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Mixing Renormalization in Majorana Neutrino Theories

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ABSTRACT

The renormalization of general theories with inter-family mixing of Dirac and/or Majorana fermions is studied at the one-loop electroweak order. The phenomenological significance of the mixing-matrix renormalization is discussed, within the context of models based on the $SU(2)_L \otimes U(1)_Y$ gauge group. The effect of radiative neutrino masses present in these models is naturally taken into account in this formulation. As an example, charged-lepton universality in pion decays is investigated in the heavy-neutrino limit. Non-decoupling heavy-neutrino effects induced by mixing renormalization are found to considerably affect the predictions in these new-physics scenarios.

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1 Introduction

Mixing effects have played a crucial rôle in the understanding of various aspects of particle phenomenology. The most well-known examples established by experiment are the $K^0-\bar{K}^0$ and $B^0-\bar{B}^0$ mixings [1], which originate from the mixing of quarks [2]. Furthermore, several astrophysical problems, including the solar-neutrino-deficit puzzle, may be explained by assuming that there is also mixing in the lepton sector leading to neutrino oscillations [3]. In the Standard Model (SM), the mixing between the photon and the Z boson is implemented at the loop level, and precision test are becoming sensitive to this effect. In addition, many of the proposed extensions of the SM predict the possibility of mixings between scalar, fermion, and vector fields. In all these cases, the origin of the mixing is related to the rotation between weak and mass eigenstates.

In order that a quantum theory yields precise quantitative predictions, it must be renormalizable. The renormalization of the SM has been established [4] and elaborated [5,6] a long time ago. It was noticed [7] that the Cabibbo-Kobayashi-Maskawa (CKM) [8] matrix must be included in the renormalization programme as well. However, this effect has been found to be insignificant in the SM [9]. The reason is that the mass differences between the down-type quarks are small compared to the electroweak scale, so that the CKM matrix can effectively be taken to be diagonal. The situation should be very different in new-physics scenarios with large inter-family mixings. An attractive solution to the problem of the smallness in mass of the known neutrinos can arise from certain $SO(10)$ grand unified models [10] and/or E_6 superstring-inspired theories [11], which predict see-saw-type neutrino mass matrices with large Dirac components [12].

The renormalization of mixing effects in extensions of the SM has not yet been addressed in the literature. In this paper, we shall take the first step in this direction by elaborating the renormalization of general theories [13] with Dirac and/or Majorana neutrinos. Our formulation will naturally include radiative neutrino-mass contributions [14,15]. These considerations will affect a number of low-energy and LEP1/SLC electroweak observables which have been utilized to establish bounds on the parameter space of these models. For illustration, we shall estimate the size of charged-lepton non-universality in pion decays. Specifically, we shall consider the heavy-neutrino limit of the observable $R_\pi = \Gamma(\pi^+ \rightarrow e^+ \nu)/\Gamma(\pi^+ \rightarrow \mu^+ \nu)$.

This paper is organized as follows. In Section 2, we shall introduce the formalism of mass, wave-function, and mixing-matrix renormalization in general models with Dirac and/or Majorana neutrinos and derive the corresponding counterterm (CT) Lagrangian. In Section 3, we shall determine the CT's for the Dirac case in the on-shell renormalization scheme. Special attention is paid to the balance of the numbers of CT's and renormalization conditions. These considerations are extended to the case of Majorana neutrinos in Section 4. In Section 5, we shall discuss the nature of the mixing matrices and their relationships in the framework of $SU(2)_L \otimes U(1)_Y$ theories. The renormalization of these mixing matrices will then be performed in Section 6. We shall see that those relationships among the mixing matrices that are enforced by the unitarity of the theory carry over to the one-loop level, while additional identities related to other symmetries are violated by

ultraviolet divergences. As an application, in Section 7, we shall quantitatively analyze $R_\pi = \Gamma(\pi^+ \rightarrow e^+\nu)/\Gamma(\pi^+ \rightarrow \mu^+\nu)$ in the heavy-neutrino limit. Our conclusions will be summarized in Section 8.

2 One-loop renormalization

We consider a general model with N_f fermions, $f = (f_1, \dots, f_{N_f})$, which may be of Dirac and/or Majorana type. As usual, we define the left- and right-handed components as $f_{L,R} = P_{L,R}f$, where $P_{L,R} = (1 \mp \gamma_5)/2$ are the corresponding chirality projection operators. We denote the weak eigenstates and their mass matrix by a prime. Bare parameters carry the superscript “0”. The bare kinetic Lagrangian in the weak basis is given by

$$\mathcal{L}_{kin}^0 = i\bar{f}_L^0 \not{\partial} f_L^0 + i\bar{f}_R^0 \not{\partial} f_R^0 - \bar{f}_L^0 M'^0 f_R^0 - \bar{f}_R^0 M'^0 f_L^0, \quad (2.1)$$

where $\mathbf{1}$ is the unity matrix in flavour space and M'^0 is a complex, non-diagonal mass matrix. M'^0 can always be diagonalized with the help of two unitary matrices, U_L^0 and U_R^0 , via

$$M'^0 = U_L^0 M^0 U_R^{0\dagger}, \quad (2.2)$$

where M^0 is a non-negative, diagonal mass matrix. Consequently, the mass eigenstates are given by

$$f_{L,R}^0 = U_{L,R}^0 f'_{L,R}. \quad (2.3)$$

Next, we study the renormalization of this theory. For this end, we write the unrenormalized parameters and fields in terms of renormalized ones and CT's as

$$U_{L,R}^0 = U_{L,R} + \delta U_{L,R}, \quad (2.4)$$

$$M'^0 = M' + \delta M', \quad (2.5)$$

$$M^0 = M + \delta M, \quad (2.6)$$

$$f_{L,R}^0 = Z_{L,R}^{1/2} f'_{L,R} = \left(1 + \frac{1}{2}\delta Z'^{L,R}\right) f'_{L,R}, \quad (2.7)$$

$$f_{L,R}^0 = Z_{L,R}^{1/2} f_{L,R} = \left(1 + \frac{1}{2}\delta Z^{L,R}\right) f_{L,R}. \quad (2.8)$$

The renormalization is arranged so that the basic structure of the theory is preserved through the order considered. $U_{L,R}$ are still unitary, so that

$$U_{L,R}\delta U_{L,R}^\dagger + \delta U_{L,R}U_{L,R}^\dagger = 0 \quad (2.9)$$

is satisfied up to terms of $\mathcal{O}(\delta U_{L,R}^2)$. Similarly to M^0 , M is diagonal with non-negative eigenvalues, and Eqs. (2.2) and (2.3) are mapped into

$$\begin{aligned} M &= U_L M' U_R^\dagger, \\ f_{L,R} &= U_{L,R} f'_{L,R}, \end{aligned} \quad (2.10)$$

respectively. It is sufficient to consider $\delta U_{L,R}$, δM , and $Z_{L,R}^{1/2}$, since the renormalization constants connected with the weak eigenstates are determined by

$$\delta M' = U_L^\dagger M \delta U_R + U_L^\dagger \delta M U_R + \delta U_L^\dagger M U_R, \quad (2.11)$$

$$Z_{L,R}^{1/2} = U_{L,R}^{0\dagger} Z_{L,R}^{1/2} U_{L,R}, \quad (2.12)$$

Equation (2.12) is nicely illustrated in the following diagrams:

$$\begin{array}{ccc} f_L^{0\dagger} & \xleftarrow{U_L^{0\dagger}} & f_L^0 \\ Z_L^{1/2} \uparrow & & \uparrow Z_L^{1/2} \\ f_L' & \xrightarrow{U_L} & f_L \end{array} \quad \begin{array}{ccc} f_R^{0\dagger} & \xleftarrow{U_R^{0\dagger}} & f_R^0 \\ Z_R^{1/2} \uparrow & & \uparrow Z_R^{1/2} \\ f_R' & \xrightarrow{U_R} & f_R \end{array} \quad (2.13)$$

In the following, we shall work in the mass basis. In order to find the appropriate CT Lagrangian, we eliminate the bare masses and fields in Eq. (2.1),

$$\begin{aligned} \mathcal{L}_{kin}^0 = & i \bar{f}_L Z_L^{1/2\dagger} Z_L^{1/2} \not{\partial} f_L + i \bar{f}_R Z_R^{1/2\dagger} Z_R^{1/2} \not{\partial} f_R - \bar{f}_L Z_L^{1/2\dagger} (M + \delta M) Z_R^{1/2} f_R \\ & - \bar{f}_R Z_R^{1/2\dagger} (M + \delta M) Z_L^{1/2} f_L. \end{aligned} \quad (2.14)$$

Up to higher-order terms, we then have

$$\mathcal{L}_{kin}^0 = \mathcal{L}_{kin} + \delta \mathcal{L}_{kin}, \quad (2.15)$$

where \mathcal{L}_{kin} is the renormalized Lagrangian and

$$\begin{aligned} \delta \mathcal{L}_{kin} = & \frac{i}{2} \bar{f}_L (\delta Z^L + \delta Z^{L\dagger}) \not{\partial} f_L + \frac{i}{2} \bar{f}_R (\delta Z^R + \delta Z^{R\dagger}) \not{\partial} f_R \\ & - \bar{f}_L \left(\delta M + \frac{1}{2} M \delta Z^R + \frac{1}{2} \delta Z^{L\dagger} M \right) f_R - \bar{f}_R \left(\delta M + \frac{1}{2} M \delta Z^L + \frac{1}{2} \delta Z^{R\dagger} M \right) f_L \end{aligned} \quad (2.16)$$

is the CT Lagrangian.

The mixing-matrix renormalization is only important if unitary matrices different from unity appear in the couplings, such as the CKM matrix in the SM charged-current interaction or the mixing matrices which give rise to flavour-changing neutral currents in new-physics scenarios. To elucidate this point, we consider the charged-current interaction in a $SU(2)_L \otimes U(1)_Y$ model with right-handed neutrinos. In this model, the bare Lagrangian for the interaction of the W boson with the charged leptons, l_i , and the neutrinos, n_i , is given by

$$\mathcal{L}_W^0 = -\frac{g^0}{\sqrt{2}} (W_\mu^-)^0 J^{0\mu} + \text{h.c.}, \quad (2.17)$$

where g is the $SU(2)_L$ coupling constant and

$$J_\mu^0 = \bar{l}_L^0 \mathbf{1} \gamma_\mu n_L^0 = \bar{l}_L^0 B^0 \gamma_\mu n_L^0. \quad (2.18)$$

Here, B^0 is a CKM-type matrix in the lepton sector defined as

$$B^0 = U_L^{0,l} U_L^{0,n\dagger}, \quad (2.19)$$

where $U_L^{0,l}$ and $U_L^{0,n}$ are the bare unitary matrices that participate in the diagonalization of the charged-lepton and neutrino mass matrices, respectively. Notice that the renormalization of the weak coupling g and the W -boson wave-function are universal and determined by other renormalization conditions [5], so that we may separately consider the renormalization of the charged current J^μ . Substituting Eq. (2.4) in Eq. (2.19), we find

$$B^0 = B + \delta B, \quad (2.20)$$

where

$$\delta B = BU_L^n \delta U_L^{n\dagger} + \delta U_L^l U_L^{l\dagger} B. \quad (2.21)$$

From Eq. (2.9) we know that $U_L^n \delta U_L^{n\dagger}$ and $\delta U_L^l U_L^{l\dagger}$ are antihermitean. Using Eqs. (2.8) and (2.20), we can write Eq. (2.18) as $J_\mu^0 = J_\mu + \delta J_\mu$, where

$$\begin{aligned} \delta J_\mu = & \bar{l}_L \gamma_\mu \left\{ \frac{1}{4} B (\delta Z^{n,L} + \delta Z^{n,L\dagger}) + \frac{1}{4} (\delta Z^{l,L} + \delta Z^{l,L\dagger}) B \right. \\ & + B \left[\frac{1}{4} (\delta Z^{n,L} - \delta Z^{n,L\dagger}) + U_L^n \delta U_L^{n\dagger} \right] \\ & \left. + \left[-\frac{1}{4} (\delta Z^{l,L} - \delta Z^{l,L\dagger}) + \delta U_L^l U_L^{l\dagger} \right] B \right\} n_L. \end{aligned} \quad (2.22)$$

Here, we have decomposed $\delta Z^{n,L}$ and $\delta Z^{l,L}$ into hermitean and antihermitean parts and collected all antihermitean CT's within square brackets. Obviously, the presence of δB is indispensable to cancel the ultraviolet (UV) divergences contained in the antihermitean parts of $\delta Z^{n,L}$ and $\delta Z^{l,L}$. This will also be illustrated in Section 7, where we shall calculate the one-loop correction induced in R_π by massive neutrinos. We may fix δU_L^n and δU_L^l by requiring that the square brackets in Eq. (2.22) vanish. This leads to

$$\begin{aligned} \delta U_L^n &= \frac{1}{4} (\delta Z^{n,L} - \delta Z^{n,L\dagger}) U_L^n, \\ \delta U_L^l &= \frac{1}{4} (\delta Z^{l,L} - \delta Z^{l,L\dagger}) U_L^l. \end{aligned} \quad (2.23)$$

A similar condition was proposed in Ref. [9] in connection with the CKM mixing of the SM. In order to enforce the UV finiteness of physical observables, it would be sufficient to replace the parentheses in Eq. (2.23) with their UV-divergent parts, evaluated at some renormalization scale μ . This would correspond to the $\overline{\text{MS}}$ renormalization prescription. By the same token, we might add finite, antihermitean matrices, c_{ij}^n and c_{ij}^l , to the terms contained within the parentheses of Eq. (2.23) and fix them by imposing some additional renormalization conditions. This would not affect the hermitean parts of $\delta Z^{n,L}$ and $\delta Z^{l,L}$, and in particular their diagonal elements, which are arranged so that the fermion propagators have unit residues. Although the use of Eq. (2.23) is not compelling, this prescription seems natural and we shall adopt it in the remainder of this paper. A detailed study of the implications of general mixing-matrix renormalization schemes will be given elsewhere.

3 Dirac case

In the following, we shall study the mass and wave-function renormalizations in a general theory involving the mixing of N_f Dirac fermions. We denote the Dirac fermions by f_i , with $i = 1, \dots, N_f$. There are two sources of imaginary contributions to the bare amplitudes. They can arise either from the possibility of on-shell cuts through the loop amplitudes (absorptive parts) or from complex mixing parameters (CKM-type couplings). Because of the hermiticity of the bare and renormalized Lagrangians, the CT Lagrangian must be hermitean, too. Therefore, only the dispersive parts of the one-loop amplitudes can participate in the renormalization procedure. Consequently, we shall only consider the dispersive parts of the two-point functions in the following. Furthermore, it will be understood that complex conjugation acts on the coupling constants in the flavour space.

We start by considering the unrenormalized $f_j \rightarrow f_i$ transition amplitude. Its most general form in compliance with hermiticity is*

$$\Sigma_{ij}(\not{q}) = \not{q} P_L \Sigma_{ij}^L(q^2) + \not{q} P_R \Sigma_{ij}^R(q^2) + P_L \Sigma_{ij}^D(q^2) + P_R \Sigma_{ji}^{D*}(q^2), \quad (3.1)$$

where

$$\Sigma_{ij}^L(q^2) = \Sigma_{ji}^{L*}(q^2), \quad \Sigma_{ij}^R(q^2) = \Sigma_{ji}^{R*}(q^2). \quad (3.2)$$

The renormalized counterparts will be denoted by a hat. As per construction, they will satisfy relations similar to Eqs. (3.1) and (3.2).

From the CT Lagrangian (2.16), we read off the relations between the bare and renormalized transition amplitudes, *viz.*

$$\begin{aligned} \hat{\Sigma}_{ij}^L(q^2) &= \Sigma_{ij}^L(q^2) + \frac{1}{2}(\delta Z_{ij}^L + \delta Z_{ji}^{L*}), \\ \hat{\Sigma}_{ij}^R(q^2) &= \Sigma_{ij}^R(q^2) + \frac{1}{2}(\delta Z_{ij}^R + \delta Z_{ji}^{R*}), \\ \hat{\Sigma}_{ij}^D(q^2) &= \Sigma_{ij}^D(q^2) - \frac{1}{2}(m_i \delta Z_{ij}^L + m_j \delta Z_{ji}^{R*}) - \delta_{ij} \delta m_i. \end{aligned} \quad (3.3)$$

Next, we evaluate the renormalization constants by imposing the on-shell renormalization conditions on the renormalized transition amplitudes. Specifically, the non-diagonal elements of δZ_{ij}^L and δZ_{ij}^R are determined by requiring that $\hat{\Sigma}_{ij}(\not{q})$ be diagonal if the external lines are put on their mass shells, while the diagonal elements are fixed in such a way that the residues of the renormalized propagators are equal to unity, *i.e.*,

$$\hat{\Sigma}_{ij}(\not{q}) u_j(q) = 0, \quad (3.4)$$

$$\bar{u}_i(q) \hat{\Sigma}_{ij}(\not{q}) = 0, \quad (3.5)$$

$$\frac{1}{\not{q} - m_i} \hat{\Sigma}_{ii}(\not{q}) u_i(q) = 0, \quad (3.6)$$

$$\bar{u}_i(q) \hat{\Sigma}_{ii}(\not{q}) \frac{1}{\not{q} - m_i} = 0. \quad (3.7)$$

*This form generalizes the one used in Ref. [9], which is specific for the SM.

Conditions (3.4)–(3.7) imply that

$$m_j \hat{\Sigma}_{ij}^L(m_j^2) + \hat{\Sigma}_{ji}^{D*}(m_j^2) = 0, \quad (3.8)$$

$$m_j \hat{\Sigma}_{ij}^R(m_j^2) + \hat{\Sigma}_{ij}^D(m_j^2) = 0, \quad (3.9)$$

$$m_i \hat{\Sigma}_{ij}^L(m_i^2) + \hat{\Sigma}_{ij}^D(m_i^2) = 0, \quad (3.10)$$

$$m_i \hat{\Sigma}_{ij}^R(m_i^2) + \hat{\Sigma}_{ji}^{D*}(m_i^2) = 0, \quad (3.11)$$

$$\begin{aligned} \hat{\Sigma}_{ii}^L(m_i^2) + \hat{\Sigma}_{ii}^R(m_i^2) + 2m_i^2 (\hat{\Sigma}_{ii}^{L'}(m_i^2) + \hat{\Sigma}_{ii}^{R'}(m_i^2)) \\ + 2m_i (\hat{\Sigma}_{ii}^{D'}(m_i^2) + \hat{\Sigma}_{ii}^{D*'}(m_i^2)) = 0, \end{aligned} \quad (3.12)$$

where $\Sigma'(q^2) = d\Sigma(q^2)/dq^2$. For $i \neq j$, we obtain from Eqs. (3.8)–(3.11)

$$\begin{aligned} \delta Z_{ij}^L &= \frac{2}{m_i^2 - m_j^2} [m_j^2 \Sigma_{ij}^L(m_j^2) + m_i m_j \Sigma_{ij}^R(m_j^2) + m_i \Sigma_{ij}^D(m_j^2) + m_j \Sigma_{ji}^{D*}(m_j^2)], \\ \delta Z_{ij}^R &= \frac{2}{m_i^2 - m_j^2} [m_i m_j \Sigma_{ij}^L(m_j^2) + m_j^2 \Sigma_{ij}^R(m_j^2) + m_j \Sigma_{ij}^D(m_j^2) + m_i \Sigma_{ji}^{D*}(m_j^2)]. \end{aligned} \quad (3.13)$$

In the diagonal case $i = j$, the number of renormalization constants to be determined may be reduced by exploiting the following symmetry present in the CT Lagrangian (2.14):

$$Z_{Lij}^{1/2} \rightarrow e^{i\theta_i} Z_{Lij}^{1/2}, \quad Z_{Rij}^{1/2} \rightarrow e^{i\theta_i} Z_{Rij}^{1/2}, \quad (3.14)$$

where θ_i are real phases. In this way, we may, *e.g.*, arrange for all δZ_{ii}^R to be real. Employing Eqs. (3.8)–(3.12), we then obtain

$$\begin{aligned} \delta Z_{ii}^L &= -\Sigma_{ii}^L(m_i^2) + \frac{1}{m_i} [\Sigma_{ii}^D(m_i^2) - \Sigma_{ii}^{D*}(m_i^2)] \\ &\quad - m_i^2 [\Sigma_{ii}^{L'}(m_i^2) + \Sigma_{ii}^{R'}(m_i^2)] - m_i [\Sigma_{ii}^{D'}(m_i^2) + \Sigma_{ii}^{D*'}(m_i^2)], \end{aligned} \quad (3.15)$$

$$\delta Z_{ii}^R = -\Sigma_{ii}^R(m_i^2) - m_i^2 [\Sigma_{ii}^{L'}(m_i^2) + \Sigma_{ii}^{R'}(m_i^2)] - m_i [\Sigma_{ii}^{D'}(m_i^2) + \Sigma_{ii}^{D*'}(m_i^2)], \quad (3.16)$$

$$\delta m_i = \frac{1}{2} m_i [\Sigma_{ii}^L(m_i^2) + \Sigma_{ii}^R(m_i^2)] + \frac{1}{2} [\Sigma_{ii}^D(m_i^2) + \Sigma_{ii}^{D*}(m_i^2)]. \quad (3.17)$$

At this stage, it is instructive to investigate the mass-degenerate limit of two fermions, f_i and f_j . We observe that δZ_{ij}^L and δZ_{ij}^R are singular in the limit $m_i \rightarrow m_j$. However, due to condition (2.23), only the hermitean parts of δZ_{ij}^L and δZ_{ij}^R occur in the renormalization of physical observables. For $m_i = m_j$, we find

$$\begin{aligned} \frac{1}{2} (\delta Z_{ij}^L + \delta Z_{ji}^{L*}) &= -\Sigma_{ij}^L(m_j^2) - m_j^2 [\Sigma_{ij}^{L'}(m_j^2) + \Sigma_{ij}^{R'}(m_j^2)] \\ &\quad - m_j [\Sigma_{ij}^{D'}(m_j^2) + \Sigma_{ij}^{D*'}(m_j^2)], \end{aligned} \quad (3.18)$$

$$\begin{aligned} \frac{1}{2} (\delta Z_{ij}^R + \delta Z_{ji}^{R*}) &= -\Sigma_{ij}^R(m_j^2) - m_j^2 [\Sigma_{ij}^{L'}(m_j^2) + \Sigma_{ij}^{R'}(m_j^2)] \\ &\quad - m_j [\Sigma_{ij}^{D'}(m_j^2) + \Sigma_{ij}^{D*'}(m_j^2)], \end{aligned} \quad (3.19)$$

which are indeed finite. For $i = j$, Eqs. (3.18) and (3.19) coincide with the hermitean parts of Eqs. (3.15) and (3.16), respectively, as they should.

Finally, we verify that the number of renormalization conditions equals the number of CT's. To be specific, in a model with N_f fermions, there are $4N_f$ real conditions for all $i = j$ (Eqs. (3.8)–(3.11) collapse to three real conditions), and $4N_f(N_f - 1)$ real conditions for all $i \neq j$ (namely Eqs. (3.8)–(3.11)). Thus, the total number of renormalization conditions is $4N_f^2$. On the other hand, counting the number of independent real CT's, we have N_f mass CT's and $4N_f^2$ wave-function renormalization constants (namely δZ_{ij}^L , δZ_{ij}^R , and their complex conjugates). From the number of CT's one has to subtract the N_f phases, θ_i , in Eq. (3.14), which can be used, *e.g.*, to render δZ_{ii}^R real. As a result, the total number of independent real CT's is $4N_f^2$ [5], which is equal to the one of the real renormalization conditions.

4 Majorana case

In this section, we shall study the renormalization of a general model with N_f Majorana neutrinos. In a way, this is a generalization of the Dirac case considered in the previous section, since a Dirac fermion may always be represented as a pair of mass-degenerate Majorana neutrinos. By the same token, it is possible to describe the mixing of Dirac and Majorana neutrinos. The kinetic Lagrangian of the model is given by

$$\mathcal{L}_{kin}^0 = \frac{1}{2} \left(i \bar{f}_L^0 \not{\partial} f_L^0 + i \bar{f}_R^0 \not{\partial} f_R^0 - \bar{f}_L^0 M^0 f_R^0 - \bar{f}_R^0 M^0 f_L^0 \right). \quad (4.1)$$

The bare and renormalized fermion fields satisfy the Majorana constraint,

$$f_L^0 = (f_R^0)^C, \quad f_L = (f_R)^C, \quad (4.2)$$

where C stands for the charge-conjugation operation. Using Eq. (2.8), we thus find that

$$Z_L^{1/2} = Z_R^{1/2*}. \quad (4.3)$$

Now, we can express the bare Lagrangian (4.1) in terms of renormalized quantities as

$$\begin{aligned} \mathcal{L}_{kin}^0 = & \frac{1}{2} \left[i \bar{f}_L Z_L^{1/2\dagger} Z_L^{1/2} \not{\partial} f_L + i \bar{f}_R Z_L^{1/2T} Z_L^{1/2*} \not{\partial} f_R - \bar{f}_L Z_L^{1/2\dagger} (M + \delta M) Z_L^{1/2*} f_R \right. \\ & \left. - \bar{f}_R Z_L^{1/2T} (M + \delta M) Z_L^{1/2} f_L \right]. \end{aligned} \quad (4.4)$$

Decomposing this as $\mathcal{L}_{kin}^0 = \mathcal{L}_{kin} + \delta \mathcal{L}_{kin}$, we obtain the CT Lagrangian,

$$\begin{aligned} \delta \mathcal{L}_{kin} = & \frac{i}{4} \bar{f}_L (\delta Z^L + \delta Z^{L\dagger}) \not{\partial} f_L + \frac{i}{4} \bar{f}_R (\delta Z^{L*} + \delta Z^{LT}) \not{\partial} f_R \\ & - \frac{1}{2} \bar{f}_L \left(\delta M + \frac{1}{2} M \delta Z^{L*} + \frac{1}{2} \delta Z^{L\dagger} M \right) f_R - \frac{1}{2} \bar{f}_R \left(\delta M + \frac{1}{2} M \delta Z^L + \frac{1}{2} \delta Z^{LT} M \right) f_L. \end{aligned} \quad (4.5)$$

The most general form of the $f_j \rightarrow f_i$ transition amplitude between fermionic Majorana states reads

$$\Sigma_{ij}(q) = \not{q} P_L \Sigma_{ij}^L(q^2) + \not{q} P_R \Sigma_{ij}^{L*}(q^2) + P_L \Sigma_{ij}^M(q^2) + P_R \Sigma_{ij}^{M*}(q^2), \quad (4.6)$$

where we have made use of the relations

$$\Sigma_{ij}^L(q^2) = \Sigma_{ij}^{R*}(q^2), \quad \Sigma_{ij}^M(q^2) = \Sigma_{ji}^M(q^2), \quad (4.7)$$

which follow from the Majorana condition (4.2).

From the CT Lagrangian (4.5), we may read off the relationships between $\Sigma_{ij}^L(q^2)$, $\Sigma_{ij}^M(q^2)$, and their renormalized counterparts, *viz.*

$$\hat{\Sigma}_{ij}^L(q^2) = \Sigma_{ij}^L(q^2) + \frac{1}{2}(\delta Z_{ij}^L + \delta Z_{ji}^{L*}), \quad (4.8)$$

$$\hat{\Sigma}_{ij}^M(q^2) = \Sigma_{ij}^M(q^2) - \frac{1}{2}(m_i \delta Z_{ij}^L + m_j \delta Z_{ji}^L) - \delta_{ij} m_i. \quad (4.9)$$

Imposing the on-shell renormalization conditions (3.4)–(3.7), we obtain the renormalization constants for $i \neq j$,

$$\delta Z_{ij}^L = \frac{2}{m_i^2 - m_j^2} [m_j^2 \Sigma_{ij}^L(m_j^2) + m_i m_j \Sigma_{ij}^{L*}(m_j^2) + m_i \Sigma_{ij}^M(m_j^2) + m_j \Sigma_{ij}^{M*}(m_j^2)], \quad (4.10)$$

and those for $i = j$,

$$\begin{aligned} \delta Z_{ii}^L &= -\Sigma_{ii}^L(m_i^2) - 2m_i^2 \Sigma_{ii}^{L'}(m_i^2) - m_i [\Sigma_{ii}^{M'}(m_i^2) + \Sigma_{ii}^{M*'}(m_i^2)] \\ &\quad + \frac{1}{2m_i} [\Sigma_{ii}^M(m_i^2) - \Sigma_{ii}^{M*}(m_i^2)], \end{aligned} \quad (4.11)$$

$$\delta m_i = m_i \Sigma_{ii}^L(m_i^2) + \frac{1}{2} [\Sigma_{ii}^M(m_i^2) + \Sigma_{ii}^{M*}(m_i^2)]. \quad (4.12)$$

A special situation arises if a Majorana neutrino is massless at tree level. In contrast to the SM Dirac case, this does not necessarily imply that the mass CT δm_i in Eq. (4.12) vanishes. In general, it will be positive and finite, *i.e.*, the Majorana neutrino receives a mass via loop effects. Various mechanisms for generating radiative neutrino masses have been suggested in the literature [14,15]; they are naturally implemented in our formulation.

Again, we may verify that the physically relevant hermitean part of δZ_{ij}^L is indeed finite in the limit $m_i \rightarrow m_j$,

$$\begin{aligned} \frac{1}{2}(\delta Z_{ij}^L + \delta Z_{ji}^{L*}) &= -\Sigma_{ij}^L(m_j^2) - m_j^2 [\Sigma_{ij}^{L'}(m_j^2) + \Sigma_{ij}^{L*'}(m_j^2)] - m_j [\Sigma_{ij}^{M'}(m_j^2) \\ &\quad + \Sigma_{ij}^{M*'}(m_j^2)]. \end{aligned} \quad (4.13)$$

Moreover, for $i = j$, we recover the hermitean part of Eq. (4.11),

$$\frac{1}{2}(\delta Z_{ii}^L + \delta Z_{ii}^{L*}) = -\Sigma_{ii}^L(m_i^2) - 2m_i^2 \Sigma_{ii}^{L'}(m_i^2) - m_i [\Sigma_{ii}^{M'}(m_i^2) + \Sigma_{ii}^{M*'}(m_i^2)]. \quad (4.14)$$

These observations reassure us of the self-consistency of our formalism.

Let us finally count the number of independent renormalization conditions and CT's in our N_f -Majorana-neutrino model. For $i \neq j$, we have only two independent complex equations or four real conditions resulting from Eqs. (3.8)–(3.11), due to the Majorana constraint (4.2). This gives $2N_f(N_f - 1)$, where we only consider the cases $i > j$, so as to avoid double-counting due to the hermiticity of the renormalization conditions. In addition, there are $3N_f$ real relations for the diagonal transitions. So, we obtain $2N_f^2 + N_f$ independent real renormalization conditions in total. On the other hand, taking the Majorana constraint (4.3) into account, we count the same number of independent CT's. In fact, there are $2N_f^2$ wave-function renormalizations, δZ_{ij}^L and δZ_{ij}^{L*} , and N_f mass CT's.

5 Mixing matrices in $SU(2)_L \otimes U(1)_Y$ theories

A minimal, renormalizable extension of the SM that can naturally accommodate heavy Majorana neutrinos is a model based on the $SU(2)_L \otimes U(1)_Y$ gauge group, in which lepton-number-violating $\Delta L = 2$ operators have been introduced in the Yukawa sector by the inclusion of a number of N_R isosinglet neutrinos. The latter are sometimes called right-handed neutrinos because they are blind under $SU(2)_L$. Here, we adopt the conventions of Ref. [15] and denote the bare isosinglet weak eigenstates by ν_R^0 (with $i = 1, \dots, N_R$). Furthermore, we assume a number of N_G weak isodoublets. The quark sector of this model is similar to that of the minimal SM. The bare Yukawa Lagrangian that describes the neutrino sector reads

$$\mathcal{L}_Y^{0,\nu} = -\frac{1}{2}(\bar{\nu}_L^0, \bar{\nu}_R^{0C}) M^{0,\nu} \begin{pmatrix} \nu_L^{0C} \\ \nu_R^0 \end{pmatrix} + \text{h.c.}, \quad (5.1)$$

where $M^{0,\nu}$ is a complex, symmetric mass matrix of the form

$$M^{0,\nu} = \begin{pmatrix} 0 & m_D^0 \\ m_D^{0T} & m_M^0 \end{pmatrix}. \quad (5.2)$$

It can always be diagonalized through the unitary transformation

$$U^{0,\nu T} M^{0,\nu} U^{0,\nu} = M^{0,\nu}. \quad (5.3)$$

The nonnegative, diagonal matrix $M^{0,\nu}$ contains the bare neutrino-mass eigenvalues. The corresponding mass eigenstates are given by

$$\begin{pmatrix} \nu_L^0 \\ \nu_R^{0C} \end{pmatrix}_i = \sum_{j=1}^{N_G+N_R} U_{ij}^{0,\nu*} n_{Lj}^0, \quad \begin{pmatrix} \nu_L^{0C} \\ \nu_R^0 \end{pmatrix}_i = \sum_{j=1}^{N_G+N_R} U_{ij}^{0,\nu} n_{Rj}^0. \quad (5.4)$$

Here, the first N_G mass eigenstates, $\nu_i \equiv n_i$ ($i = 1, \dots, N_G$), are identified with the ordinary light neutrinos (if $N_G = 3$), and the remaining N_R states, $N_i \equiv n_{i+N_G}$ ($i = 1, \dots, N_R$), are the new neutral leptons predicted by the model. These neutral leptons are the heavy

Majorana neutrinos, which should be heavier than the Z boson, as they have escaped detection in production experiments at LEP1/SLC. The diagonalization of the charged-lepton mass matrix proceeds as outlined in Section 2.

In our minimal model, quantum mixing effects enter via the interactions of the Majorana neutrinos, n_i , and charged leptons, l_i , with the intermediate bosons, W^\pm and Z , and the Higgs boson, H . In the 't Hooft-Feynman gauge, the bare Lagrangians of these interactions are [15]

$$\mathcal{L}_W^0 = -\frac{g^0}{\sqrt{2}}(W_\mu^-)^0 \sum_{l=1}^{N_G} \sum_{j=1}^{N_G+N_R} \bar{l}^0 B_{lj}^0 \gamma^\mu P_L n_j^0 + \text{h.c.}, \quad (5.5)$$

$$\mathcal{L}_{G^\pm}^0 = -\frac{g^0}{\sqrt{2}M_W^0}(G^\pm)^0 \sum_{l=1}^{N_G} \sum_{j=1}^{N_G+N_R} \bar{l}^0 B_{lj}^0 (m_l^0 P_L - m_j^0 P_R) n_j^0 + \text{h.c.}, \quad (5.6)$$

$$\mathcal{L}_Z^0 = -\frac{g^0}{4c_w^0} Z_\mu^0 \sum_{i,j=1}^{N_G+N_R} \bar{n}_i^0 \gamma^\mu (i\Im m C_{ij}^0 - \gamma_5 \Re C_{ij}^0) n_j^0, \quad (5.7)$$

$$\mathcal{L}_{G^0}^0 = \frac{ig^0}{4M_W^0} (G^0)^0 \sum_{i,j=1}^{N_G+N_R} \bar{n}_i^0 [\gamma_5 (m_i^0 + m_j^0) \Re C_{ij}^0 + i(m_j^0 - m_i^0) \Im m C_{ij}^0] n_j^0, \quad (5.8)$$

$$\mathcal{L}_H^0 = -\frac{g^0}{4M_W^0} H^0 \sum_{i,j=1}^{N_G+N_R} \bar{n}_i^0 [(m_i^0 + m_j^0) \Re C_{ij}^0 + i\gamma_5 (m_j^0 - m_i^0) \Im m C_{ij}^0] n_j^0, \quad (5.9)$$

where G^\pm and G^0 are the charged and neutral Higgs-Kibble ghosts, respectively, g is the $SU(2)_L$ coupling constant, c_w is the cosine of the weak mixing angle, and B and C are $N_G \times (N_G + N_R)$ and $(N_G + N_R) \times (N_G + N_R)$ mixing matrices, respectively. The bare matrices are defined as

$$B_{lj}^0 = \sum_{k=1}^{N_G} V_{lk}^{0,l} U_{kj}^{0,\nu*}, \quad (5.10)$$

$$C_{ij}^0 = \sum_{k=1}^{N_G} U_{ki}^{0,\nu} U_{kj}^{0,\nu*}. \quad (5.11)$$

Note that C_{ij}^0 is hermitean. Comparing Eqs. (2.19) and (5.10), we may identify

$$V^{0,l} = U_L^{0,l}, \quad U^{0,\nu} = U_L^{0,\nu T}. \quad (5.12)$$

Furthermore, B^0 and C^0 satisfy a number of identities, which will turn out to be crucial to warranty the renormalizability of our model, namely [15–17]

$$\sum_{k=1}^{N_G+N_R} B_{l_1 k}^0 B_{l_2 k}^{0*} = \delta_{l_1 l_2}, \quad (5.13)$$

$$\sum_{k=1}^{N_G+N_R} C_{ik}^0 C_{kj}^0 = C_{ij}^0, \quad (5.14)$$

$$\sum_{k=1}^{N_G+N_R} B_{lk}^0 C_{ki}^0 = B_{li}^0, \quad (5.15)$$

$$\sum_{l=1}^{N_G} B_{li}^{0*} B_{lj}^0 = C_{ij}^0. \quad (5.16)$$

In addition, there are identities involving the Majorana-neutrino masses,

$$\sum_{k=1}^{N_G+N_R} m_k^0 C_{ik}^0 C_{jk}^0 = 0, \quad \sum_{k=1}^{N_G+N_R} m_k^0 B_{lk}^0 C_{ik}^0 = 0, \quad \sum_{k=1}^{N_G+N_R} m_k^0 B_{l_1 k}^0 B_{l_2 k}^0 = 0. \quad (5.17)$$

These relations signify the presence of lepton-number-violating interactions, *e.g.*, in the possible neutrinoless double-beta decay of a nucleus [18].

6 Renormalization of the mixing matrices

Inserting Eq. (2.23) into Eq. (2.21), we obtain a closed expression for the CT matrix of B ,

$$\delta B = \frac{1}{4}(\delta Z^{l,L} - \delta Z^{l,L\dagger}) B - \frac{1}{4} B (\delta Z^{n,L} - \delta Z^{n,L\dagger}). \quad (6.1)$$

By analogy, writing $C_{ij}^0 = C_{ij} + \delta C_{ij}$ and requiring that Eq. (5.16) also holds true for the renormalized quantities, *i.e.*, that

$$\sum_{l=1}^{N_G} B_{li}^* B_{lj} = C_{ij} \quad (6.2)$$

is correct to one loop, we find

$$\delta C = \frac{1}{4}(\delta Z^{n,L} - \delta Z^{n,L\dagger}) C - \frac{1}{4} C (\delta Z^{n,L} - \delta Z^{n,L\dagger}). \quad (6.3)$$

With the help of Eqs. (6.1) and (6.3), we may verify that the renormalized versions of Eqs. (5.13)–(5.15),

$$\sum_{k=1}^{N_G+N_R} B_{l_1 k} B_{l_2 k}^* = \delta_{l_1 l_2}, \quad \sum_{k=1}^{N_G+N_R} C_{ik} C_{kj} = C_{ij}, \quad \sum_{k=1}^{N_G+N_R} B_{lk} C_{ki} = B_{li}, \quad (6.4)$$

are valid at the one-loop level. On the other hand, the relations in Eq. (5.17) become UV divergent if we replace the bare parameters with their renormalized counterparts. This may be attributed to the fact that, in contrast to Eqs. (5.13)–(5.16), the relations in Eq. (5.17) are not enforced by unitarity.

7 The observable R_π

In order to illustrate the phenomenological significance of mixing-matrix renormalization, we shall now study possible violations of charged-current universality in the leptonic pion decays $\pi^+ \rightarrow l^+ \nu$ within the Majorana-neutrino mixing models described in Section 5. To simplify matters, we shall consider the limit where the novel Majorana neutrinos are much heavier than the intermediate bosons. We may then exploit the Goldstone-boson equivalence theorem [19] formulated in the 't Hooft-Feynman gauge and include in the loop amplitudes only the massive Higgs boson, H , the massless Goldstone bosons, G^\pm and G^0 , and the heavy Majorana neutrinos, N_i . In this way, we may extract the leading electroweak radiative corrections of $O(G_F m_{N_i}^2)$ and $O(G_F M_H^2)$, which occur because Majorana neutrinos and the Higgs boson do not decouple in the high-mass limit.

The decay $\pi^+(P) \rightarrow l^+(k) \nu(p)$ is described by the parton-model process $u\bar{d} \rightarrow l^+ \nu$ in connection with the hadronic matrix element $\langle 0 | \bar{d}(0) \gamma^\mu \gamma_5 u(0) | \pi^+(P) \rangle = f_\pi P^\mu$, where f_π is the pion decay constant. The corresponding vector-current matrix element is taken to be zero. The tree-level transition-matrix element reads

$$T_0 = -\frac{\pi\alpha}{s_w^2} f_\pi V_{ud}^* B_{l\nu}^* \frac{m_l}{M_W^2 - m_\pi^2} \bar{u}_\nu(p) P_R v_l(k), \quad (7.1)$$

where α is the fine-structure constant, V is the CKM matrix, and $c_w^2 = 1 - s_w^2 = M_Z^2/M_W^2$. In a full electroweak one-loop analysis, one needs to include the corrections related to the W -boson propagator, the Wud and $Wl\nu$ vertices, and the $udl\nu$ box as well as the renormalizations of M_W , α , s_w , the external quark and lepton fields, and the mixing matrices, V and B . For the time being, we ignore photon bremsstrahlung and corrections due to strong interactions. Neglecting the pion, quark, and external-lepton masses, the corrected matrix element takes the form

$$\mathcal{T} = T_0(1 + \delta_{ct}^{univ} + \delta_{ct}^l) + \mathcal{T}_v + \mathcal{T}_b, \quad (7.2)$$

where \mathcal{T}_v and \mathcal{T}_b are the one-particle-irreducible vertex and box amplitudes, respectively, and we distinguish between universal and lepton-flavour-dependent CT contributions. To simplify the notation, we include the propagator corrections in the universal CT. Specifically, we have

$$\begin{aligned} \delta_{ct}^{univ} = & \frac{\Pi_{WW}(0) - \delta M_W^2}{M_W^2} + 2\frac{\delta e}{e} - 2\frac{\delta s_w}{s_w} \\ & + \frac{1}{V_{ud}^*} \left[\delta V_{ud}^* + \frac{1}{2} (\delta Z_{d;d}^{L*} V_{ud_i}^* + V_{u_i;d}^* \delta Z_{u_i;u}^L) \right], \end{aligned} \quad (7.3)$$

$$\delta_{ct}^l = \frac{1}{B_{l\nu}^*} \left[\delta B_{l\nu}^* + \frac{1}{2} (\delta Z_{n_i;\nu}^{L*} B_{l n_i}^* + B_{l i;\nu}^* \delta Z_{l i;l}^L) \right], \quad (7.4)$$

where summation over the quark and lepton flavours u_i , d_i , l_i , and n_i is implied and [5]

$$\delta M_W^2 = \Re e \Pi_{WW}(M_W^2),$$

$$\begin{aligned}\frac{\delta e}{e} &= \frac{1}{2}\Pi'_{AA}(0) - \frac{s_w}{c_w} \frac{\Pi_{ZA}(0)}{M_Z^2}, \\ \frac{\delta s_w}{s_w} &= -\frac{1}{2} \frac{c_w^2}{s_w^2} \Re \left[\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2} \right].\end{aligned}\quad (7.5)$$

The transverse gauge-boson vacuum-polarization contributions induced by the leptons of our $SU(2)_L \otimes U(1)_Y$ model may be found in Ref. [17]. The calculation considerably simplifies in the heavy-neutrino limit, $m_{N_i} \gg M_W$. According to the equivalence theorem, it is then sufficient to compute the vertex and lepton-mixing diagrams depicted in Fig. 1, while $T_b = 0$. Furthermore, we may project out the relevant vertex form factor by

$$T_0^* T_\nu = |T_0|^2 (1 + \delta_\nu^l). \quad (7.6)$$

The present knowledge of the radiative corrections to $\Gamma(\pi^+ \rightarrow l^+ \nu)$ in the SM has been nicely summarized by Marciano and Sirlin [20]. The short-distance corrections and most uncertainties cancel in the ratio $R_\pi = \Gamma(\pi^+ \rightarrow e^+ \nu)/\Gamma(\pi^+ \rightarrow \mu^+ \nu)$. Apart from helicity-suppressed terms of $O[(\alpha/\pi)(m_\mu^2/m_\rho^2) \ln(m_\rho^2/m_\mu^2)]$, where the typical hadronic mass scale m_ρ is used as a demarcation between short- and long-distance loop corrections, only QED corrections survive. The latter consist of a pointlike-pion contribution [21] and a structure-dependent bremsstrahlung correction, which is suppressed by m_π^4/m_ρ^4 . The leading term of the pointlike-pion contribution is given by $-(3\alpha/2\pi) \ln(m_\mu^2/m_e^2) \approx -3.7\%$ and may be summed via the renormalization group; it makes up the bulk of the SM correction to R_π . According to Ref. [20], the current SM prediction is

$$R_\pi^{SM} = (1.2352 \pm 0.0005) \times 10^{-4}, \quad (7.7)$$

where the error is mainly due to model dependence. As we shall see in the following, the presence of heavy Majorana neutrinos might be manifested by a significant shift in the theoretical prediction of R_π relative to R_π^{SM} . In turn, confrontation of the modified prediction with experiment, together with similar analyses for other low-energy and LEP1/SLC observables, will allow one to improve the constraints on the parameter space of the Majorana-neutrino models under consideration. In our approximation, the one-loop-corrected value of R_π in the Majorana-neutrino scenario is given by

$$R_\pi = R_\pi^0 (1 + C), \quad (7.8)$$

where R_π^0 emerges from the SM tree-level result by multiplication with $|B_{e\nu}|^2/|B_{\mu\nu}|^2$ and

$$C = 2\Re(\delta_\nu^e + \delta_{ct}^e - \delta_\nu^\mu - \delta_{ct}^\mu). \quad (7.9)$$

The universal contribution δ_{ct}^{univ} has dropped out. Analytic results for δ_ν^l and δ_{ct}^l are listed in the Appendix. We note that δ_{ct}^{univ} and $\delta_\nu^l + \delta_{ct}^l$ are separately UV finite. The significance of mixing-matrix renormalization becomes apparent by observing that $\delta_\nu^l + \delta_{ct}^l$ contains the UV-divergent contribution from $\delta B_{l\nu}^*$. We may refine Eq. (7.8) by including the SM radiative corrections of Ref. [20]. For this end, we write $R_\pi^0 = (|B_{e\nu}|^2/|B_{\mu\nu}|^2) R_\pi^{SM}$, where R_π^{SM} is given by Eq. (7.7).

For clarity, we shall henceforth consider a very simple realization of a $SU(2)_L \otimes U(1)_Y$ model with Majorana neutrinos, which is of phenomenological interest and displays the essential features of mixing renormalization. We shall assume that the SM is extended by two heavy Majorana neutrinos, N_1 and N_2 , which only mix with the leptons of one family, the first one, say. The other two families are assumed to be standard. In the notation of Section 5, this corresponds to $N_G = 1$ and $N_R = 2$. The relevant mixing-matrix elements are determined by solving Eqs. (5.13)–(5.17), with the result that

$$\begin{aligned} |B_{e\nu}|^2 &= 1 - (s_L^{\nu_e})^2, & B_{eN_1} &= \frac{\rho^{1/4} s_L^{\nu_e}}{\sqrt{1 + \rho^{1/2}}}, & B_{eN_2} &= i\rho^{-1/4} B_{eN_1}, \\ C_{N_1 N_1} &= \frac{\rho^{1/2} (s_L^{\nu_e})^2}{1 + \rho^{1/2}}, & C_{N_2 N_2} &= \rho^{-1/2} C_{N_1 N_1}, & C_{N_1 N_2} &= -C_{N_2 N_1} = i\rho^{-1/4} C_{N_1 N_1}, \end{aligned} \quad (7.10)$$

where $\rho = m_{N_2}^2/m_{N_1}^2$ and $s_L^{\nu_e}$ measures the degree of light-heavy neutrino mixing [22]. Without loss of generality, we may assume that $\rho \geq 1$. As mentioned above, we set $s_L^{\nu_\mu} = s_L^{\nu_\tau} = 0$. Then, Eq. (7.9) becomes $C = 2\Re(\delta_v^e + \delta_{et}^e)$. This pattern of mixing may be motivated by the non-observation of the decay $\mu \rightarrow e\gamma$ or the absence of μ - e conversion in nuclei. Furthermore, a global analysis of low-energy data gives the following upper limits [23]:

$$(s_L^{\nu_e})^2 < 0.010, \quad (s_L^{\nu_\mu})^2 < 0.0020 \quad (7.11)$$

at the 95% confidence level. We should note that electroweak corrections, mainly those originating from the SM, are taken into account only for a limited number of low-energy observables, such as the muon decay width, in the evaluation of the bounds given in Eq. (7.11). Therefore, the so-derived bounds should be considered to be of tree-level accuracy.

The results for δ_v^l and δ_{et}^l given in the Appendix are valid for M_H arbitrary as long as $m_{N_1}, m_{N_2} \gg M_W$. If we are interested in the non-decoupling effects of $O(G_F m_{N_i}^2)$, we may put $M_H = 0$. In this limit, we have

$$C = -\frac{G_F (s_L^{\nu_e})^4 m_{N_2}^2}{4\pi^2 \sqrt{2} (1 + \sqrt{\rho})^2} \left[3 + \left(1 - \frac{1}{\sqrt{\rho}}\right) f(\rho) + (1 - \sqrt{\rho}) f\left(\frac{1}{\rho}\right) \right], \quad (7.12)$$

where

$$f(x) = \frac{1}{2} \left(1 - \frac{1}{x^2}\right) \ln |1 - x| - \frac{1}{4} - \frac{1}{2x}. \quad (7.13)$$

Notice that, to the order considered, we may introduce G_F in Eq. (7.8) by substituting $G_F = (\pi\alpha/\sqrt{2}s_w^2 M_W^2)$; the adjustment proportional to Δr [6] will only appear at the two-loop order because the tree-level expression for R_π does not contain α . The correction C is throughout negative, takes on the simple form

$$C = -\frac{3G_F (s_L^{\nu_e})^4 m_{N_1}^2}{16\pi^2 \sqrt{2}} \quad (7.14)$$

for $\rho = 1$, and exhibits the limiting behaviour

$$C = -\frac{G_F (s_L^{\nu_e})^4 m_{N_1}^2}{8\pi^2 \sqrt{2}} \left[\left(1 - \frac{1}{\sqrt{\rho}}\right) \ln \rho + \frac{11}{2} + \frac{7}{6\sqrt{\rho}} + O\left(\frac{1}{\rho}\right) \right] \quad (7.15)$$

for $\rho \gg 1$. It is interesting to observe that C is of $O(G_F m_{N_1}^2)$, i.e., it scales with the squared mass of the lighter neutrino.

We are now in a position to estimate how the value of $s_L^{\nu_e}$ extracted from the measurement of R_π is affected by the inclusion of loop effects due to heavy Majorana neutrinos. If we neglect these loop effects, we obtain the value $(s_{L0}^{\nu_e})^2$ by equating $R_\pi^{exp} = R_\pi^{SM}[1 - (s_{L0}^{\nu_e})^2]$. On the other hand, including these effects, we extract the corrected value $(s_L^{\nu_e})^2 = (s_{L0}^{\nu_e})^2 + \Delta(s_L^{\nu_e})^2$ from $R_\pi^{exp} = R_\pi^{SM}[1 - (s_L^{\nu_e})^2](1 + C)$, where C is evaluated with $s_L^{\nu_e}$. Consequently, the relative shift in $(s_L^{\nu_e})^2$ is

$$\frac{\Delta(s_L^{\nu_e})^2}{(s_L^{\nu_e})^2} = \frac{1 - (s_L^{\nu_e})^2}{(s_L^{\nu_e})^2} C \approx \frac{1}{(s_L^{\nu_e})^2} \left(\frac{R_\pi}{R_\pi^0} - 1 \right). \quad (7.16)$$

In the degenerate case, $m_{N_1} = m_{N_2} = m_N$, we then have

$$\frac{\Delta(s_L^{\nu_e})^2}{(s_L^{\nu_e})^2} \approx -16\% \times \left(\frac{s_L^{\nu_e} m_N}{1 \text{ TeV}} \right)^2. \quad (7.17)$$

Inserting in Eq. (7.17) the perturbative upper limit [24]

$$s_L^{\nu_e} m_N \approx m_D \lesssim 1 \text{ TeV}, \quad (7.18)$$

we find that $(s_L^{\nu_e})^2$ is reduced by 16%.

With the help of Eq. (A.2), we shall now investigate the dependence of $-\Delta(s_L^{\nu_e})^2/(s_L^{\nu_e})^2 \approx (1 - R_\pi/R_\pi^{SM})/(s_L^{\nu_e})^2$ on m_{N_1} , m_{N_2} , and M_H . We first consider the degenerate case, with $\rho = 1$. In Fig. 2, we show $-\Delta(s_L^{\nu_e})^2/(s_L^{\nu_e})^2$ as a function of m_N for $M_H = 100, 300, 500$, and 1000 GeV, assuming (a) $(s_L^{\nu_e})^2 = 0.01$ and (b) 0.007 . We observe that, at low values of m_N , the relative reduction of $(s_L^{\nu_e})^2$ strongly depends on M_H ; it increases by almost one order of magnitude as M_H runs from 100 GeV to 1 TeV. Comparing Figs. 2(a) and (b), we see that, for m_N in the TeV range, the relative reduction of $(s_L^{\nu_e})^2$ is slightly smaller for the lower value of $(s_L^{\nu_e})^2$. The high- m_N regime is well described by Eq. (7.14). Next, we study the non-degenerate case for $M_H = 200$ GeV. In Fig. 3, we display the ρ dependence of $-\Delta(s_L^{\nu_e})^2/(s_L^{\nu_e})^2$ for $m_{N_1} = 0.5, 1, 2$, and 5 GeV, again assuming (a) $(s_L^{\nu_e})^2 = 0.01$ and (b) 0.007 . We see that, in compliance with Eq. (7.15), $-\Delta(s_L^{\nu_e})^2/(s_L^{\nu_e})^2$ grows quadratically with m_{N_1} for ρ fixed, while it grows logarithmically with ρ for m_{N_1} fixed. From Fig. 3(a), we read off a 20% effect for $m_{N_1} = 5$ TeV and $m_{N_2} = 50$ TeV. On the experimental side, the measurements at the Tri-University Meson Facility (TRIUMF) [25] and the Paul Scherrer Institute (PSI) [26] yielded

$$\begin{aligned} R_\pi^{exp}(\text{TRIUMF}) &= (1.2265 \pm 0.0034(\text{stat}) \pm 0.0044(\text{syst})) \times 10^{-4}, \\ R_\pi^{exp}(\text{PSI}) &= (1.2346 \pm 0.0035(\text{stat}) \pm 0.0036(\text{syst})) \times 10^{-4}, \end{aligned} \quad (7.19)$$

respectively. This represents a remarkable reduction in error, by a factor of 3, relative to the previous value [27], $R_\pi^{exp} = (1.218 \pm 0.014) \times 10^{-4}$. It is therefore reasonable to expect that, in the near future, experiments will become sensitive to 10% effects in $(s_L^{\nu_e})^2$.

8 Conclusions

The renormalization of general theories with inter-family mixing between fermionic Dirac and/or Majorana states was studied to one loop in the electroweak on-shell scheme. Special attention was paid to the renormalization of the mixing matrix [7], which plays a central rôle in such theories. Similarly to the renormalization prescription for the CKM matrix of the SM proposed in Ref. [9], we adjusted the mixing-matrix CT's in such a way that they precisely cancel the antihermitean parts of the wave-function renormalization constants. Our formulation naturally takes possible radiative-neutrino-mass contributions into account.

The phenomenological implications of mixing renormalization for Majorana-neutrino mass models with large $SU(2)_L$ -breaking Dirac components [16] were analyzed in Section 7. In such scenarios, low-energy observables may receive sizeable corrections due to the non-decoupling of heavy neutrinos. As an example, the electroweak corrections to the observable R_π were estimated in the heavy-neutrino limit. It was found that they may reduce the tree-level values of R_π predicted in these models by up to 0.2%. This has to be contrasted with the present theoretical error in the SM prediction, which is $\pm 0.04\%$ [20]. Future experiments to be performed at TRIUMF and PSI may be sensitive to such new-physics phenomena.

Finally, we wish to emphasize that our formalism for the mixing renormalization of fermionic states may straightforwardly be extended to theories which involve the mixing of scalar or vector particles, such as the mixing of scalar quarks in supersymmetric theories.

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A Analytic expressions

In this appendix, we list the lepton transition amplitudes as well as the corrections δ_v^l and δ_{ct}^l defined in Eqs. (7.4) and (7.6), respectively, to one loop in the Majorana-neutrino models of Section 5 assuming $m_{N_i} \gg M_W$ and M_H arbitrary. We adopt the Passarino-Veltman [28] conventions for the standard one-loop integrals in dimensional regularization, implemented with the Minkowskian metric, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, as in Appendix A of Ref. [29].

The charged-lepton and Majorana-neutrino mixing amplitudes depicted in Figs. 1(b) and (c) are found to be

$$\begin{aligned}\Sigma_{ij}^l(\not{p}) &= -\frac{\alpha}{8\pi s_w^2 M_W^2} B_{l_i k} B_{l_j k}^* \left[\not{p} (m_k^2 P_L + m_{l_i} m_{l_j} P_R) B_1(p^2, m_k^2, M_W^2) \right. \\ &\quad \left. + (m_{l_i} P_L + m_{l_j} P_R) m_k^2 B_0(p^2, m_k^2, M_W^2) \right], \\ \Sigma_{ij}^n(\not{p}) &= -\frac{\alpha}{16\pi s_w^2 M_W^2} \left\{ \left[\not{p} P_L (m_i C_{ik}^* + m_k C_{ik}) (m_k C_{kj} + m_j C_{kj}^*) \right. \right. \\ &\quad \left. + \not{p} P_R (m_i C_{ik} + m_k C_{ik}^*) (m_k C_{kj}^* + m_j C_{kj}) \right] [B_1(p^2, m_k^2, M_Z^2) + B_1(p^2, m_k^2, M_H^2)] \\ &\quad \left. + [P_L m_k (m_i C_{ik} + m_k C_{ik}^*) (m_k C_{kj} + m_j C_{kj}^*) + P_R m_k (m_i C_{ik}^* + m_k C_{ik}) \right. \\ &\quad \left. \times (m_k C_{kj}^* + m_j C_{kj}) \right] [B_0(p^2, m_k^2, M_Z^2) - B_0(p^2, m_k^2, M_H^2)] \right\},\end{aligned}\quad (\text{A.1})$$

respectively. Here and in the following, the Majorana indices n_i are abbreviated by i , and it is summed over the indices of the heavy Majorana neutrinos. For the $Wl\nu$ vertex correction due to Fig. 1(a) and the lepton-flavour-dependent CT contribution to $\Gamma(\pi^+ \rightarrow l^+ \nu)$ we find

$$\begin{aligned}\delta_v^l &= \frac{\alpha}{8\pi s_w^2 M_W^2} C_{ii} m_i^2 [C_{24}(0, 0, 0, M_H^2, m_i^2, 0) + C_{24}(0, 0, 0, 0, m_i^2, 0)], \\ \delta_{ct}^l &= \frac{\alpha}{32\pi s_w^2 M_W^2} \left\{ C_{ii} m_i^2 [B_1(0, m_i^2, M_H^2) + 3B_1(0, m_i^2, 0)] \right. \\ &\quad \left. + |C_{ij}|^2 m_j^2 [B_1(m_i^2, m_j^2, M_H^2) + B_1(m_i^2, m_j^2, 0) - B_1(0, m_j^2, M_H^2) - B_1(0, m_j^2, 0) \right. \\ &\quad \left. - B_0(m_i^2, m_j^2, M_H^2) + B_0(m_i^2, m_j^2, 0) + B_0(0, m_j^2, M_H^2) - B_0(0, m_j^2, 0)] \right. \\ &\quad \left. + C_{ij}^2 m_i m_j [B_1(m_i^2, m_j^2, M_H^2) + B_1(m_i^2, m_j^2, 0) - \frac{m_j^2}{m_i^2} (B_0(m_i^2, m_j^2, M_H^2) \right. \\ &\quad \left. - B_0(m_i^2, m_j^2, 0) - B_0(0, m_j^2, M_H^2) + B_0(0, m_j^2, 0))] \right\}.\end{aligned}\quad (\text{A.2})$$

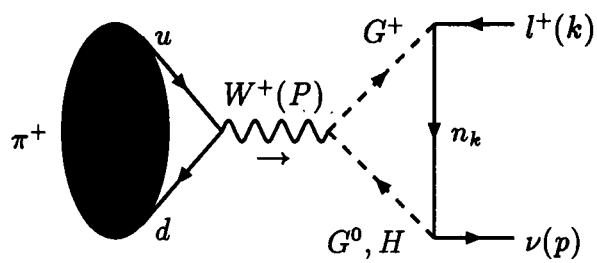
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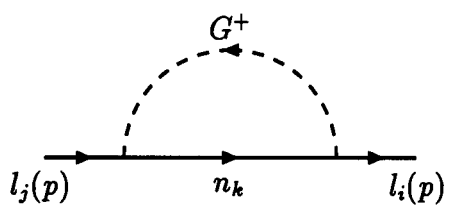
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Figure Captions

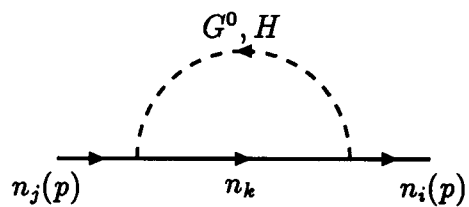
- Fig. 1:** Feynman graphs contributing to the observable $R_\pi = \Gamma(\pi^+ \rightarrow e^+ \nu) / \Gamma(\pi^+ \rightarrow \mu^+ \nu)$ in the heavy-neutrino limit: (a) $Wl\nu$ vertex corrections, (b) charged-lepton transition amplitudes, (c) neutrino transition amplitudes.
- Fig. 2:** $-\Delta(s_L^{\nu_e})^2 / (s_L^{\nu_e})^2 \approx (1 - R_\pi / R_\pi^0) / (s_L^{\nu_e})^2$ as a function of $m_N = m_{N_1} = m_{N_2}$ for $M_H = 100, 300, 500, 1000$ GeV, assuming $(s_L^{\nu_\mu})^2 = 0$ and (a) $(s_L^{\nu_e})^2 = 0.01$ or (b) $(s_L^{\nu_e})^2 = 0.007$.
- Fig. 3:** $-\Delta(s_L^{\nu_e})^2 / (s_L^{\nu_e})^2 \approx (1 - R_\pi / R_\pi^0) / (s_L^{\nu_e})^2$ as a function of $\rho = m_{N_2}^2 / m_{N_1}^2$ for $m_{N_1} = 0.5, 1, 2, 5$ TeV, assuming $M_H = 200$ GeV, $(s_L^{\nu_\mu})^2 = 0$, and (a) $(s_L^{\nu_e})^2 = 0.01$ or (b) $(s_L^{\nu_e})^2 = 0.007$.



(a)



(b)



(c)

Fig. 1

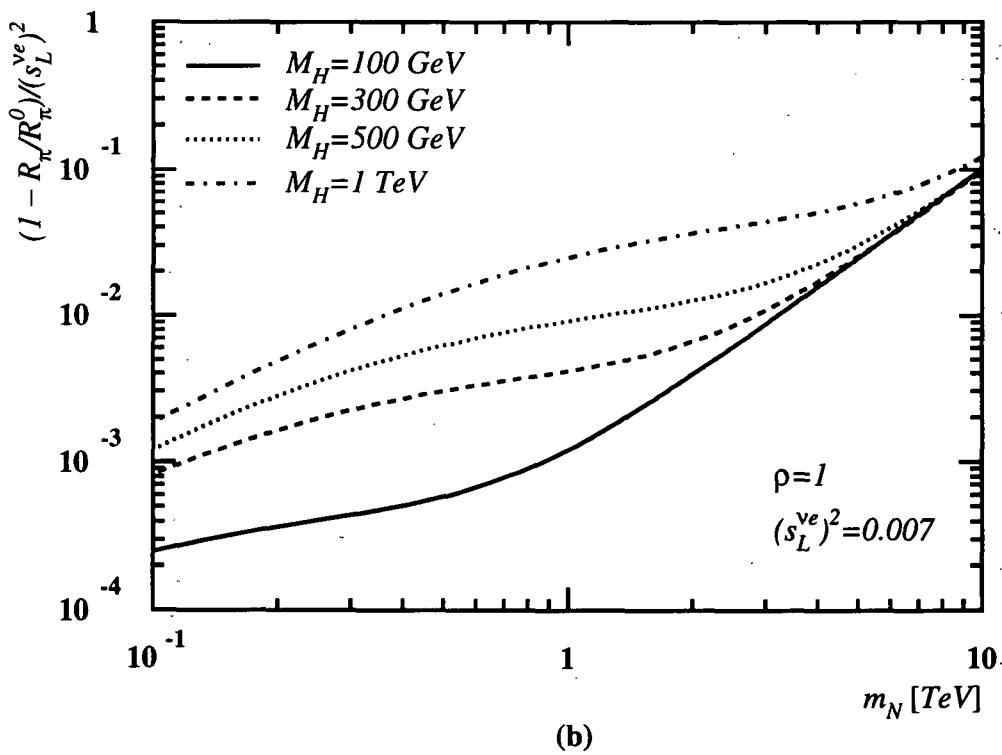
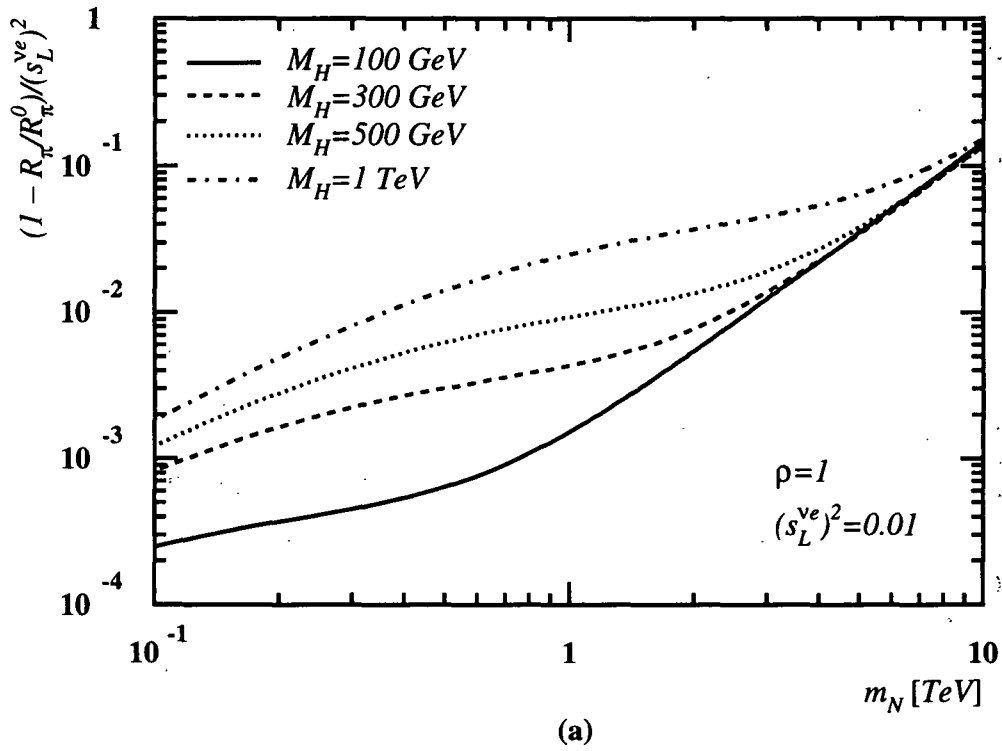


Fig. 2

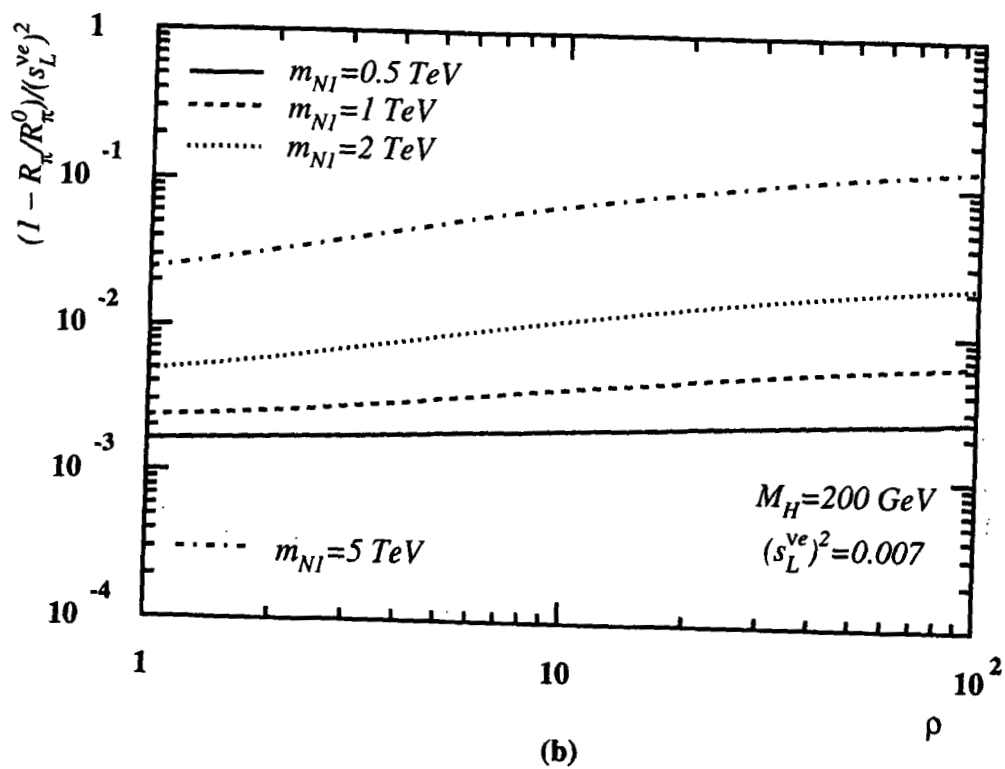
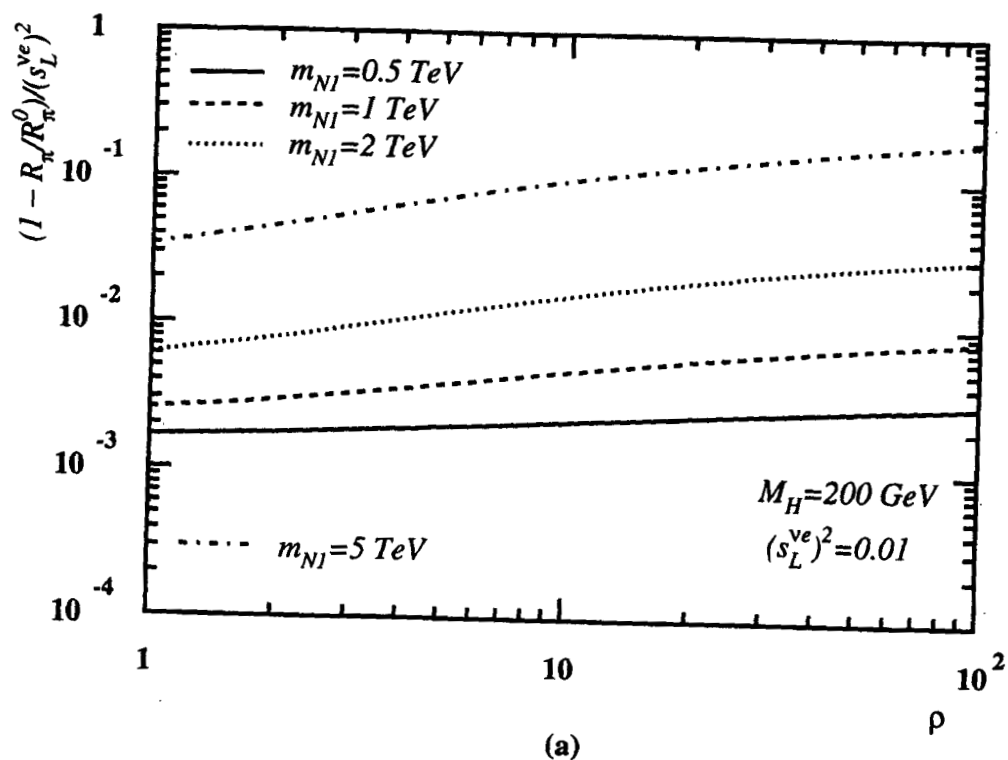


Fig. 3