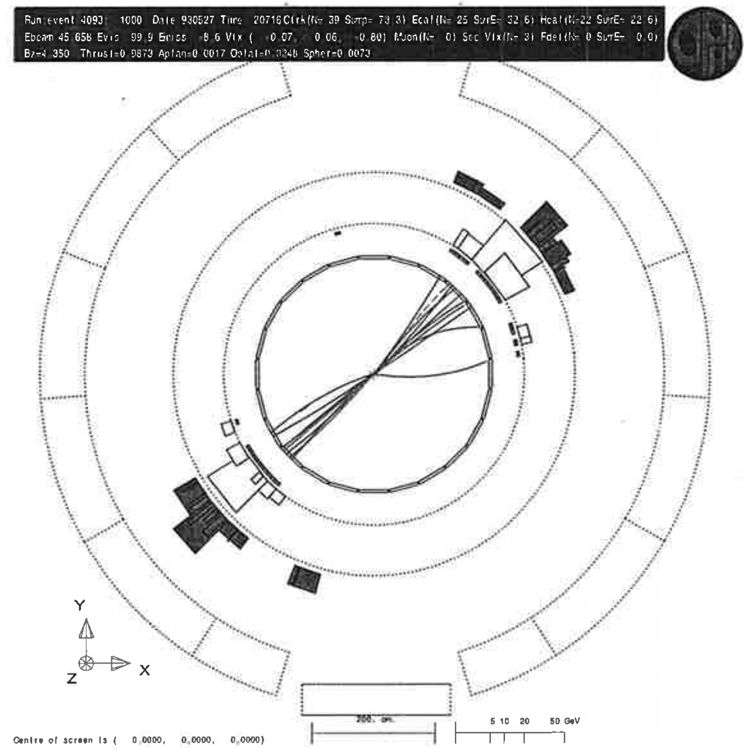


• BN & KLN apply also to more exclusive quantities: e.g.

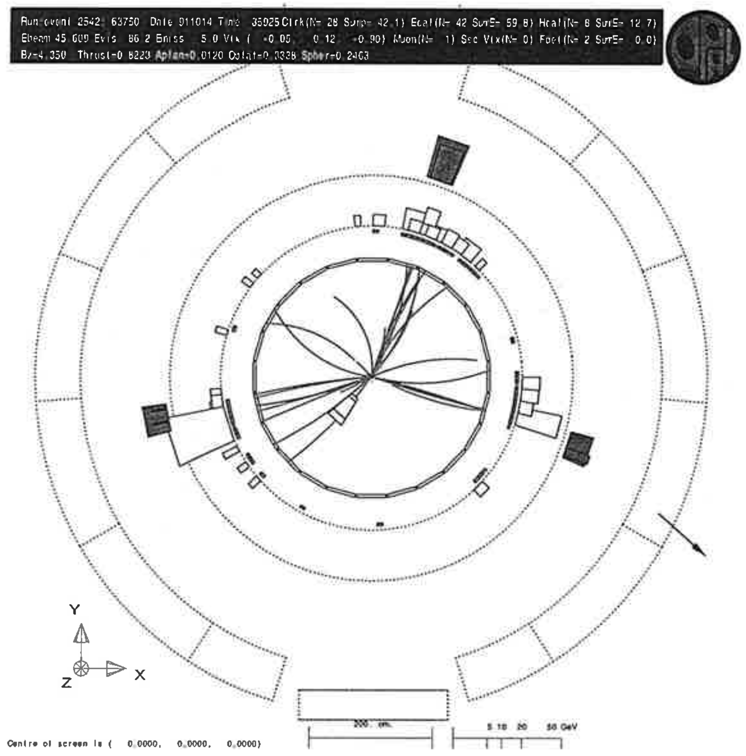
1. n-jets cross section, $n = 2, 3, ...$
2. Event-shape variables like <u>thrust</u>

Jets

Naively expect most events to look like:



with a fraction $\sim \alpha_s$ more like:



(and even a fraction $\sim \alpha_s^2$ four-jet like, etc)

Jet definition

- Intuitively, jet is a spray of collimated particles.
- Need a procedure: in e^+e^- use clustering algorithms.
- Start with a list of momenta $p_1^\mu, p_2^\mu, ..., p_n^\mu$.

(In perturbative calculations, they are parton momenta.)

- Three ingredients:

1. A measure of inter-jet distance: y_{ij}.

   $\rightarrow$ for each pair of final state momenta calculate, e.g.

$$y_{ij} = m_{ij}^2/s \text{ (Invariant Mass)}$$
$$y_{ij} = 2E_i\, E_j\, (1 - \cos\theta_{ij})/s \text{ (JADE)}$$
$$y_{ij} = 2\min\{E_i^2, E_j^2\}\, (1 - \cos\theta_{ij})/s \text{ (Durham)}$$

2. A resolution for the latter: y_{cut}.

   $\rightarrow$ $\min\{y_{ij}, ...\} < y_{\text{cut}}$ combine i and j into k

3. A recombination procedure: e.g.

$$p_k^\mu = p_i^\mu + p_j^\mu \text{ (E-scheme)}$$
$$p_k^\mu = (|\mathbf{p}_i + \mathbf{p}_j|, \mathbf{p}_i + \mathbf{p}_j) \text{ (p-scheme)}$$
$$p_k^\mu = (|E_i + E_j|, \tfrac{E_i+E_j}{|\mathbf{p}_i+\mathbf{p}_j|}\mathbf{p}_i + \mathbf{p}_j) \text{ (E0-scheme)}$$

- Repeat till $\min\{y_{kl}, ...\} > y_{\text{cut}}$: remaining objects are jets.
- A n-parton final state can give any number of jets between n (all partons well-separated) and 2 (e.g. two energetic quarks accompanied by soft and collinear gluons).

Jet rates

- Define n-jet fraction $f_n(y)$ by $(y \equiv y_{\rm cut})$.

$$f_n(y) = \frac{\sigma_n(y)}{\sum_m \sigma_m(y)} = \frac{\sigma_n(y)}{\sigma_{\rm tot}},$$

- If $\sigma_{\rm tot} = \sigma_0(1 + \alpha_s/\pi + \ldots)$, then

$$\sum_n f_n(y) = 1.$$

- For $\mu = \sqrt{s}$ and $n = 2, 3$ and 4:

$$f_2(y) = 1 - \left(\frac{\alpha_s}{2\pi}\right) A(y) + \left(\frac{\alpha_s}{2\pi}\right)^2 (2A(y) - B(y) - C(y)) + \ldots,$$

$$f_3(y) = \left(\frac{\alpha_s}{2\pi}\right) A(y) + \left(\frac{\alpha_s}{2\pi}\right)^2 (B(y) - 2A(y)) + \ldots,$$

$$f_4(y) = \left(\frac{\alpha_s}{2\pi}\right)^2 C(y) + \ldots,$$

- Coupling constant α_s and functions $A(y), B(y)$ and $C(y)$ defined in some renormalisation scheme (e.g. $\overline{\rm MS}$ scheme).

- Terms of order $\mathcal{O}(\alpha_s^2)$ involving $A(y)$ take account of the normalisation to $\sigma_{\rm tot}$ rather than to σ_0.

- Example 1 of n-jet event rates (α_s^2 vs. OPAL)

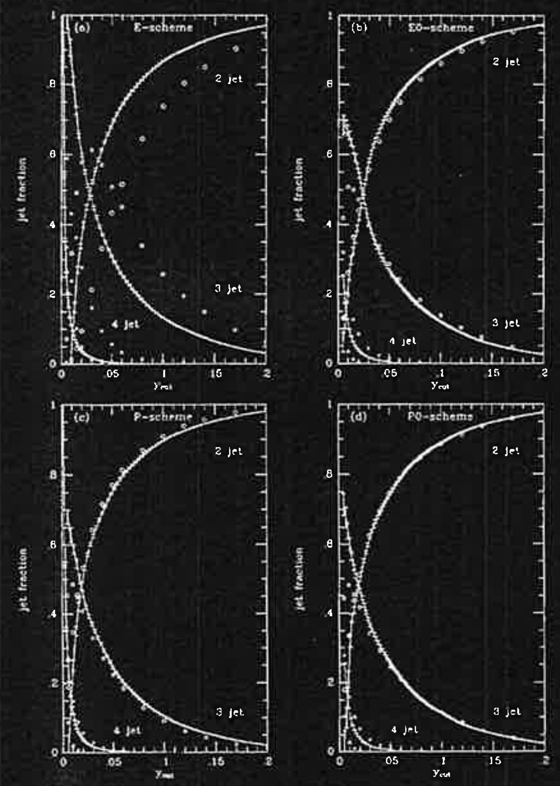
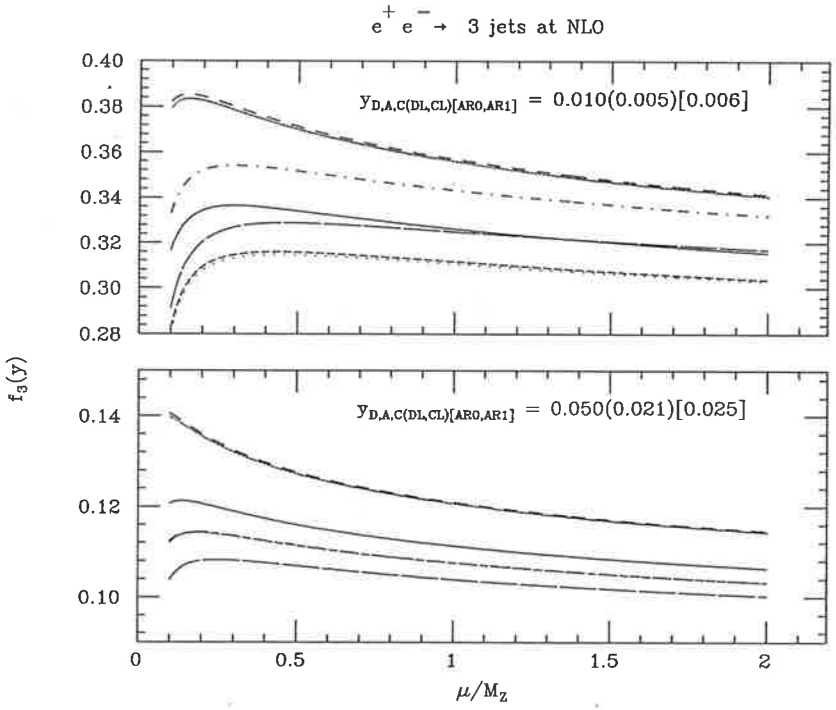


Fig. 5. Comparison of the $\mathcal{O}(\alpha_s^2)$ parton level two, three and four jet fractions (solid lines with statistical error bars) with the hadronic data (points) as given by the OPAL collaboration [3] in the (a) E-scheme, (b) E0-scheme, (c) P-scheme and the (d) P0-scheme.

- The μ-dependence of the three-jet rate is introduced by

$$\alpha_s \to \alpha_s(\mu), \qquad B(y) \to B(y) - A(y)\beta_0 \ln \frac{Q}{\mu}.$$



Event shape variables

- Attempt to find a more global measure of 2/3-jet separation
- E.g. Thrust (T):

$$T = \max_{\hat{\mathbf{n}}} \frac{\sum |\mathbf{p_i} \cdot \hat{\mathbf{n}}|}{\sum |\mathbf{p_i}|}$$

- Through order α_s^2:

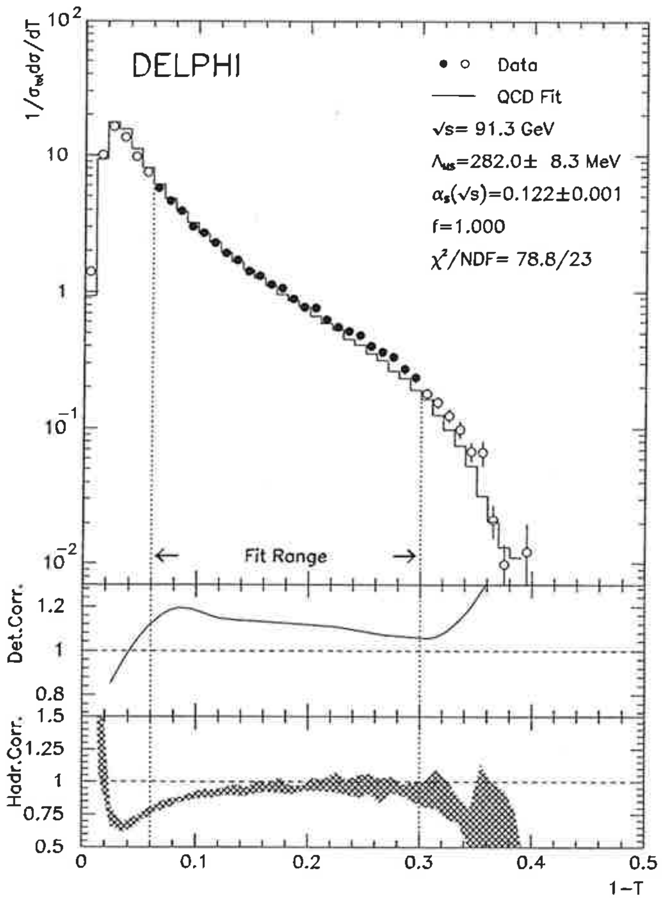
$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{\alpha_s(\mu)}{2\pi} A(T) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \Bigg[\underbrace{2\pi A(T)\beta_0 \log\frac{\mu^2}{s}}_{\text{renormalisation scale dependence}} + B(T) \Bigg]$$

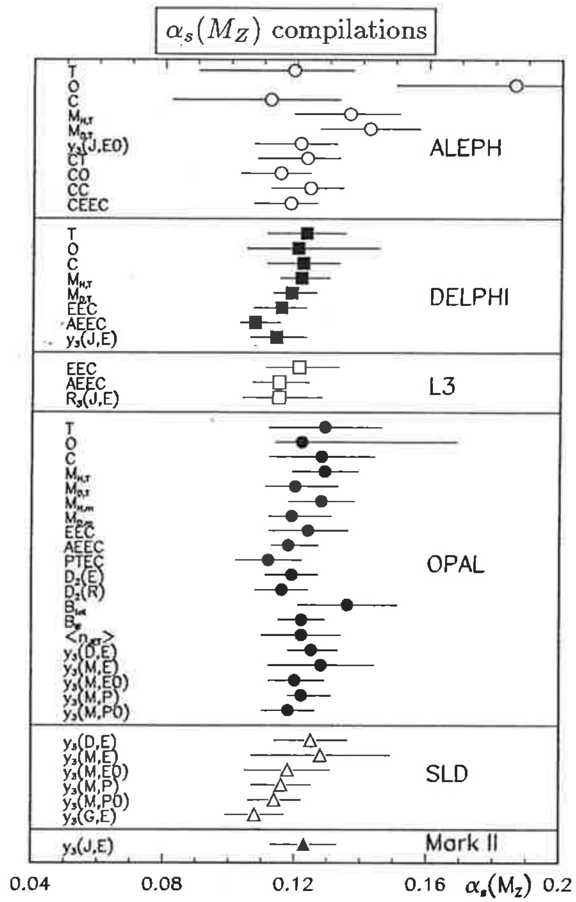
- LO term:

$$A(T) = C_F \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \log\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right]$$

$$\stackrel{T\to 1}{\sim} C_F \left[\frac{4}{1-T} \log\frac{1}{1-T} - \frac{3}{1-T} \right].$$

- NLO term $B(T)$ computed numerically !



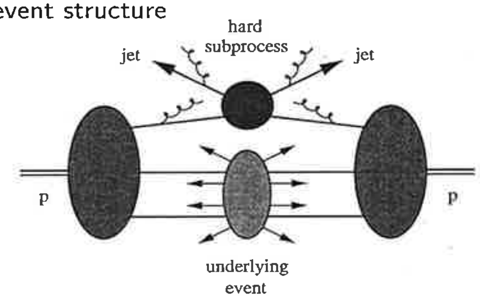


$\mathcal{O}(\alpha_s^2)$!

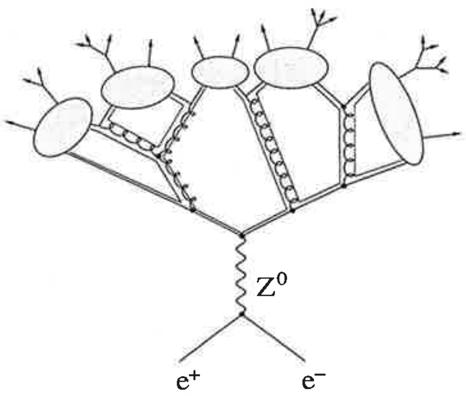
Parton shower

QCD Event Generators

- Basic event structure



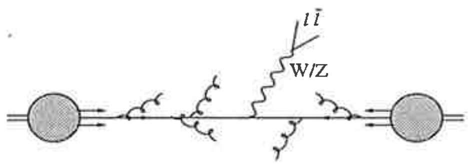
- Parton showers and hadronization



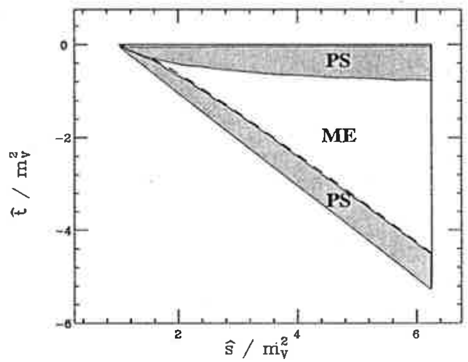
Matrix Element Corrections

- Parton showers inside cones do not populate whole phase space. We also have to include (less singular) matrix element corrections

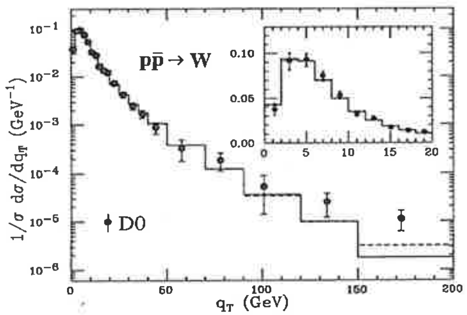
- For example, in W/Z hadroproduction

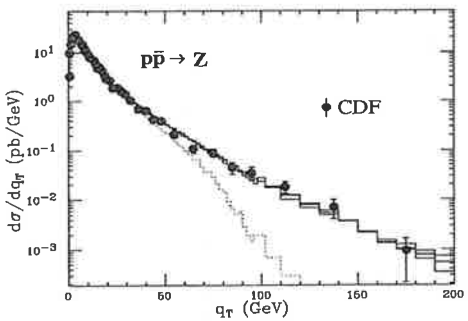


- Phase space for W + jet



- Comparisons with Tevatron data:





Hadronisation

- The formation of hadrons (long distance physics) is not described by perturbative QCD
- Space-time picture:
  - e^+ and e^- form γ (or Z) with virtual mass $Q = \sqrt{s}$, which fluctuates into q and $\bar{q}$.
    - By the uncertainty principle, fluctuation occurs at short distance/timescale $\sim 1/Q$.
    - At large Q, the rate $e^+e^- \to q\bar{q}(g)$ is given by perturbation theory.
  - At much later times $\sim 1/\Lambda$, quarks form hadrons.
    - Hadronisation modifies the outgoing state, but occurs too late to change the original probability for the event to happen.

$\Rightarrow \sigma(e^+e^- \to \text{hadrons}) =$

$\sigma(e^+e^- \to \text{partons}) \times (1 + \mathcal{O}(\Lambda/Q)^n)$ (power corrections)

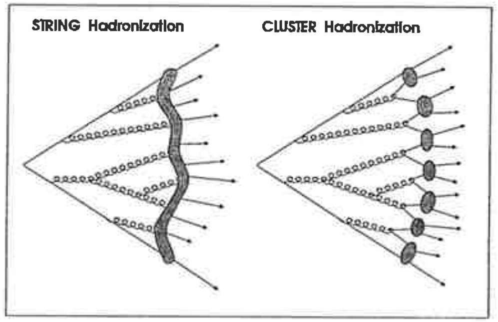
$\Rightarrow \sigma(e^+e^- \to \text{hadrons})$ can be calculated in perturbative QCD for $Q \gg \Lambda$.

$\Rightarrow$ Need Monte Carlo (MC) approach for $Q \leq \Lambda$.

• Quarks and gluons produced in a short-distance process form themselves into hadrons: hadronisation.

• Hadronisation modelled to data in MC programs like

1. HERWIG (cluster hadronisation)
2. PYTHIA, ARIADNE (string hadronisation)



• General approach to hadronisation based on "parton-hadron duality": the flow of momentum and quantum numbers at hadron level follows that established at the partonic stage.

• E.g. flavour of quark initiating a jet found in hadron near the jet axis.

• Approach works because hadronisation is long-distance process which only involves small momentum transfers.

Deeply Inelastic Scattering (DIS)

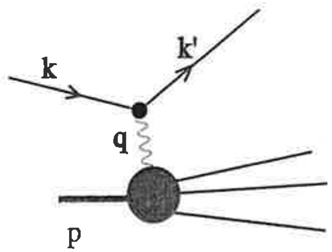
• First test of perturbative QCD was breaking of Bjorken scaling in deeply inelastic lepton hadron-scattering (DIS).

• DIS structure functions provide among most precise tests of QCD & determine Parton Distribution Functions (PDFs) of hadrons:

→ can be used in predicting hadronic cross sections.

• Kinematics of DIS:

→ Consider $l(k) + h(p) \to l'(k') + X$ (via γ, W or Z):



• Standard DIS variables are defined by

$$Q^2 = -q^2; \qquad x = \frac{-q^2}{2p \cdot q} \qquad \text{and} \qquad y = \frac{q \cdot p}{k \cdot p}.$$

• Scattering is called deeply inelastic if $Q^2 \gg \Lambda^2$.

• Structure functions parametrise target as 'seen' by γ, W, Z.

• Consider photon only:

$\rightarrow$ cross section can be written as $d\sigma \propto L^{\mu\nu}(k,q)\, W_{\mu\nu}(p,q)$.

• Structure of lepton tensor is determined by QED:

$$L^{\mu\nu} = \mathrm{Tr}\left(k \cdot \gamma\, \gamma^\mu k' \cdot \gamma\, \gamma^\nu\right)/2.$$

• Hadronic tensor $W^{\mu\nu}$ contains instead information about photon interaction with hadronic target and cannot be calculated in perturbation theory !

• Symmetry properties give restrictions on $W^{\mu\nu}$ form.

• Define two scalar structure functions, F_1 and F_2, dependent only on (invariants) x and Q^2:

$$\begin{aligned} W_{\mu\nu} &= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) \\ &+ \left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) \frac{1}{p\cdot q}\, F_2(x,Q^2). \end{aligned}$$

• Neglecting hadron mass w.r.t. Q^2, DIS cross section is

$$\frac{d\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{2Q^2}\left[yF_1 + \frac{1-y}{xy}F_2\right].$$

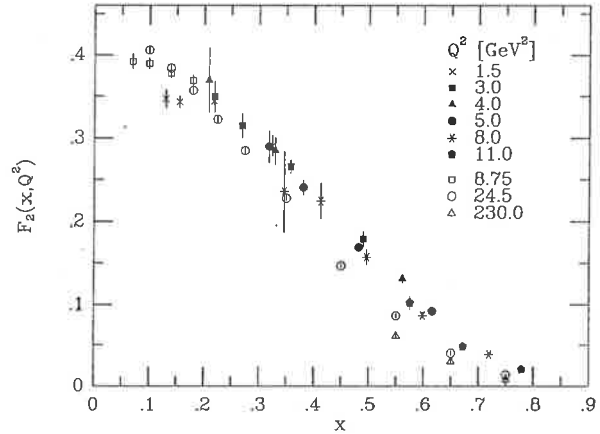
• In principle, can use y dependence to determine structure functions F_i in a DIS experiment.

• (Additional structure function, F_3, needed for W, Z.)

• Bjorken scaling limit defined as $Q^2 \rightarrow \infty$ with x fixed:

$\rightarrow$ structure functions obey approximate scaling law, i.e.

$$F_i(x,Q^2) \rightarrow F_i(x).$$



• Even though the Q^2 values vary by three orders of magnitude, data approximately lie on universal curve.

• Scaling implies γ^*-scattering off point-like constituents.

[Otherwise (dimensionless) structure functions would depend on Q/Q_0, with $1/Q_0$ some length scale characterising size of constituents.]

• Observation of scaling was the motivation for parton model.

• Parton model: p made of point-like constituents $\to$ partons.

• Their interactions are over time scales of $\mathcal{O}(1/\Lambda)$: longer w.r.t. time it takes e^- to traverse Lorentz contracted proton.

• Can therefore consider partons as (approximately) free particles over the very short interaction time.

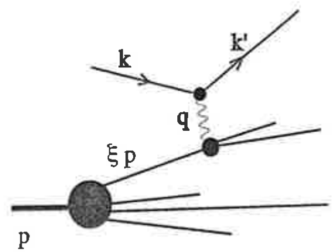
• Model leads to intuitive formula:

$$\frac{d\sigma^{(lh)}}{dx dQ^2} = \sum_a \int_0^1 d\xi\, f_{a/h}(\xi)\, \frac{d\sigma^{(la)}}{dx dQ^2},$$

$d\sigma^{(lh}$ $\to$ inclusive cross section for lepton-nucleon scattering;

$d\sigma^{(la)}$ *to* parton-electron one;

ξp, $0 < \xi < 1$, $\to$ parton momentum.

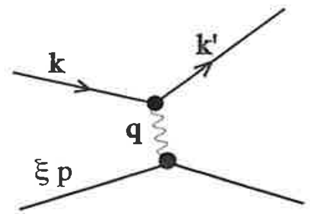


• Function f is called PDF:

$\to$ $f_{a/h}(\xi)\, d\xi$ gives the probability to find a parton with flavour $a = g, u, \bar{u}, d, \ldots$ in hadron h, carrying momentum fraction within $d\xi$ of ξ.

• PDFs are universal: i.e. independent of particular hard scattering process and can be determined from experiment.

• Hard scattering cross sections from perturbation theory:



• Using QED Feynman rules:

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right].$$

• Mass-shell constraint for outgoing quark

$$(\xi p + q)^2 = q^2 + 2\xi p \cdot q = -2p \cdot q(x - \xi) = 0$$

implies $x = \xi$.

• Write $\int_0^1 dx \delta(x - \xi) = 1$ and obtain

$$\frac{d\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[1 + (1-y)^2\right] \frac{1}{2} e_q^2 \delta(x - \xi).$$

• At lowest order, structure functions are given by

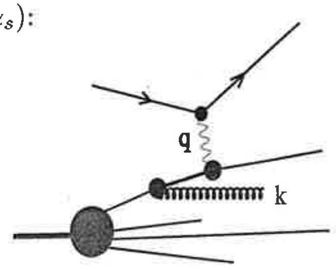
$$F_2(x, Q^2) = \sum_q e_q^2 x f_{/h}(x) = 2x F_1(x, Q^2).$$

$\to$ Callan-Gross relation: from spin of partons !

• Do not confuse structure functions and PDFs !

- In higher order QCD, structure functions F_i are Q^2-dependent and scaling is broken by logarithms of Q^2.

- Through $\mathcal{O}(\alpha_s)$:



- Quark acquires large transverse momentum k_T with probability $\sim \alpha_s dk_T^2/k_T^2$ at large k_T.
- Integral extends up to the kinematic limit $k_T^2 \sim Q^2$ and gives rise to contributions $\propto \alpha_s \ln Q^2$ which break scaling.
- Also, k_T integral logarithmically divergent as $|k_T| \to 0$.
- Introducing k_T cut-off λ:

$$F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_{q/h}(\xi) \left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\lambda^2}\right) + C\left(\frac{x}{\xi}\right)\right\}\right].$$

- $P_{qq}(\xi) = C_F(1+\xi^2)/(1-\xi)$ called splitting function and C is finite term due to virtual gluon exchange.
- Limit $k_T \to 0$ ($\lambda \to 0$) corresponds to long-range non-perturbative QCD: however,

$\to$ factorisation theorem: can separate from hard scattering.

QCD factorisation theorem

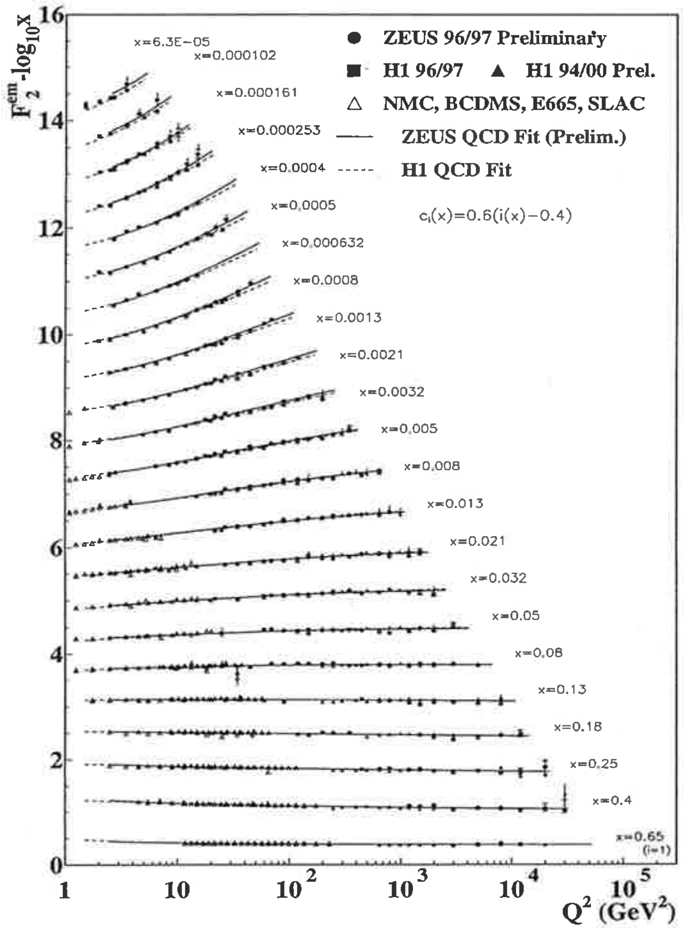
- Perturbative expansion can be rearranged such that contributions from long-range physics appear in PDFs while those short-distance appear in the hard-scattering cross section (Collins, Soper, Sterman).
- Separation requires introduction of factorisation scale μ_F.
- E.g. gluon emission with $k_T^2 \le \mu_F^2$ is part of $f_{q/h}$ while with $k_T^2 \ge \mu_F^2$ is part of perturbative scattering.
- Through $\mathcal{O}(\alpha_s)$:

$$F_2(x,Q^2) = x\sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} f_{q/h}(\xi,\mu_F^2) \left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu_F^2}\right) + C_{FS}\left(\frac{x}{\xi}\right)\right\}\right]$$

(C_{FS} factorisation-scheme dependent finite correction).

- Arbitrariness in how much of C_{FS} is factored into PDFs defines so-called 'factorisation scheme'.
- While PDFs and hard scattering cross section depend on μ_F, physical cross section does not.
- The more terms included in the perturbative expansion the weaker the dependence on μ_F.
- Factorisation turns QCD into a reliable calculational tool !

• QCD scaling violation observed experimentally:



• PDFs can be defined in terms of quark- and gluon-field operators.

• PDFs appear in QCD formulae for any process with $n \geq 1$ hadrons in initial state.

• PDFs could (in principle) be calculated in lattice QCD, yet determined from experiment.

• Dependence of PDFs on μ_F determined by Renormalisation Group Equation (RGE) [Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation]:

$$\frac{d}{d\ln\mu_F} f_{a/h}(x,\mu_F) = \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu_F))\, f_{b/h}(\xi,\mu_F).$$

• Splitting function P_{ab} has perturbative expansion:

$$P_{ab}(x/\xi, \alpha_s(\mu_F)) = P_{ab}^{(1)}(x/\xi)\,\frac{\alpha_s(\mu_F)}{\pi} + P_{ab}^{(2)}(x/\xi)\left(\frac{\alpha_s(\mu_F)}{\pi}\right)^2 + \cdots$$

• First two terms known and used in numerical solutions.

• DGLAP-equation:

→ enables to relate PDFs measured at one scale to other scales and make corresponding predictions

PDFs

- Constrain PDFs by using many different beams/targets:

Nomenclature/isospin:
$f_{u/n}(x,Q^2) = f_{d/p}(x,Q^2) = f_{\bar{d}/\bar{p}}(x,Q^2) \equiv f_d(x,Q^2)$, etc.

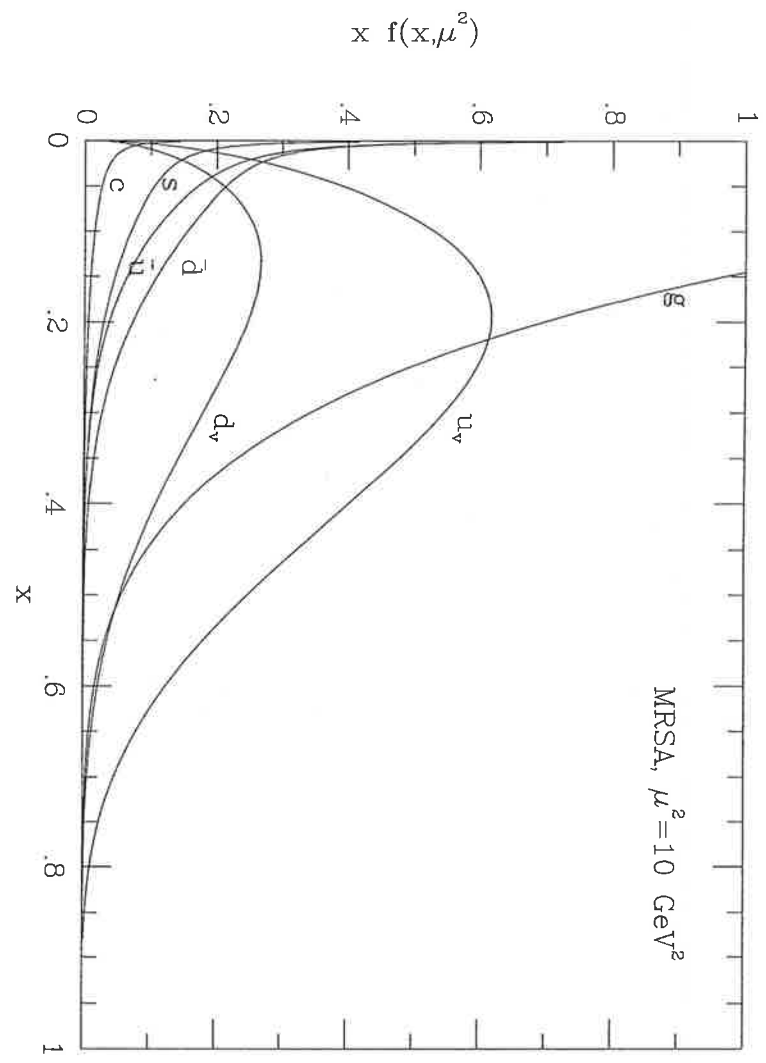
$$
\begin{aligned}
F_2^{ep}(x,Q^2) &= \frac{1}{9}xf_d + \frac{4}{9}xf_u + \frac{1}{9}xf_{\bar{d}} + \frac{4}{9}xf_{\bar{u}} + \frac{1}{9}xf_s + \ldots \\
F_2^{en}(x,Q^2) &= \frac{4}{9}xf_d + \frac{1}{9}xf_u + \frac{4}{9}xf_{\bar{d}} + \frac{1}{9}xf_{\bar{u}} + \frac{1}{9}xf_s + \ldots \\
F_2^{\nu p}(x,Q^2) &= 2xf_d + 2xf_{\bar{u}} + 2xf_s + 2xf_{\bar{c}} + \ldots \\
F_3^{\nu p}(x,Q^2) &= 2xf_d - 2xf_{\bar{u}} + 2xf_s - 2xf_{\bar{c}} + \ldots \\
F_2^{\bar{\nu} p}(x,Q^2) &= 2xf_u + 2xf_{\bar{d}} + 2xf_c + 2xf_{\bar{s}} + \ldots \\
F_3^{\bar{\nu} p}(x,Q^2) &= 2xf_u - 2xf_{\bar{d}} + 2xf_c - 2xf_{\bar{s}} + \ldots
\end{aligned}
$$

etc ...

→ global fits → fig

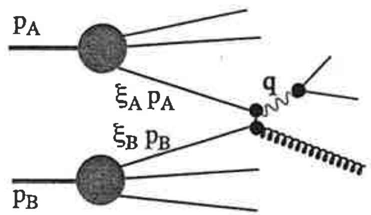
- E.g. MRS(T), CTEQ, GRV, etc.,

→ `http://durpdg.dur.ac.uk/hepdata/pdf.html`



Hadron-hadron collisions

- Cross section for e.g. Z boson production



can be factored:

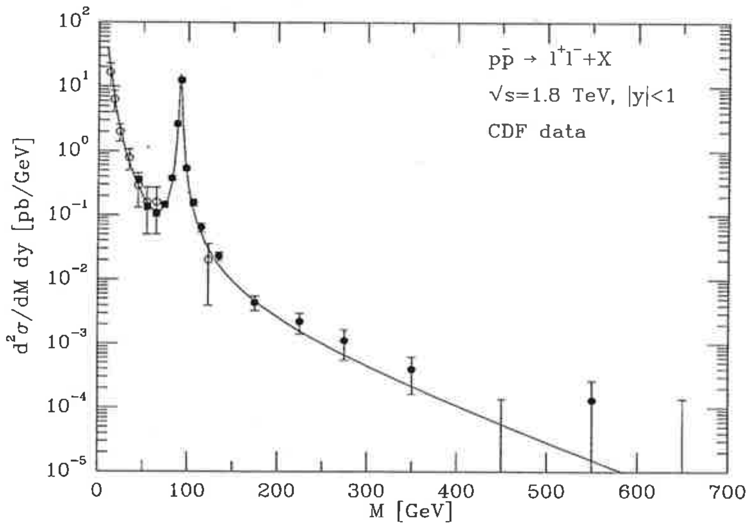
$$d\sigma(p_A, p_B) = \sum_{a,b} \int d\xi_A d\xi_B \; f_{a/A}(\xi_A, \mu_F) \; f_{b/B}(\xi_B, \mu_F)$$
$$\times d\hat{\sigma}_{ab}(\xi_A p_A, \xi_B p_B, \mu_F).$$

- Characteristic scale of hard scattering $Q^2 \gg \Lambda^2$ could be e.g. M_Z or p_Z^T.

- Factorisation formula holds up to Λ^2/Q^2 corrections.

- To prove factorisation one needs to sum over graphs and use unitarity, causality and gauge invariance (Collins, Soper and Sterman).

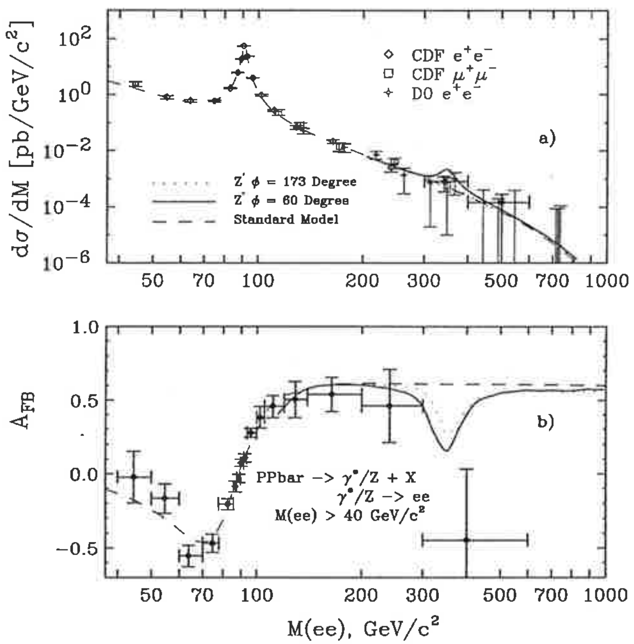
- Historically, confirmed by lepton-pair hadro-production ($A + B \to l^+l^- + X$, or <u>'Drell-Yan'</u> process, DY), using the parton picture and the PDFs from DIS:



(Distribution is lepton pair invariant mass squared.)

NP in DY processes

CDF Preliminary



(a) $d\sigma/dM$ distribution of e^+e^- (CDF and DØ) and $\mu^+\mu^-$ (CDF). SM (dashed) normalized ($\times 1.11$) to CDF data in Z mass region. (b) CDF A_{FB} versus mass compared to SM (dashed). Also shown are theoretical curves ($\times 1.11$) for $d\sigma/dM$ and A_{FB} for extra E_6 boson with $M_{Z'} = 350$ GeV and $\Gamma_{Z'} = 0.1\, M_{Z'}$, for $\phi = 60^0$ (solid) and 173^o (dotted).

Top mass

- Measured by Tevatron experiments (CDF & DØ).
- Hadro-production modes:

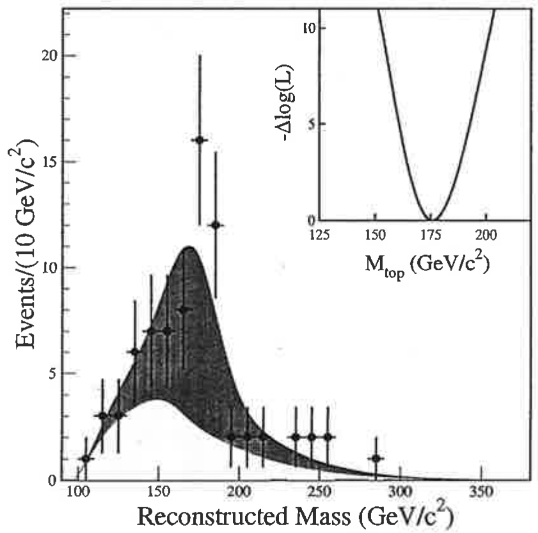
$$gg \to t\bar{t} \text{ (dominates at LHC)},$$

$$q\bar{q} \to t\bar{t} \text{ (dominates at Tevatron)}.$$

- Top decays (e.g. semi-leptonic, j = jet & $\ell = e, \mu$):

$$t\bar{t} \to (bW^+)(\bar{b}W^-) \to (bjj)(\bar{b}\ell\nu_\ell) \ + \ \text{C.C.}$$

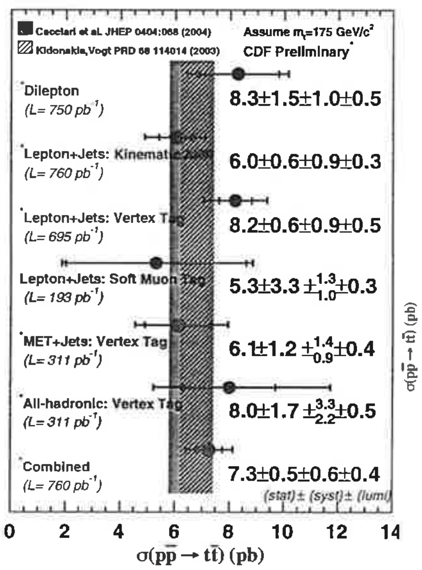
- 6-jet signatures has worse combinatorics !
- Reconstruct m_t from bjj invariant mass (e.g. DØ):

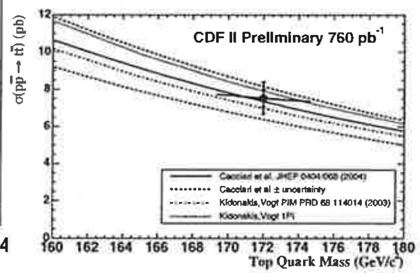


Top pair production cross section

- Measured by both Tevatron experiments (CDF & DØ).
- Achieved consistency with SM prediction (required NLO and resummation)

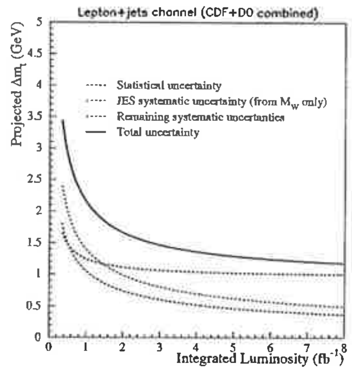
1. Many different channels
2. Compare to look for NP
3. Currently statistics limited (750 pb^{-1})
4. Assume $m_t = 175$ GeV.



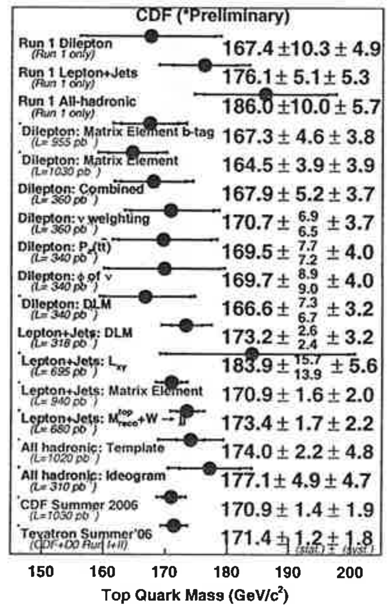


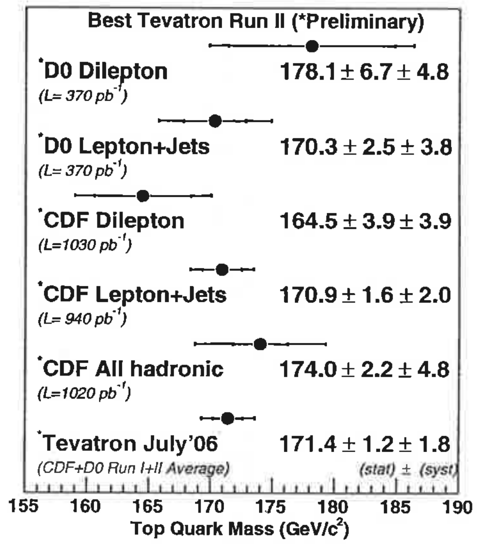
Top mass: Tevatron Summary

- Run 2 initial goal was $\delta m_t = 2.0$–2.5 GeV (per experiment)



- Large systematics from jet energy scale





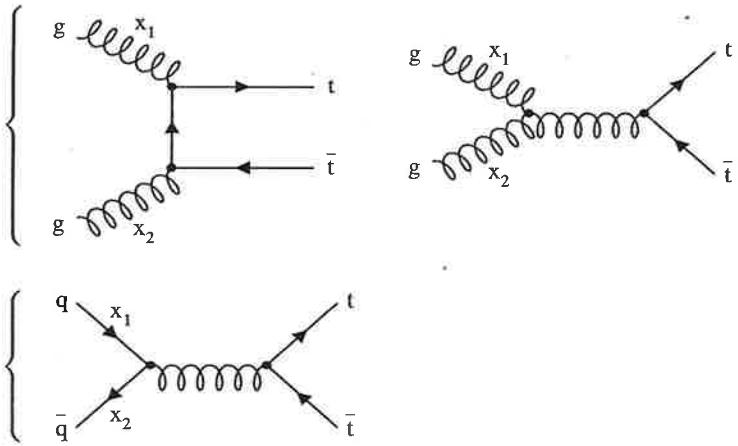
$t\bar{t}$ production at the LHC

$\sqrt{s} = 14\,\text{TeV} \quad \sigma(pp \to t\bar{t}) \simeq 800\,\text{pb} \quad \Rightarrow 8 \cdot 10^6$ events @ $\mathcal{L}_{\text{low}} = 10\,\text{fb}^{-1}$

$\Rightarrow 8 \cdot 10^7$ events @ $\mathcal{L}_{\text{high}} = 10^5\,\text{pb}^{-1}$

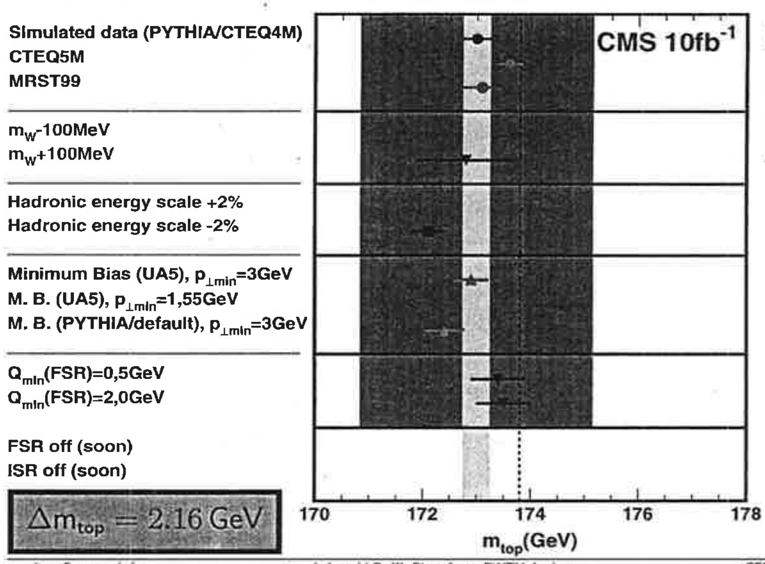
90 %

10 %



Systematical uncertainties of the top mass

(only hadronic decay in semileptonic channel)



Crucial cuts:

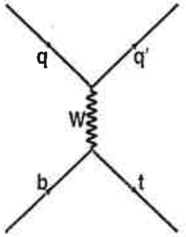
$p_\perp(\ell) > 20\,\text{GeV}$

$\not{p}_\perp > 20\,\text{GeV}$

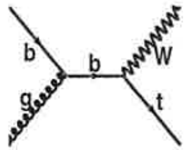
4 jets:
$p_\perp > 40\,\text{GeV}$
$|\eta_{\text{jet axis}}| < 2.4$

demand 2 b jets

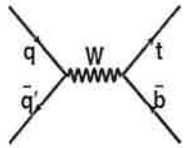
- At LHC it is possible to produce top quarks singly via the weak interaction



Wg
large LHC x-sec $\approx$ 245 pb
high rate, V_{tb}, polarized tops, etc.



Wt
LHC x-sec $\approx$ 50 pb
V_{tb}, new theoretical results recently.....



W*
LHC x-sec $\approx$ 10 pb
low th. errors, V_{tb}

- "New Physics" can affect each rate differently
- Single top provides the best opportunity to study W-t-b vertex:
  - cross-section $\propto |V_{tb}|^2$
  - source of polarized tops (precise prediction)

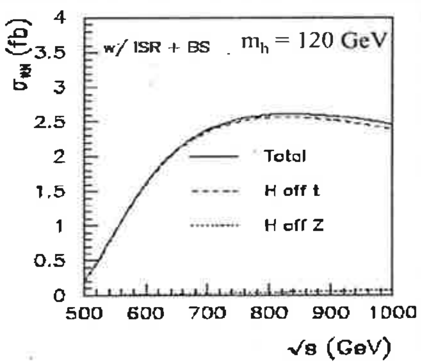
1

Threshold Results

- Mass: $\Delta m_t = 16$ MeV, $\Delta\alpha_S = 0.0011$
  - Using cross section only: $\Delta m_t = 24$ MeV, $\Delta\alpha_S = 0.0017$.
  - Γ_t, g_{tth} fixed at SM values; assume m_h=120 GeV, $\alpha_s(M_Z)$=0.120.
  - Theory error: ~100 MeV.
- Width: allow to vary in a 3–parameter fit.
  - $\Delta\Gamma_t = 32$ MeV, $\Delta m_t = 18$ MeV, $\Delta\alpha_S = 0.0015$
  - 2% exp. uncertainty on width

$t\bar{t}H$ production and the Top Yukawa Coupling

- $e^+e^- \to$ ttH→ WbWb bb
- Very complicated final state:
  - Up to 8 jets
  - 4 b's
  - Many kinematic constraints
- Tiny cross section (~2 fb), with backgrounds ~3 orders of magnitude higher.
- Interfering backgrounds from EWK (ttZ), QCD (g→bb)
- Non–interfering backgrounds
  - Dominantly $e^+e^- \to$ tt
  - Formally smaller number of partons, but can enter the selection due to hard gluon radiation, detector effects, and their very large cross sections

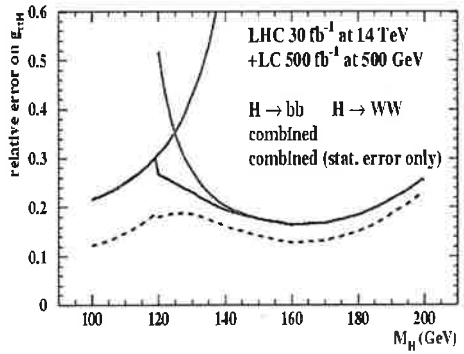


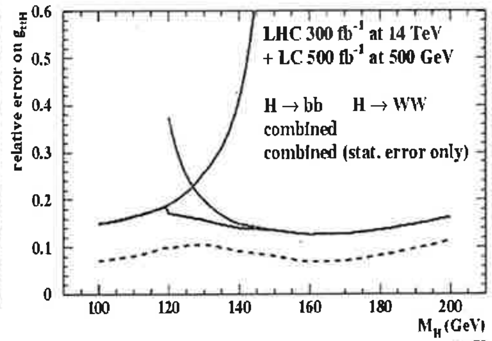
Top Yukawa Coupling

K. Desch
M. Schumacher
hep-ph/0407159

SM prediction is $g_{ttH} = \frac{\sqrt{2}m_{top}}{246\ GeV} = 1.02 \pm 0.02$

- Important to test coupling between Higgs and top quark
- Combine LHC and LC for model independent measurement
  - LHC: pp →ttH+X – measure σ(ttH)xBR(H→WW) to 20-50%
  - ILC: $e^+e^- \to$ZH - measure BR(H→WW) to 2% $\sigma(ttH) \propto g^2_{ttH}$
    - Can do with 500 GeV Linear Collider

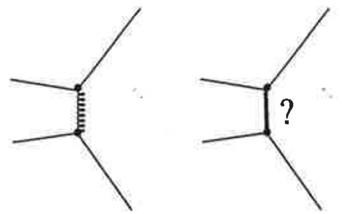




p. 59

New physics searches

- NP can introduce new terms in SM (effective) Lagrangian.
- Imagine quarks scattering by gluon exchange to produce two jets supplemented by quarks exchanging new object with mass $M \sim \mathcal{O}(\text{TeV})$:



- At $\sqrt{s} \ll M$, NP details cannot be resolved.
- Effect can be emulated by new terms in QCD Lagrangian:

$$\Delta\mathcal{L} = \frac{\tilde{g}^2}{M^2}\,\bar{\psi}\gamma^\mu\psi\;\bar{\psi}\gamma_\mu\psi.$$

($\tilde{g}^2 \to$ strength of coupling between q and NP).

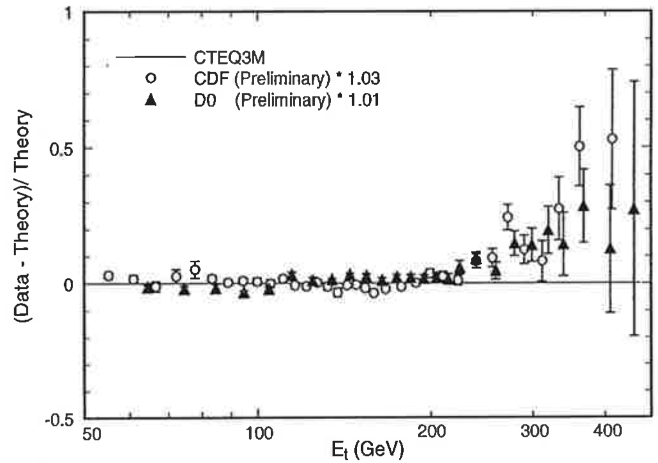
- Factor $1/M^2$ needed for dimensional reasons and implies that effect of NP is small.
- To observe deviation from SM need:

1. high-precision experiment, or
2. experiment looking for some effect forbidden in SM, or
3. an experiment at $\sqrt{s} \approx M$.

- E.g. consider $p\bar{p} \to \text{jet} + X$ as a function of $\approx E_T^{\text{jet}}$ of jet.
- For $E_T^{\text{jet}} \ll M$:

$$\frac{\text{Data} - \text{Theory}}{\text{Theory}} \propto \tilde{g}^2\frac{E_T^2}{M^2}$$

- Compare experimental jet cross section to NLO QCD:



- Beware: observed effect can most likely be explained by theoretical uncertainty on gluon PDF at large x !

Test of SM: EW Physics

Outline

- Weak interactions from unitarity
- SM renormalisation
- e^+e^- annihilation near Z pole
- W production
- Indirect search for top and Higgs

Weak interactions from unitarity

- Weak interactions discovered in β-decay and described by effective Lagrangian (Fermi theory).
- For $\mu^- \to e^- \bar{\nu}_e \nu_\mu$, Lagrangian is:

$$\mathcal{L} = \frac{G_F}{\sqrt{2}}[\bar{\nu}_\mu \gamma_\lambda (1-\gamma_5)\mu][\bar{e}\gamma^\lambda (1-\gamma_5)\nu_e]$$

with $G_F \approx 1.17 \times 10^{-5}$ GeV^{-2} (Fermi coupling)

- Fermi theory as an effective low-energy theory and cannot be extended to arbitrarily high energies.
- Applying effective Lagrangian at high energies,

$$\mathcal{M}[\bar{\nu}_\mu e^- \to \mu^- \nu_e] \sim \frac{G_F s}{2\sqrt{2}\pi}$$

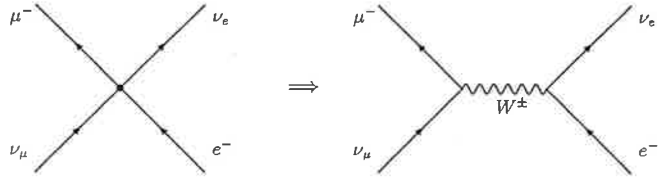
- Scattering amplitude must respect unitarity bound

$$|Re\mathcal{M}| \leq 1/2.$$

$\Rightarrow$ Theory cannot be applied at $s \gtrsim (600 \text{ GeV})^2$.

$\Rightarrow$ Can deduce structure of weak interactions from unitarity constraints (Llewellyn Smith and Cornwall, Levin and Tiktopoulos).

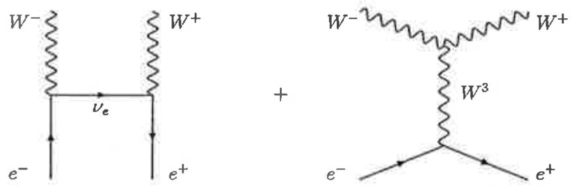
• Unitarity problem in $\bar{\nu}_\mu e^- \to \mu^- \nu_e$ solved by assuming weak interactions mediated by heavy charged vector bosons:



• W propagator dampens rise of scattering amplitudes as $\sqrt{s} \to \infty$ if $M_W \approx 100$ GeV:

$$\mathcal{M}[\bar{\nu}_\mu e^- \to \mu^- \nu_e] \to \frac{G_F s}{2\sqrt{2}\pi} \frac{M_W^2}{M_W^2 - s}$$

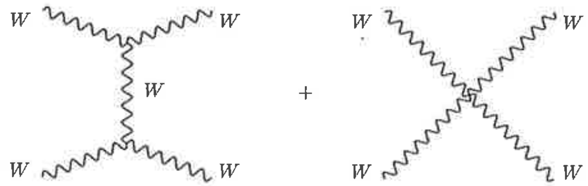
• Consider production of W^+W^- pairs in e^+e^- annihilation.



• Neutrino term grows quadratically and violates unitarity.

• Bad high-energy behaviour cured by exchange of a new neutral vector boson W^3 in s-channel !

• Amplitude for $W_L W_L \to W_L W_L$, as mediated by virtual W exchange and quadrilinear W boson coupling,

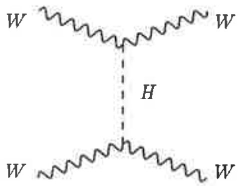


grows quadratically with energy !

• WW scattering amplitude can be damped by new interactions between W bosons at high-energy.

• If theory is to remain weakly interacting up to high energies, a new scalar particle, Higgs boson, must be introduced, which couples to a particle with a strength proportional to particle mass.

• Higgs boson exchange cancels bad high-energy behaviour so that amplitude fulfills unitarity requirement if $M_H \lesssim 1$ TeV.

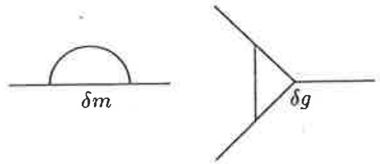


$$\mathcal{M} \to \frac{G_F M_H^2}{4\sqrt{2}\pi}$$

• (Unitarity requirements can be exploited further to determine quartic W-Higgs interactions and Higgs self-interaction potential.)

SM renormalisation

- Structure of EW interactions emerged from requirement of unitarity at high energies.
- Theoretically, SM is a non-Abelian gauge field theory.
- SM observables can be calculated to arbitrarily high precision in a systematic expansion after a few basic parameters are fixed experimentally.
- Quantum corrections in interacting field theories modify particle masses and couplings, i.e. interactions renormalise the fundamental parameters.
- Described by Feynman diagrams including loops

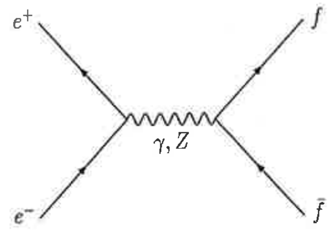


$$\sim \int \frac{d^4k}{k^4} \sim \ln \Lambda_{\text{cut}}^2$$

- Self-energy and vertex corrections are logarithmically divergent for large loop momenta and lead to contributions $\sim \ln \Lambda_{\text{cut}}^2$ where Λ_{cut} is energy scale up to which SM is valid.
- Quantum corrections add to unobservable bare mass m_0 and bare coupling g_0 to generate the observable physical mass m and coupling g, i.e. $m_0 + \delta m = m$ and $g_0 + \delta g = g$.
- Renormalisation is sufficient to absorb all divergences and render all observables finite if $\Lambda_{\text{cut}} \to \infty$.
- SM is renormalisable ('t Hooft and Veltman).
- Once masses/couplings are fixed experimentally, all other observables are calculable to arbitrarily high precision.

e^+e^- annihilation near Z pole

- LEP1 and SLC experiments allowed tests of EW theory at quantum level.
- Consider $e^+e^- \to f\bar{f}$ $(f = q, \ell, \nu)$ in SM:



$$\sigma_\gamma(s) = \frac{4\pi\alpha^2}{3s} Q_f^2 N_f \qquad (N_q = N_C,\ N_{\ell,\nu} = 1)$$

$$\sigma_Z(s) = \frac{4\pi\alpha^2}{3s} \frac{s^2}{(s - M_Z^2)^2 + M_Z^2\Gamma_Z^2} \mathcal{A}_f \mathcal{A}_e N_f$$

with

$$\mathcal{A}_f = v_f^2 + a_f^2 = \frac{(t_{3f} - 2Q_f \sin^2\theta_W)^2 + t_{3f}^2}{4\sin^2\theta_W \cos^2\theta_W}$$

- Include leading logarithmic radiative corrections

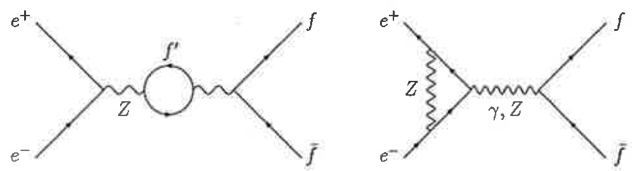
$$\alpha \to \alpha(s) \text{ (improved Born approximation)}$$

- SM cross section:

$$\sigma(s) = \frac{4\pi\alpha^2(s)}{3s}\frac{s^2}{(s-M_Z^2)^2+(s^2/M_Z^2)\Gamma_Z^2}\left[1+\underbrace{\Delta_Z}\right] + \frac{4\pi\alpha^2(s)}{3s}Q_f^2 N_f \qquad \gamma - Z \text{ interference}$$

(also use running width).

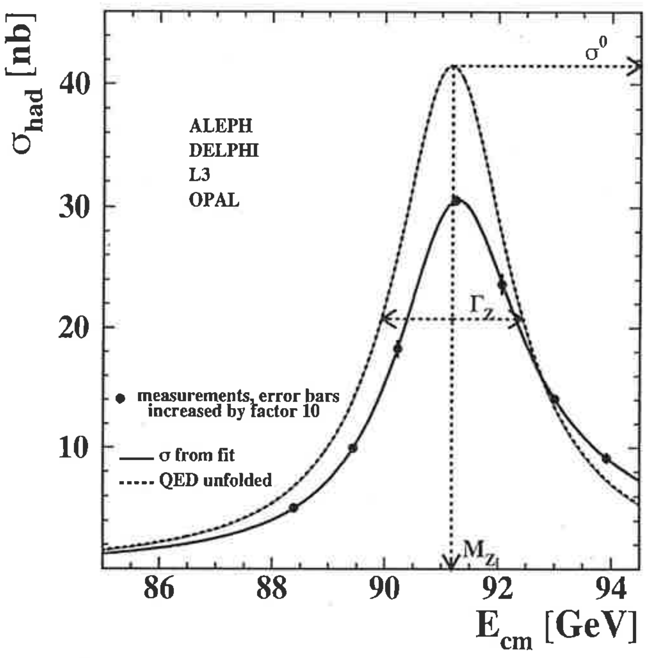
- Most important EW corrections near Z resonance:



(Lead to ultraviolet divergences which have to be absorbed into renormalised masses and couplings.)

- In addition, QCD corrections have to be included in $q\bar{q}$.

- QED corrections due to photon Initial State Radiation (ISR) are crucial near resonance:



- First non-trivial SM test:

→ given measurements of α, M_Z, G_F and Γ_Z predicts

$$\sigma_{SM}(M_Z) = \frac{4\pi\alpha^2}{3\Gamma_Z^2}\mathcal{A}_f\mathcal{A}_e$$

Total Z Width

- Consider width to given final state fermion:

$$\Gamma_f = \frac{1}{3}\alpha\, M_Z\, \mathcal{A}_f$$

- Total width comes all possible final states:

$$\Gamma_Z = \sum_f \Gamma_f = \sum_\ell \Gamma_\ell + \sum_\nu \Gamma_\nu + \sum_q \Gamma_q$$

$$\rightarrow \Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV}$$

→ Gives further non-trivial test of SM !

- Gives measurement of number of neutrino species:

$$N_\nu = 2.993 \pm 0.011$$

- Or limit on width to additional invisible particles:

$$\Gamma_{\text{inv}} = 499.0 \pm 1.5 \text{ MeV}$$

Forward-Backward asymmetry

- Line-shape and widths only sensitive to combinations of:

$$\mathcal{A}_f = v_f^2 + a_f^2$$

- $\cos\theta$-dependence also contains

$$\mathcal{B}_f = 2 v_f a_f$$

- Construct forward-backward asymmetry:

$$\begin{aligned} A_{FB} &\equiv \frac{\sigma_{\text{SM}}(\theta < 90^\circ) - \sigma_{\text{SM}}(\theta > 90^\circ)}{\sigma_{\text{SM}}(\theta < 90^\circ) + \sigma_{\text{SM}}(\theta > 90^\circ)} \\ &= \frac{3}{4}\frac{\mathcal{B}_e \mathcal{B}_f}{\mathcal{A}_e \mathcal{A}_f} \end{aligned}$$

→ more and complementary tests !

Left-right asymmetry

- SLC had unique feature: highly polarized electrons

$$P_{e^-} \sim 69\%$$

- New asymmetry:

$$A_{LR} \equiv \frac{\sigma_{\rm SM}(e^+e^-_L) - \sigma_{\rm SM}(e^+e^-_R)}{\sigma_{\rm SM}(e^+e^-_L) + \sigma_{\rm SM}(e^+e^-_R)} = -\frac{\mathcal{B}_e}{\mathcal{A}_e}$$

- Note:

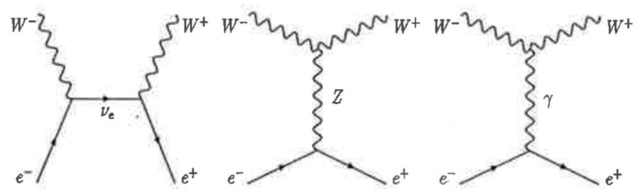
1. independent of final state
2. independent of angular range
3. much larger than A_{FB}

→ almost systematically error-free !
- Just need to measure polarisation well ...

- (SLD: world's best $\sin^2\theta_W$.)

W production

- Consider $e^+e^- \to W^+W^-$ (LEP2):



- Large cancellations at high energies (ditto):

each diagram $\sim \frac{G_F^2 s}{48\pi}$, $s \gg M_W^2$

but sum $\sim \frac{G_F^2 m_w^4}{s\pi} \log \frac{s}{M_W^2}$, $s \gg M_W^2$

- Very sensitive to Triple Gauge Couplings (TGCs)

→ other very powerful SM test

W mass in e^+e^-

- Predicted by SM once α, G_F and M_Z measured

$\Rightarrow$ another strong SM test (symmetry breaking mechanism)

- Near threshold:

$$\sigma_{WW} \sim \frac{G_F^2 M_W^2}{2\pi} \underbrace{\sqrt{1 - \frac{4M_W^2}{s}}}_{\text{velocity of } W}$$

rapidly varying for $\sqrt{s} \sim 2M_W$

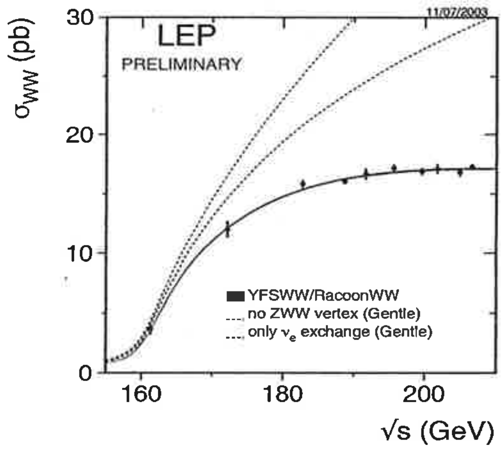
Very clean theoretically, but few events
$\Rightarrow$ large statistical errors $\rightarrow$ fig

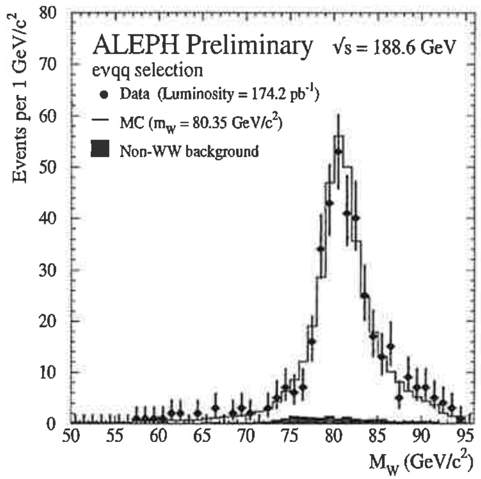
- Above threshold:

$\rightarrow$ Measure invariant mass of W decay products $\rightarrow$ fig

- LEP average:

$$M_W = 80.412 \pm 0.042 \text{ GeV}$$





W mass in hadron-hadron

- W boson mass measured at hadron colliders (Tevatron).
- $W^{\pm} \rightarrow \ell\bar{\nu}_\ell/\bar{\ell}\nu_\ell$ decays provide small but clean sample.
- Neutrino lost $\Rightarrow p^\nu$ reconstructed from rest of event.
- Many hadrons lost in beam directions.

$\Rightarrow$ only transverse momentum conservation can be used

- Use:

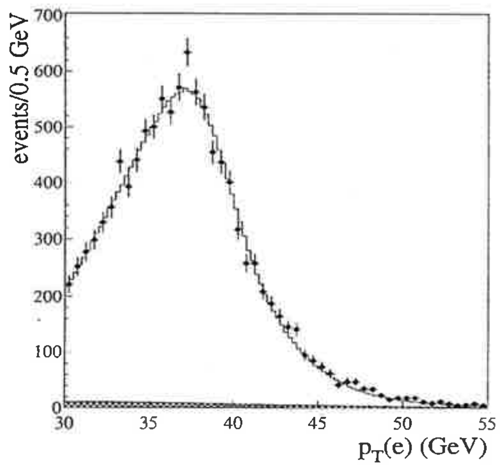
1. lepton transverse momentum: $p_T(\ell) \rightarrow$ fig
2. transverse mass: $M_T^2 \equiv 2p_T^e p_T^\nu(1 - \cos\phi) \rightarrow$ fig

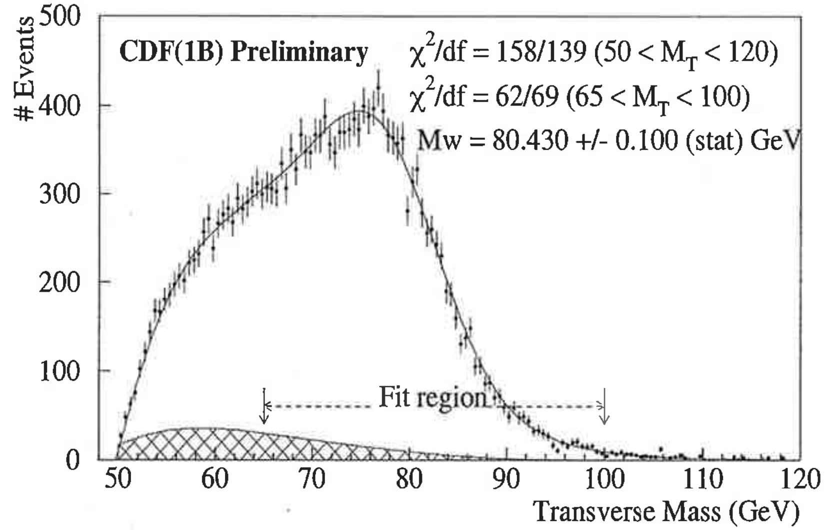
- (Insensitive to W transverse momentum !)
- Tevatron average:

$$M_W = 80.452 \pm 0.059 \text{ GeV}$$

- World average:

$$M_W = 80.425 \pm 0.034 \text{ GeV}$$

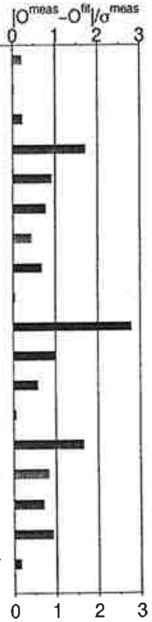




Precision observables

Summer 2006

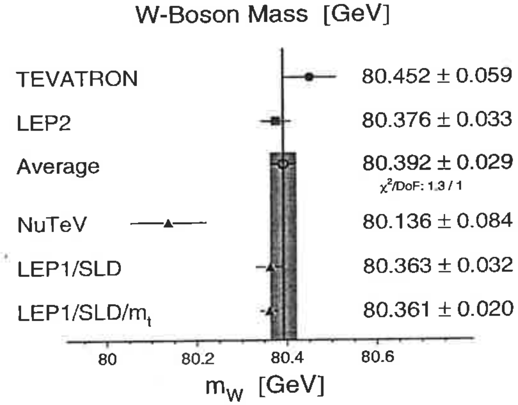
	Measurement	Fit
$\Delta\alpha^{(5)}_{had}(m_Z)$	0.02758 ± 0.00035	0.02766
m_Z [GeV]	91.1875 ± 0.0021	91.1874
Γ_Z [GeV]	2.4952 ± 0.0023	2.4957
σ^0_{had} [nb]	41.540 ± 0.037	41.477
R_l	20.767 ± 0.025	20.744
$A^{0,l}_{fb}$	0.01714 ± 0.00095	0.01640
$A_l(P_\tau)$	0.1465 ± 0.0032	0.1479
R_b	0.21629 ± 0.00066	0.21585
R_c	0.1721 ± 0.0030	0.1722
$A^{0,b}_{fb}$	0.0992 ± 0.0016	0.1037
$A^{0,c}_{fb}$	0.0707 ± 0.0035	0.0741
A_b	0.923 ± 0.020	0.935
A_c	0.670 ± 0.027	0.668
A_l(SLD)	0.1513 ± 0.0021	0.1479
$\sin^2\theta^{lept}_{eff}(Q_{fb})$	0.2324 ± 0.0012	0.2314
m_W [GeV]	80.392 ± 0.029	80.371
Γ_W [GeV]	2.147 ± 0.060	2.091
m_t [GeV]	171.4 ± 2.1	171.7



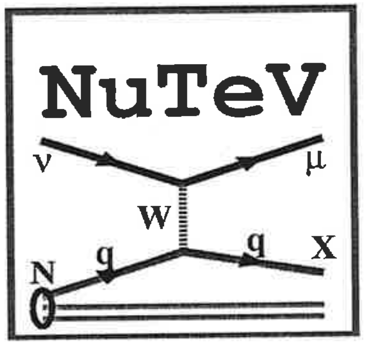
- (Pull is defined as deviation from theoretical prediction in units of corresponding one-standard deviation experimental uncertainty.)
- Includes latest top mass !

M_W (and NuTeV anomaly ?)

- Direct vs. indirect M_W determinations:



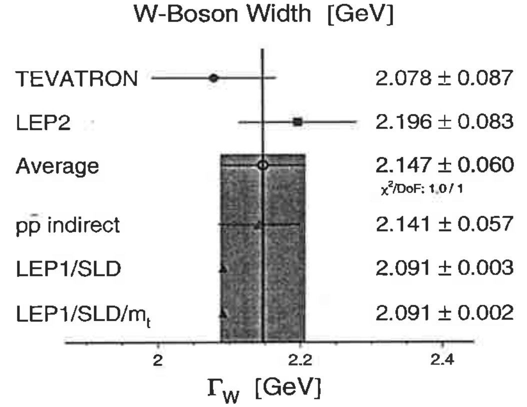
- NuTeV:



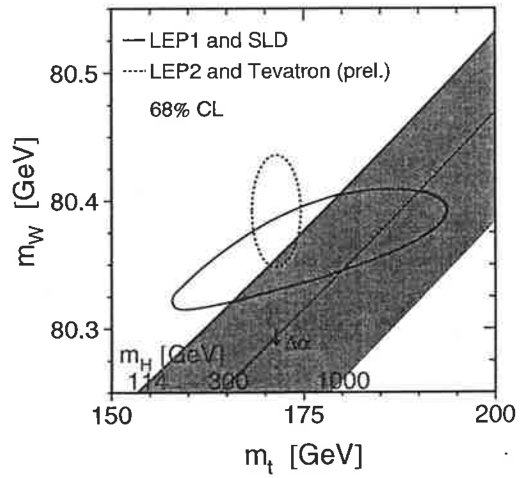
$\rightarrow$ ratio of neutral to charged currents in neutrino-nucleon

- Measurement from NuTeV collaboration (when interpreted as a measurement of M_W) shows 2.6–2.8σ deviation.
- (Some sort of) SM inconsistency ?
- (Can be viewed as PDF problem, etc.)

- Also W width:



- Can correlate m_t and M_W in global EW fit:



(Summer 2006 combination for $m_t = 171.4 \pm 2.1$ GeV)

Indirect search for top and Higgs

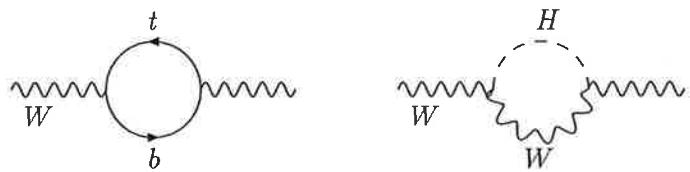
- Precision observables are affected by quantum fluctuations:

$\rightarrow$ give access to two high mass SM scales: m_t and M_H

- t, H enter in loop corrections to EW observables.
- E.g. radiative corrections to M_W, M_Z vs. $\sin^2 \vartheta_W$ relation:

$$\sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2}$$

- Quadratic dependence on m_t and logarithmic on M_H:



$\propto m_t^2$ $\qquad\qquad$ $\propto \log M_H$

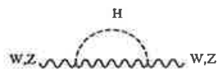
- (Sensitivity also to BSM physics.)

EW precision fits: perturbatively calculate observables in terms of few parameters:

$$M_Z, G_F, \alpha(M_Z), M_W, m_f, (\alpha_s(M_Z))$$

extracted from experiments with high accuracy.

- SM needs Higgs boson to cancel infinities, e.g.

$M_W, M_Z \longrightarrow$ 

- Finite logarithmic contributions survive, e.g. radiative corrections to $\rho = M_W^2/(M_Z^2 \cos^2\theta_W)$:

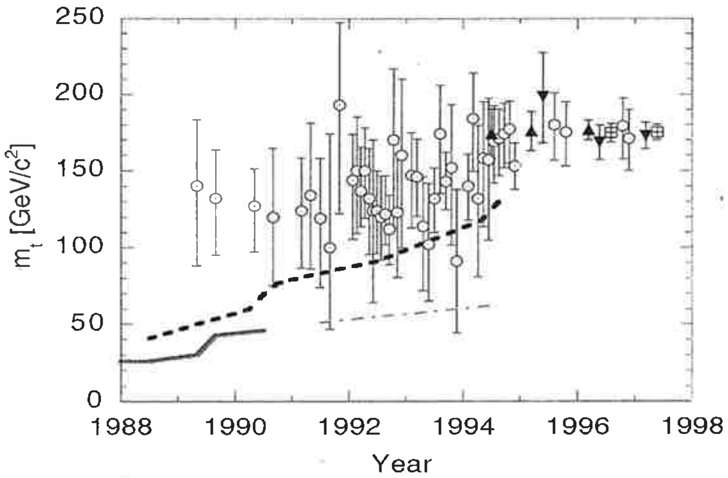
$$\rho = 1 - \frac{11g^2}{96\pi^2 \tan^2\theta_W} \ln\left(\frac{M_H}{M_W}\right)$$

  Main effects in oblique radiative corrections (S,T-parameters)

- New physics at the scale Λ will appear as higher dimension effective operators.

- EW precision observables led to m_t prediction !
- Determinations of m_t from

1. fits to EW observables (open circles)
2. 95% confidence-level (CL) lower bounds on m_t from
   - direct searches in e^+e^- annihilations (solid line)
   - direct searches in $\bar{p}p$ collisions (broken line)
3. from Γ_W in $\bar{p}p \to (W \text{ or } Z) + X$ (dot-dashed line)
4. direct measurements of m_t by CDF (triangles) and DØ (inverted triangles)



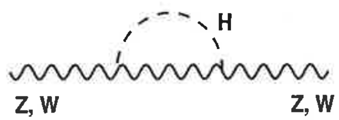
- Compare to latest PDG compilation (2005):

$$m_t = 172.7 \pm 2.9 \text{ (direct observation)},$$

$$m_t = 178.1^{+10.4}_{-8.3} \text{ (SM EW fits)}.$$

Precision Indirect Higgs Mass

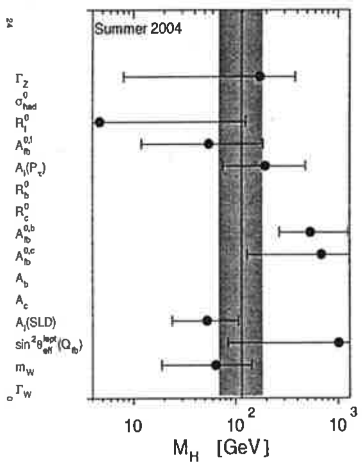
Try same trick to find Higgs mass:



$$\delta M_W^2 \sim \ln \frac{M_H}{M_W}$$

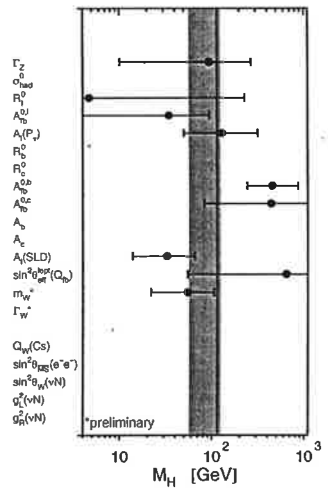
Much weaker dependence on M_H than on m_t.

➥Task is harder and requires as much EW precision data as you can get your hands on...

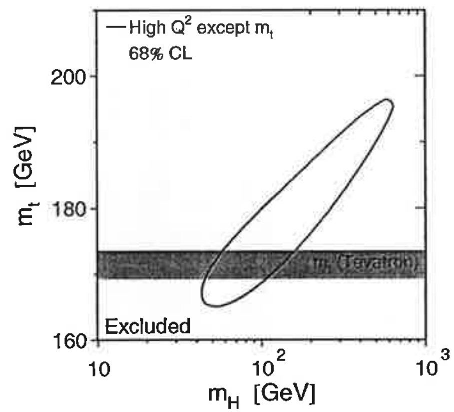


LEP+SLD EW Working Group '04 (adapted)

• Sensitivity to M_H only logarithmic, still limits available.



• Allowed region in $m_t - M_H$ plane:



• A SM Higgs boson with mass $M_{\mathrm{Higgs}} < 114$ GeV is already excluded from direct LEP searches at 95% CL (see later).

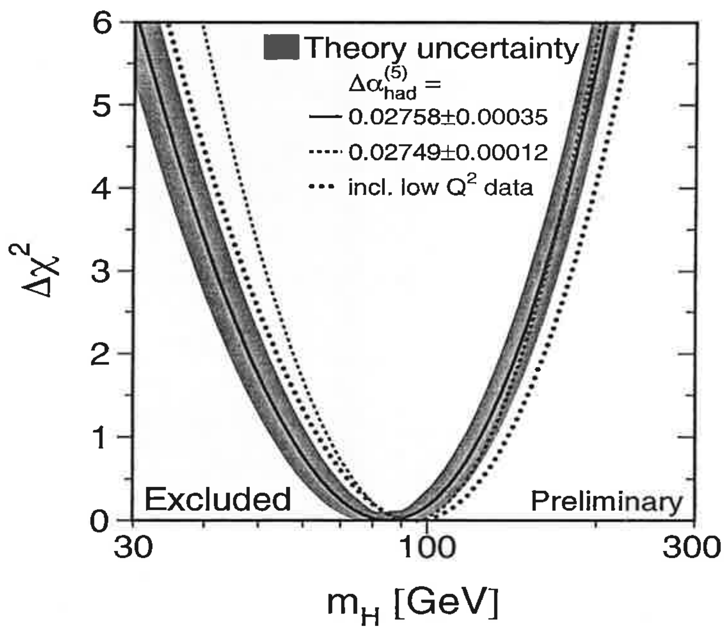
• Possible LEP evidence of SM Higgs at 115 GeV (see later).

- Fixing m_t to experimental value from direct measurement at Tevatron, precision data lead to:

$$M_H^{\rm SM} < 166 \text{ GeV (approx)} \quad \text{at } 95\% \text{ CL } (\Delta\chi^2 = 2.7)$$

$$\text{Best fit}: M_H = 85\pm^{39}_{28} \text{ GeV}$$

(Summer 2006 combination for $m_t = 171.4 \pm 2.1$ GeV)



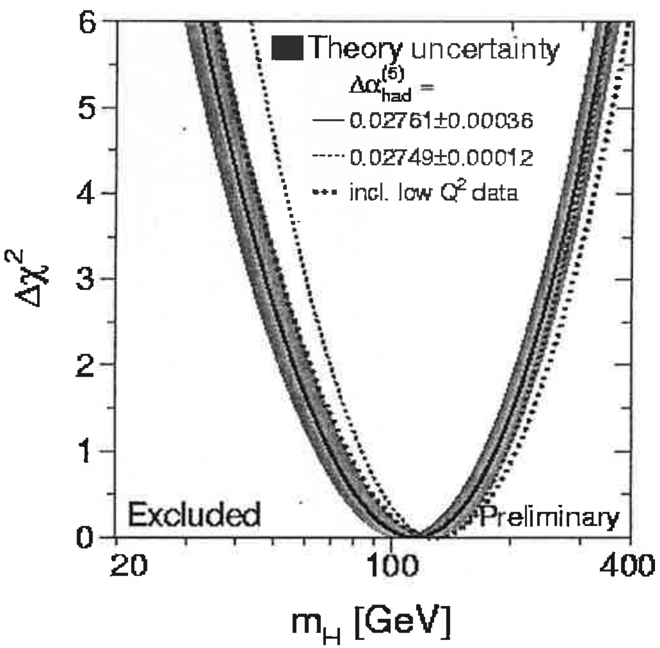
- Minimal impact of NuTeV anomaly.

- Recall: $M_W^2 = M_Z^2 \cos^2\theta_W + \mathcal{O}(m_t^2) - \mathcal{O}\left(\log \frac{M_H^2}{M_W^2}\right)$.

- Compare to 2004 plot (larger m_t, limited Run 2 stats)

$$M_H^{\rm SM} < 260 \text{ GeV} \quad \text{at } 95\% \text{ CL } (\Delta\chi^2 = 2.7)$$

(LEPEWWG+LEPHWG Winter'04)



- Dramatic shift downwards of best M_H fit and upper limit !

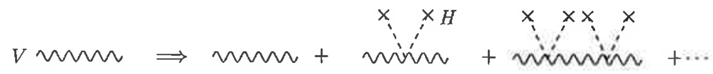
Higgs Boson search

Outline

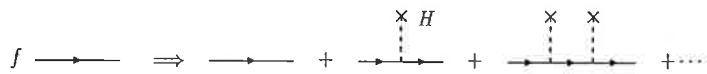
- The Higgs mechanism
- The Higgs picture
- The Higgs profile
- Collider searches

The Higgs mechanism

- Unitarity of EW interactions requires existent of a scalar Higgs field which couples to other particles proportional to their mass.
- In Higgs sector of theory, scalar fields interact with each other in such a way that ground state acquires a non-zero field strength, breaking EW symmetry spontaneously.
- Masses of gauge bosons V and fermions f build up by (infinitely) repeated interactions with the background Higgs field.
- Such interactions masses from zero to finite values:

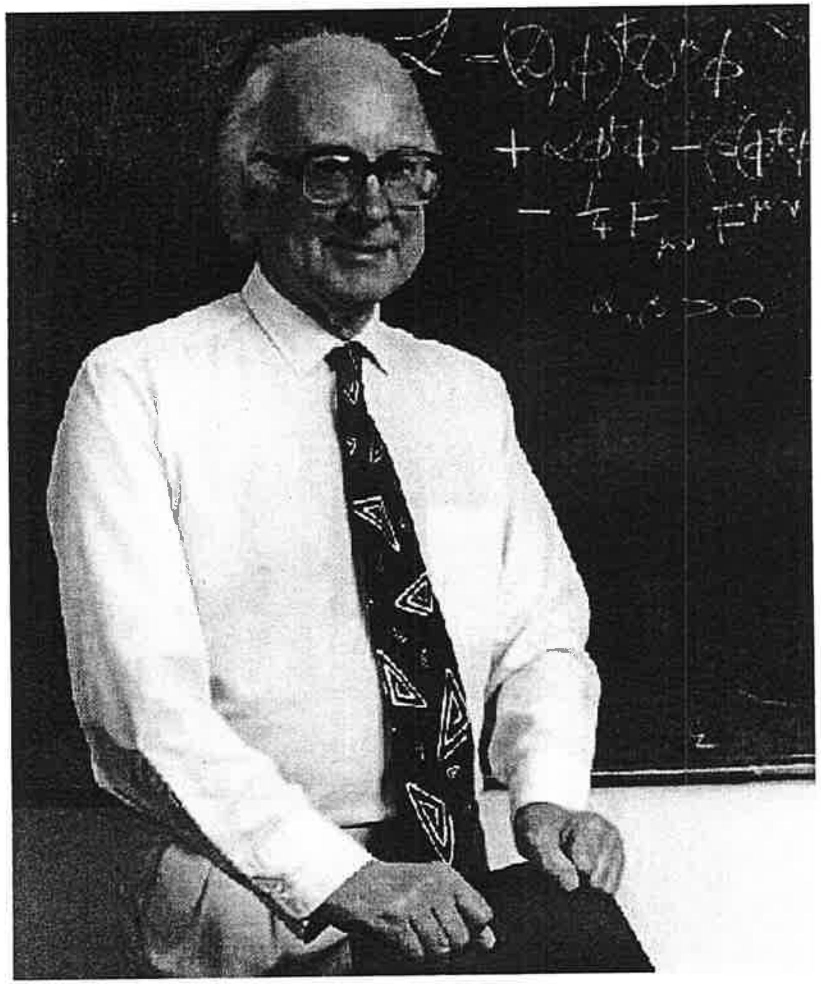


$$\frac{1}{q^2} \Rightarrow \frac{1}{q^2} + \sum_j \frac{1}{q^2}\left[\left(\frac{g_W v}{\sqrt{2}}\right)^2 \frac{1}{q^2}\right]^j = \frac{1}{q^2 - M_V^2} \; : \; M_V^2 = g_W^2 \frac{v^2}{4}$$



$$\frac{1}{\not{q}} \Rightarrow \frac{1}{\not{q}} + \sum_j \frac{1}{\not{q}}\left[\frac{g_f v}{\sqrt{2}} \frac{1}{\not{q}}\right]^j = \frac{1}{\not{q} - m_f} \; : \; m_f = g_f \frac{v}{\sqrt{2}}$$

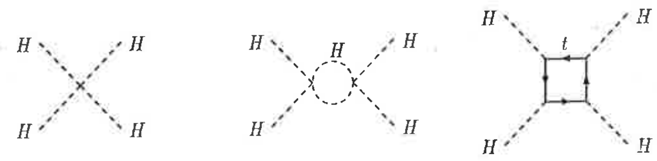
The Higgs picture



The Higgs profile

- M_H from curvature of self-energy potential V, $M_H^2 = \lambda v^2$.
- SM cannot predicted it since quartic coupling λ is unknown.
- Nevertheless, restrictive bounds on M_H follow from hypothetical assumptions on energy scale Λ up to which SM is valid before NP emerges.

$\rightarrow$ quantum fluctuations introduce energy dependence $\lambda(\mu)$.



- $\lambda(\mu)$ running from renormalisation group equation (RGE):

$$\frac{d\lambda}{d\ln \mu^2} = \frac{3}{8\pi^2}[\lambda^2 + \lambda g_t^2 - g_t^4]$$

with $\lambda(v^2) = M_H^2/v^2$ and $g_t(v^2) = \sqrt{2}m_t/v$.

- For moderate m_t large M_H,

$$d\lambda/d\ln \mu^2 \sim +\lambda^2,$$

and becomes strong shortly before Landau pole:

$$\lambda(\mu^2) = \frac{\lambda(v^2)}{1 - \frac{3\lambda(v^2)}{8\pi^2} \ln \frac{\mu^2}{v^2}}.$$

- Requirement SM be perturbative up to a scale Λ: $\lambda(\Lambda) < \infty$.
- Can be translated into upper bound on M_H

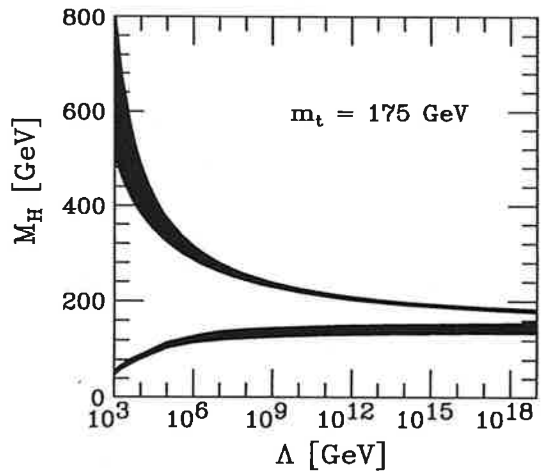
$$M_H^2 \lesssim \frac{8\pi^2 v^2}{3 \ln(\Lambda^2/v^2)}$$

- Lower bound on M_H derived from requirement of vacuum stability:
$\rightarrow$ top-loop corrections reduce λ for increasing m_t, λ becomes negative if m_t too large.

- Self-energy potential would become deeply negative and ground state would no longer be stable.

- To avoid instability, M_H must exceed minimal value for given m_t to balance negative contribution.

- Lower bound depends on cut-off value Λ.



- For $m_t = 175$ GeV:

Λ	M_H
1 TeV	55 GeV $\leq M_H \leq$ 700 GeV
10^{19} GeV	130 GeV $\leq M_H \leq$ 190 GeV

- If SM valid up to grand unification theory (GUT) scale, 130 GeV $< M_H <$ 190 GeV !

- Observation of M_H outside this range would demand a new strong interaction scale below GUT scale.

Unitarity: longitudinal gauge boson scattering cross section at high energy grows with M_H.

Electroweak Equivalence Theorem:
in the high energy limit ($s \gg M_V^2$)

$$\mathcal{A}(V_L^1 \ldots V_L^n \to V_L^1 \ldots V_L^m) = (i)^n(-i)^m \mathcal{A}(\omega^1 \ldots \omega^n \to \omega^1 \ldots \omega^m) + O\left(\frac{M_V^2}{s}\right)$$

(V_L^i=longitudinal weak gauge boson; ω^i=associated Goldstone boson).

Example: $\boxed{W_L^+ W_L^- \to W_L^+ W_L^-}$

$$\mathcal{A}(W_L^+ W_L^- \to W_L^+ W_L^-) \sim -\frac{1}{v^2}\left(-s - t + \frac{s^2}{s - M_H^2} + \frac{t^2}{t - M_H^2}\right)$$

$$\mathcal{A}(\omega^+\omega^- \to \omega^+\omega^-) = -\frac{M_H^2}{v^2}\left(\frac{s}{s - M_H^2} + \frac{t}{t - M_H^2}\right)$$

$$\Downarrow$$

$$\mathcal{A}(W_L^+ W_L^- \to W_L^+ W_L^-) = \mathcal{A}(\omega^+\omega^- \to \omega^+\omega^-) + O\left(\frac{M_W^2}{s}\right)$$

Using partial wave decomposition:

$$\mathcal{A} = 16\pi \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)a_l$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|\mathcal{A}^2| \longrightarrow \sigma = \frac{16\pi}{s}\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{1}{s}\mathrm{Im}\left[\mathcal{A}(\theta=0)\right]$$

$$\Downarrow$$

$$\boxed{|a_l|^2 = \mathrm{Im}(a_l)} \longrightarrow \boxed{|\mathrm{Re}(a_l)| \leq \tfrac{1}{2}}$$

Most constraining condition for $W_L^+ W_L^- \to W_L^+ W_L^-$ from

$$a_0(\omega^+\omega^- \to \omega^+\omega^-) = -\frac{M_H^2}{16\pi v^2}\left[2 + \frac{M_H^2}{s - M_H^2} - \frac{M_H^2}{s}\log\left(1 + \frac{s}{M_H^2}\right)\right) \xrightarrow{s \gg M_H^2} -\frac{M_H^2}{8\pi v^2}$$

$$\boxed{|\mathrm{Re}(a_0)| < \tfrac{1}{2}} \longrightarrow \boxed{M_H < 870 \text{ GeV}}$$

Best constraint from coupled channels ($2W_L^+ W_L^- + Z_L Z_L$):

$$a_0 \xrightarrow{s \gg M_H^2} -\frac{5M_H^2}{32\pi v^2} \longrightarrow \boxed{M_H < 780 \text{ GeV}}$$

Observe that: if there is no Higgs boson, i.e. $M_H \gg s$:

$$a_0(\omega^+\omega^- \to \omega^+\omega^-) \overset{M_H^2 \gg s}{\longrightarrow} -\frac{s}{32\pi v^2}$$

Imposing the unitarity constraint $\longrightarrow$ $\boxed{\sqrt{s_c} < 1.8 \text{ TeV}}$

Most restrictive constraint $\longrightarrow$ $\boxed{\sqrt{s_c} < 1.2 \text{ TeV}}$

$\Downarrow$

New physics expected at the TeV scale

Exciting !!
this is the range of energies of both Tevatron and LHC

Triviality: a $\lambda\phi^4$ theory cannot be perturbative at all scales unless $\lambda = 0$.

In the SM the scale evolution of λ is more complicated:

$$32\pi^2 \frac{d\lambda}{dt} = 24\lambda^2 - (3g'^2 + 9g^2 - 24y_t^2)\lambda + \frac{3}{8}g'^4 + \frac{3}{4}g'^2g^2 + \frac{9}{8}g^4 - 24y_t^4 + \cdots$$

($t = \ln(Q^2/Q_0^2)$, $y_t = m_t/v \to$ top quark Yukawa coupling).

Still, for large λ ($\leftrightarrow$ large M_H) the first term dominates and (at 1-loop):

$$\lambda(Q) = \frac{\lambda(Q_0)}{1 - \frac{3}{4\pi^2}\lambda(Q_0)\ln\left(\frac{Q^2}{Q_0^2}\right)}$$

$\boxed{\text{when } Q \text{ grows}}$ $\longrightarrow$ $\boxed{\lambda(Q) \text{ hits a pole} \to \text{triviality}}$

Imposing that $\lambda(Q)$ is finite, gives a scale dependent bound on M_H:

$$\boxed{\frac{1}{\lambda(\Lambda)} > 0 \quad \longrightarrow \quad M_H^2 < \frac{8\pi^2 v^2}{3\log\left(\frac{\Lambda^2}{v^2}\right)}}$$

where we have set $Q \to \Lambda$ and $Q_0 \to v$.

Vacuum stability: $\boxed{\lambda(Q) > 0}$

For small λ ($\leftrightarrow$ small M_H) the last term in $d\lambda/dt = \ldots$ dominates and:

$$\lambda(\Lambda) = \lambda(v) - \frac{3}{4\pi^2} y_t^2 \log\left(\frac{\Lambda^2}{v^2}\right)$$

from where a first rough lower bound is derived:

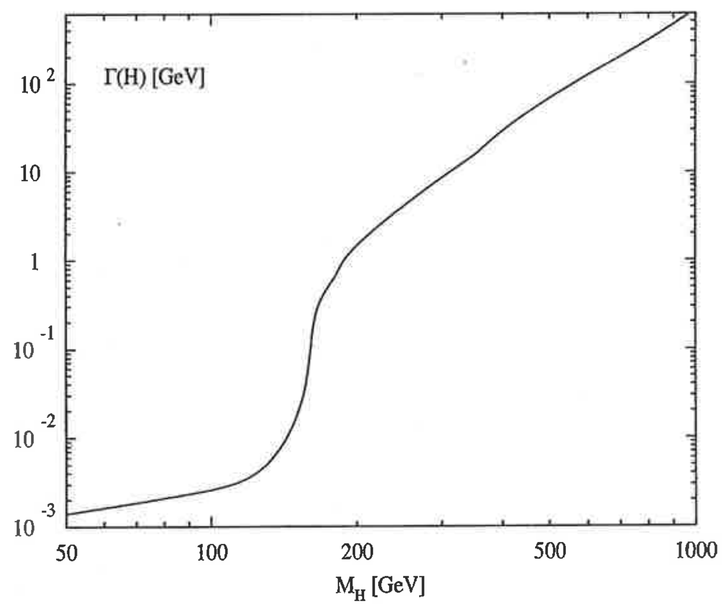
$$\boxed{\lambda(\Lambda) > 0 \quad \longrightarrow \quad M_H^2 > \frac{3v^2}{2\pi^2} y_t^2 \log\left(\frac{\Lambda^2}{v^2}\right)}$$

More accurate analyses use 2-loop renormalization group improved V_{eff}.

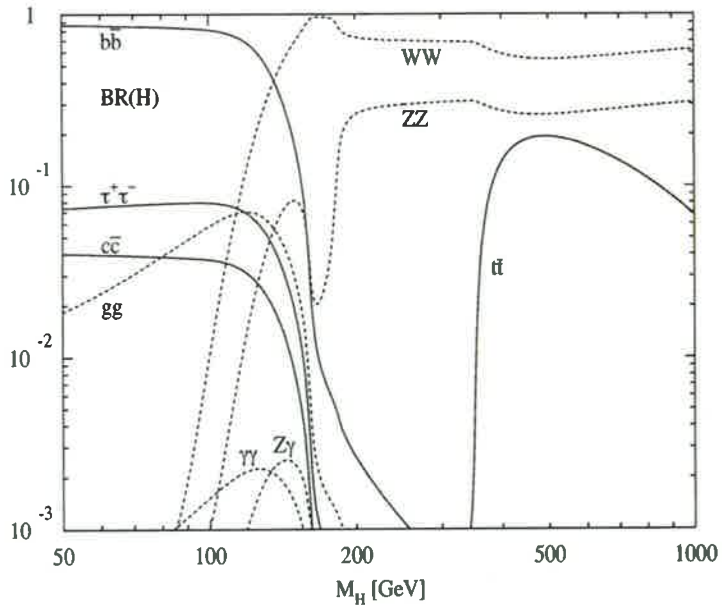
- H couplings to EW gauge bosons and fermions:

$$g_{ffH} = \left[\sqrt{2}G_F\right]^{1/2} m_f \,,$$
$$g_{VVH} = 2\left[\sqrt{2}G_F\right]^{1/2} M_V^2 \,.$$

- Γ_H and Branching Ratios (BRs) determined by these:



Collider searches



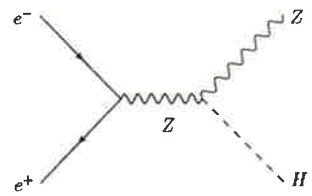
- $M_H \geq 250$ GeV: Higgs too wide to resolve experimentally.
- Best decay channel depends on collider environment:

1. leptonic/photonic decays needed at hadron colliders !

2. can also use hadronic decays at lepton colliders !

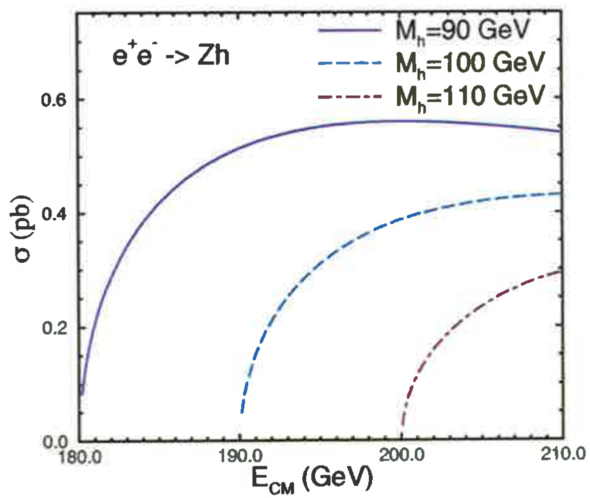
- (Muon colliders could scan resonance, like LEP with Z).

- LEP, SLC:



Higgs-strahlung

- Inclusive Higgs cross section (LEP2):



- Look for $H \to b\bar{b}$ (use b-tagging) and $Z \to X$: no evidence,

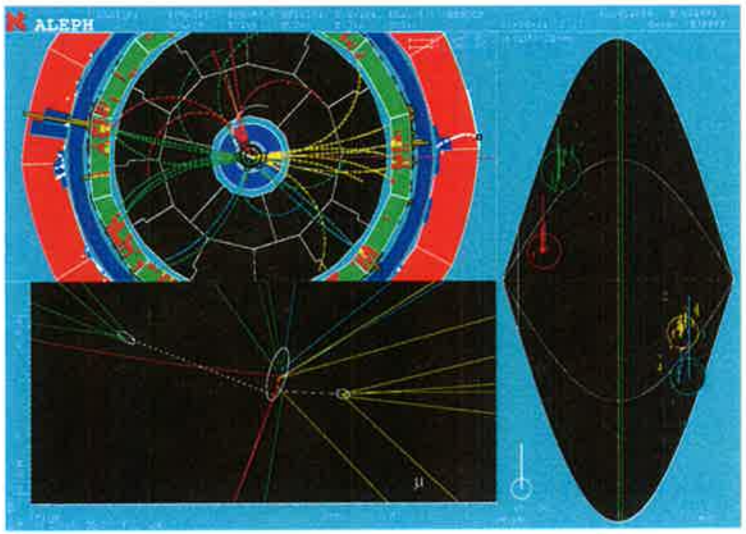
$$M_H < 114.4 \text{ GeV} \quad \text{at } 95\%\,\text{CL} \to \text{ fig}$$

- Small excess can be interpreted as production of a SM Higgs boson with $M_H \approx 115$ GeV. → fig

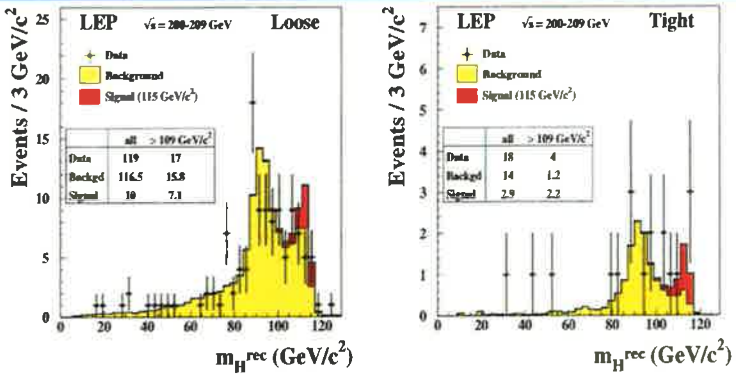
- Significance insufficient to claim Higgs observation/discovery.

- Most candidates from four-jet final states:

$$e^+e^- \to Z(\to q\bar{q})H(\to b\bar{b}).$$



Data v. expected signal & background



	all	> 109 GeV/c²
Data	119	17
Backgd	116.5	15.8
Signal	10	7.1

	all	> 109 GeV/c²
Data	18	4
Backgd	14	1.2
Signal	2.9	2.2

LEP Higgs WG conclusions:

statistical analysis: signal at 1.7 standard dev., corresponding to $M_H \simeq 116$ GeV

- Hadron colliders, Tevatron and Large Hadron Collider (LHC):

(a) gluon fusion : $gg \to H$

(b) WW, ZZ fusion : $W^+W^-, ZZ \to H$

(c) Higgs-strahlung off W, Z : $q\bar{q} \to W, Z \to W, Z + H$

(d) Higgs bremsstrahlung off b, t : $q\bar{q}, gg \to (b\bar{b})t\bar{t} + H$

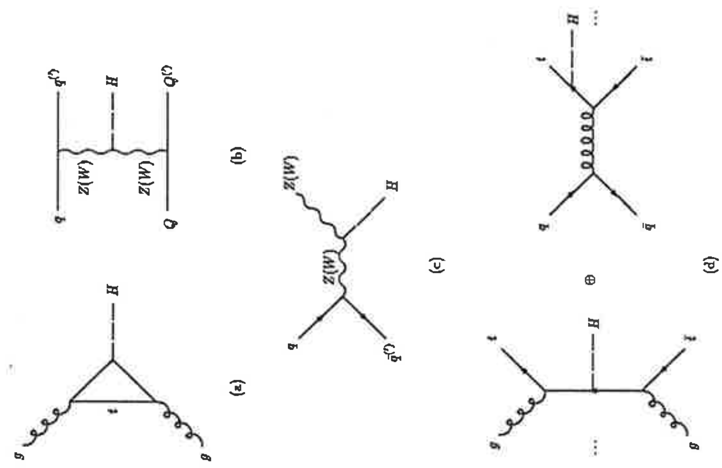


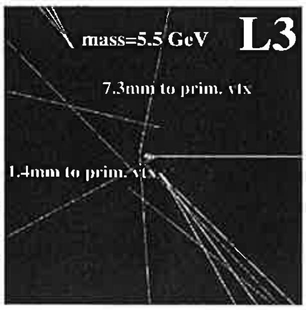
Fig. 4

Missing energy channel

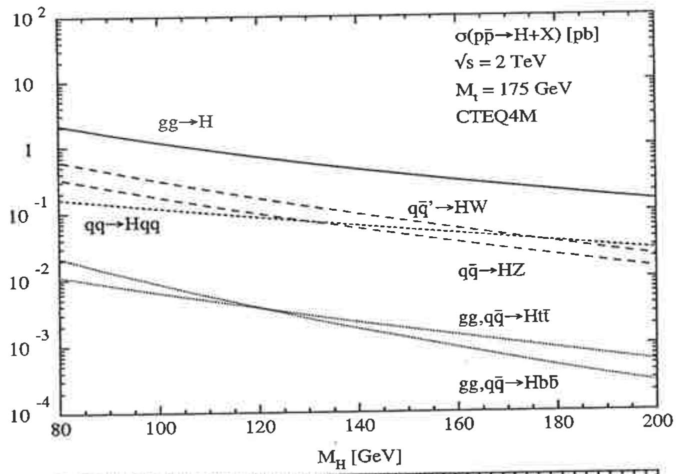
most significant Hνν candidate

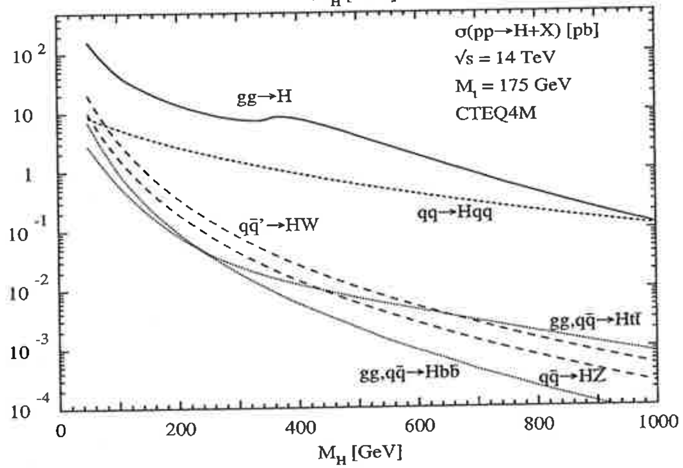
measured H mass=115 GeV
H mass resolution ~3 GeV

Secondary vtx's view



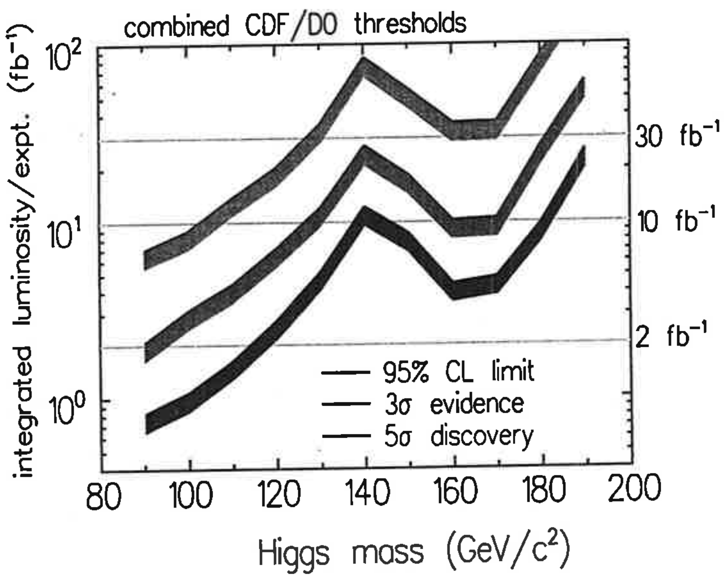
• Inclusive production cross sections:





• Sensitivity for Higgs searches at Tevatron best for Higgs-strahlung with $H \to b\bar{b}$ and $Z \to \ell^+\ell^-$ (other channels too). $\to$ figs

• Integrated luminosity per experiment required for SM Higgs boson exclusion and evidence, as function of M_H.



(Note: legends in reverse order with respect to curves.)

• Can do better than LEP ? (Luminosity problems ?)

Final State Modes and Backgrounds

Signal Production and Final State:	Primary Background Processes:
$gg \to H \to b\bar{b}$	QCD Dijet Background…Huge
$p\bar{p} \to WH \to q\bar{q}'b\bar{b}$	QCD Jet Background/W+jets
$p\bar{p} \to WH \to \nu b\bar{b}$	W+$b\bar{b}$/$c\bar{c}$, Single top, $t\bar{t}$
$p\bar{p} \to ZH \to q\bar{q}b\bar{b}$	QCD Jet Background/W+jets
$p\bar{p} \to ZH \to {}^{+}\;{}^{-}b\bar{b}$	W/Z+$b\bar{b}$/$c\bar{c}$, $t\bar{t}$ (Poor BR)
$p\bar{p} \to ZH \to \nu\bar{\nu}b\bar{b}$	W/Z+$b\bar{b}$/$c\bar{c}$, $t\bar{t}$, QCD Jets

Essentials:
Lepton Acceptance, b-tagging eff/Acceptance, dijet Mass Resolution

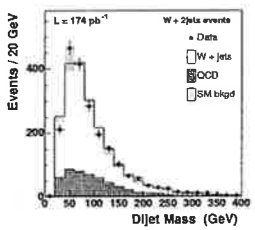
Event Rates/fb^{-1}

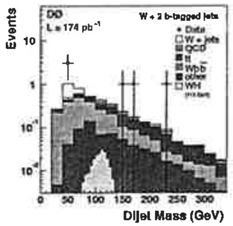
Rates determined from a combination of MC and data.

	No Mass Window	Mass Window
WH Signal(115)	1.7	1.5
ZH Signal(115)	2.5	2.3
Total Signal	4.2	3.8
tt	8.8	2.2
t(W*)	3.3	0.7
t(Wg)	2.4	0.5
W/Z bb	22.3	3.3
WZ/ZZ	16.5	2.7
QCD	61.2	10.2
Total Bkg	114	19.6
$S/\sqrt{B}$	0.39	0.85
S/B	0.037	0.19

Missed Chg Lepton

Leptons + jets: WH→eνbb

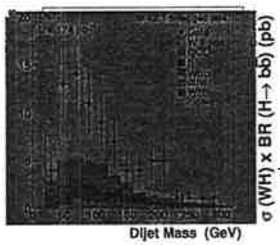


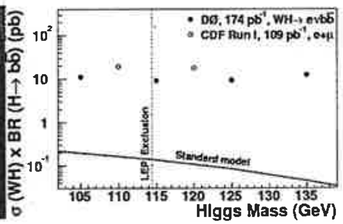


DØ : 6 double b-tag events in 174 pb⁻¹
spectra well described by SM

- σ(Wbb) < 6.6 pb (for P_T(b)>20 GeV, ΔR(bb) > 0.75)
- σ(WH) < 9.0-12.2 pb (for M_H=105-135 GeV)

hep-ex/0410062





Stefan Grünendahl Tevatron Searches @ Aspen 2005

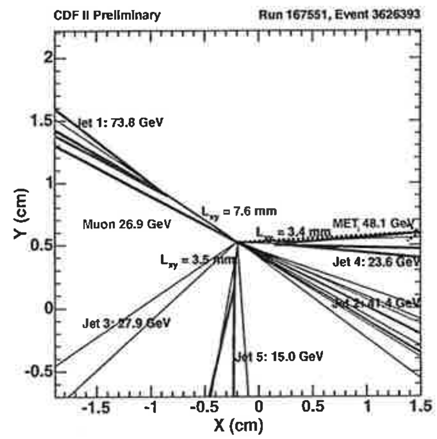
15

- Now CDF can do also:

$ttH \to lvjjbbbb$ (lepton, missing E_T, ≥ 5 jets with ≥ 3 b-tag)

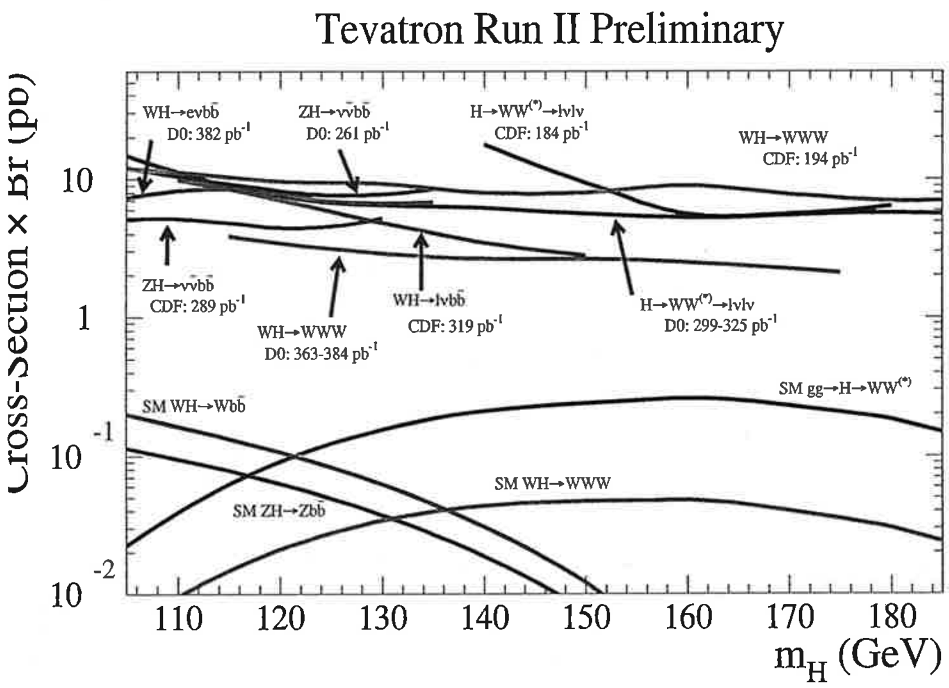
Observed Event - Kinematics

Event with an identified, central muon and three b-tagged jets:

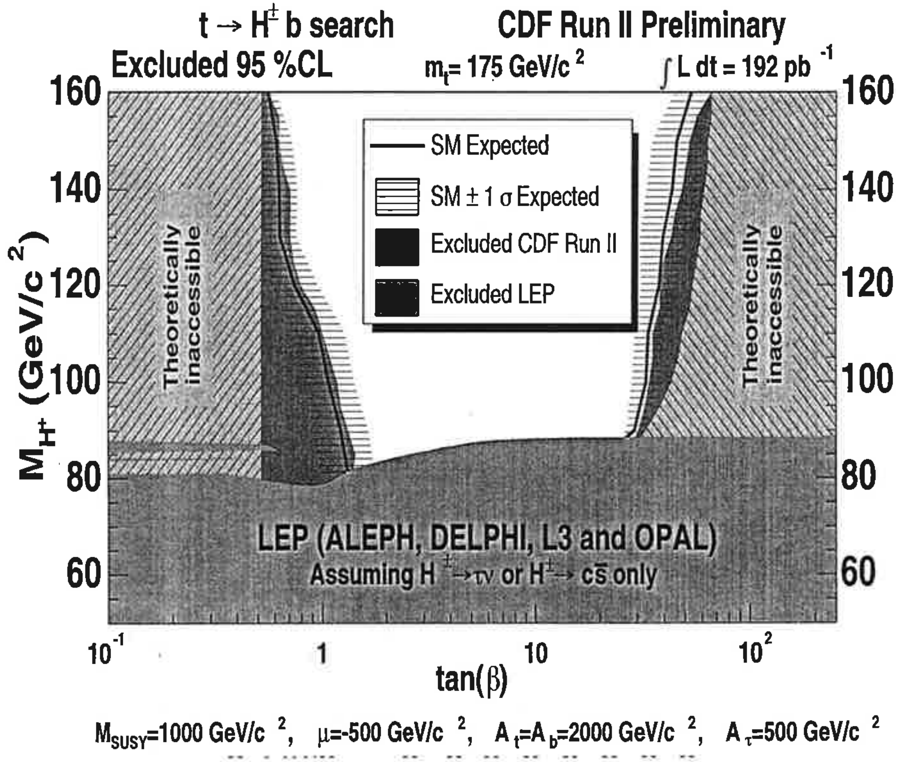


(APS06)

9



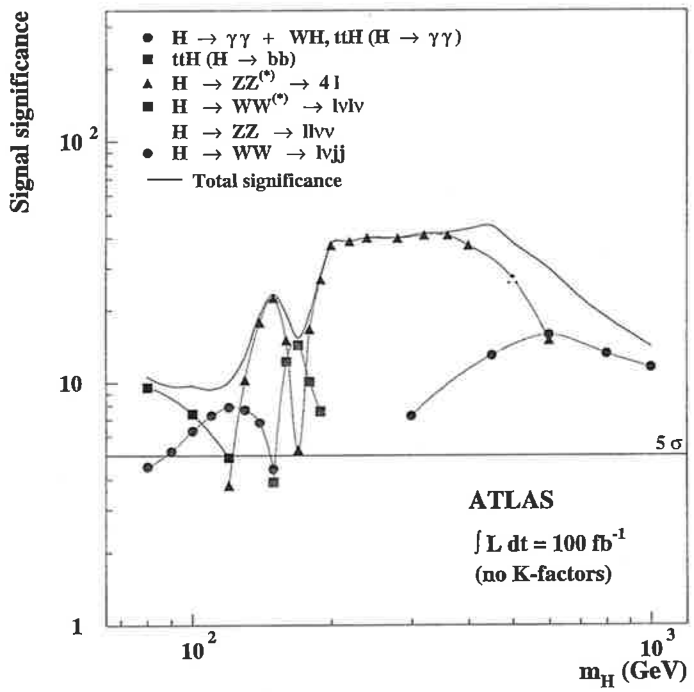
Tevatron Run II Preliminary
Cross-Section × Br (pb)
WH→eνbb̄ D0: 382 pb⁻¹
ZH→ννbb̄ D0: 261 pb⁻¹
H→WW(*)→lνlν CDF: 184 pb⁻¹
WH→WWW CDF: 194 pb⁻¹
ZH→ννbb̄ CDF: 289 pb⁻¹
WH→WWW D0: 363-384 pb⁻¹
WH→lνbb̄ CDF: 319 pb⁻¹
H→WW(*)→lνlν D0: 299-325 pb⁻¹
SM WH→Wbb̄
SM gg→H→WW(*)
SM WH→WWW
SM ZH→Zbb̄
m_H (GeV)

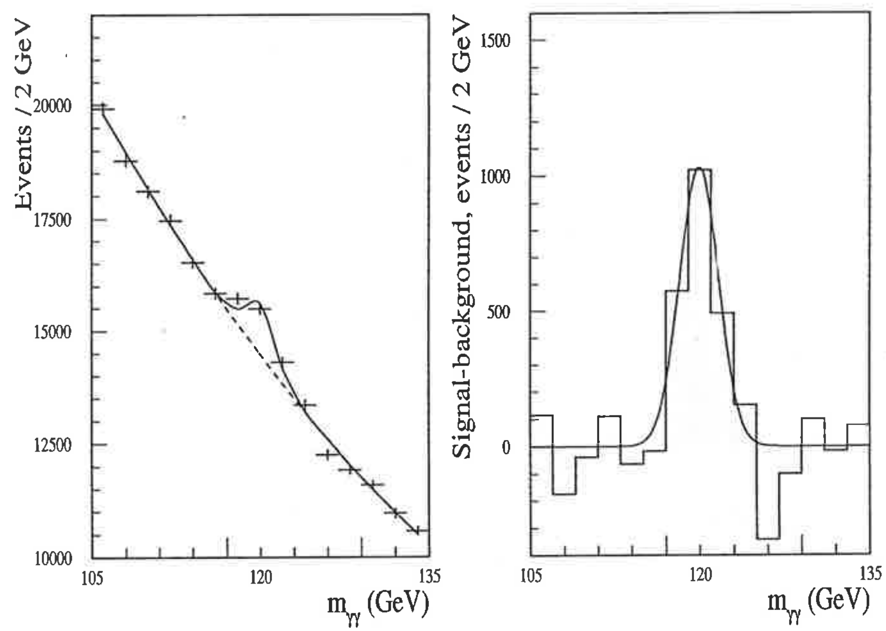


t → H± b search
CDF Run II Preliminary
Excluded 95 %CL
m_t= 175 GeV/c²
∫L dt = 192 pb⁻¹
SM Expected
SM ± 1 σ Expected
Excluded CDF Run II
Excluded LEP
Theoretically inaccessible
LEP (ALEPH, DELPHI, L3 and OPAL)
Assuming H±→τν or H±→ cs̄ only
M_H± (GeV/c²)
tan(β)
M_SUSY=1000 GeV/c², μ=-500 GeV/c², A_t=A_b=2000 GeV/c², A_τ=500 GeV/c²

• Huge QCD background at LHC requires triggering on leptonic decays of W, Z and t and exploiting $H \to \gamma\gamma$ and $H \to ZZ \to 4\ell^{\pm}$ resonances. $\to$ fig

• Expected 5σ discovery limits from ATLAS:

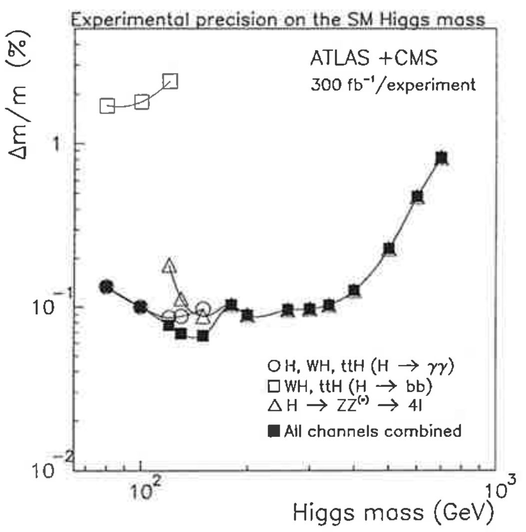




• LHC expected to cover entire canonical SM Higgs mass range: $M_H \lesssim 700$ GeV !

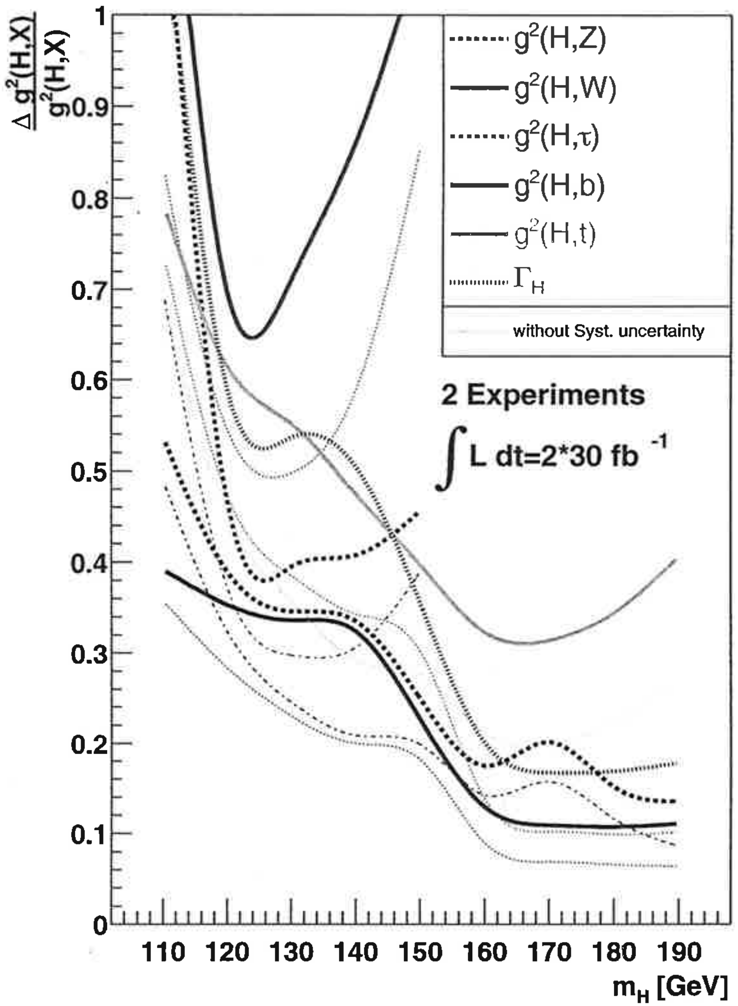
- To firmly establish Higgs nature need measuring:

1. mass:

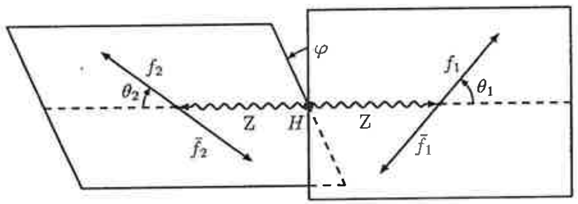


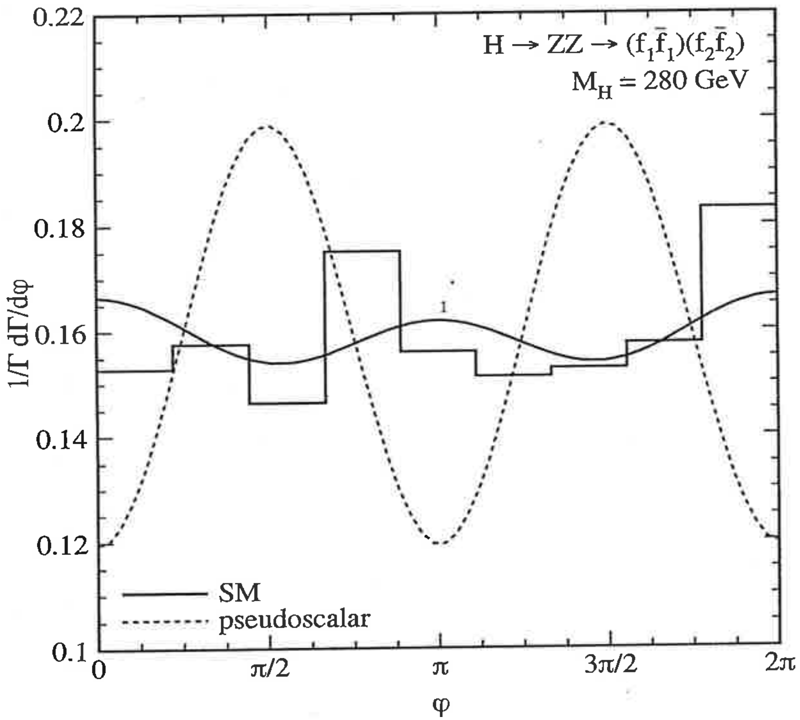
2. possibly lifetime/width
3. couplings to gauge bosons and fermions $\rightarrow$ fig
4. Higgs self-couplings
5. spin/parity quantum numbers (e.g., $H \rightarrow ZZ \rightarrow 4\ell$) $\rightarrow$ fig

Higgs couplings at LHC



Higgs Parity at LHC

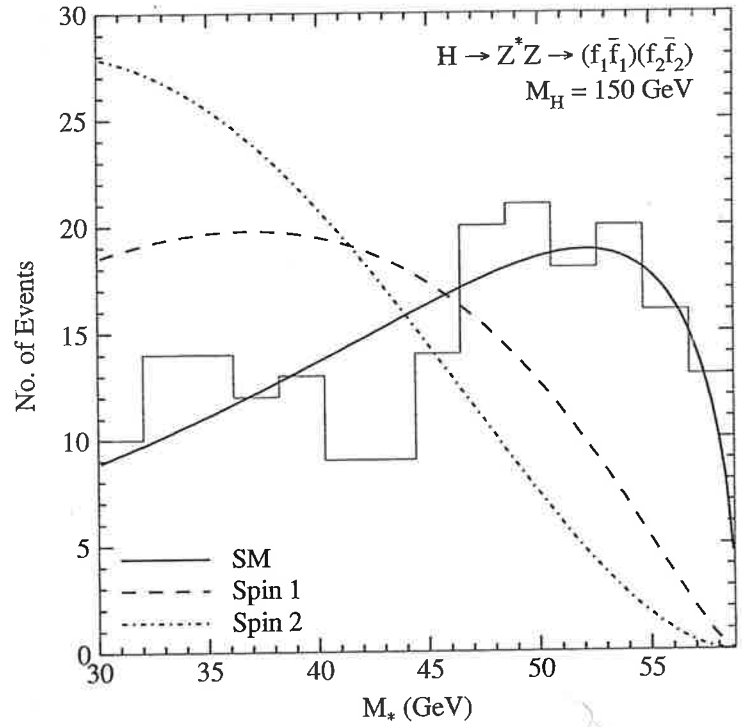




Higgs Spin at LHC

$$\frac{d\Gamma_H}{dM_*^2} = \frac{3G_F^2 M_Z^4 \delta_Z}{16\pi^3 M_H} \frac{12M_*^2 M_Z^2 + M_H^4 \beta^2}{(M_*^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \beta$$

where β is the Z^*/Z three-momentum in H rest frame in units of Higgs particle mass M_H.



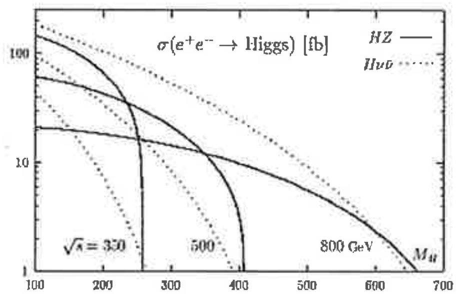
Example of Precision of Higgs Measurements at The next Linear Collider

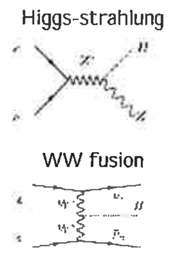
For M_H = 140 GeV, 500 fb^{-1} @ 500 GeV

Mass Measurement	$\delta M_H \approx 60$ MeV $\approx 5 \times 10^{-4} M_H$
Total width	$\delta \Gamma_H / \Gamma_H \approx 3$ %
Particle couplings	
tt	(needs higher √s for 140 GeV, except through H → gg)
bb	$\delta g_{Hbb} / g_{Hbb} \approx 2$ %
cc	$\delta g_{Hcc} / g_{Hcc} \approx 22.5$ %
$\tau^+\tau^-$	$\delta g_{H\tau\tau} / g_{H\tau\tau} \approx 5$ %
WW*	$\delta g_{HWW} / g_{HWW} \approx 2$ %
ZZ*	??
gg	$\delta g_{Hgg} / g_{Hgg} \approx 12.5$ %
γγ	$\delta g_{H\gamma\gamma} / g_{H\gamma\gamma} \approx 10$ %
Spin-parity-charge conjugation	establish $J^{PC} = 0^{++}$
Self-coupling	$\delta\lambda_{HHH} / \lambda_{HHH} \approx 32$ % (statistics limited)

If Higgs is lighter, precision is often better

The Higgs Production Cross section at The next Linear Collider

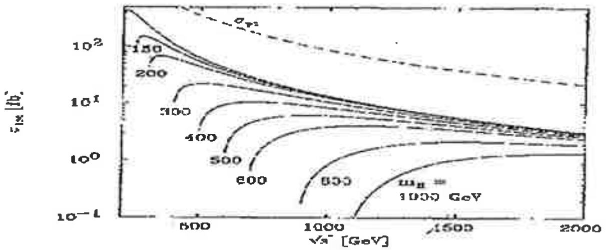




Recall, σ_{pt} = **87** nb / $(E_{cm})^2$ ~ **350** fb @ 500 GeV

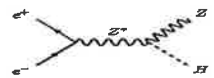
The Higgs Production Cross section at The next Linear Collider

Higgs-strahlung



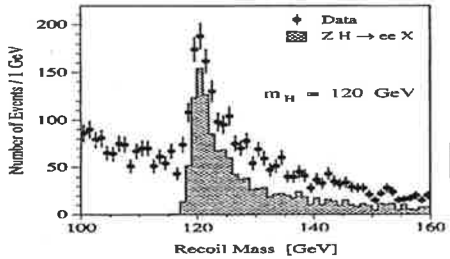
$e^+e^- \to ZH \to l^+l^- X$ @ 500 GeV	
M_H (GeV)	events/ 500 fb^{-1}
120	2020
140	1910
160	1780
180	1650
200	1500
250	1110

Higgs studies - The power of simple reactions



The LC can produce the Higgs recoiling from a Z, with known CM energy⇓, which provides a powerful channel for unbiassed tagging of Higgs events, allowing measurement of even invisible decays (⇓ - some beamstrahlung)

- Tag Z $l^+ l$
- Select $M_{recoil} = M_{Higgs}$



500 fb^{-1} @ 500 GeV, TESLA TDR, Fig 2.1.4

Invisible decays are included

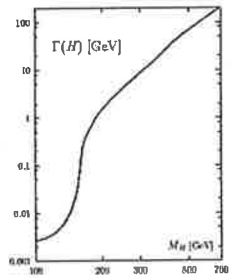
Higgs studies - The Mass Measurement

M_H	δM_H(Recoil)	δM_H(Recon & fit)
120 GeV		40 MeV (3.3×10^{-4})
150 GeV	90 MeV	70 MeV (2×10^{-4})
180 GeV	100 MeV	80 MeV (4×10^{-4})

500 fb^{-1} @ 350 GeV, TESLA TDR, Table 2.2.1

J.Brau, Snowmass, July 3, 2001 14

Total width



$\Gamma_{TOT} = \Gamma_X / BR(H \to X)$

- $BR(H \to WW^*) = \Gamma_{WW} / \Gamma_{TOT}$
- Γ_{WW} from WW fusion cross section

M_H	WW fusion	Higgs-strahlung
120 GeV	6.1%	5.6%
140 GeV	4.5%	3.7%
160 GeV	13.4%	3.6%

500 fb^{-1} @ 350 GeV, TESLA TDR, Table 2.2.4

Γ_{TOT} to few%

J.Brau, Snowmass, July 3, 2001 15

Higgs Couplings - the branching ratios

M_H	$H \to b\bar{b}$	$H \to c\bar{c}$	$H \to gg$	$H \to \tau^+\tau^-$
120 GeV	2.9 %	39 %	18 %	7.9 %
140 GeV	4.1 %	45 %	23 %	10 %

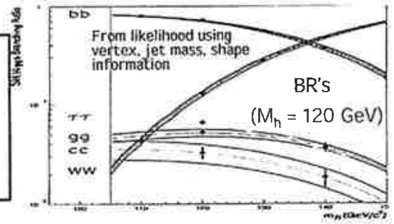
(through Higgs-strahlung, only)

500 fb^{-1} @ 500 GeV, LC Physics Resource Book, Table 3.1

At lower energy, including $e^+e^- \to H\nu\nu$, along with $e^+e^- \to ZH$

M_H	$H \to bb$	$H \to cc$	$H \to gg$	$H \to \tau^+\tau^-$
120 GeV	2.4 %	8.3 %	5.5 %	5.0 %
140 GeV	2.6 %	19.0 %	14.0 %	8.0 %
160 GeV	6.5 %			

500 fb^{-1} @ 350 GeV, TESLA TDR, Table 2.2.5



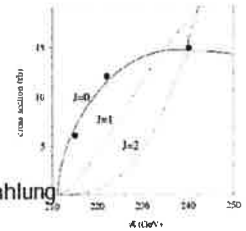
Measurement of BR's is powerful indicator of new physics

e.g. in MSSM, these differ from the SM in a characteristic way. Higgs BR must agree with MSSM parameters from many other measurements.

Higgs spin parity and charge conjugation (J^{PC})

$H \to \gamma\gamma$ or $\gamma\gamma \to H$ rules out J=1 and indicates C=+1

Threshold cross section ($e^+e^- \to ZH$) for J=0
$\sigma \sim \beta$, while for J > 0, generally higher power of β (assuming $n = (-1)^J P$)

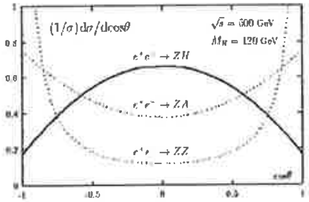


LC Physics Resource Book, Fig 3.23(a)

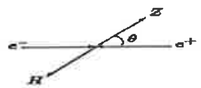
Production angle (θ) and Z decay angle in Higgs-strahlung reveals J^P ($e^+e^- \to Z H \to f\bar{f}H$)

	$J^P = 0^+$	$J^P = 0^-$
$d\sigma/d\cos\theta$	$\sin^2\theta$	$(1 - \sin^2\theta)$
$d\sigma/d\cos\phi$	$\sin^2\phi$	$(1 +/- \cos\phi)^2$

ϕ is angle of the fermion, relative to the Z direction of flight, in Z rest frame



TESLA TDR, Fig 2.2.8



Higgs self couplings

Measures Higgs potential λ

$V(\Phi)=\lambda(\Phi^2-\frac{1}{2}v^2)^2$ $v \sim$ 246 GeV

$m_h^2=4\lambda v^2$

double Higgs-strahlung: $e^+e^- \to Zhh$

Study ZHH production and decay to 6 jets (4 b's). Cross section is small; premium on very good jet energy resolution. Can enhance x5 with positron polarization.

m_h (GeV/c²)	σ_{hhZ} (fb)	N^{500}_{hhZ}	ϵ_{hhZ}	$\mathcal{L}$= 500 fb⁻¹	1000 fb⁻¹	2000 fb⁻¹
120	0.186	93.	43%	24.1%	17.3%	11.6%
130	0.149	74.	43%	26.6%	19%	17.7%
140	0.115	57.	39%	32%	23 %	17%

Δλ/λ error 36% → 18%

Supersymmetry

Outline

- Why Supersymmetry ?
- The hierarchy problem and gauge coupling unification
- The Minimal Supersymmetric Standard Model (MSSM)
- Indirect Searches: $g-2$
- Collider Searches

Why supersymmetry?

- No direct experimental evidence of SUSY exists to date ...
- Nonetheless, prime candidate for BSM physics !

1. SUSY is a generalisation of space-time symmetries of QFT transforming fermions into bosons and vice versa.
2. It also provides a framework for unification of particle physics and gravity, at a scale $M_{\text{Planck}} \approx 10^{19}$ GeV.
3. If SUSY were an exact symmetry of Nature, particles and superpartners would be degenerate in mass.
4. Thus, SUSY cannot be an exact symmetry of nature: if it is realised, it must be <u>broken</u>.
5. Crucial question for phenomenology: at what scale might SUSY be broken ? Is there a reason why masses of superpartners should not be as heavy as M_{Planck} ?

- Most compelling motivation for existence of TeV scale SUSY particles at TeV scale is linked to so-called hierarchy problem: instability of Higgs mass under quadratically divergent radiative corrections.
- Additional support for TeV scale SUSY comes from unification of gauge couplings in SUSY GUTs.

- <u>Hierarchy problem</u>: why is Higgs mass so much smaller than Planck scale ?

Quantum corrections to Higgs mass are quadratically divergent as internal momentum in loops becomes very large.

$\rightarrow \delta m_H^2 \sim \frac{\alpha}{\pi}(\Lambda^2 + m_F^2)$

- Cutoff Λ represents scale up to which SM remains valid.
- If $\Lambda \sim M_{\text{Planck}}$ extreme fine-tuning between bare Higgs mass and quantum fluctuations δM_H^2 would be needed to generate a physical Higgs mass of order $\mathcal{O}(100)$ GeV.

- Most elegant solution is to introduce additional symmetry that transforms fermions into bosons: SUSY !
- <u>Pauli's principle</u>: additional Higher Order (HO) corrections due to superpartners enter δM_H^2 with sign opposite to SM contributions

$\rightarrow$ divergent terms cancel:

$\rightarrow \delta m_H^2 \sim -\frac{\alpha}{\pi}(\Lambda^2 + \widetilde{m}_F^2)$

- Since $\delta m_H^2 \sim \frac{\alpha}{\pi}(m_F^2 - \widetilde{m}_F^2)$, any fine-tuning is avoided for SUSY particle (sparticle) masses $\widetilde{m} \lesssim \mathcal{O}(1\text{ TeV})$.
- Argument is qualitative and does not tell where SUSY is !
- In renormalisable theories all infinities can be absorbed in bare parameters: one might not need worry about SM fine-tuning.
- Argument tells that large hierarchy is intrinsically unstable:

$\rightarrow$ SUSY very plausible way of stabilising it !

Coupling unification

- Additional support for SUSY $\tilde{m} \lesssim \mathcal{O}(1\ \text{TeV})$ follows from gauge coupling unification.
- GUTs seek gauge group including $SU(3)$, $SU(2)$ and $U(1)$.
- Apparent obstacle is $\alpha_s \gg \alpha$ at EW scale, yet quantum corrections introduce energy dependence:

$$\frac{d\alpha_i(\mu)}{d\ln\mu^2} = \beta_i(\alpha_i(\mu)) \qquad \beta_i = -\beta_{i,0}\alpha_i^2 + \mathcal{O}(\alpha_i^3).$$

- β-functions depend on gauge group and matter multiplets to which gauge bosons couple.
- Only particles with mass $< \mu$ contribute to β_i and to coupling evolution at $Q \leq \mu$.
- SM couplings evolve with μ according to

$$\begin{aligned} SU(3) &: \beta_{3,0} = (33 - 4n_g)/(12\pi) \\ SU(2) &: \beta_{2,0} = (22 - 4n_g - n_h/2)/(12\pi) \\ U(1) &: \beta_{1,0} = (-4n_g - 3n_h/10)/(12\pi) \end{aligned}$$

($n_g = 3$ quark/lepton generations; $n_h = 1$ Higgs doublets).

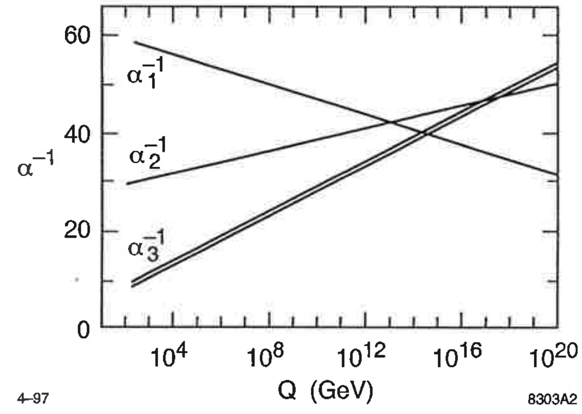
- $SU(3)$, $SU(2)$ β-functions negative ($\beta_0 > 0 \to \beta < 0$):

$\to$ α_3 and α_2 decrease as μ increases (asymptotic freedom).

- $U(1)$ β-function negative and α_1 increases with μ:

$\to$ extrapolated to high energy, couplings must converge !

- SM coupling evolution:



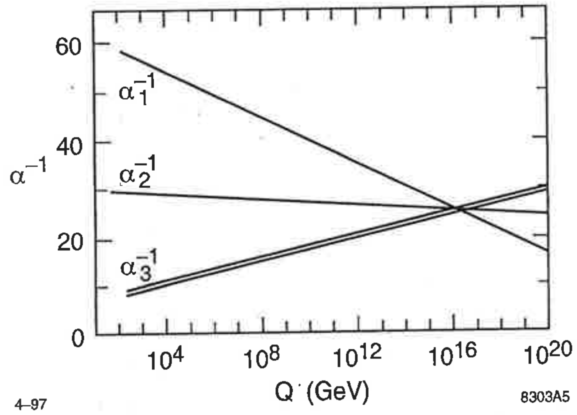
- Couplings do not come to a common value at any scale.
- Loop contributions of superpartners change β-functions hence evolution of gauge couplings in SUSY.
- Most economical model (MSSM):

$$\begin{aligned} SU(3) &: \beta_{3,0}^{\rm SUSY} = (27 - 6n_g)/(12\pi) \\ SU(2) &: \beta_{2,0}^{\rm SUSY} = (18 - 6n_g - 3n_h/2)/(12\pi) \\ U(1) &: \beta_{1,0}^{\rm SUSY} = (-6n_g - 9n_h/10)/(12\pi) \end{aligned}$$

- Couplings evolution changes also because SUSY requires two Higgs doublets:

$\to$ $n_h = 2$ in above equations above ...

- After including SUSY:



- Coupling implies SUSY masses $\widetilde{m} \lesssim \mathcal{O}(1 \text{ TeV})$.

- Theoretical uncertainties (e.g. model-dependent thresholds) are such that one cannot constrain $\widetilde{m}$ very tightly ...

- Higgs mechanism: Supergravity realisations of SUSY with universal scalar masses at GUT scale, one mass evolves down to negative values inducing EWSB !

MSSM

- MSSM based upon:

1. minimal particle content;
2. Poincare invariance;
3. gauge invariance;
4. SUSY.

- MSSM particle content:

Gauge bosons $S=1$	Gauginos $S=1/2$
gluon, $W^{\pm}, Z, \gamma$	gluino, $\widetilde{W}, \widetilde{Z}, \widetilde{\gamma}$
Fermions $S=1/2$	Sfermions $S=0$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}$	$\begin{pmatrix} \widetilde{u}_L \\ \widetilde{d}_L \end{pmatrix} \begin{pmatrix} \widetilde{\nu}_L^e \\ \widetilde{e}_L \end{pmatrix}$
u_R, d_R, e_R	$\widetilde{u}_R, \widetilde{d}_R, \widetilde{e}_R$
Higgs	Higgsinos
$\begin{pmatrix} H_2^0 \\ H_2^- \end{pmatrix} \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix}$	$\begin{pmatrix} \widetilde{H}_2^0 \\ \widetilde{H}_2^- \end{pmatrix} \begin{pmatrix} \widetilde{H}_1^+ \\ \widetilde{H}_1^0 \end{pmatrix}$

- Charged and neutral higgsinos mix with non-coloured gauginos to form physical mass eigenstates, so-called charginos $\widetilde{\chi}_{1,2}^{\pm}$ neutralinos $\widetilde{\chi}_{1,\dots 4}^0$.
- Mixing also expected in b, t and τ sfermion sector.

• Introduce additional discrete symmetry, R-parity:

→ forbid $B-L$ violating interactions (i.e., no proton decay).

• SM particles are R-even while SUSY ones are R-odd !

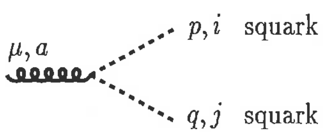
• MSSM with R-parity conservation:

→ SUSY in pairs and Lightest SUSY Particle (LSP) is stable.

• Sparticles interactions fixed by gauge symmetries and SUSY:

→ no adjustable parameters, i.e. predictive !

• $q, g, \widetilde{q}, \widetilde{g}$ interactions determined by α_s only:

gluon 

$$= -i\, g_s\, (T_a)_{ij} (p+q)^\mu$$

• SUSY is <u>non-exact symmetry of nature</u>:

→ SM particle and SUSY sparticles non-degenerate masses !

• Mechanism of SUSY breaking not understood:

$$\mathcal{L} = \mathcal{L}(\text{SUSY}) + \mathcal{L}(\text{SUSY-breaking}).$$

(see Sacha's course).

• (<u>Soft-breaking terms terms</u> are consistent with Poincare and SM gauge invariance) and <u>do not reintroduce</u> quadratic divergences for scalar particles !)

• >100 free parameters to parametrise SUSY breaking: → assume universality of parameters at Plank scale

• Introduce SUSY-breaking framework, e.g. Supergravity (mSUGRA):

→ SUSY breaking parameters become related and number of MSSM free parameters in Supergravity models is reduced to only five (mSUGRA)

• At $\mathcal{O}(M_{\text{Planck}})$:

1. scalars (Higgs bosons, $\tilde{\ell}$ and $\tilde{q}$) have common mass, $\widetilde{M}_0$;
2. gauginos ($\tilde{B}$, $\tilde{W}$, and $\tilde{g}$) have common mass, $\widetilde{M}_{1/2}$;
3. trilinear couplings have common value, A_0;

<u>plus</u>

4. Ratio of two Higgs vacuum expectation values, $\tan\beta$,
5. Higgs mass parameter in Superpotential, sign(μ).

• Evolving universal masses from M_{Planck} to EW scale (RGEs):

→ entire spectrum of SUSY particles can be generated !

• Two-doublet Higgs models are anomaly-free, e.g. MSSM.

• SUSY structure also requires (at least) two Higgs doublets to generate masses for both "up"-type and "down"-type quarks and charged leptons.

• MSSM Higgs sector consists of five physical Higgs particles:
→ two CP-even neutral Higgses, h^0 and H^0 ($M_{h^0} \leq M_{H^0}$)
→ one CP-odd neutral Higgs boson, A^0
→ a charged Higgs boson pair, $H^\pm$.

• AT LO, $\tan\beta$ (ratio of VEVs) and one Higgs mass (M_{A^0}) completely determine MSSM Higgs sector.

• E.g. at LO: $M_{h^0} < M_Z$, $M_{A^0} < M_{H^0}$ and $M_W < M_{H^\pm}$!

• Sparticle (virtual) effects enter in higher orders via

$$\epsilon = \frac{3G_F}{\sqrt{2}\pi^2}\frac{M_t^4}{\sin^2\beta}\log\left(1+\frac{M_S^2}{M_t^2}\right) .$$

• When radiative corrections are included (NLO):

$$M_{h^0}^2 \leq M_Z^2 \cos^2 2\beta + \epsilon \sin^2\beta .$$

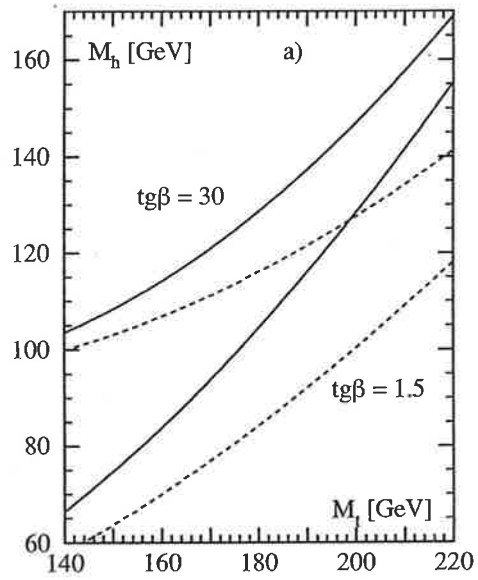
• NNLO (almost): → fig

$$M_{h^0} \lesssim 130 \text{ GeV ! } \qquad \text{(MSSM)}$$

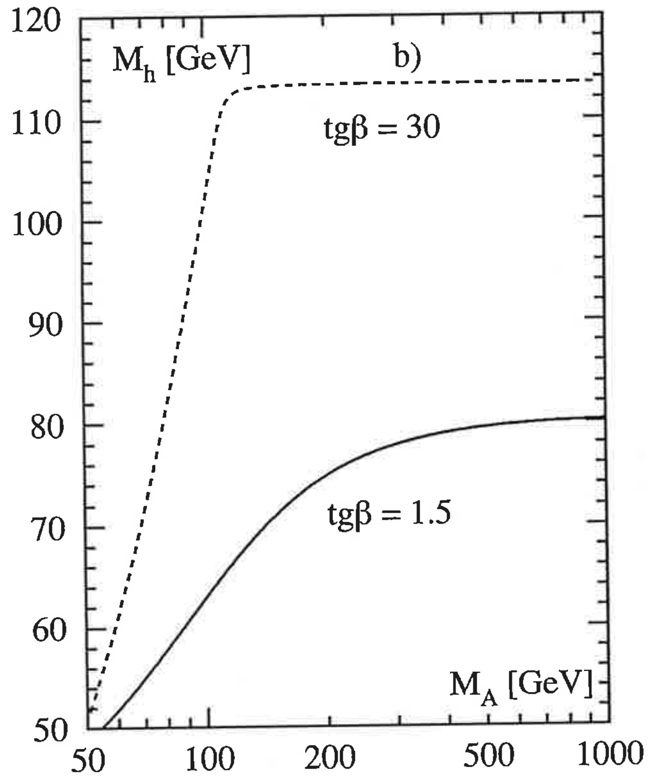
• Generalise: existence of light Higgs boson with $M_{h^0} \lesssim 200$ GeV is generic prediction of SUSY and its search is most important test of SUSY theories.

• Assume two scenarios:

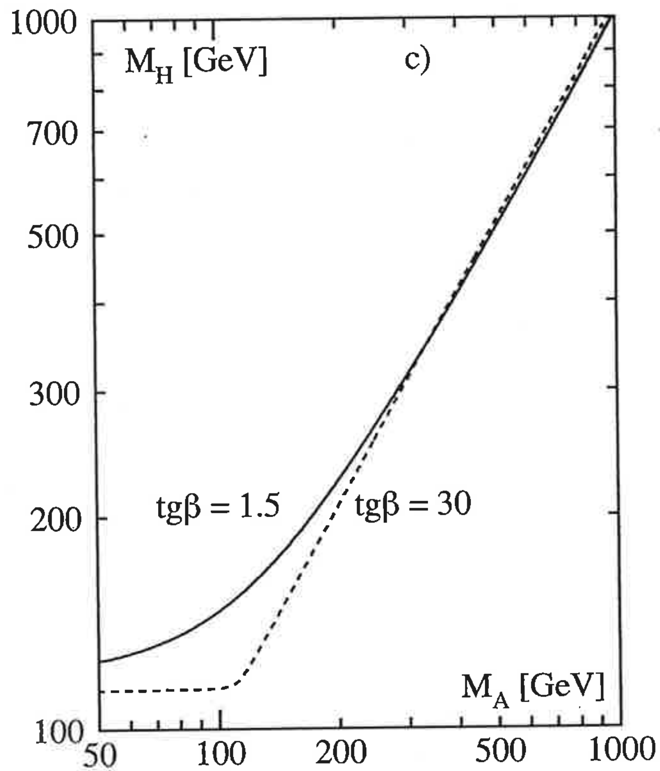
1. Minimal mixing $\mu = A_t = A_b = 0$ (dash);
2. Maximal mixing $\mu = 0$, $A_b = 0$, $A_t = \sqrt{6}M_{\text{SUSY}}$ (solid).



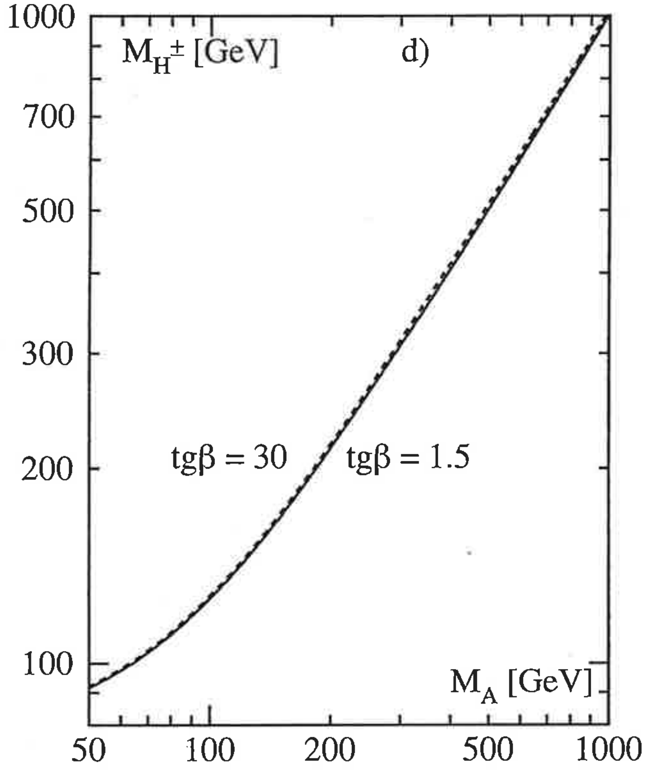
• M_{h^0} rather sensitive to top mass:



M_h [GeV]
b)
tgβ = 30
tgβ = 1.5
M_A [GeV]
120
110
100
90
80
70
60
50
50
100
200
500
1000



M_H [GeV]
c)
tgβ = 1.5
tgβ = 30
M_A [GeV]
1000
700
500
300
200
100
50
100
200
500
1000

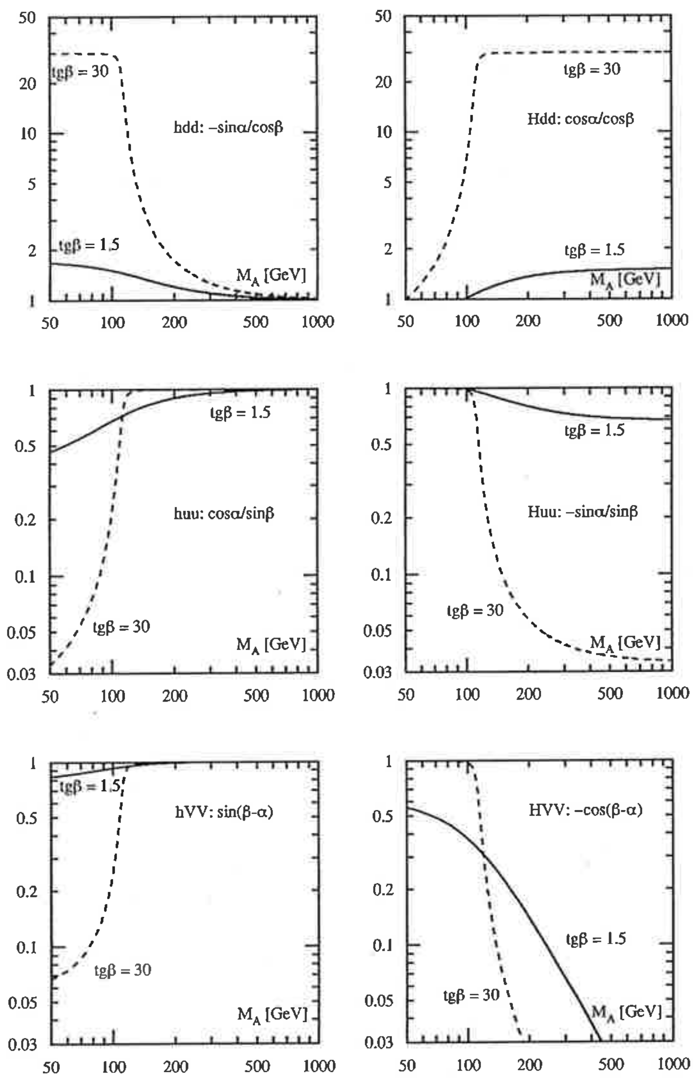


- Higgs mixing parameter α is fixed by $\tan\beta$ and M_{A^0},

$$\tan 2\alpha = \tan 2\beta \frac{M_{A^0}^2 + M_Z^2}{M_{A^0}^2 - M_Z^2 + \epsilon/\cos 2\beta} \qquad \text{with} \qquad -\frac{\pi}{2} < \alpha < 0\,.$$

- Higgs couplings to ordinary matter: $\rightarrow$ fig

	Φ	g_u^Φ	g_d^Φ	g_V^Φ
SM	H	1	1	1
MSSM	h	$\cos\alpha/\sin\beta$	$-\sin\alpha/\cos\beta$	$\sin(\beta-\alpha)$
	H	$\sin\alpha/\sin\beta$	$\cos\alpha/\cos\beta$	$\cos(\beta-\alpha)$
	A	$1/\tan\beta$	$\tan\beta$	0



Indirect searches: $g-2$

- SUSY can appear via HO effects in precision observables:

$\rightarrow \mu$ magnetic moment provides limits on BSM physics !

- Dirac equation for μ in EM field A is given by

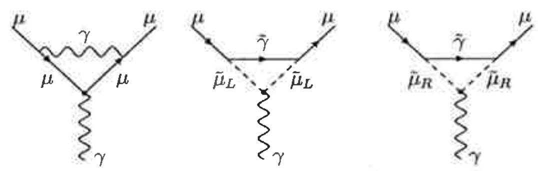
$$(\partial\!\!\!/ - e\, A\!\!\!/ - m_\mu)\psi_\mu = 0.$$

- For external magnetic field, Hamiltonian for a μ is given by

$$\mathcal{H} = \frac{e}{m}\vec{S}(\mu)\cdot\vec{B},$$

($\vec{S} \rightarrow$ spin vector; $\vec{B} \rightarrow$ magnetic field)

- Bohr magneton of electron $\mu_B = \frac{e}{2m_e}$ and magnetic moment of muon $\mu_\mu \equiv g_\mu \mu_B (m_e \rightarrow m_\mu)$.
- Dirac equation predicts $g_\mu = 2$ (tree − level).
- One-loop corrections from SM (QED) and SUSY (MSSM)



(a) photon, (b) photino+left-handed smuon, (c) photino+right-handed smuon.

- SM QED at one-loop:

$$\Delta\left(\frac{g_\mu - 2}{2}\right) = \frac{\alpha_{QED}(Q^2=0)}{2\pi} = 0.0011614.$$

- SUSY QED at one-loop

$$\left(\frac{g-2}{2}\right)_\mu^{susy} = -\frac{m_\mu^2 e^2}{8\pi^2}\int_0^1 dx \frac{x^2(1-x)}{m_\mu^2 x^2 + (m_{\tilde{\mu}_L}^2 - M_{\tilde{\gamma}}^2 - m_\mu^2)x + M_{\tilde{\gamma}}^2}.$$

- Latter is quadratic in m_μ^2 !

- Consider integral in limit $m_{\tilde{\mu}_L} = m_\mu$ and $M_{\tilde{\gamma}} = 0$,

$$\left(\frac{g-2}{2}\right)_\mu^{\text{susy}} = -\frac{\alpha_{\text{em}}}{6\pi}.$$

- SM and SUSY contribution have opposite sign !

- Assume light photino and heavy smuon ($m_{\tilde{\mu}_L} \gg M_{\tilde{\gamma}}, m_\mu$):

$$\left(\frac{g-2}{2}\right)_\mu^{\text{susy}} = -\frac{\alpha_{\text{em}}}{6\pi}\frac{m_\mu^2}{m_{\tilde{\mu}_L}^2}.$$

- Realistic scenario for SUSY breaking

- SUSY contribution decouples rapidly !

- SUSY correction $\propto$ fermion mass squared,

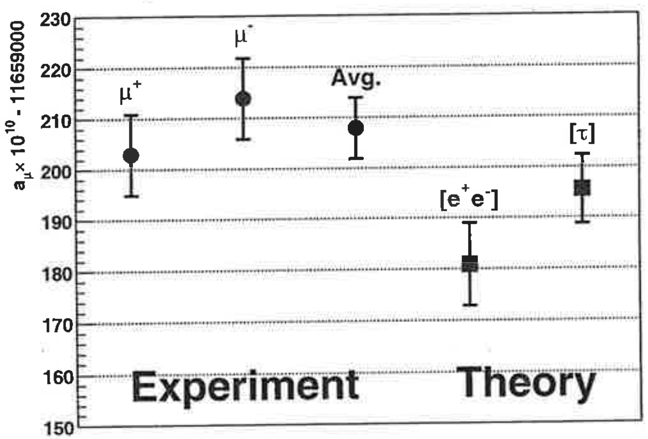
$\rightarrow$ suppressed for electron !

- BNL E821 experiment has reported measurement of

$$a_\mu \equiv \frac{1}{2}(g_\mu - 2) =$$

$\rightarrow$ 2.7σ from SM based $e^+e^- \rightarrow$ hadrons data.

- Discrepancy is $\approx 0.8\sigma$ if one uses $\tau \rightarrow$ hadrons SM data:



(http://www.g-2.bnl.gov/index.shtml)

- Though situation is not conclusive, a_μ does provide a significant constraint on low-energy SUSY.

MSSM Higgs collider searches

- There exist limits from LEP2 and Tevatron, again assume:

  1. Minimal mixing scenario → fig
  2. Maximal mixing scenario → fig

- MSSM parameter space available is reducing !

- LHC prospects from ATLAS and CMS: → figs

→ no-lose theorem but large decoupling area !

- Ought to observe second Higgs state or make precision measurements of BRs, Γs, etc. → fig

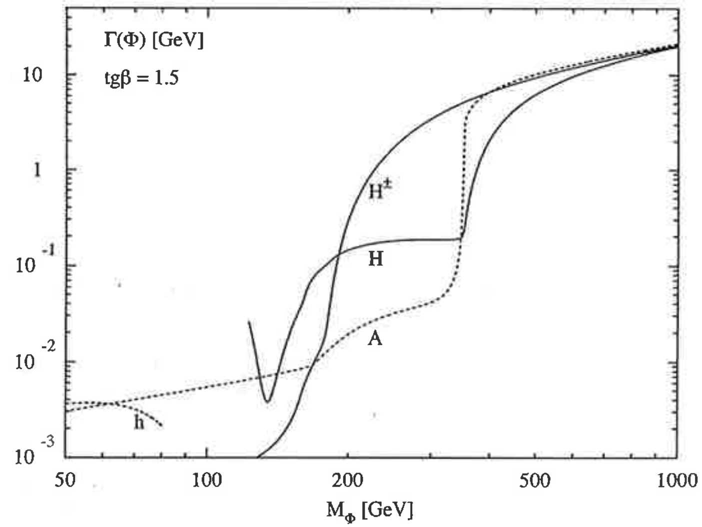
- Can construct modifications of MSSM: e.g.

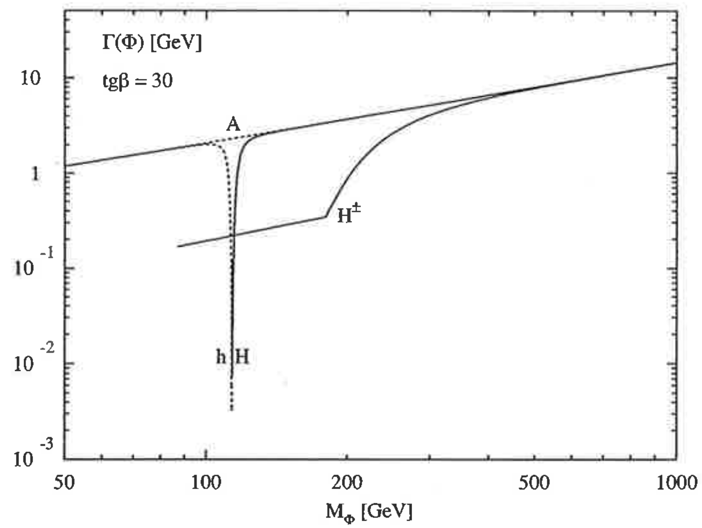
  1. Next-to-MSSM (NMSSM) → fig

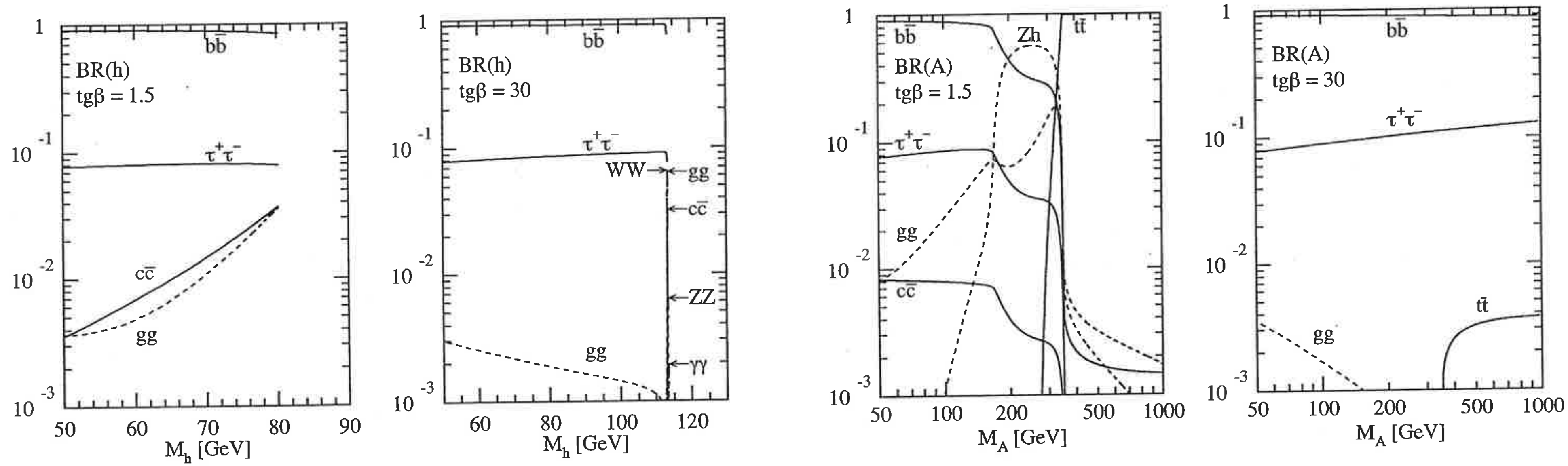
- Ought to include interaction between Higgs/SUSY sectors:

→ Higgs → SUSY and SUSY → Higgs signatures ! → figs

- MSSM Higgs decay rates:

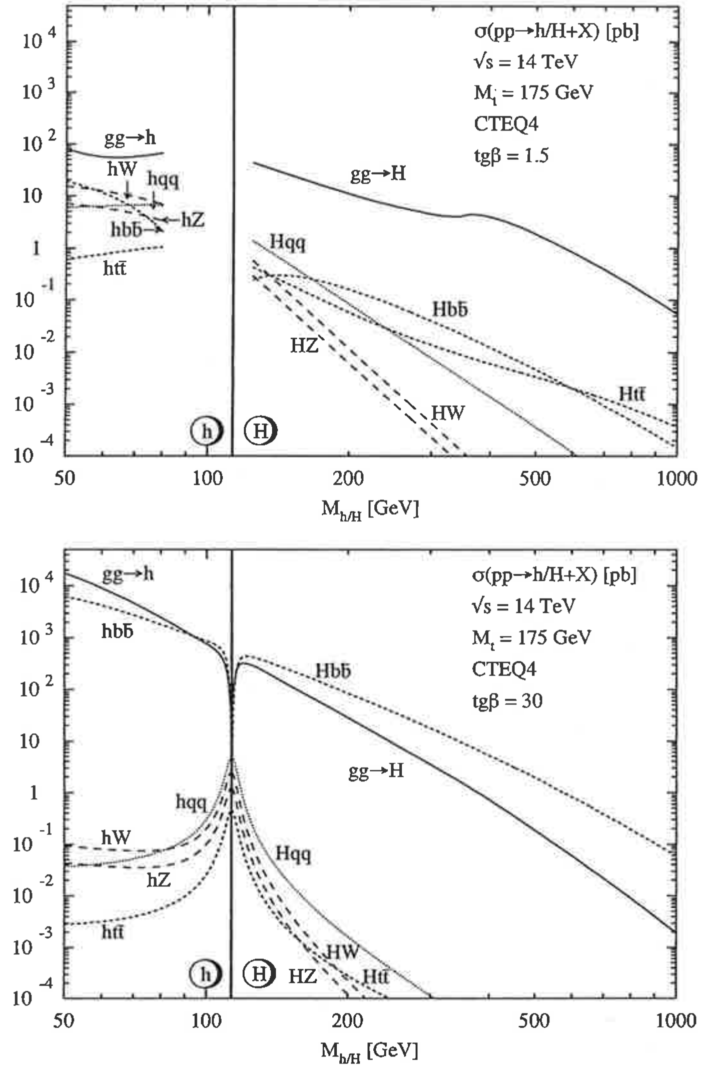




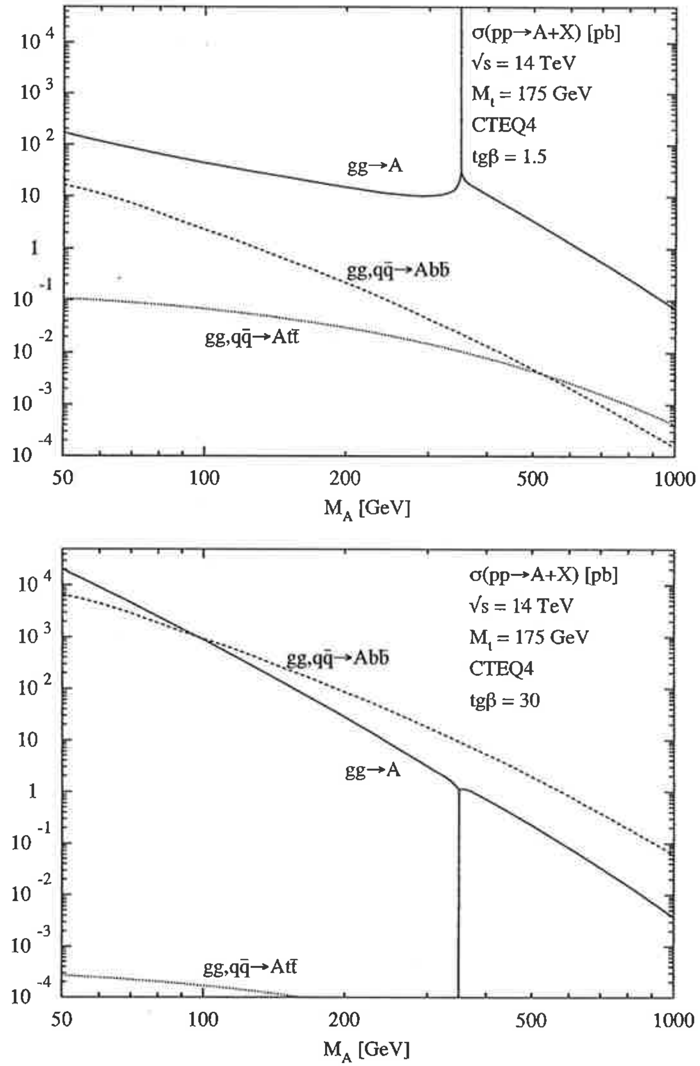


BR(h)
tgβ = 1.5
bb̄
τ⁺τ⁻
cc̄
gg
M_h [GeV]
BR(h)
tgβ = 30
bb̄
τ⁺τ⁻
WW
gg
cc̄
ZZ
γγ
gg
M_h [GeV]
BR(A)
tgβ = 1.5
bb̄
Zh
tt̄
τ⁺τ⁻
gg
cc̄
M_A [GeV]
BR(A)
tgβ = 30
bb̄
τ⁺τ⁻
gg
tt̄
M_A [GeV]

• MSSM Higgs production rates at LHC (h^0, H^0):



• MSSM Higgs production rates at LHC (A^0):



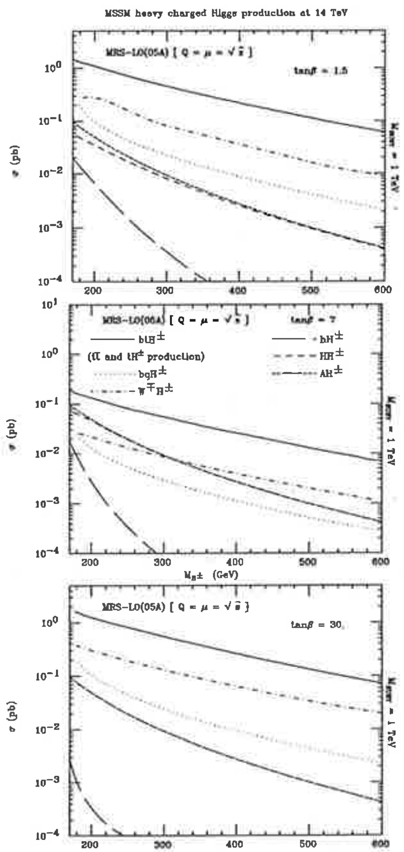
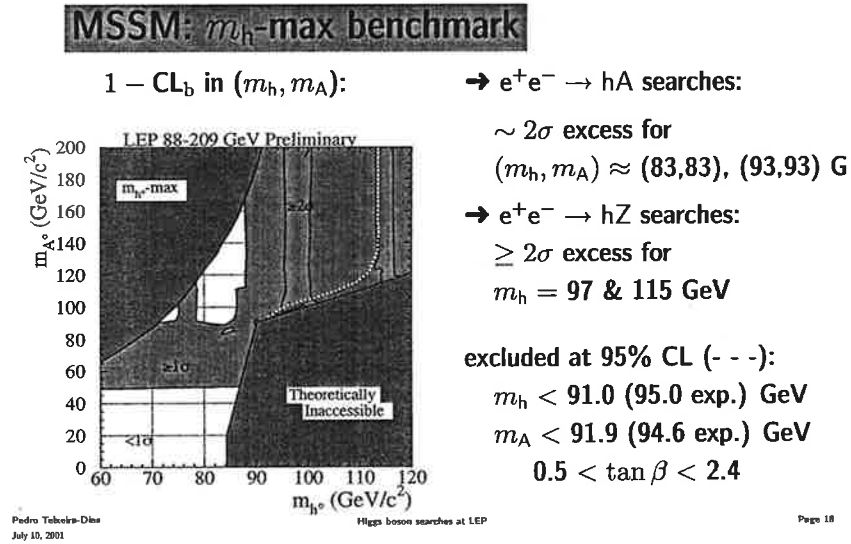
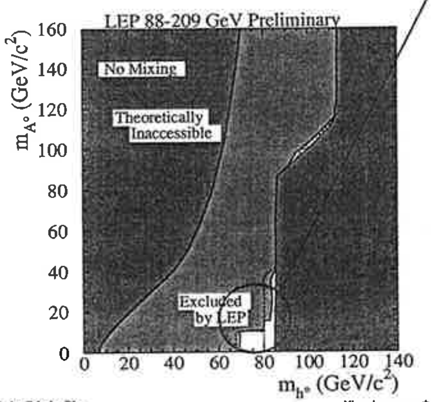


Figure 5: Cross sections in picobarns at the LHC for the production mechanisms of a single charged Higgs boson, for $\tan\beta = 1.5$ (top), 7 (middle) and 30 (bottom). (The $q\bar{q}' \to \Phi H^{\pm}$ rates, with $\Phi = h, A$, visually coincide for $\tan\beta = 30$.)



MSSM: no-mixing benchmark

(m_h, m_A) 95% CL exclusion:



- $e^+e^- \to hZ \to AAZ$, but no $A \to b\bar{b}$!

  occurs for

  $\tan\beta < 0.7$ & $m_{H^\pm} < 74$ GeV

- Use $H^\pm$ direct searches to exclude this .

  ... but: $B(H^\pm \to W^\pm A) \neq 0$

  $\Rightarrow$ under investigation

excluded at 95% CL (for $\tan\beta > 0.7$):

$m_h < 91.5$ (95.0 exp.) GeV

$m_A < 92.2$ (95.3 exp.) GeV

$0.7 < \tan\beta < 10.5$

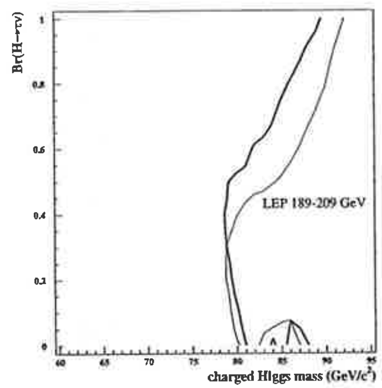
Charged Higgs, $H^\pm$

Assume $B(H^+ \to c\bar{s}) + B(H^+ \to \tau^+\nu) \doteq 1$

➜ $e^+e^- \to H^+H^- \to c\bar{s}s\bar{c},\ c\bar{s}\tau^-\bar{\nu},\ \tau^+\nu\tau^-\bar{\nu}$

- L3 observe a large excess in the 4-jets channel

➜ compatibility with ALEPH, DELPHI, OPAL is being investigated.



LEP combined search excludes (95%CL)

$m_{H^\pm} < 78.6$ (78.8 exp.) GeV

for any $B(H^+ \to \tau^+\nu)$

$H^{\pm}$ at Tevatron from Run 2 report)

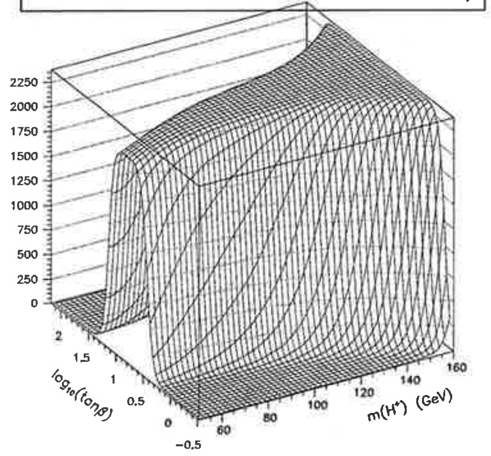


FIGURE 101. The likelihood of $n_{obs} = 600$ (in arbitrary units), as a function of m_{H^+} and $\tan\beta$ (assuming $m_t = 175$ GeV and parameters given in Table 50).

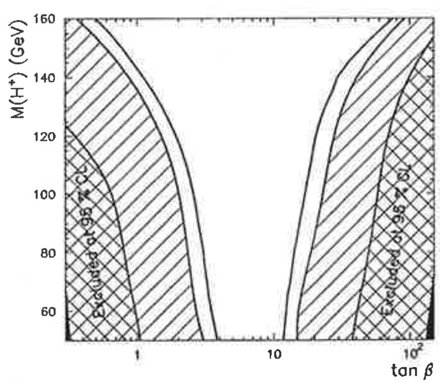


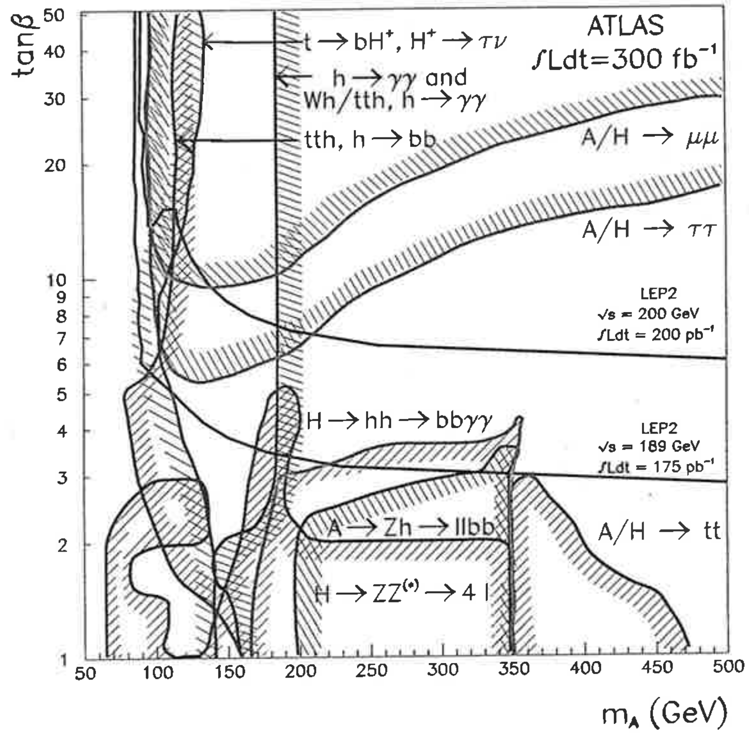
FIGURE 102. The 95% CL exclusion boundaries in the $[m_{H^+}, \tan\beta]$ plane for $m_t = 175$ GeV and several values of the integrated luminosity: 0.1 fb^{-1} (at $\sqrt{s} = 1.8$ TeV, cross-hatched), 2.0 fb^{-1} (at $\sqrt{s} = 2.0$ TeV, single-hatched), and 10 fb^{-1} (at $\sqrt{s} = 2.0$ TeV, hollow), if the n_{obs} continues to be where the SM-based prediction peaks.

146

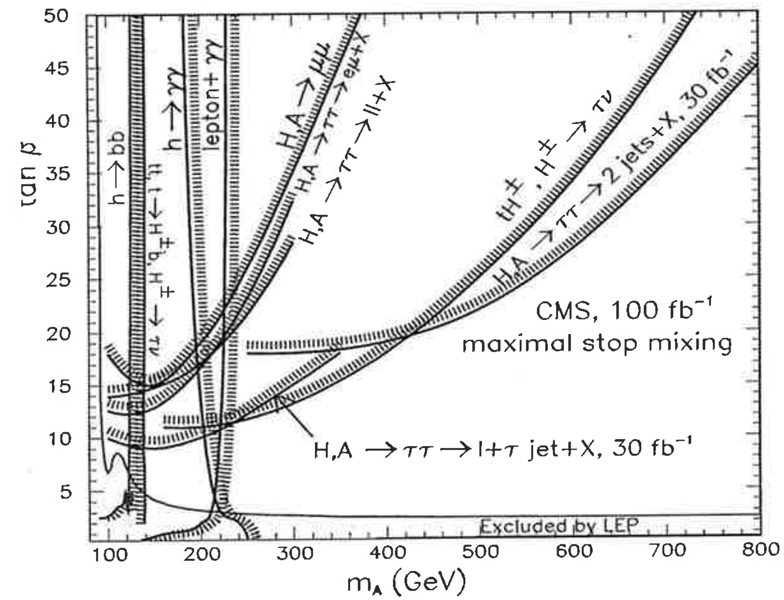
LHC search channels for MSSM Higgses

Typical discovery channels are:

- $h \to \gamma\gamma$, inclusive production and production in association with an isolated lepton in Wh^0 and $t\bar{t}h^0$ final states
- $h \to b\bar{b}$ in association with an isolated lepton and b-jets in Wh^0 and $t\bar{t}h^0$
- $A^0, H^0 \to \mu\mu$, inclusive and in $b\bar{b}H^0/A^0$ final states
- $A^0, H^0 \to \tau\tau$ with 2ℓ, $\ell + \tau$-jet and 2τ-jet final states
- $H^{\pm} \to \tau\nu$ in $gb \to tH^{\pm}$ and in $q\bar{q}' \to H^{\pm}$
- $H^{\pm} \to tb$ in $gb \to tH^{\pm}$



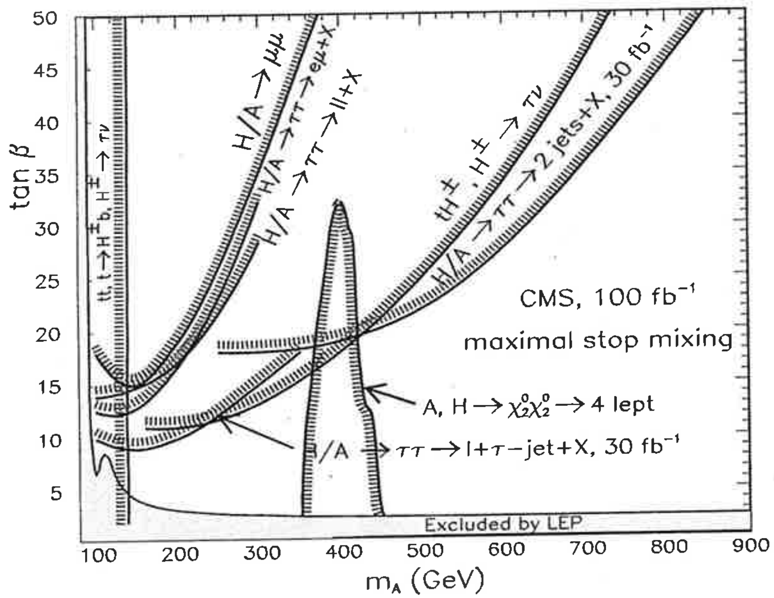
ATLAS after 300 fb^{-1} of luminosity (MaxMix)



CMS after 100 fb^{-1} of luminosity (MaxMix)

Higgses → SUSY

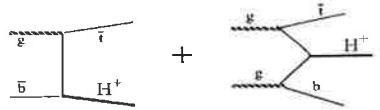
- Clean channel: $A, H \to \chi_2^0 \chi_2^0 \to 4\ell^\pm + X$

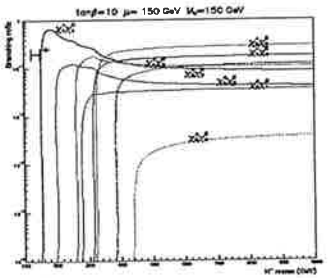


CMS after 100 fb^{-1} assuming $M_1 = 90$ GeV, $M_2 = 180$ GeV, $\mu = 500$ GeV, $M_{\tilde{\ell}} = 250$ GeV, $M_{\tilde{q},\tilde{g}} = 1000$ GeV.

CCCCCCC2 CCCCCCCCCCCCCC

d dddlddddtldddddtlddndddddddmddddm± d





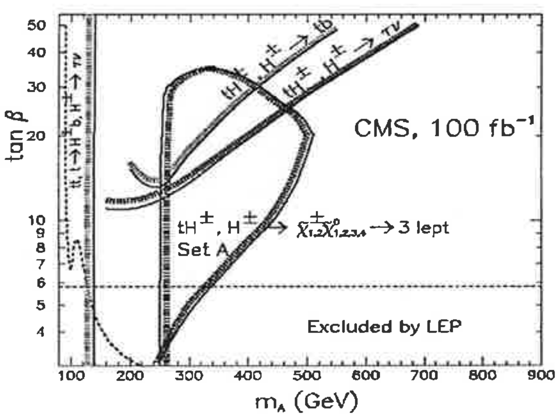
d dddlddddlddddndddd

m± → χm: : χ: :m± → dl + $\not{E}_T^{miss}$

→ dddldldtlddtlddddlddddtldtldddddtldddlddddtlddddtlddtl→ddtl

Les H H H H

DIDDDDDDIDDDDD



2 SS2 SSSSmSSSSS

2 $_2$ SS2SSSS SS S
μ SSS5SS SS S
2 wwwwSSSSSS SS S
2 wwwwwwwwSSSS SS

Les H H H H

SUSY → Higgses

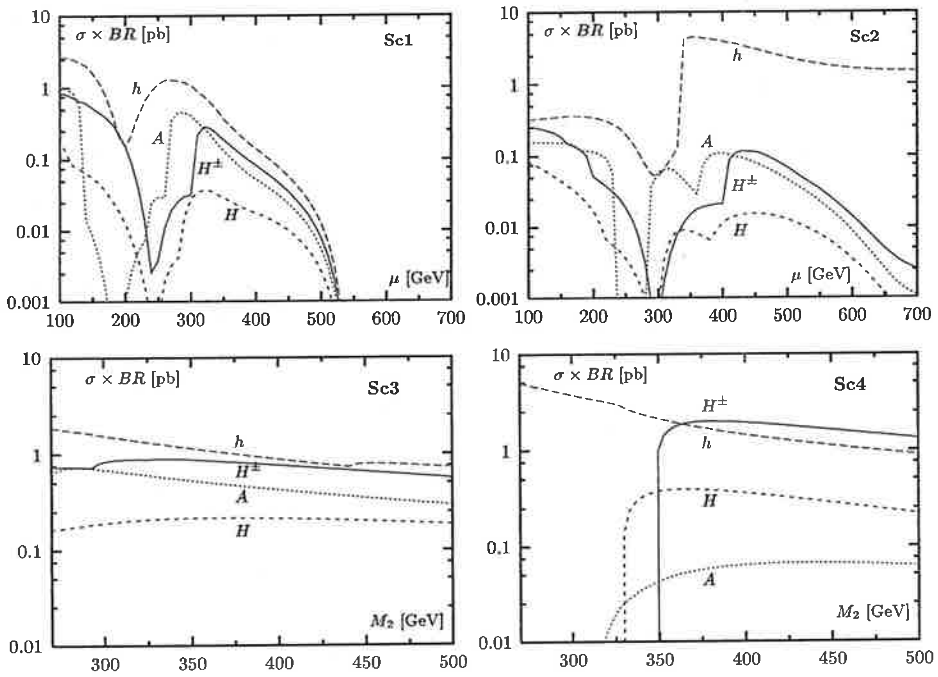
- Possible channels:

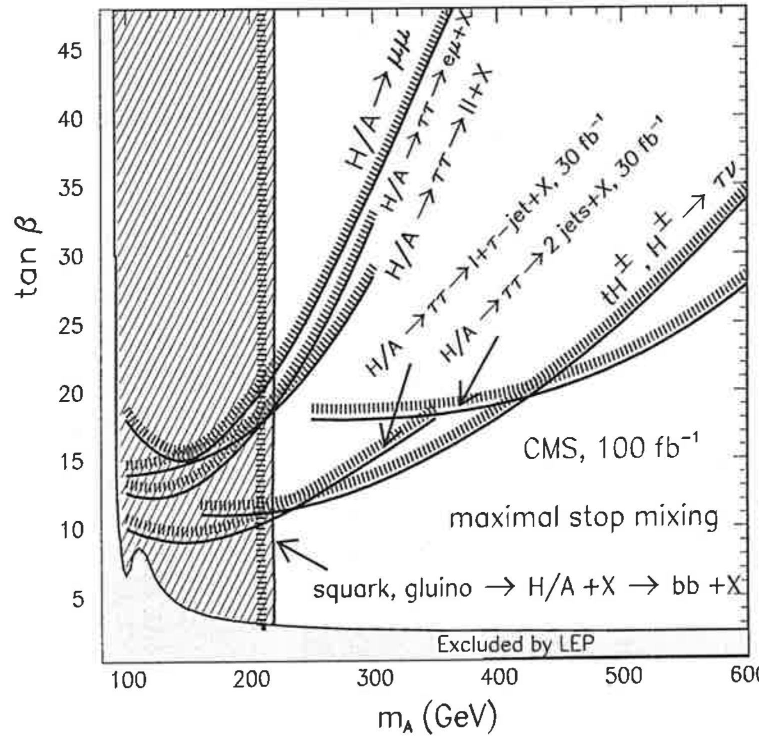
$$pp \to \tilde{g}\tilde{g}, \tilde{q}\tilde{q}, \tilde{q}\tilde{q}^*, \tilde{q}\tilde{g} \quad \to \quad \chi_2^\pm, \chi_3^0, \chi_4^0 X$$
$$\to \quad \chi_1^\pm, \chi_2^0, \chi_1^0 h^0, H^0, A^0, H^\pm \; X \; (1)$$

$$pp \to \tilde{g}\tilde{g}, \tilde{q}\tilde{q}, \tilde{q}\tilde{q}^*, \tilde{q}\tilde{g} \quad \to \quad \chi_1^\pm, \chi_2^0 X$$
$$\to \quad \chi_1^0 H^\pm, h^0, H^0, A^0 \; X \qquad (2)$$

$$pp \quad \to \quad \tilde{t}_2\tilde{t}_2^*, \tilde{b}_2\tilde{b}_2^* \;\text{ with }\; \tilde{t}_2(\tilde{b}_2) \to \tilde{t}_1(\tilde{b}_1)h^0, H^0, A^0 \;\text{ or }\; \tilde{b}_1(\tilde{t}_1)H^\pm$$

$$pp \to \tilde{g}\tilde{g}, \tilde{q}\tilde{q}, \tilde{q}\tilde{q}^*, \tilde{q}\tilde{g} \quad \to \quad t/\bar{t}X \to H^\pm X$$





CMS after 100 fb^{-1} assuming $M_2 = 175$ GeV, $M_2 = 350$ GeV, $\mu = 150$ GeV, $M_{\tilde{\ell}} = 175$ GeV, $M_{\tilde{q},\tilde{g}} = 800$ GeV

MSSM sparticle collider searches

- To date no experimental evidence for SUSY.
- Assuming R-parity conservation and LSP (e.g. neutralino $\tilde{\chi}_1^0$) identification, SUSY production signatures:

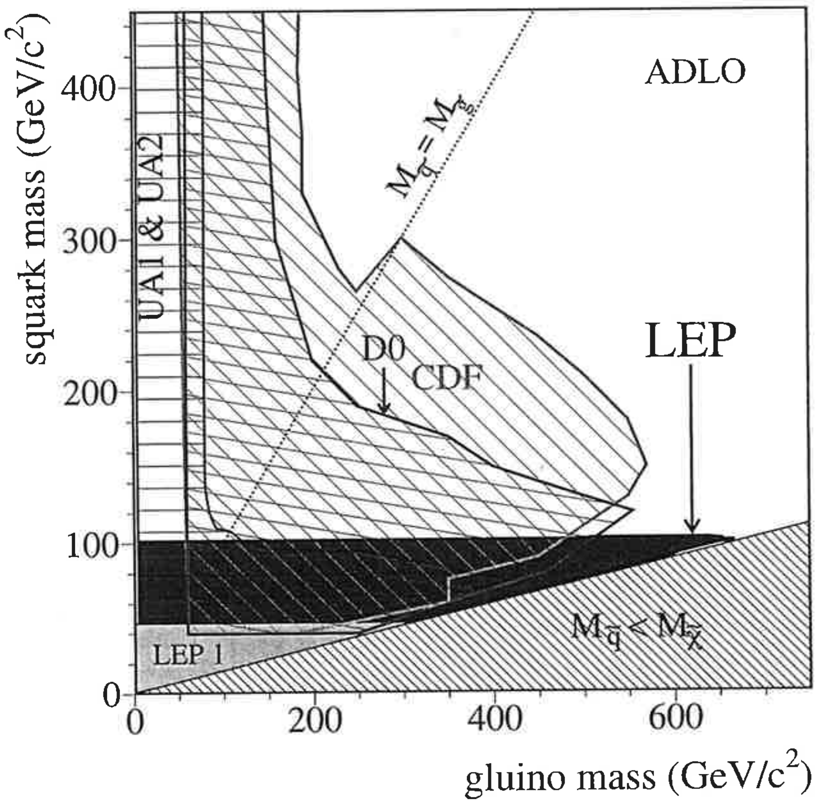
$\rightarrow$ multi-jets/leptons plus missing transverse momentum.

- Limits from e^+e^- and $p\bar{p}$ colliders. $\rightarrow$ figs
- SUSY searches at Tevatron and LHC: $\rightarrow$ fig
- At LHC $\tilde{q}, \tilde{g}$ up to masses $\lesssim 2.5$ TeV can be discovered !
- Since $M_{\text{SUSY}} \sim 1$ TeV:

$\rightarrow$ LHC will either confirm or disprove (low-energy) SUSY ?!

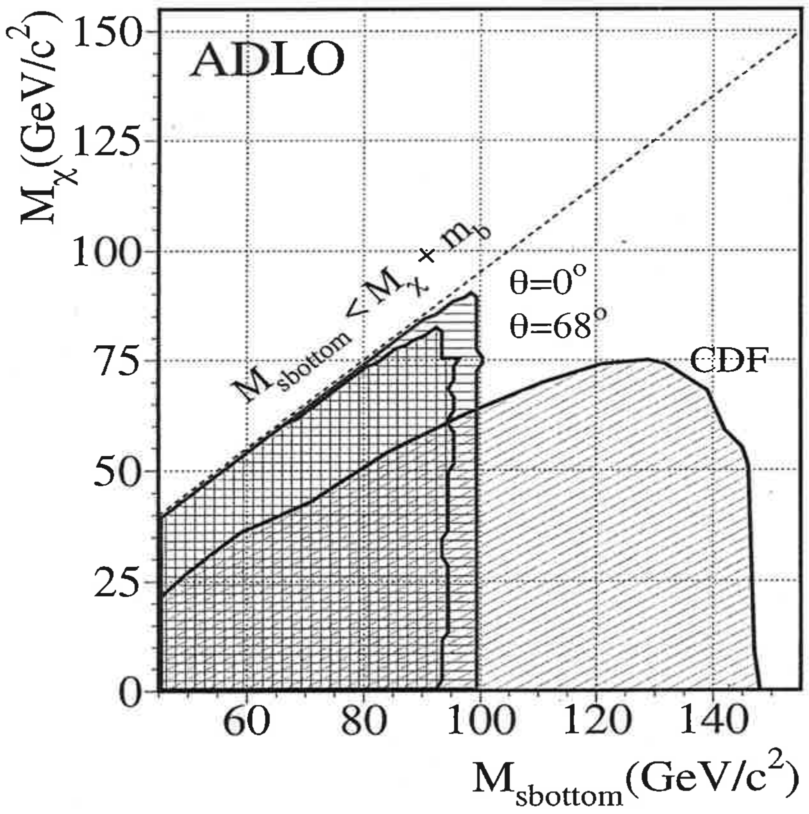
- Measurements of gross features of SUSY particle production will allow to determine typical mass scale of coloured SUSY particles at the LHC.
- SUSY cascade decays with favorable BRs can be exploited to determine mass differences of sparticles. $\rightarrow$ fig
- It is in general difficult to observe heavy weakly interacting particles such as $\tilde{\chi}_i^0, \tilde{\chi}_i^{\pm}, \tilde{\ell}$ at LHC: need an International Linear Collider (ILC).
- Future LCs tool for precise measurements of masses and couplings of SUSY particles (especially non-coloured ones). $\rightarrow$ fig

- Gluino mass from $\tilde{g} \rightarrow q\chi$ (LEP & Tevatron combined)



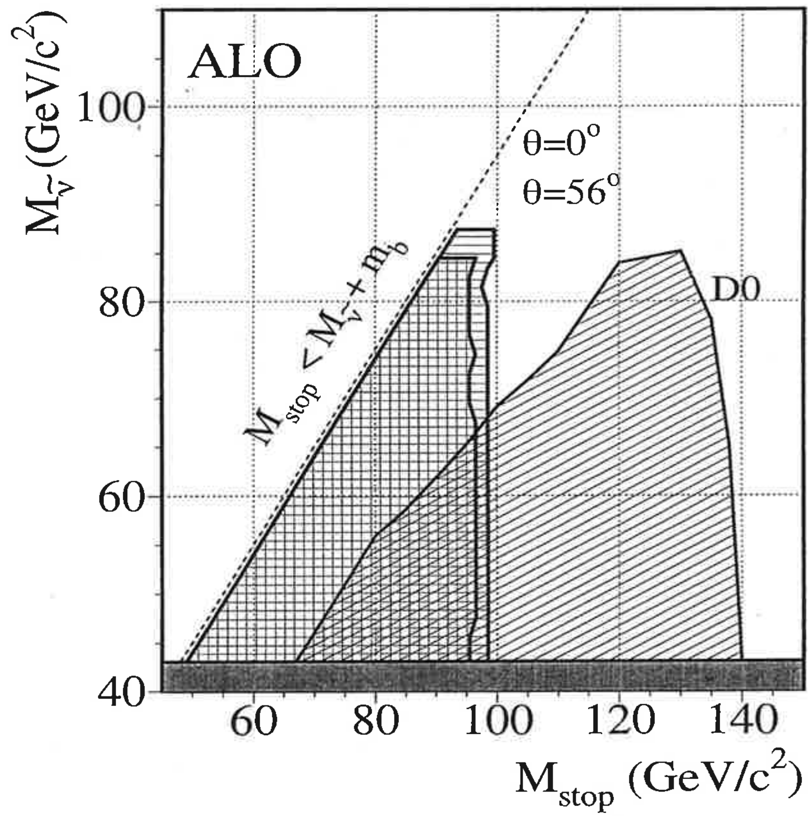
- Sbottom mass from $\tilde{b} \to b\chi$ (LEP & Tevatron combined)

($\theta = 0(68)$ horizontal(vertical) hatching)

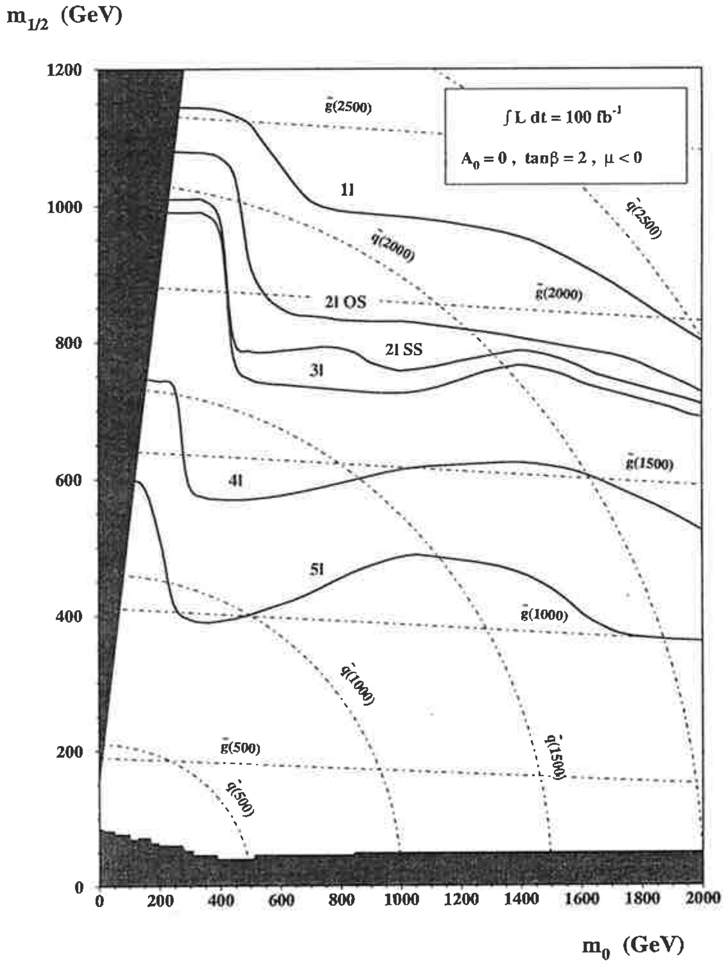


- Stop mass from $\tilde{t} \to b\ell\tilde{\nu}$ (LEP & Tevatron combined)

($\theta = 0(56)$ horizontal(vertical) hatching)



• SUSY searches via ≥ 2 jets, $n\ell^{\pm}$ + missing energy (LHC):



• SUSY processes reconstruction at LHC:

→ kinematically solves for neutralino momenta and masses of heavier sparticles using measured jet and lepton momenta and a few mass inputs.

• E.g.:

$$\tilde{g} \to \tilde{b}b \to \tilde{\chi}_2^0 bb \to \tilde{\ell}bb\ell \to \tilde{\chi}_1^0 bb\ell\ell.$$

(Decay chain basically free from SM background after cuts.)

• Five mass shell conditions:

$$\begin{aligned} m_{\tilde{\chi}_1^0}^2 &= p_{\tilde{\chi}_1^0}^2, \\ m_{\tilde{\ell}}^2 &= (p_{\tilde{\chi}_1^0} + p_{\ell_1})^2, \\ m_{\tilde{\chi}_2^0}^2 &= (p_{\tilde{\chi}_1^0} + p_{\ell_1} + p_{\ell_2})^2, \\ m_{\tilde{b}}^2 &= (p_{\tilde{\chi}_1^0} + p_{\ell_1} + p_{\ell_2} + p_{b_1})^2, \\ m_{\tilde{g}}^2 &= (p_{\tilde{\chi}_1^0} + p_{\ell_1} + p_{\ell_2} + p_{b_1} + p_{b_2})^2. \end{aligned}$$

• Assume $m_{\tilde{\chi}_1^0}$,$m_{\tilde{\chi}_2^0}$ and $m_{\tilde{\ell}}$ measured at LHC using first generation squark cascade decays with $\sim 10\%$ accuracy.

• For two $bb\ell\ell$ events, we have ten equations and ten unknowns (two neutralino four momenta, $m_{\tilde{g}}$ and $m_{\tilde{\chi}_1^0}$) !

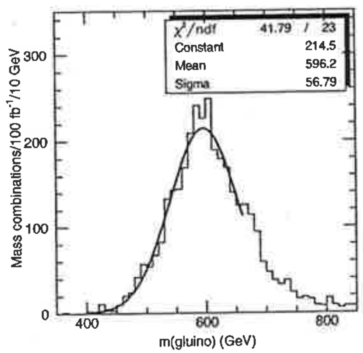
• Solution of above equations can be written:

$$\begin{aligned} m_{\tilde{g}}^2 &= F_0 + F_1 m_{\tilde{b}}^2 \pm F_2 D, \\ \text{where}\quad D^2 &\equiv D_0 + D_1 m_{\tilde{b}}^2 + D_2 m_{\tilde{b}}^4. \end{aligned}$$

where F_i and D_i depend upon p_{ℓ_i}, p_{b_i}, $\tilde{\chi}_1^0$ and $\tilde{\ell}$ masses.

- Procedure:
  - In the event, there are two b jets: assume highest p_T b-jet from $\tilde{b}$ decay.
  - Two leptons must come from $\tilde{\chi}_2^0$ and $\tilde{\ell}$ decay.
  - Four sets of gluino and sbottom mass solutions together with two lepton assignments for each decay, because cannot determine from which decay the lepton originates.
  - To reduce combinatorics take event pair which satisfies:

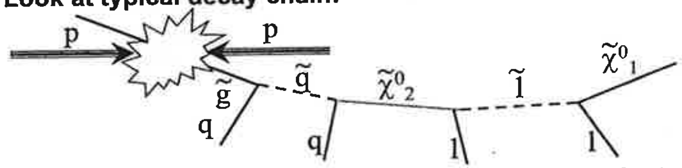
1. Only one lepton assignment has solution to equations.
2. For a pair of events there are only two solutions and there is a difference ~ 100 GeV between two gluino mass solutions.



- (Input was $m_{\tilde{g}} = 595.2$ GeV.)

SUSY Signatures

- Q: What do we expect SUSY events @ LHC to look like?
- A: Look at typical decay chain:



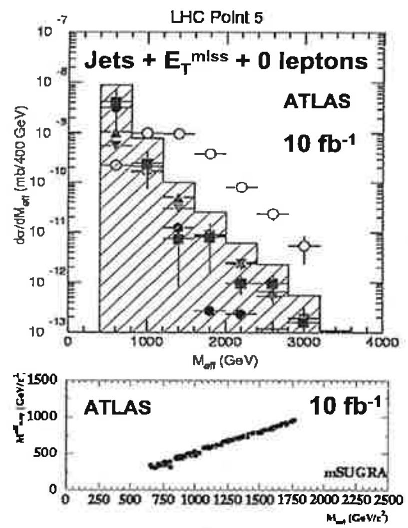
- Strongly interacting sparticles (squarks, gluinos) dominate production.
- Heavier than sleptons, gauginos etc. → cascade decays to LSP.
- Long decay chains and large mass differences between SUSY states
  - Many high p_T objects observed (leptons, jets, b-jets).
- If R-Parity conserved LSP (lightest neutralino in mSUGRA) stable and sparticles pair produced.
  - Large E_T^{miss} signature (c.f. $W \to l\nu$).
- Closest equivalent SM signature $t \to Wb$.

Dan Tovey 7 Kyoto, January 2005

SUSY Mass Scale

- First measured SUSY parameter likely to be mass scale:
  - Defined as weighted mean of masses of initial sparticles.
- Calculate distribution of 'effective mass' variable defined as scalar sum of masses of all jets (or four hardest) and E_T^{miss}:

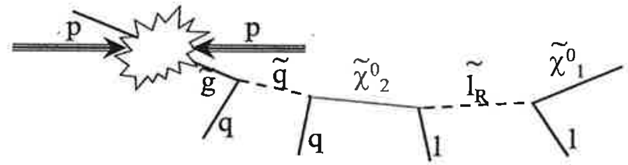
  $M_{eff}=\Sigma|p_T^i| + E_T^{miss}$.
- Distribution peaked at ~ twice SUSY mass scale for signal events.
- Pseudo 'model-independent' measurement.
- Typical measurement error (syst+stat) ~10% for mSUGRA models for 10 fb^{-1}.



Dan Tovey 9 Kyoto, January 2005

Exclusive Studies

- With more data will attempt to measure weak scale SUSY parameters (masses etc.) using exclusive channels.
- Different philosophy to TeV Run II (better S/B, longer decay chains) → aim to use model-independent measures.

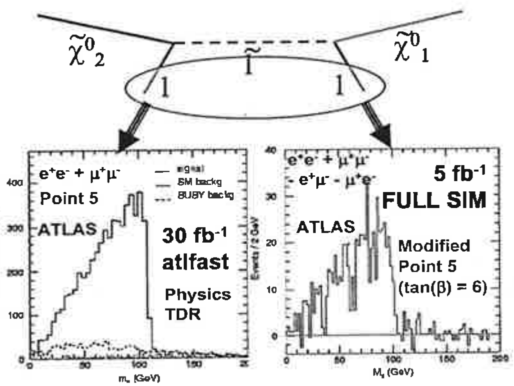


- Two neutral LSPs escape from each event
  - Impossible to measure mass of each sparticle using one channel alone
- Use kinematic end-points to measure combinations of masses.
- Old technique used many times before (ν mass from β decay spectrum, W (transverse) mass in W→lν).
- Difference here is we don't know mass of neutral final state particles.

Dan Tovey 10 Kyoto, January 2005

Dilepton Edge Measurements

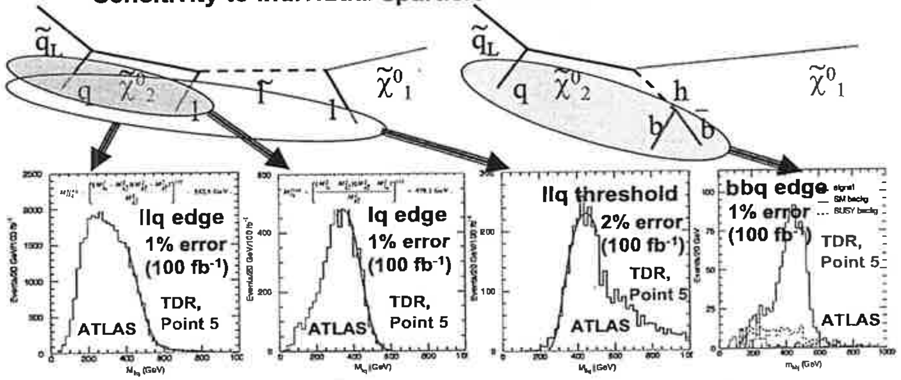
- When kinematically accessible $\tilde{\chi}^0_2$ can undergo sequential two-body decay to $\tilde{\chi}^0_1$ via a right-slepton (e.g. LHC Point 5).
- Results in sharp OS SF dilepton invariant mass edge sensitive to combination of masses of sparticles.
- Can perform SM & SUSY background subtraction using OF distribution
  $e^+e^- + \mu^+\mu^- - e^+\mu^- - \mu^+e^-$
- Position of edge measured with precision ~ 0.5% (30 fb^{-1}).



$$M_{ll}^{max} = M(\tilde{\chi}_2^0)\sqrt{1-\frac{M^2(\tilde{l}_R)}{M^2(\tilde{\chi}_2^0)}}\sqrt{1-\frac{M^2(\tilde{\chi}_1^0)}{M^2(\tilde{l}_R)}} = 108.93 \text{ GeV}$$

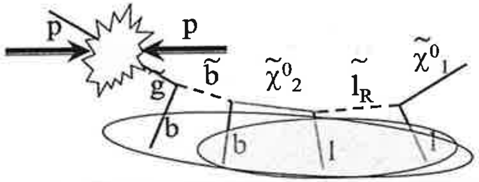
Measurements With Squarks

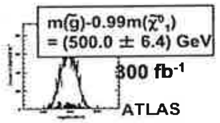
- Dilepton edge starting point for reconstruction of decay chain.
- Make invariant mass combinations of leptons and jets.
- Gives multiple constraints on combinations of four masses.
- Sensitivity to individual sparticle masses.

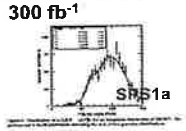


Sbottom/Gluino Mass

- Following measurement of squark, slepton and neutralino masses move up decay chain and study alternative chains.
- One possibility: require b-tagged jet in addition to dileptons.
- Give sensitivity to sbottom mass (actually two peaks) and gluino mass.
- Problem with large error on input $\tilde{\chi}^0_1$ mass remains → reconstruct difference of gluino and sbottom masses.
- Allows separation of $\tilde{b}_1$ and $\tilde{b}_2$ with 300 fb^{-1}.

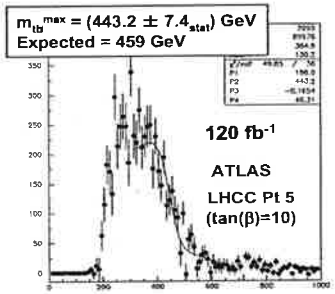


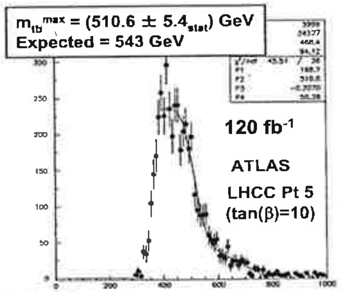




Stop Mass

- Look at edge in tb mass distribution.
- Contains contributions from
  - $\tilde{g} \rightarrow t\tilde{t}_1 \rightarrow tb\tilde{\chi}^+_1$
  - $\tilde{g} \rightarrow b\tilde{b}_1 \rightarrow bt\tilde{\chi}^+_1$
  - SUSY backgrounds
- Measures weighted mean of end-points
- Require m(jj) ~ m(W), m(jjb) ~ m(t)

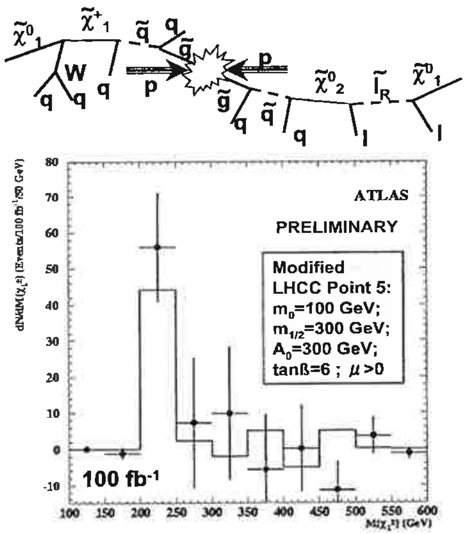




- Subtract sidebands from m(jj) distribution
- Can use similar approach with $\tilde{g} \rightarrow t\tilde{t}_1 \rightarrow tt\tilde{\chi}^0_i$
  - Di-top selection with sideband subtraction
- Also use 'standard' bbll analyses (previous slide)

Chargino Mass Measurement

- Mass of lightest chargino very difficult to measure as does not participate in standard dilepton SUSY decay chain.
- Decay process via ν+slepton gives too many extra degrees of freedom - concentrate instead on decay $\tilde{\chi}^+_1 \rightarrow W \tilde{\chi}^0_1$.
- Require dilepton $\tilde{\chi}^0_2$ decay chain on other 'leg' of event and use kinematics to calculate chargino mass analytically.
- Using sideband subtraction technique obtain clear peak at true chargino mass (218 GeV).
- ~ 3 σ significance for 100 fb^{-1}.



Measuring Model Parameters

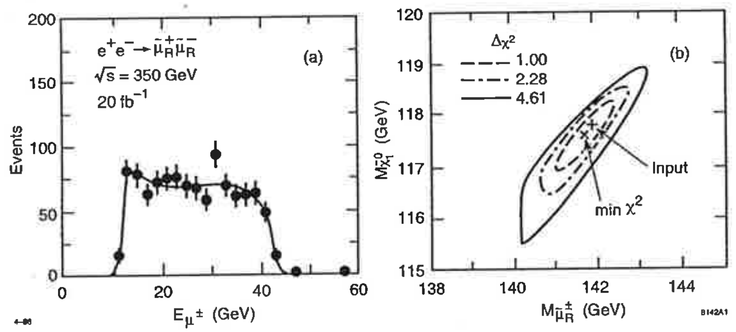
- Alternative use for SUSY observables (invariant mass end-points, thresholds etc.).
- Here assume mSUGRA/CMSSM model and perform global fit of model parameters to observables
  - So far mostly private codes but e.g. SFITTER, FITTINO now on the market;
  - c.f. global EW fits at LEP, ZFITTER, TOPAZ0 etc.

Point	m_0	$m_{1/2}$	A_0	tan(β)	sign(μ)
LHC Point 5	100	300	300	2	+1
SPS1a	100	250	-100	10	+1

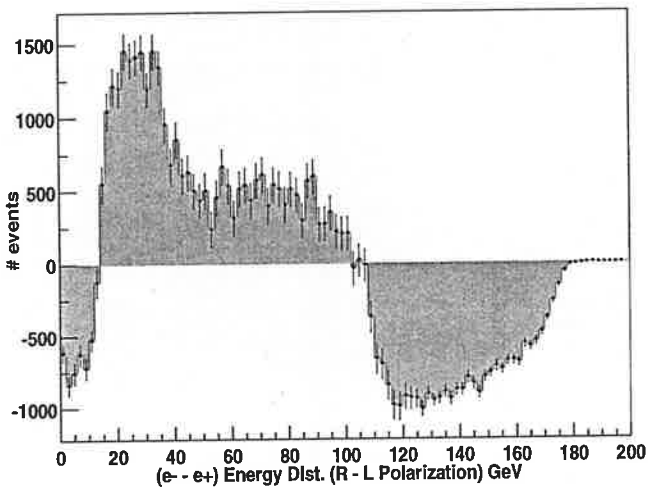
Variable	Value (GeV)	Errors Stat. (GeV)	Scale (GeV)	Total
$m_{\ell\ell}^{max}$	77.07	0.03	0.08	0.08
$m_{\ell\ell q}^{max}$	428.5	1.4	4.3	4.5
$m_{\ell q}^{low}$	300.3	0.9	3.0	3.1
$m_{\ell q}^{high}$	378.0	1.0	3.8	3.9
$m_{\ell\ell q}^{min}$	201.9	1.6	2.0	2.6
$m_{\ell\ell b}^{min}$	183.1	3.6	1.8	4.1
$m(\ell_L) - m(\tilde{\chi}^0_1)$	106.1	1.6	0.1	1.6
$m_{\ell\ell}^{max}(\tilde{\chi}^0_4)$	280.9	2.3	0.3	2.3
$m_{\tau\tau}^{max}$	80.6	5.0	0.8	5.1
$m(\tilde{g}) - 0.99 \times m(\tilde{\chi}^0_1)$	500.0	2.3	6.0	6.4
$m(\tilde{q}_R) - m(\tilde{\chi}^0_1)$	424.2	10.0	4.2	10.9
$m(\tilde{g}) - m(\tilde{b}_1)$	103.3	1.5	1.0	1.8
$m(\tilde{g}) - m(\tilde{b}_2)$	70.6	2.5	0.7	2.6

Parameter	Expected precision (300 fb^{-1})
m_0	± 2%
$m_{1/2}$	± 0.6%
tan(β)	± 9%
A_0	± 16%

ILC Potential in the non-coloured SUSY sector

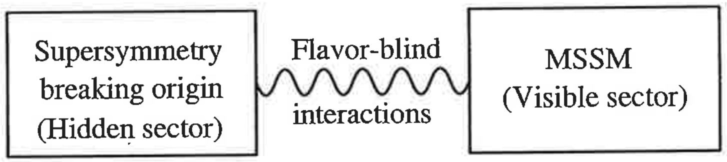


An example of the possible accuracy of the determination of the smuon mass and neutralino mass, for an e^+e^- linear collider with $\sqrt{s} = 500$ GeV and $\mathcal{L} = 100$ fb^{-1}.

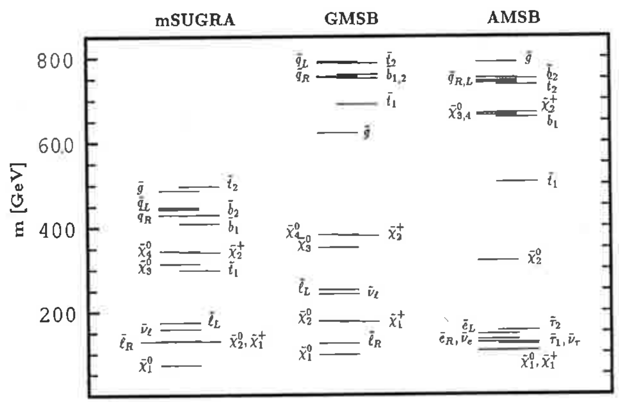


Energy spectrum $(E_{e^-} - E_{e^+})_{e_R^-} - (E_{e^-} - E_{e^+})_{e_L^-}$ for $e_{R,L}^- e^+ \to \tilde{e}_R \tilde{e}_L$ in the model SPS 1a at $\sqrt{s}$=500 GeV, $\mathcal{L} = 500$ fb^{-1}.

- SUSY discovery beginning of extensive and exciting experimental research program.
- Crucial to make accurate measurements of SUSY masses and couplings to constrain SUSY-breaking mechanism.



- Different models of SUSY-breaking can give rise to very different sparticle mass spectra:



- (GMSB: Gauge Mediated SUSY Breaking; AMSB: Anomaly Mediated SUSY Breaking)
- Understanding SUSY-breaking mechanism will open window onto physics at Planck or GUT scales.

Epilogue

The theoretical speculations about physics beyond the Standard Model discussed in this lecture course have centered around supersymmetry and grand unified theories. However, even much more radical changes could happen as we approach a new frontier of high-energy physics:

- qualitatively new degrees of freedom could appear, like strings or extra dimensions.
- symmetries of the Standard Model, like baryon and lepton number conservation could be broken.
- "sacred principles" like locality, micro-causality or CPT invariance could be violated.
- theoretical frameworks like general relativity and quantum mechanics might break down under certain conditions.

Physics beyond the Standard Model will be complex and maybe confusing, with new interactions and a rich phenomenological structure. The accelerators and experiments planned and envisaged for the next generation will provide the tools required to unravel the structure of fundamental physics at the TeV scale and – hopefully – beyond.

Bibliography

- General textbooks
  - M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," *Addison-Wesley (1995).*
  - V. D. Barger and R. J. Phillips, "Collider Physics," *Addison-Wesley (1987).*
- QCD
  - R. K. Ellis, W. J. Stirling and B. R. Webber, "QCD And Collider Physics," Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. **8** (1996) 1.
  - D. E. Soper, "Basics of QCD perturbation theory," arXiv:hep-ph/0011256.
  - G. Sterman, "Partons, factorization and resummation," arXiv:hep-ph/9606312.
  - R. Brock *et al.* [CTEQ Collaboration], "Handbook of perturbative QCD; Version 1.1: September 1994,"
  - M. H. Seymour, "Topics in standard model phenomenology," *In *Chilton 1999, School for young high energy physicists**.
- Electroweak Physics
  - H. Spiesberger, M. Spira and P. M. Zerwas, "The standard model: Physical basis and scattering experiments," arXiv:hep-ph/0011255.
- Higgs Physics
  - M. Spira and P. M. Zerwas, "Electroweak symmetry breaking and Higgs physics," arXiv:hep-ph/9803257.
  - S. Dawson, "Introduction to electroweak symmetry breaking," arXiv:hep-ph/9901280.

- Supersymmetry
  - J. R. Ellis, "Beyond the standard model for hill-walkers," arXiv:hep-ph/9812235; "Supersymmetry for Alp hikers," arXiv:hep-ph/0203114.
  - M. E. Peskin, "Beyond the standard model," arXiv:hep-ph/9705479.
  - H. Dreiner, "Hide and seek with supersymmetry," arXiv:hep-ph/9902347.