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The Evolution of Parton Distributions Beyond Leading Order: The Singlet Case

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The evolution of parton distributions beyond leading order:

the singlet case

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Abstract

A complete description of the calculation of anomalous dimensions (GLAP splitting functions) is given for parton distributions which appear in space-like processes. The calculation is performed in the light-cone gauge. The results are in agreement with the previous results of Furmanski and Petronzio.

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1 Introduction

The light-cone gauge has always occupied at special role in the description of hard processes at high energy. It belongs to a class of physical gauges in which many of the precepts of the QCD parton model are true, because in this gauge collinear divergences occur in diagrams corresponding to the parton cascade. We thus retain the probabilistic interpretation of a hard scattering event, which is obscured in covariant gauges. In fact, by introducing an additional gauge vector we obtain many of the advantages of an infinite momentum frame formulation in a covariant notation.

However, doubts have been raised about the utility of light-cone gauge in practical calculations[1,2]. One refutation of these misgivings is provided by the classic calculation of the two loop splitting functions or anomalous dimensions as given by Curci, Furmanski and Petronzio[3] for the non-singlet case and the calculation of Furmanski and Petronzio[4] for the singlet case. The calculation of ref. [4], however, has never been fully documented. It could be be that this lack of complete documentation has acted as a barrier to further developments along this line. One example of a further application of this method is the calculation of the polarized two loop splitting functions, recently presented in ref. [7]. It therefore seemed a valuable addition to the literature to provide a more complete description of the calculation of the singlet evolution probabilities. This is the modest aim of this paper.

In our calculation n is a light-like vector which serves to define the longitudinal direction. The momentum of the incoming parton (taken to be massless) is denoted by p. Thus we have two light-like vectors which are defined such that,

$$n^2 = p^2 = 0, \quad n \cdot p \equiv pn \neq 0, \quad n \cdot t = p \cdot t = 0 \,, \tag{1}$$

where t is any vector in the transverse plane. In addition to specifying the longitudinal direction we also use the vector n to fix the light-cone gauge:

$$n \cdot A = 0 \,. \tag{2}$$

Following reference [3] we use the principal value (PV) prescription to regulate the divergences which occur in the light-cone gauge propagator in loop and phase space integrals, i.e.

$$\frac{1}{n \cdot k} \rightarrow \frac{1}{2}\left(\frac{1}{n \cdot k + i\delta(pn)} + \frac{1}{n \cdot k - i\delta(pn)}\right) = \frac{n \cdot k}{(n \cdot k)^2 + \delta^2 (pn)^2} \,. \tag{3}$$

This prescription is at variance with the Mandelstam-Leibrandt (ML) treatment of the $1/(n \cdot k)$ singularities[5,1]. The ML prescription, since it permits the Wick rotation in virtual diagrams, leads to power counting rules and a proof of renormalizability of the theory in this gauge. Indeed the one loop non-singlet splitting function has been investigated in this gauge[6]. We have chosen not to follow the ML prescription. The technical reason is that it leads to a proliferation of graphs, because of ghost-like contributions associated with the $n \cdot k$ propagator[6]. The physical reason is that we wanted to stay as close as possible to old-fashioned perturbation theory in which manifest Lorentz covariance is sacrificed in order to have a simple form for unitarity. The advantage of Eq. (3) is that the unitarity of the theory is explicit. In this way we hoped to gain a greater physical understanding of the two loop anomalous dimensions. It would be interesting to repeat the calculation of the two loop anomalous dimensions using the ML prescription.

Of course, the calculation of any gauge invariant quantity such as the two loop splitting function is independent of the gauge in which the calculation is performed and of the method of calculation. However, the discovery of complications in the covariant gauge calculations[8,9] makes the method outlined in ref. [10] and implemented in refs. [3,4] even more attractive. Not only is this method close to the parton model, but it also leads to compact answers and may be the most efficient method from a calculational point of view. It might present a viable method for the analytic calculation of the three loop splitting functions[2].

2 Calculation of anomalous dimensions

2.1 Factorization

In this section we shall explain the method of factorization of the two-particle irreducible (2PI) diagram in the light-cone gauge. It is not our intention to repeat the discussion which is clearly provided in ref. [3]. We only include those details which are necessary to present the structure of the calculation or to define the notation. Following ref. [10] we define a generalized ladder expansion by introducing the 2PI kernel K_0:

$$M = C_0(1 + K_0 + K_0^2 + K_0^3 + \ldots) \equiv \frac{C_0}{1 - K_0} . \tag{4}$$

Factorization occurs by introducing the projector onto physical states, $\mathcal{P}$,

$$\begin{aligned} \frac{1}{1-K_0} &= \frac{1}{1-(1-\mathcal{P})K_0 - \mathcal{P}K_0} \\ &\equiv \left[\frac{1}{1-(1-\mathcal{P})K_0}\right]\left[\frac{1}{1-\mathcal{P}K_0[1-(1-\mathcal{P})K_0]^{-1}}\right] . \end{aligned} \tag{5}$$

Defining the modified kernel K,

$$K = \frac{K_0}{1-(1-\mathcal{P})K_0} , \tag{6}$$

[2]Results for low moments of the three loop splitting functions are given in Ref. [11]

we can thus write M as

$$\begin{aligned} M &= C_0 \frac{1}{1-(1-\mathcal{P})K_0} \frac{1}{1-\mathcal{P}K} \\ &\equiv C\,\Gamma \end{aligned} \tag{7}$$

where

$$C = C_0 \frac{1}{1-(1-\mathcal{P})K_0}\,, \tag{8}$$

$$\Gamma = \frac{1}{1-\mathcal{P}K}\,. \tag{9}$$

At this stage the factorized structure becomes apparent. In the light-cone gauge the 2PI kernels K_0 are finite before the integration over the sides of the ladder is performed. Collinear singularities appear only after integrating over the lines connecting the rungs of the ladders[10]. All collinear singularities are contained in Γ, whereas C is interpreted as the (finite) short distance cross section. Re-expanding we find that

$$K = K_0\,(1 + (1-\mathcal{P})K_0 + (1-\mathcal{P})(K_0(1-\mathcal{P})K_0) + \ldots)\ , \tag{10}$$

$$\begin{aligned} \Gamma &= 1 + \mathcal{P}K + (\mathcal{P}K)(\mathcal{P}K) + \ldots \\ &\equiv 1 + \mathcal{P}K_0 + \mathcal{P}K_0(1-\mathcal{P})K_0 + (\mathcal{P}K_0)(\mathcal{P}K_0) + \ldots \end{aligned} \tag{11}$$

Restoring the indices and regulating collinear singularities by going to $d = 4-2\epsilon$ dimensions we have that

$$\Gamma_{ij} = Z_j \frac{1}{1-\mathcal{P}K} = Z_j \left[1 + \mathcal{P}K_0 + \mathcal{P}(K_0^2) - \mathcal{P}(K_0 \mathcal{P} K_0) + \ldots \right] \tag{12}$$

and explicitly

$$\Gamma_{ij}(z, \alpha_s, \frac{1}{\epsilon}) = Z_j \left[\delta(1-z)\delta_{ij} + z\ \mathrm{PP} \int \frac{d^d k}{(2\pi)^d} \delta(z - \frac{n\cdot k}{pn}) U_i K \frac{1}{1-\mathcal{P}K} L_j \right] \tag{13}$$

where 'PP' extracts the pole part of the expression on its right and Z_j ($j = q(g)$) is the residue of the pole of the full quark (gluon) propagator, contributing to the diagonal splitting functions. Furthermore the spin averaged projection operators onto physical states are given by,

$$\begin{aligned} U_q &= \frac{1}{4n\cdot k} \not{n}, \quad L_q = \not{p} \\ U_g &= -g^{\mu\nu}, \quad L_g = \frac{1}{d-2} \left[-g^{\mu\nu} + \frac{n^\mu p^\nu + n^\nu p^\mu}{pn} \right]. \end{aligned} \tag{14}$$

2.2 Derivation of GLAP equation

The property of factorization allows us to separate the low momentum physics from the high momentum physics in a multiplicative way. This separation is performed at a scale μ, which is completely arbitrary, and no physical prediction can depend on it. In this section we investigate the constraints provided by this condition. For simplicity, we will consider a non-singlet cross section which can only be initiated by a quark. We therefore have the factorized result,

$$\sigma(\frac{Q^2}{\mu^2}, \alpha_S(\mu^2), \epsilon) = \tilde{\sigma}_q(\frac{Q^2}{\mu^2}, \alpha_S(\mu^2)) \otimes \Gamma_{qq}(\alpha_S(\mu^2), \epsilon) \tag{15}$$

where we have indicated that Γ_{qq} does *not* depend on Q^2 (i.e. Q^2/μ^2), which is a consequence[3] of the finiteness of the kernel K_0 in the light-cone gauge. The symbol $\otimes$ indicates a convolution integral over longitudinal momentum fractions of the type

$$f \otimes g \equiv \int_0^1 dy\, dz\; f(y)g(z)\; \delta(x - yz)\; . \tag{16}$$

If we take moments,

$$f(j) = \int_0^1 dx x^{j-1} f(x) \tag{17}$$

on both sides of Eq. (15), it reduces to a simple product:

$$\sigma(j, \frac{Q^2}{\mu^2}, \alpha_S(\mu^2), \epsilon) = \tilde{\sigma}_q(j, \frac{Q^2}{\mu^2}, \alpha_S(\mu^2)) \Gamma_{qq}(j, \alpha_S(\mu^2), \epsilon)\; . \tag{18}$$

$\tilde{\sigma}_q$ is the short distance cross section from which all singularities have been factorized. Γ_{qq} contains the mass singularities which manifest themselves as poles in ϵ. The independence of the full cross section of μ implies that

$$\frac{d}{d\ln\mu^2}\sigma = 0 \tag{19}$$

and hence that

$$\frac{d}{d\ln\mu^2} \ln \Gamma_{qq}(j, \alpha_S(\mu^2), \epsilon) = -\frac{d}{d\ln\mu^2} \ln \tilde{\sigma}_q(j, \frac{Q^2}{\mu^2}, \alpha_S(\mu^2)) = \gamma_{qq}(j, \alpha_S(\mu^2))\; . \tag{20}$$

The function γ_{qq} is known as the anomalous dimension, because it measures the deviation of $\tilde{\sigma}_q$ from its naive scaling dimension. It must be finite and can only depend on $\alpha_S(\mu^2)$ because these are the only variables common to both Γ_{qq} and $\tilde{\sigma}_q$. The anomalous dimension is extracted from Eq. (20) in the following way: Because the μ dependence of Γ_{qq} enters only through the running coupling we have that,

$$\gamma_{qq}(j, \alpha_S(\mu^2)) \equiv \beta(\alpha_S, \epsilon) \frac{d}{d\alpha_S} \ln \Gamma_{qq}(j, \alpha_S(\mu^2), \epsilon)\; , \tag{21}$$

where $\beta(\alpha_S, \epsilon)$ is the d-dimensional QCD β function in the $\overline{\text{MS}}$ scheme,

$$\beta(\alpha_S, \epsilon) = \frac{d\alpha_S}{d\ln\mu^2} = -\epsilon\alpha_S + \beta(\alpha_S) \ . \tag{22}$$

In the $\overline{\text{MS}}$ scheme Γ_{qq} is given by a series of the form

$$\Gamma_{qq}(j, \alpha_S, \epsilon) = 1 + \sum_{i=1}^{\infty} \frac{\Gamma_{qq}^{(i)}(j, \alpha_S)}{\epsilon^i} \ . \tag{23}$$

Comparing the coefficient of the term of order ϵ^0 we find that

$$\gamma_{qq}(j, \alpha_S) = -\frac{d}{d\ln\alpha_S}\Gamma_{qq}^{(1)}(j, \alpha_S) \ . \tag{24}$$

Integrating Eqs. (20,21) one obtains

$$\Gamma_{qq}(j, \alpha_S, \epsilon) = \exp\left\{ \int_0^{\alpha_S} d\lambda \frac{\gamma_{qq}(j, \lambda)}{\beta(\lambda) - \epsilon\lambda} \right\} \tag{25}$$

and

$$\sigma(j, \frac{Q^2}{\mu^2}, \alpha_S(\mu^2), \epsilon) = \tilde{\sigma}_q(j, 1, \alpha_S(Q^2)) \exp\left\{ \int_0^{\alpha_S(Q^2)} d\lambda \frac{\gamma_{qq}(j, \lambda)}{\beta(\lambda) - \epsilon\lambda} \right\} \tag{26}$$

with the running coupling $\alpha_S(Q^2)$. In order to obtain the hadronic cross section, $\sigma(Q^2/\mu^2, \alpha_S, \epsilon)$ has to be convoluted with 'bare' ('unrenormalized') quark densities $\tilde{q}(\alpha_S, \epsilon)$ which contain mass singularities that must exactly cancel those in Γ_{qq}. The resulting 'dressed' ('renormalized') quark distribution function

$$q(j, Q^2) = \exp\left\{ \int_0^{\alpha_S(Q^2)} d\lambda \frac{\gamma_{qq}(j, \lambda)}{\beta(\lambda) - \epsilon\lambda} \right\} \tilde{q}(j, \alpha_S(\mu^2), \epsilon) \tag{27}$$

is free of mass singularities and satisfies the non-singlet evolution equation

$$\frac{dq(j, Q^2)}{d\ln Q^2} = \gamma_{qq}(j, \alpha_S(Q^2)) q(j, Q^2) \tag{28}$$

which, defining

$$\int_0^1 dz z^{j-1} P_{qq}(z, \alpha_S) \equiv \gamma_{qq}(j, \alpha_S) \ , \tag{29}$$

in x space takes the form of the GLAP equation[12,13]:

$$\frac{dq(x, Q^2)}{d\ln Q^2} = \int_0^1 dy \int_0^1 dz \ P_{qq}(y, \alpha_S(Q^2)) q(z, Q^2) \, \delta(x - yz) \ . \tag{30}$$

Expanding

$$P_{qq}(x,\alpha_S) = \left(\frac{\alpha_S}{2\pi}\right) P_{qq}^{(0)}(x) + \left(\frac{\alpha_S}{2\pi}\right)^2 P_{qq}^{(1)}(x) + \dots \tag{31}$$

one has

$$\Gamma_{qq}(x,\alpha_S,\epsilon) = \delta(1-x) - \frac{1}{\epsilon}\left(\frac{\alpha_S}{2\pi} P_{qq}^{(0)}(x) + \frac{1}{2}\left(\frac{\alpha_S}{2\pi}\right)^2 P_{qq}^{(1)}(x) + \dots\right) + O\left(\frac{1}{\epsilon^2}\right) . \tag{32}$$

The generalization to the singlet case is straightforward.

2.3 Non-singlet and singlet equations

The separation into singlet and non-singlet parts depends on the properties of the kernel. Using $SU(f)$ flavour symmetry we may define the following combinations of qq and $q\bar{q}$ matrix elements:

$$\begin{aligned} P_{q_i q_k} &= \delta_{ik} P_{qq}^V + P_{qq}^S \\ P_{q_i \bar{q}_k} &= \delta_{ik} P_{q\bar{q}}^V + P_{q\bar{q}}^S \\ P^{\pm} &= P_{qq}^V \pm P_{q\bar{q}}^V . \end{aligned} \tag{33}$$

In addition, because of charge conjugation invariance, we have that

$$\begin{aligned} P_{q_i q_j} &= P_{\bar{q}_i \bar{q}_j} \\ P_{q_i \bar{q}_j} &= P_{\bar{q}_i q_j} \\ P_{q_i g} &= P_{\bar{q}_i g} \\ P_{g q_i} &= P_{g \bar{q}_i} . \end{aligned} \tag{34}$$

At two loop order, there is a non-zero contribution from P_{qq}^S and $P_{q\bar{q}}^S$, but we have the additional relation

$$P_{qq}^S = P_{q\bar{q}}^S . \tag{35}$$

which simplifies the treatment of the non-singlet pieces.

For each flavour we define the sum and difference of the quark and anti-quark distributions as

$$q_i^{\pm} = q_i \pm \bar{q}_i . \tag{36}$$

One then finds that the combinations

$$V_i = q_i^- \tag{37}$$

and

$$T_l = \sum_{i=1}^{k} q_i^+ - k q_k^+ \tag{38}$$

(where $i, k = 1, \ldots, n_f;\ l = k^2 - 1$) are non-singlets, i.e., evolve according to Eq. (30) with the kernels P^- and P^+, respectively.

The singlet Altarelli-Parisi equation is[3][12,13]

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\Sigma(j,Q^2)\\ G(j,Q^2)\end{pmatrix} = \begin{pmatrix}P_{qq}(j,\alpha_S(Q^2)) & P_{qg}(j,\alpha_S(Q^2))\\ P_{gq}(j,\alpha_S(Q^2)) & P_{gg}(j,\alpha_S(Q^2))\end{pmatrix}\begin{pmatrix}\Sigma(j,Q^2)\\ G(j,Q^2)\end{pmatrix} \quad (39)$$

where $G(j)$ is the moment of the gluon distribution and $\Sigma(j)$ is the singlet quark combination,

$$\Sigma(j,Q^2) = \sum_f q_i^+(j,Q^2) \equiv \sum_f \left[q_i(j,Q^2) + \bar{q}_i(j,Q^2)\right] . \quad (40)$$

The elements of the anomalous dimension matrix are given in terms of the kernels defined in Eqs. (33-35) as,

$$\begin{aligned} P_{qq} &= P^+ + n_f(P^S_{qq} + P^S_{q\bar{q}}) \\ P_{qg} &= 2n_f P_{q_ig} \\ P_{gq} &= P_{gq_i} . \end{aligned} \quad (41)$$

2.4 Renormalization constants

The notation for the renormalization constants is shown in Fig. 1. We define the integral

$$I_0 = \int_0^1 du \frac{u}{u^2 + \delta^2} \quad (42)$$

which contains the divergences in the PV regulator δ (see Eq. (3)) arising from the light-cone gauge propagator. As already noted in ref. [3], use of the PV prescription (3) in the light-cone gauge entails the disagreeable feature that the renormalization constants depend on the longitudinal momentum fractions x.

$$\begin{aligned} Z_q(x) &= 1 + \frac{\alpha_S}{2\pi}\frac{1}{2\epsilon}\left[C_F(-3 + 4I_0 + 4\ln x)\right] \\ Z_g(x) &= 1 + \frac{\alpha_S}{2\pi}\frac{1}{2\epsilon}\left[\frac{4T_f}{3} + N_C(-\frac{11}{3} + 4I_0 + 4\ln x)\right] \\ Z_q^{(1)}(x_1,x_2,x_3) &= 1 + \frac{\alpha_S}{2\pi}\frac{1}{2\epsilon}\left[C_F(3 - 4I_0 - 2\ln x_1 - 2\ln x_2) - 2N_C(I_0 + \ln x_3)\right] \\ Z_g^{(1)}(x_1,x_2,x_3) &= 1 + \frac{\alpha_S}{2\pi}\frac{1}{2\epsilon}\left[N_C(\frac{11}{3} - 6I_0 - 2\ln x_1 - 2\ln x_2 - 2\ln x_3) - \frac{4T_f}{3}\right] \end{aligned} \quad (43)$$

[3]Note that the notation for the off-diagonal terms is different than in ref. [4].

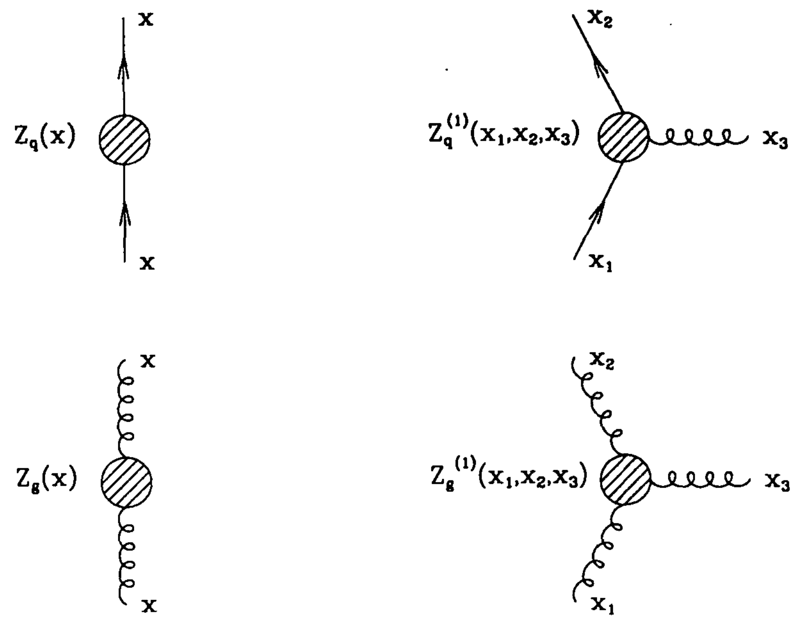

Figure 1: Renormalization constants and the vertices which they renormalize

where

$$C_F = \frac{4}{3},\ N_C = 3,\ T_f = T_R n_f = \tfrac{1}{2} n_f\ . \tag{44}$$

When combined in the appropriate combinations to investigate the renormalization of the charge, the momentum dependent terms and the divergent integrals I_0 cancel. Thus the relationship between the bare and renormalized couplings is

$$\begin{aligned} \alpha_S^{(0)} &= \alpha_S \mu^{2\epsilon} Z_q^{(1)}(x_1, x_2, x_3)\sqrt{Z_q(x_1) Z_q(x_2) Z_g(x_3)} \\ &= \alpha_S \mu^{2\epsilon} Z_g^{(1)}(x_1, x_2, x_3)\sqrt{Z_g(x_1) Z_g(x_2) Z_g(x_3)} \\ &= \alpha_S \mu^{2\epsilon}\Big[1 - \frac{\alpha_S}{2\pi}\frac{1}{2\epsilon}\big(\frac{11 N_C}{6} - \frac{2T_f}{3}\big) + \ldots\Big] \equiv \alpha_S \mu^{2\epsilon}\Big[1 - \frac{\alpha_S}{2\pi}\frac{1}{4\epsilon}\beta_0 + \ldots\Big]\ . \end{aligned} \tag{45}$$

2.5 Topologies of NLO graphs

The basic topologies of all 2PI diagrams which occur in two loops are shown in Fig. 2. The notation of the topologies (b)-(i) is determined by the labelling of the diagrams for the non-singlet calculation given in ref. [3]. We have not included diagrams which can be obtained by reflection about the vertical axis which occur in cases (c),(d),(e) and (j). Topologies (hi) correspond to the terms $\mathcal{P}(K_0^2) - \mathcal{P}(K_0 \mathcal{P} K_0)$ in Eq. (12), all other topologies belong to $\mathcal{P}K_0$. As an example, the diagrams corresponding to P_{qq}^V are given explicitly in Fig. 3. Fig. 4 shows the diagrams for $P_{q\bar{q}}^V$ (b) and for $P_{q\bar{q}}^S$ (h,i). The appendices give the necessary ingredients needed for the evaluation of the real and virtual graphs in Fig. 2.

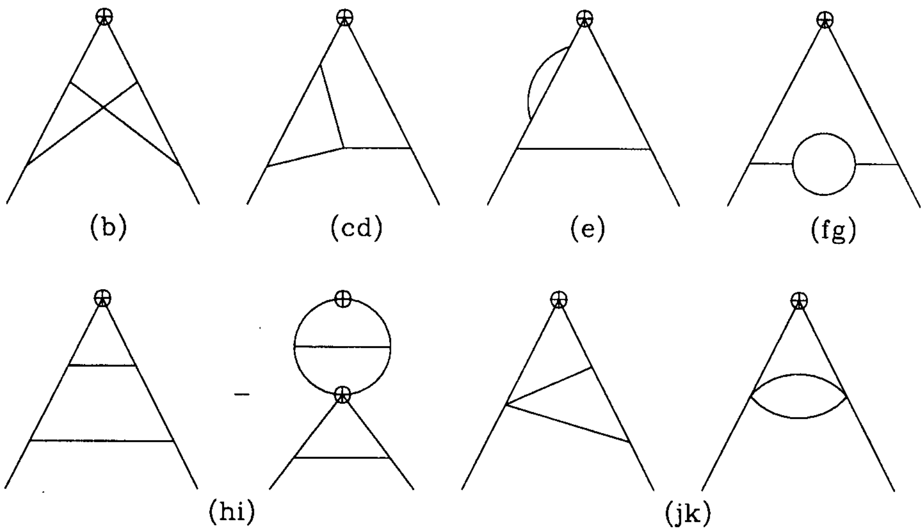

Figure 2: Basic topologies of the diagrams

3 Results

As in Eq. (31) we define the perturbative expansion

$$P_{ij}(x,\alpha_S) = \left(\frac{\alpha_S}{2\pi}\right)P_{ij}^{(0)}(x) + \left(\frac{\alpha_S}{2\pi}\right)^2 P_{ij}^{(1)}(x) + \dots \tag{46}$$

The full one loop results are included for completeness[12]

$$P_{qq}^{(0)}(x) = C_F\Big\{\frac{2}{[1-x]_+} - 1 - x + \frac{3}{2}\delta(1-x)\Big\} \tag{47}$$

$$P_{qg}^{(0)}(x) = 2T_f\Big\{x^2 + (1-x)^2\Big\} \tag{48}$$

$$P_{gq}^{(0)}(x) = C_F\Big\{\frac{1+(1-x)^2}{x}\Big\} \tag{49}$$

$$P_{gg}^{(0)}(x) = 2N_C\Big\{\frac{1}{[1-x]_+} + \frac{1}{x} - 2 + x(1-x)\Big\} + \frac{\beta_0}{2}\delta(1-x)\,. \tag{50}$$

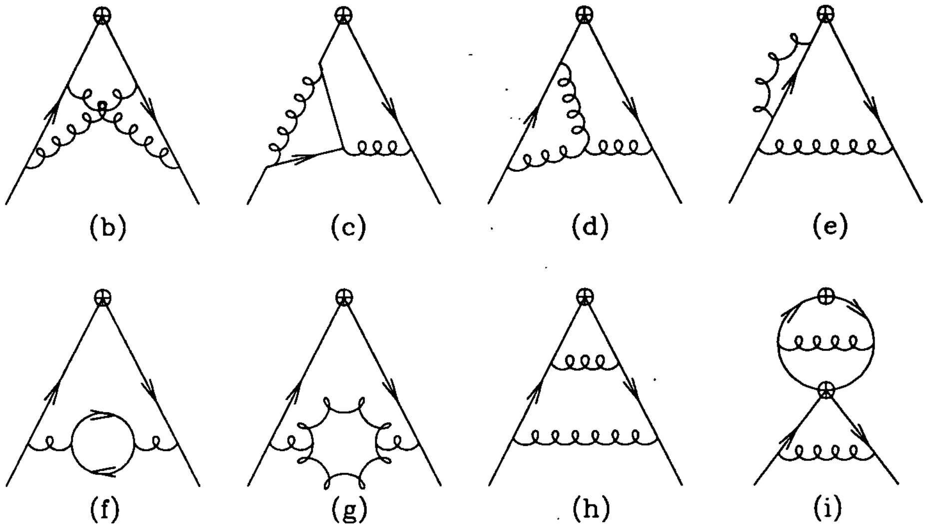

Figure 3: Diagrams for the qq part of the NLO non-singlet splitting functions

In order to write the full result for the two loop splitting functions $P_{ij}^{(1)}$ we introduce the notation

$$p_{qq}(x) = \frac{2}{1-x} - 1 - x \tag{51}$$

$$p_{qg}(x) = x^2 + (1-x)^2 \tag{52}$$

$$p_{gq}(x) = \frac{1+(1-x)^2}{x} \tag{53}$$

$$p_{gg}(x) = \frac{1}{1-x} + \frac{1}{x} - 2 + x(1-x)\,. \tag{54}$$

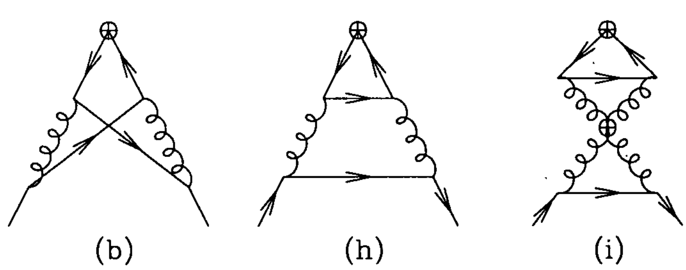

Figure 4: The $q\bar{q}$ diagrams

	$C_F T_R$
Terms	(hi)
$x\ \ln^2(x)$	-1
$\ln^2(x)$	-1
$x^2\ \ln(x)$	8/3
$x\ln(x)$	5
$\ln(x)$	1
x^2	-56/9
x	6
1	-2
$1/x$	20/9

Table 1: Results for the $q\bar{q}$-Singlet diagram

The graph-by-graph results are given in Tables 1-4, where we only list the contributions to the singlet splitting functions since those for the non-singlet case have been presented in a similar table in ref. [3]. Our final full results for the two loop non-singlet splitting functions read for $x \neq 1$ [3]:

$$P_{qq}^{V,(1)} = C_F^2\Big\{ - [2\ln x \ln(1-x) + \frac{3}{2}\ln x] p_{qq}(x)$$

$$-(\frac{3}{2} + \frac{7}{2}x)\ln x - \frac{1}{2}(1+x)\ln^2 x - 5(1-x)\Big\}$$

$$+C_F N_C\Big\{[\frac{1}{2}\ln^2 x + \frac{11}{6}\ln x + \frac{67}{18} - \frac{\pi^2}{6}] p_{qq}(x) + (1+x)\ln x + \frac{20}{3}(1-x)\Big\}$$

$$+C_F T_f\Big\{ - [\frac{2}{3}\ln x + \frac{10}{9}] p_{qq}(x) - \frac{4}{3}(1-x))\Big\} \tag{55}$$

$$P_{q\bar{q}}^{V,(1)} = C_F(C_F - \frac{N_C}{2})\Big\{2p_{qq}(-x)S_2(x) + 2(1+x)\ln x + 4(1-x)\Big\}\ . \tag{56}$$

Our results for the singlet terms are[4],

$$P_{qq}^{(1)} = P^{+,(1)} \tag{57}$$

$$+2C_F T_f\Big\{\frac{20}{9x} - 2 + 6x - \frac{56}{9}x^2 + (1 + 5x + \frac{8}{3}x^2)\ln x - (1+x)\ln^2 x\Big\}$$

$$P_{qg}^{(1)} = C_F T_f\Big\{4 - 9x - (1-4x)\ln x - (1-2x)\ln^2 x + 4\ln(1-x)$$

$$+[2\ln^2(\frac{1-x}{x}) - 4\ln(\frac{1-x}{x}) - \frac{2}{3}\pi^2 + 10] p_{qg}(x)\Big\}$$

$$+N_C T_f\Big\{\frac{182}{9}+\frac{14}{9}x+\frac{40}{9x}+(\frac{136}{3}x-\frac{38}{3})\ln x-4\ln(1-x)-(2+8x)\ln^2 x$$
$$+\big[-\ln^2 x+\frac{44}{3}\ln x-2\ln^2(1-x)+4\ln(1-x)+\frac{\pi^2}{3}-\frac{218}{9}\big]p_{qg}(x)$$
$$+2p_{qg}(-x)S_2(x)\Big\} \qquad (58)$$

$$P^{(1)}_{gq}=C_F^2\Big\{-\frac{5}{2}-\frac{7}{2}x+(2+\frac{7}{2}x)\ln x-(1-\frac{1}{2}x)\ln^2 x-2x\ln(1-x)$$
$$-\big[3\ln(1-x)+\ln^2(1-x)\big]p_{gq}(x)\Big\}$$
$$+C_F N_C\Big\{\frac{28}{9}+\frac{65}{18}x+\frac{44}{9}x^2-(12+5x+\frac{8}{3}x^2)\ln x+(4+x)\ln^2 x+2x\ln(1-x)$$
$$+\big[-2\ln x\ln(1-x)+\frac{1}{2}\ln^2 x+\frac{11}{3}\ln(1-x)+\ln^2(1-x)-\frac{\pi^2}{6}+\frac{1}{2}\big]p_{gq}(x)$$
$$+S_2(x)p_{gq}(-x)\Big\}$$
$$+C_F T_f\Big\{-\frac{4}{3}x-\big[\frac{20}{9}+\frac{4}{3}\ln(1-x)\big]p_{gq}(x)\Big\} \qquad (59)$$

$$P^{(1)}_{gg}=C_F T_f\Big\{-16+8x+\frac{20}{3}x^2+\frac{4}{3x}-(6+10x)\ln x-2(1+x)\ln^2 x\Big\}$$
$$+N_C T_f\Big\{2-2x+\frac{26}{9}(x^2-\frac{1}{x})-\frac{4}{3}(1+x)\ln x-\frac{20}{9}p_{gg}(x)\Big\}$$
$$+N_C^2\Big\{\frac{27}{2}(1-x)+\frac{67}{9}(x^2-\frac{1}{x})-(\frac{25}{3}-\frac{11}{3}x+\frac{44}{3}x^2)\ln x+4(1+x)\ln^2 x$$
$$+\big[\frac{67}{9}-4\ln x\ln(1-x)+\ln^2 x-\frac{\pi^2}{3}\big]p_{gg}(x)+2p_{gg}(-x)S_2(x))\Big\} \qquad (60)$$

where the function $S_2(x)$ is defined as[4]

$$S_2(x)=\int_{\frac{x}{1+x}}^{\frac{1}{1+x}}\frac{dz}{z}\ln\big(\frac{1-z}{z}\big)\,. \qquad (61)$$

In the small-x limit S_2 becomes

$$S_2=\frac{1}{2}\ln^2 x-\frac{\pi^2}{6}+O(x)\,. \qquad (62)$$

All results in Eqs. (55-60) are in complete agreement with the corresponding results in [3,4]. They can be extended to all values of x using a trick to evaluate the endpoint contributions in Eqs. (55,60). The sum rule from the conservation of fermion number is

$$\int_0^1 dx\,\Big(P_{qq}(x)-P_{q\bar q}(x)\Big)\equiv\int_0^1 dx\,P^-(x)=0\,. \qquad (63)$$

[4]Note that the definition of S_2 in ref. [4] contains a typographical mistake.

The conservation of momentum leads to the following two relations,

$$\int_0^1 dx\, x\Big(P_{qq}(x) + P_{gq}(x)\Big) = 0\,, \tag{64}$$

$$\int_0^1 dx\, x\Big(P_{qg}(x) + P_{gg}(x)\Big) = 0\,. \tag{65}$$

These results for the integrals of the splitting functions are satisfied if one makes the substitutions

$$\frac{1}{1-x} \to \frac{1}{[1-x]_+} \tag{66}$$

in Eqs. (51,54) and adds in the end-point contributions to Eqs. (55,60) [14],

$$P_{qq}^{V,(1)} \to P_{qq}^{V,(1)} + \Bigg[C_F^2\Big\{\frac{3}{8} - \frac{\pi^2}{2} + 6\zeta(3)\Big\} + C_F N_C\Big\{\frac{17}{24} + \frac{11\pi^2}{18} - 3\zeta(3)\Big\} - C_F T_f\Big\{\frac{1}{6} + \frac{2\pi^2}{9}\Big\}\Bigg]\delta(1-x) \tag{67}$$

$$P_{gg}^{(1)} \to P_{gg}^{(1)} + \Bigg[N_C^2\Big\{\frac{8}{3} + 3\zeta(3)\Big\} - C_F T_f - \frac{4}{3}N_C T_f\Bigg]\delta(1-x) \tag{68}$$

where $\zeta(3) \approx 1.202057$. The substitution in Eq. (66) is obviously not necessary if the factor of $1/(1-x)$ has a coefficient which vanishes at $x = 1$, such as $\ln(x)$.

	$C_F T_f$					$N_C T_f$			
Terms	(cd)	(e)	(fg)	(hi)	Sum	(b)	(cd)	(hi)	Sum
$p_{qg}(x)\ \ln^2(1-x)$	2		-2	2	2			-2	-2
$p_{qg}(x)\ \ln^2(x)$	2	-4		4	2	-1			-1
$p_{qg}(x)\ \ln(x)\ln(1-x)$	4	-8	-4	4	-4		4	-4	
$p_{qg}(x)\ I_0\ (\ln(1-x)+\ln(x))$	8	-8	-4	4			4	-4	
$p_{qg}(x)\ \ln(x)$	-4	8	3	-3	4	3		35/3	44/3
$p_{qg}(x)\ \ln(1-x)$	-2	6		-8	-4	4			4
$p_{qg}(x)\ \mathrm{Li}_2(1-x)$	-8			8			8	-8	
$p_{qg}(x)\ \pi^2/3$	6	-4	-2	-2	-2		-1	2	1
$p_{qg}(-x)\ S_2$						2			2
$p_{qg}(x)\ I_0$		8		-8		4	-4		
$p_{qg}(x)\ I_1$	-8	8	4	-4			-4	4	
$p_{qg}(x)$	-14	8	7	9	10	-13	-4	-65/9	-218/9
$x\ \ln^2(x)$				2	2			-8	-8
$\ln^2(x)$				-1	-1			-2	-2
$x\ln(x)$				4	4	20	10	46/3	136/3
$\ln(x)$	2	-8		5	-1	1	-4	-29/3	-38/3
$\ln(1-x)$	4		-4	4	4			-4	-4
I_0	8	-8	-4	4			4	-4	
x	2		1	-12	-9	-4	4	14/9	14/9
1	-8	6	3	3	4	17	2	11/9	182/9
$1/x$								40/9	40/9

Table 2: Results for the qg diagrams

	C_F^2				$C_F N_C$					$C_F T_f$
Terms	(cd)	(fg)	(hi)	Sum	(b)	(cd)	(e)	(hi)	Sum	(e)
$p_{gq}(x)\ \ln^2(1-x)$	1	-1	-1	-1				1	1	
$p_{gq}(x)\ \ln^2(x)$					-1/2	1	-2	2	1/2	
$p_{gq}(x)\ \ln(x)\ln(1-x)$	4	-2	-2				-4	2	-2	
$p_{gq}(x)\ I_0\ (\ln(1-x)+\ln(x))$	4	-2	-2			2	-4	2		
$p_{gq}(x)\ \ln(x)$	-3	3/2	3/2		-3/2			3/2		
$p_{gq}(x)\ \ln(1-x)$	1	-2	-2	-3	2		11/3	-2	11/3	-4/3
$p_{gq}(x)\ \mathrm{Li}_2(1-x)$	4		-4			-4		4		
$p_{gq}(x)\ \pi^2/3$		-1	1			5/2	-2	-1	-1/2	
$p_{gq}(-x)\ S_2$					1				1	
$p_{gq}(x)\ I_0$	4	-2	-2		2			-2		
$p_{gq}(x)\ I_1$	-4	2	2			-2	4	-2		
$p_{gq}(x)$	-10	5	5		-13/2	-2	67/9	14/9	1/2	-20/9
$x\ \ln^2(x)$			1/2	1/2				1	1	
$\ln^2(x)$			-1	-1				4	4	
$x^2\ \ln(x)$								-8/3	-8/3	
$x\ln(x)$	1		5/2	7/2	-1/2	4	-4	-9/2	-5	
$\ln(x)$			2	2	-10	-5		3	-12	
$x\ \ln(1-x)$	2	-2	-2	-2				2	2	
$x\ I_0$	4	-2	-2			2	-4	2		
x^2								44/9	44/9	
x	-4	3/2	-1	-7/2	17/2	1	11/3	-86/9	65/18	-4/3
1	1	1/2	-4	-5/2	-2	2		28/9	28/9	

Table 3: Results for the gq diagrams

	$C_F T_f$			$N_C T_f$					N_C^2						
Terms	(b)	(hi)	Sum	(b)	(cd)	(e)	(fg)	Sum	(b)	(cd)	(e)	(fg)	(hi)	(jk)	Sum
$p_{gg}(x)\ \ln^2(1-x)$										2		-2			
$p_{gg}(x)\ \ln^2(x)$									-1	2	-4		4		1
$p_{gg}(x)\ \ln(x)\ln(1-x)$										8	-8	-4			-4
$p_{gg}(x)\ I_0\ (\ln(1-x)+\ln(x))$										12	-8	-4			
$p_{gg}(x)\ \ln(x)$					4/3		-4/3			-11/3		11/3			
$p_{gg}(x)\ \ln(1-x)$					8/3	-8/3			4	2/3	22/3	-4	-8		
$p_{gg}(x)\ \pi^2/3$										5	-4	-2			-1
$p_{gg}(-x)\ S_2$									2						2
$p_{gg}(x)\ I_0$									4	8		-4	-8		
$p_{gg}(x)\ I_1$										-12	8	4			
$p_{gg}(x)$					40/9	-40/9	-20/9	-20/9		-170/9	134/9	103/9			67/9
$x\ \ln^2(x)$		-2	-2										4		4
$\ln^2(x)$		-2	-2										4		4
$x^2\ \ln(x)$	8/3	-8/3		-4/3	4/3				-11/3	-11/3			-22/3		-44/3
$x\ln(x)$	8	-18	-10	-4	8/3			-4/3	-17/2	-23/6			16		11/3
$\ln(x)$	8	-14	-6	-4	8/3			-4/3	-17/2	-23/6			4		-25/3
$\ln(x)/x$	8/3	-8/3		-4/3	4/3				-11/3	-11/3			22/3		
x^2	-76/9	136/9	20/3	38/9	-8/3		4/3	26/9	136/9	22/3		1/3	-46/3		67/9
x	4	4	8	-2	5/3		-5/3	-2	-105/8	-25/3		-5/12	19/2	-9/8	-27/2
1	-4	-12	-16	2	-5/3		5/3	2	105/8	25/3		5/12	-19/2	9/8	27/2
$1/x$	76/9	-64/9	4/3	-38/9	8/3		-4/3	-26/9	-136/9	-22/3		-1/3	46/3		-67/9

Table 4: Results for the gg diagrams

4 Summary

This paper has presented a recalculation of the two loop anomalous dimension for space-like processes. The results given in this paper are therefore not new. The new features are the presentation of the results in a coherent notation, the description of some of the integrals which are required to derive the results and a detailed description of the contributions of the sub-diagrams to the results.

A Virtual Integrals

A.1 Two point function

The evaluation of the virtual integrals involving non-covariant denominators of the form $1/(n \cdot k)$ requires some care. We define

$$\begin{aligned} l_+ &= n \cdot l \\ l_- &= p \cdot l \\ d^d l &= dl_+ dl_- d^{d-2} l_\perp \,. \end{aligned} \tag{A.1}$$

We shall evaluate the integrals by explicitly performing the integrals over $l_-, l_\perp$ keeping l_+ fixed. This formulation will be useful when $f(l_+)$ contains poles in $1/l_+$ coming from non-covariant denominators as long as the method used to regulate the l_+ singularity does not involve l_-. The general two-point function then reads

$$\begin{aligned} &\int \frac{d^d l}{(2\pi)^d} \frac{f(l_+)}{(l^2 + i\varepsilon)((l-k)^2 + i\varepsilon)} \\ &= \frac{i}{16\pi^2} \left(\frac{4\pi}{-k^2} \right)^{\epsilon} \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dz \, f(l_+) \, z^{-\epsilon} (1-z)^{-\epsilon} \end{aligned} \tag{A.2}$$

where f is an arbitrary function, $d = 4 - 2\epsilon$ and $z = l_+/k_+$ the boost invariant rescaled value of l_+. If $f(l_+) = 1$ we recover the normal covariant result,

$$J_2 \equiv \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + i\varepsilon)((l-k)^2 + i\varepsilon)} = \frac{i}{16\pi^2} \left(\frac{4\pi}{-k^2} \right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \frac{1}{\epsilon} \,. \tag{A.3}$$

The result for the integral with one non-covariant denominator is

$$\begin{aligned} J_{2,n} &\equiv \int \frac{d^d l}{(2\pi)^d} \mathrm{PV}\left(\frac{k_+}{l_+ - k_+} \right) \frac{1}{(l^2 + i\varepsilon)((l-k)^2 + i\varepsilon)} \\ &= \frac{i}{16\pi^2} \left(\frac{4\pi}{-k^2} \right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{I_0 + \ln x}{\epsilon} \right. \\ &\quad \left. -I_0 \ln x + I_1 - \frac{1}{2} \ln^2 x - \frac{\pi^2}{6} + O(\epsilon) \right\} \end{aligned} \tag{A.4}$$

where $x = n \cdot k/pn$ and we have indicated that the PV prescription defined by Eq. (3) has been used. I_0 is defined as in Eq. (42); in the small-δ limit it reduces to

$$I_0 = \int_0^1 du \frac{u}{u^2+\delta^2} \approx -\ln|\delta| \, . \tag{A.5}$$

Furthermore,

$$I_1 = \int_0^1 du \frac{u \ln u}{u^2+\delta^2} \approx -\frac{1}{2}\ln^2|\delta| - \frac{\pi^2}{24} \, . \tag{A.6}$$

For the case without an endpoint singularity in the integral over the plus component we may take the limit $\delta \to 0$ and hence obtain,

$$\int \frac{d^d l}{(2\pi)^d} \frac{k_+}{l_+ - p_+} \frac{1}{(l^2+i\varepsilon)((l-k)^2+i\varepsilon)}$$
$$= \frac{i}{16\pi^2}\left(\frac{4\pi}{-k^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ \frac{\ln(1-x)}{\epsilon} + 2\,\mathrm{Li}_2(1-x) - \frac{\pi^2}{3} \right.$$
$$\left. +2\ln x \ln(1-x) - \frac{1}{2}\ln^2(1-x) + O(\epsilon) \right\} \tag{A.7}$$

where $\mathrm{Li}_2(x)$ is the usual dilogarithm function,

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt \, . \tag{A.8}$$

We note that one also needs the vector two point function with one non-covariant denominator,

$$J^\mu_{2,n} \equiv \int \frac{d^d l}{(2\pi)^d} \mathrm{PV}\left(\frac{k_+}{l_+}\right) \frac{l^\mu}{(l^2+i\varepsilon)((l-k)^2+i\varepsilon)} \, . \tag{A.9}$$

Assuming Lorentz covariance of the momentum integral one finds

$$J^\mu_{2,n} = \left(k^\mu - \frac{k^2}{n\cdot k} n^\mu\right) J_2 + \frac{k^2}{2n\cdot k} n^\mu \left(-J_{2,n} + \frac{1}{k^2} J_{1,n}\right) \tag{A.10}$$

where

$$J_{1,n} = \int \frac{d^d l}{(2\pi)^d} \mathrm{PV}\left(\frac{k_+}{l_+}\right) \frac{1}{((l-k)^2+i\varepsilon)} \, . \tag{A.11}$$

It turns out that the integral $J_{1,n}$ always cancels out in the final answer.

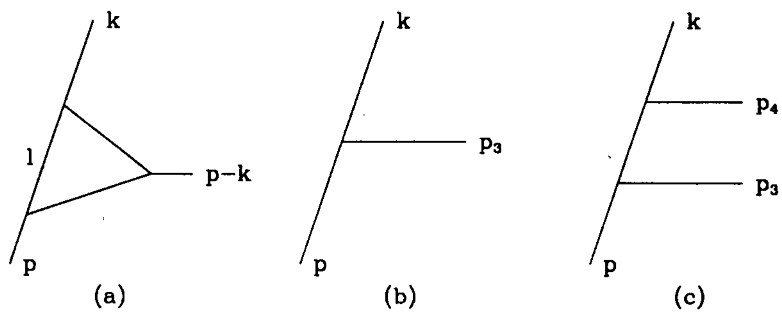

Figure 5: (a) Vertex correction graph (b) One parton emission (c) Two parton emission.

A.2 Three point function

We shall only consider the special case which is needed for our purpose. We employ the momentum assignments $p^2 = (p-k)^2 = 0$ and define the boost invariant quantities, $x = k_+/p_+, y = l_+/p_+, z = y/x = l_+/k_+$. The corresponding diagram is shown in Fig. 5a. One finds

$$\int \frac{d^dl}{(2\pi)^d} \frac{f(l_+)}{(l^2+i\varepsilon)((l-k)^2+i\varepsilon)((l-p)^2+i\varepsilon)} = \frac{-i}{16\pi^2k^2}\left(\frac{4\pi}{-k^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon}$$
$$\left[\int_0^x dy\, f(l_+)\, z^{-\epsilon}(1-z)^{-1-\epsilon}\, {}_2F_1\left(1+\epsilon,1;1-\epsilon;\frac{z(1-x)}{z-1}\right)\right.$$
$$\left.+2\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(1-x)^\epsilon \int_x^1 dy\, f(l_+)\, (1-y)^{-1-2\epsilon}\right] \qquad \text{(A.12)}$$

where ${}_2F_1$ is the hypergeometric function. With a little work one can show that for the special case $f(l_+) = 1$ one recovers the normal covariant result,

$$\int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2+i\varepsilon)((l-k)^2+i\varepsilon)((l-p)^2+i\varepsilon)}$$
$$= \frac{i}{16\pi^2k^2}\left(\frac{4\pi}{-k^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \qquad \text{(A.13)}$$

The explicit result for the three point function with one light-cone gauge denominator using the PV prescription is (we have performed a shift of the integration variables relative to Eq. (A.12)):

$$\int \frac{d^dl}{(2\pi)^d}\,\text{PV}\left(\frac{p_+}{l_++p_+}\right)\frac{1}{((l+p)^2+i\varepsilon)(l^2+i\varepsilon)((l+p')^2+i\varepsilon)} =$$

$$\frac{i}{16\pi^2 k^2}\left(\frac{4\pi}{-k^2}\right)^{\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{\frac{1}{\epsilon^2}+\frac{\ln x - I_0}{\epsilon}\right.$$

$$\left.+I_1 - I_0\ln x - 2\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2 x - \frac{\pi^2}{6} + O(\epsilon)\right\} . \qquad \text{(A.14)}$$

The other integral which we need can be obtained from Eq. (A.14) by exchange of p and p'

$$\int \frac{d^d l}{(2\pi)^d}\,\mathrm{PV}\left(\frac{p'_+}{l_+ + p'_+}\right)\frac{1}{((l+p)^2+i\varepsilon)(l^2+i\varepsilon)((l+p')^2+i\varepsilon)} =$$

$$\frac{i}{16\pi^2 k^2}\left(\frac{4\pi}{-k^2}\right)^{\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{\frac{1}{\epsilon^2}+\frac{\ln x - 2\ln(1-x) - I_0}{\epsilon}\right.$$

$$\left.+I_1 - I_0\ln x + 2\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2 x + \ln^2(1-x) - \frac{5\pi^2}{6} + O(\epsilon)\right\} . \qquad \text{(A.15)}$$

A useful relation in comparing these results with the real diagrams is

$$\Gamma(1+\epsilon)\Gamma(1-\epsilon) = 1 + \epsilon^2\frac{\pi^2}{6} + O(\epsilon^4) . \qquad \text{(A.16)}$$

B Real integrals

In this appendix we will describe some of the integrals which occur in diagrams involving the emission of real partons. As illustrated in Figs. 5(b,c), we will denote the momenta of the emitted particles by p_3 and p_4 and the momentum of the 'observed' parton line by k. The phase space for one and two parton emission, and keeping k^2 and $n \cdot k$ fixed, is given by (we set $pn \equiv 1$ in the following)

$$\begin{aligned} PS^{(1)} &= \int \frac{d^d p_3}{(2\pi)^{d-1}}\delta^+(p_3^2) \\ &\quad \int d^d k\; \delta(x - n\cdot k)\; \delta(|k^2| + (p-p_3)^2)\delta^d(p - p_3 - k) \end{aligned} \qquad \text{(B.1)}$$

$$\begin{aligned} PS^{(2)} &= \int \frac{d^d p_3}{(2\pi)^{d-1}}\delta^+(p_3^2)\int \frac{d^d p_4}{(2\pi)^{d-1}}\delta^+(p_4^2) \\ &\quad \int d^d k\; \delta(x - n\cdot k)\; \delta(|k^2| + (p-p_3-p_4)^2)\delta^d(p - p_3 - p_4 - k) . \end{aligned} \qquad \text{(B.2)}$$

Integrating over irrelevant angles in d dimensions we have for the transverse phase space,

$$\int d^{d-2}k_T = \frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)} \int dk_T^2\, k_T^{-2\epsilon} \int_0^\pi d\theta_1 \sin^{-2\epsilon}\theta_1\ . \tag{B.3}$$

If the integrand is independent of θ_1 we can integrate further to obtain

$$\int d^{d-2}k_T = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int_0^{|k^2|(1-x)} dk_T^2\, k_T^{-2\epsilon}\ . \tag{B.4}$$

B.1 Crossed ladder diagrams: topology b

Here we shall describe the integrals needed for the evaluation of the crossed ladder diagram (topology b) as shown in Fig. 6. We introduce a notation for the real parton momenta such that,

$$\begin{aligned} p_3^\mu &= z_1 p^\mu + \frac{\mathbf{t_1^2}}{2z_1} n^\mu - \mathbf{t_1}^\mu \\ p_4^\mu &= z_2 p^\mu + \frac{\mathbf{t_2^2}}{2z_2} n^\mu - \mathbf{t_2}^\mu \end{aligned} \tag{B.5}$$

with transverse momenta $\mathbf{t_1}$, $\mathbf{t_2}$. In terms of these variables the denominators which occur in the diagram in Fig. 6 can be written as

$$p_1^2 = -\frac{\mathbf{t_1}^2}{z_1},\ p_2^2 = -\frac{\mathbf{t_2}^2}{z_2},\ k^2 = (p-p_3-p_4)^2 = -a_1\mathbf{t_1}^2 - a_2\mathbf{t_2}^2 - 2\mathbf{t_1}\cdot\mathbf{t_2} \tag{B.6}$$

where

$$a_1 = \frac{(1-z_2)}{z_1},\ a_2 = \frac{(1-z_1)}{z_2}\ . \tag{B.7}$$

The general form of the matrix element which has to be integrated over the phase space of Eq. (B.2) is

$$A(z_1,z_2) + B(z_1,z_2)\frac{\mathbf{t_1}\cdot\mathbf{t_2}}{\mathbf{t_1^2}} + C(z_1,z_2)\frac{\mathbf{t_1}\cdot\mathbf{t_2}}{\mathbf{t_2^2}} + D(z_1,z_2)\frac{(\mathbf{t_1}\cdot\mathbf{t_2})^2}{\mathbf{t_1^2 t_2^2}}\ . \tag{B.8}$$

The integrations over $\mathbf{t_1}$ and $\mathbf{t_2}$ are finite at small transverse momenta, so that before the k^2 integration the expression is finite. Introducing the constants

$$\begin{aligned} F &= \frac{(4\pi)^\epsilon}{16\pi^2\Gamma(1-\epsilon)} \\ f &= \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \end{aligned} \tag{B.9}$$

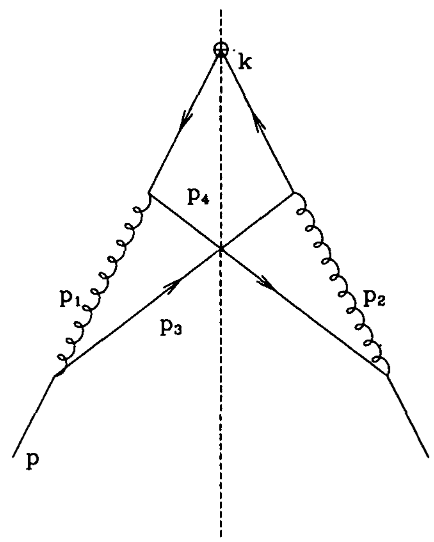

Figure 6: An example of the graph of topology (b)

we have in the frame specified by Eq. (B.5):

$$PS^{(2)} = F^2 \int \frac{dz_1}{z_1} \frac{dz_2}{z_2} \delta(1-x-z_1-z_2) \frac{d^{d-2}\mathbf{t_1}}{f} \frac{d^{d-2}\mathbf{t_2}}{f}$$
$$\delta\left(|k^2| = a_1 \mathbf{t_1}^2 + a_2 \mathbf{t_2}^2 + 2\mathbf{t_1} \cdot \mathbf{t_2} \right) . \qquad \text{(B.10)}$$

Since the integrals over the transverse momenta are finite we may take the limit $d \to 4$. The values of the integrals for the integrands which occur in Eq. (B.8) and the phase space weight given in Eq. (B.10) are collected in Table B.1. The remaining one dimensional integrals are easily performed.

B.2 Real diagrams: topology cd

The calculation of the topology (b) graphs was facilitated by the fact that before integration over k^2 the kernel was finite. The situation is more complicated for topology (cd), which has two cuts involving either one or two real partons. The cut with two real partons (gluons) is shown in Fig. 7 which also shows the definition of the kinematics.

Integrand	Integral in units of $F^2\frac{\|k^2\|}{x}\int dz_1 dz_2\, \delta(1-z_1-z_2-x)\theta(z_1)\theta(z_2)$
$<1>$	1
$<\frac{\mathbf{t_1}.\mathbf{t_2}}{\mathbf{t_1^2}}>$	$-\frac{z_1}{1-z_2}$
$<\frac{\mathbf{t_1}.\mathbf{t_2}}{\mathbf{t_2^2}}>$	$-\frac{z_2}{1-z_1}$
$<\frac{(\mathbf{t_1}.\mathbf{t_2})^2}{\mathbf{t_1^2}\mathbf{t_2^2}}>$	$\left[1+\frac{1}{2}\frac{x}{z_1 z_2}\ln\left(\frac{x}{(1-z_1)(1-z_2)}\right)\right]$

Table 5: Real integrals for crossed ladders

We perform a light-cone decomposition of the light-like momenta p_3 and p_4:

$$\begin{aligned} p_3^\mu &= (1-z)p^\mu + \frac{\mathbf{t_1^2}}{2(1-z)}n^\mu - \mathbf{t_1}^\mu \\ p_4^\mu &= z(1-y)p^\mu + \frac{\mathbf{t_2^2}}{2z(1-y)}n^\mu - \mathbf{t_2}^\mu \end{aligned} \tag{B.11}$$

where $zy = x$. It is expedient to perform a change of variables,

$$\begin{aligned} \mathbf{t_1} &\to \sqrt{\frac{1-z}{1-y}}\mathbf{r_1} \\ \mathbf{t_2} &\to \sqrt{\frac{1-y}{1-z}}(\mathbf{r_2}-(1-z)\mathbf{r_1})\ , \end{aligned} \tag{B.12}$$

so that p_3 and p_4 become

$$p_3^\mu = (1-z)p^\mu + \frac{\mathbf{r_1^2}}{2(1-y)}n^\mu - \sqrt{\frac{1-z}{1-y}}\mathbf{r_1}^\mu \tag{B.13}$$

$$p_4^\mu = z(1-y)p^\mu + \frac{(\mathbf{r_2}-(1-z)\mathbf{r_1})^2}{2z(1-z)}n^\mu - \sqrt{\frac{1-y}{1-z}}(\mathbf{r_2}^\mu-(1-z)\mathbf{r_1}^\mu)\ .$$

With this choice the propagators of the diagram in Fig. 7 can be written as

$$|k^2| = -(p-p_3-p_4)^2 = \frac{\mathbf{r_1^2}y}{(1-y)} + \frac{\mathbf{r_2^2}}{(1-z)} \tag{B.14}$$

$$|p_1^2| = -(p-p_3)^2 = \frac{\mathbf{r_1^2}}{1-y} \tag{B.15}$$

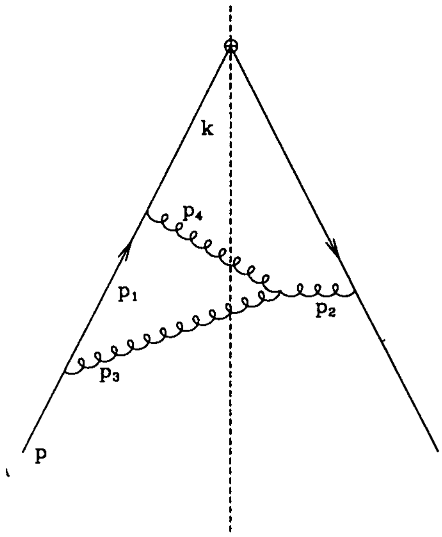

Figure 7: An example of the graph of topology (cd)

$$|p_2^2| = (p_3+p_4)^2 = \frac{\mathbf{r_1^2}+\mathbf{r_2^2}-2\mathbf{r_1}\cdot\mathbf{r_2}}{z} \tag{B.16}$$

$$n\cdot p_3 = 1-z, \qquad n\cdot p_4 = z(1-y)\ . \tag{B.17}$$

Hence we have that in this frame,

$$\begin{aligned} PS^{(2)} &= F^2 \int \frac{dz}{(1-z)}\,\frac{dy}{(1-y)}\,\delta(x-yz)\frac{d^{d-2}\mathbf{r_1}}{f}\frac{d^{d-2}\mathbf{r_2}}{f} \\ &\quad \delta\left(|k^2| = \frac{\mathbf{r_1^2}y}{(1-y)}+\frac{\mathbf{r_2^2}}{(1-z)}\right)\ . \end{aligned} \tag{B.18}$$

If we are integrating over quantities which do not depend on angles we may write Eq. (B.18) in the form

$$PS^{(2)} = F^2\,|k^2|^{1-2\epsilon}\int_x^1 \frac{dz}{x}\left(\frac{x}{(1-z)(z-x)}\right)^{\epsilon}\int_0^1 d\omega\,\omega^{-\epsilon}(1-\omega)^{-\epsilon} \tag{B.19}$$

where the rescaled transverse momentum ω is defined as

$$|p_1^2| = \frac{\mathbf{r_1^2}}{(1-y)} = \frac{\omega z|k^2|}{x}\ . \tag{B.20}$$

The results for the integrals are given in Table 6.

Integrand	Value of integral in units of $F^2\lvert k^2\rvert^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$
$<1>$	$\frac{\lvert k^2\rvert}{1-2\epsilon}\int_x^1 \frac{dz}{x}\left(\frac{x}{(1-z)(z-x)}\right)^\epsilon$
$<\frac{1}{\lvert p_1^2\rvert}>$	$-\frac{1}{\epsilon}\int_x^1 \frac{dz}{z}\left(\frac{x}{(1-z)(z-x)}\right)^\epsilon$
$<\frac{1}{\lvert p_2^2\rvert}>$	$-\frac{1}{\epsilon}\int_x^1 \frac{dz}{(1-x)}\left(\frac{x}{(1-z)(z-x)}\right)^\epsilon$
$<\frac{1}{\lvert p_1^2\rvert\lvert p_2^2\rvert}>$	$-\frac{2}{\epsilon}\frac{1}{\lvert k^2\rvert}\int_x^1 \frac{dz}{(1-z)^{1+2\epsilon}}\left(1+\epsilon\ln\left(\frac{z(1-x)}{z-x}\right)+\epsilon^2\frac{\pi^2}{6}+O\left(\epsilon^2(1-z)\right)\right)$

Table 6: Real integrals

We note at this point that Eq. (B.19) is also suitable for dealing with a light-cone gauge denominator term like $1/(n\cdot p_3)$ in the matrix element. For instance, we obtain

$$< \frac{1}{n\cdot p_3} >= F^2|k^2|^{1-2\epsilon}\frac{1}{x}\int_x^1 dz\mathrm{PV}\left(\frac{1}{1-z}\right) = F^2|k^2|^{1-2\epsilon}\frac{1}{x}\left(I_0+\ln(1-x)\right) \tag{B.21}$$

where I_0 is as defined in Eq. (42).

If we have a denominator which depends on the angle between $\mathbf{r_1}$ and $\mathbf{r_2}$, the integral is more complicated. For example when we have the denominator as given by Eq. (B.16) the angular integration splits into two regions, $\mathbf{r}_1^2 > \mathbf{r}_2^2$ and $\mathbf{r}_2^2 > \mathbf{r}_1^2$:

$$< \frac{1}{|p_2^2|} >= F^2\,|k^2|^{-2\epsilon}\int_x^1 \frac{dz}{(1-z)}\,\frac{z}{x}\left(\frac{x}{(1-z)(z-x)}\right)^\epsilon I(\alpha) \tag{B.22}$$

$$\begin{aligned} I(\alpha) \quad &= \quad \left\{\int_0^{\frac{1}{1+\alpha}} d\omega\,\omega^{-\epsilon}(1-\omega)^{-1-\epsilon}{}_2F_1(1,1+\epsilon;1-\epsilon;\frac{\alpha\omega}{1-\omega})\right.\\ &\quad \left.+\frac{1}{\alpha}\int_{\frac{1}{1+\alpha}}^1 d\omega\,\omega^{-1-\epsilon}(1-\omega)^{-\epsilon}{}_2F_1(1,1+\epsilon;1-\epsilon;\frac{1-\omega}{\alpha\omega})\right\} \end{aligned} \tag{B.23}$$

where

$$\alpha = \frac{(z-x)}{x(1-z)}\,. \tag{B.24}$$

Now by redefinition of variables the two integrals give

$$I(\alpha) = J(\alpha) + \frac{1}{\alpha} J(\frac{1}{\alpha}) \tag{B.25}$$

where

$$\begin{aligned} J(\alpha) &= \alpha^{-\epsilon} \int_0^1 dv v^{-\epsilon} (1+\alpha v)^{-1+2\epsilon} {}_2F_1(1, 1+\epsilon; 1-\epsilon; v) \\ &\equiv -\frac{1}{2\epsilon} \alpha^{-\epsilon} (1+\alpha)^{-1+2\epsilon} {}_2F_1(1, -2\epsilon; 1-\epsilon; \frac{\alpha}{1+\alpha}) . \end{aligned} \tag{B.26}$$

Hence combining using the identity,

$${}_2F_1(1, -2\epsilon; 1-\epsilon; z) + {}_2F_1(1, -2\epsilon; 1-\epsilon; 1-z) = 2 \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} z^{\epsilon} (1-z)^{\epsilon} \tag{B.27}$$

one obtains

$$I(\alpha) = -\frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{1+\alpha} . \tag{B.28}$$

Thus the final result is as given in Table 6. This result can be obtained much more easily by performing a shift of the transverse momenta so that $|p_2^2|$ only depends on a single transverse momentum.

If we now add a second denominator such a shift is no longer useful. The scalar integral with two denominators is given by a simple modification of Eq. (B.22),

$$\begin{aligned} < \frac{1}{|p_1^2||p_2^2|} > &= F^2 \, |k^2|^{-1-2\epsilon} \int_x^1 \frac{dz}{(1-z)^{1+2\epsilon}} \\ & \left\{ \alpha^{-\epsilon} \int_0^{\frac{1}{1+\alpha}} d\omega \, \omega^{-1-\epsilon} (1-\omega)^{-1-\epsilon} {}_2F_1(1, 1+\epsilon; 1-\epsilon; \frac{\alpha\omega}{1-\omega}) \right. \\ & \left. + \alpha^{-1-\epsilon} \int_{\frac{1}{1+\alpha}}^1 d\omega \, \omega^{-2-\epsilon} (1-\omega)^{-\epsilon} {}_2F_1(1, 1+\epsilon; 1-\epsilon; \frac{1-\omega}{\alpha\omega}) \right\} \end{aligned} \tag{B.29}$$

where α is as given in Eq. (B.24). By change of variables this integral may be further written as

$$< \frac{1}{|p_1^2||p_2^2|} > = F^2 \, |k^2|^{-1-2\epsilon} \int_x^1 \frac{dz}{(1-z)^{1+2\epsilon}} K(\alpha) \tag{B.30}$$

where

$$\begin{aligned} K(\alpha) &= \left\{ \int_0^1 dv \, v^{-(1+\epsilon)} \left(\frac{v}{\alpha} + 1\right)^{2\epsilon} {}_2F_1(1, 1+\epsilon; 1-\epsilon; v) \right. \\ & \left. + \int_0^1 dv \, v^{-\epsilon} \left(\frac{1}{\alpha} + v\right)^{2\epsilon} {}_2F_1(1, 1+\epsilon; 1-\epsilon; v) \right\} . \end{aligned} \tag{B.31}$$

The partial result for $K(\alpha)$ which is sufficient for our purposes is

$$K(\alpha) = -\frac{2}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(1 + \epsilon \ln\left(\frac{z(1-x)}{z-x}\right) + \epsilon^2\frac{\pi^2}{6} + O\left(\epsilon^2(1-z)\right)\right) . \quad \text{(B.32)}$$

So the final result for the integral is as given in Table 6.

References

1) G. Leibbrandt, *Phys. Rev.* **D29** (1984) 1699;
For an overview see also: G. Leibbrandt, *Rev. Mod. Phys.* **59** (1987) 1067.

2) A. Andrasi and J. C. Taylor, *Nucl. Phys.* **B310** (1988) 222.

3) G. Curci, W. Furmanski and R. Petronzio, *Nucl. Phys.* **B175** (1980) 27.

4) W. Furmanski and R. Petronzio, *Phys. Lett.* **97B** (1980) 437.

5) S. Mandelstam, *Nucl. Phys.* **B213** (1983) 149.

6) A. Bassetto, *Phys. Rev.* **D47** (1993) 727.

7) W. Vogelsang, Rutherford preprint, RAL-TR-95-071, hep-ph/9512218.

8) R. Hamberg, PhD Thesis, University of Leiden, 1991;
R. Hamberg and W. Van Neerven, *Nucl. Phys.* **B379** (1992) 143.

9) J.C. Collins and R.J. Scalise, *Phys. Rev.* **D50** (1994) 4117.

10) R.K. Ellis *et al.*, *Nucl. Phys.* **B152** (1979) 285.

11) S.A. Larin *et al.*, *Nucl. Phys.* **B427** (1994) 41.

12) G. Altarelli and G. Parisi, *Nucl. Phys.* **B126** (1977) 298.

13) Yu. L. Dokshitzer *Sov. Phys. JETP* **46** (1977) 641;
cf. L. N. Lipatov, *Sov. J. Nucl. Phys.* **20** (1975) 95;
V.N. Gribov and L.N. Lipatov, *Sov. J. Nucl. Phys.* **15** (1972) 438.

14) R.T. Herrod and S. Wada, *Phys. Lett.* **96B** (1980) 195.