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Technical Report
RAL-TR-96-018

Proceedings of the School for Young High Energy Physicists

Dr D Dunbar Dr J Flynn Dr E W N Glover and Dr G M Shore

March 1996

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ISSN 1358-6254

RAL-TR-96-018

Central Laboratory of the Research Councils

Rutherford Appleton Laboratory
Chilton DIDCOT Oxon OX11 0QX

Proceedings of the School for Young High Energy Physicists

Editor: K J Peach

Rutherford Appleton Laboratory
September 10 - 22 1995

HEP SUMMER SCHOOL FOR YOUNG HIGH ENERGY PHYSICISTS

RUTHERFORD APPLETON LABORATORY/THE COSENER'S HOUSE, ABINGDON:

10 - 22 SEPTEMBER 1995

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RAL Summer School for Young Experimental High Energy Physicists

Cosener's House, 10 - 22 September 1995

Preface

Fifty-two young experimental particle physicists students attended the 1995 Summer school, held as usual in Cosener's House in Abingdon in mid September. This year, the weather was mild and sunny, and a number of tutorials and impromptu seminars were held on the spacious lawns leading down to the Thames, adding to and enhancing the relaxed atmosphere which disguises the very real intellectual challenge of the material.

The lectures reproduced here were given by Dave Dunbar (Relativistic Quantum Field Theory), Jonathan Flynn (Relativistic Quantum Mechanics), Graham Shore (The Standard Model and Beyond) and Nigel Glover (Phenomenology). They were delivered and received enthusiastically, providing material for lively discussions in tutorials and elsewhere.

Michael Berry (Bristol) gave an interesting seminar on the physics of the Levitron, solving on the way some Christmas present problems. Mike Whalley introduced the Durham HEP database with its impressive facilities now available through the World Wide Web. George Kalmus gave an informed and entertaining after dinner speech, finishing (as all such speeches should) with a challenging, relevant and politically correct joke.

Each student gave a ten minute seminar in one of the evening sessions; I am consistently amazed by the quality of these talks. In many ways, the shorter the talk the more difficult the task of communicating a coherent message. It is a real achievement that so many did so with style and evident good humour.

The work of the school was helped enormously by the hard work of the tutors - Susan Cartwright (Sheffield), Paul Dauncey (RAL), Jeff Forshaw (RAL), Paul Harrison (QMWC) and Bill Scott (RAL). Cosener's House provided its customary welcome; the calmness of the house and grounds - largely undisturbed by the bustle of Abingdon - and the excellent food are important factors contributing to the success of the school. The hard work and good humour of all of the Cosener's staff are much appreciated. The School also owes a debt of gratitude to Ann Roberts, who once again organised the director efficiently, and whose quick thinking and lively anticipation ensured that potential disasters were avoided.

The ingredients for a successful summer school are few - an interesting topic, excellent lecturers and tutors, pleasant surroundings and above all committed and enthusiastic participants. This year, all came together to create a superb atmosphere. The school is intellectually and physically demanding, but also rewarding. I have enjoyed my three years as director, none more than this year's school. To all who helped make it so enjoyable - lecturers, tutors, staff at both Cosener's and RAL and above all the students - I extend my thanks and my good wishes. In particular, I wish to acknowledge the efforts and support of Susan Cartwright, Ann Roberts, Jonathan Flynn and Bill Scott who have been with me throughout these three years, and to wish Graham Shore (who is also leaving after only two years) the best of luck with BUSSTEPP next year.

Finally, I hope that Steve Lloyd, my successor as Director of the School, has as much fun as I have had. To all who made it fun, many thanks and best wishes.

Ken Peach (Director)
Department of Physics & Astronomy,
University of Edinburgh

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INTRODUCTION TO QUANTUM FIELD THEORY AND GUAGE THEORIES

By Dr D C Dunbar
University of Wales, Swansea

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1995

Introduction to Quantum Field Theory and Gauge Theories

David C. Dunbar

University of Wales, Swansea

Contents

Acknowledgments:

In preparing these lectures I have extensively "borrowed" ideas from the equivalent courses given by previous speakers especially those of Ian Halliday. In places, this "borrowing" is close to complete. These notes are more extensive in places than what was actually discussed during the lecture course. In particular the issue of Gauge fixing was not mentioned during the lectures although I include it here. (I had *intended* to discuss it!) Jonathon Flynn covered gauge fixing during his course in a slightly different manner from these notes. Hopefully we will complement each other rather than interfere.

Finally, I would like to thank Ken Peach for his relentless enthusiasm throughout the school, Ann Roberts for organising things impecably and the students for "hanging in" through the rather fast schedule.

Jan 5th 1996

Introduction

The purpose of this course is twofold.

Firstly, it is provide a simple introduction to quantum field theory starting from, roughly, your undergraduate quantum mechanics course. Since you undoubtably come from a very varied background this is not particularly easy and I guess the beginning material will be fairly familiar to many of you. To ensure a level "playing field" I will assume only that you are all familiar with the distributed prerequisites. I hope you are! The intended endpoint will be to enable you to take a general field theory and write down the appropriate Feynman rules which are used to evaluate scattering amplitudes. There are two formalisms commonly used for this. The simplest *for a simple theory* is the "Canonical quantisation" wheras the more modern approach is to use the "Path Integral Formulation". I will cover both during the course although the Path Integral Formulation will be done rather hueristically.

The second theme will be to consider the quantisation of gauge theories. For various reasons this is not completely a trivial application of general quantum field theory methods. Hopefully this will connect up to the other courses at this school.

1. Classical Formulations of Dynamics

There are three "equivalent" but different formulations of classical mechanics which I will consider here,

- Newtonian
- Lagrangian
- Hamiltonian

I will illustrate these formulations with a specific example - the simple pendulum, which approximates to a harmonic oscillator when the perturbations are small. The ideal pendulum which we consider here is an object of mass m described by its positions x and y connected to the point $(0,0)$ by a rigid string. This is an example of a constrained system because x and y are forced to satisfy the constraint $x^2+y^2=L^2$ where L is the length of the string. The object could equivalently be described by the angle θ which is a function of x,y given by $\tan\theta=-x/y$.

• Firstly consider Newtonian Mechanics. Newtonian mechanics are only valid if we consider inertial coordinates. In this case good coordinates are $\underline{x}=(x,y)$ and *not* θ whence we have Newtons equations

$$m\frac{d^2\underline{x}}{dt^2}=\underline{F} \tag{1.1}$$

Newton's equation's reduce to a pair of second order coordinates. To these equations we have to explicitly insert the forces applied by the string.

• Next we consider the Lagrangian method. For Lagrange an important difference is that any coordinates will do not merely inertial ones. Thus we are free to describe the pendulum using θ. In general a system will be described by coordinates q_r. We construct the *Lagrangian* from the kinetic (T) and potential (V) energy terms $L=T-V$. Lagrange's equations in terms of L are

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_r}\right]-\frac{\partial}{\partial q_r}L=0 \tag{1.2}$$

For the simple pendulum, if we use θ as a coordinate Lagranges equation produces a single second order equation. The advantage over Newton's method lies in the simplicity in the way which constraints may be applied.

• We now turn to the Hamiltonian method. The idea is to work with first order differential equations rather than second order equations. Suppose we define

$$p_r\equiv\frac{\partial L}{\partial \dot{q}_r} \tag{1.3}$$

then we can write Lagrange's equations as

$$\frac{dp_r}{dt}=\frac{\partial L}{\partial q_r} \tag{1.4}$$

For a system with Kinetic term

$$T=\sum_r\frac{1}{2}m_r\dot{q}_r^2 \tag{1.5}$$

then p_r is just the normal momentum. The Lagrangian is a function of q_r and $\dot{q}_r$. We wish to change variables from $q, \dot{q}$ to q, p. (This is a very close analogy to what happens in a thermodynamic system when changing variables from V, S to V, T.) Examine the response of L to a small change in q_r and $\dot{q}_r$,

$$\begin{aligned} \delta L &= \sum_r \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) \\ &= \sum_r \left(\dot{p}_r \delta q_r + p_r \delta \dot{q}_r \right) \end{aligned} \tag{1.6}$$

by eqs.(1.3)and (1.4). We can, by adding and subtracting $\sum_r \dot{q}_r \delta p_r$, rewrite this as

$$= \delta \left(\sum_r p_r \dot{q}_r \right) + \sum_r \left(\dot{p}_r \delta q_r - \dot{q}_r \delta p_r \right) \tag{1.7}$$

So that by shuffling terms we obtain

$$\delta \left(-L + \sum_r p_r \dot{q}_r \right) = \sum_r \dot{q}_r \delta p_r - \sum_r \dot{p}_r \delta q_r \tag{1.8}$$

So we have obtained a quantity whose responses are in terms of δp_r and δq_r. This is the Hamiltonian. It is given, in general, in terms of the Lagrangian by

$$H = \sum_r p_r \dot{q}_r - L \tag{1.9}$$

The Hamiltonian is to be thought of as a function of q_r and p_r only. If $T \sim \dot{q}^2$ and $V = V(q)$, as is the case in many situations, then $H = T + V$. However the above expresion is the more general. The Hamiltonian equations are then, from (1.8)

$$\begin{aligned} \dot{q}_r &= \frac{\partial H}{\partial p_r} \\ \dot{p}_r &= - \frac{\partial H}{\partial q_r} \end{aligned} \tag{1.10}$$

This is a very similar to the situation in thermodynamics if we change from the energy, E, satisfying $dE = TdS - PdV$ where E is thought of as a function of S, V to the Free energy F which is thought of as a function of T, V and $dF = -SdT - PdV$. Recall that the relationship betwen E and F is $F = E - ST$. In fact, the correct way of thinking about this is to regard thermodynamics as a dynamical system whence the change from E to F is precisely a change such as from L to H. The Hamiltonian system is particularly useful when we consider quantum mechanics because q and p become non-commuting operators - something which makes sense if we use $H(p, q)$ but which requires more thought if we use $L(q, \dot{q})$. For our simple pendulum, Hamiltonian dynamics will produce a pair of first order equations.

Before leaving Hamiltonian mechanics, let is define the Poisson Bracket of any two functions of p and q. Let f and g be any functions of p, q then

$$\left\{f, g\right\} = \sum_r \left(\frac{\partial f}{\partial q_r} \frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r} \frac{\partial g}{\partial q_r} \right) \tag{1.11}$$

The Poisson bracket of the variables q_i and p_j are then

$$\begin{aligned} \left\{q_i, q_j\right\} &= 0 \\ \left\{q_i, p_j\right\} &= \delta_{ij} \\ \left\{p_i, p_j\right\} &= 0 \end{aligned} \tag{1.12}$$

A *Canonical* change of coordinates is a change from p, q to coordinates $Q(p, q)$ and $P(p, q)$ which maintain the above Poisson brackets. Hamiltonian dynamics is invariant under such canonical transformations. (As an extremely nasty technical point, Quantum mechanics is *not.* Thus there are many quantisations of the same classical system , in principle.)

The best known way of quantising a classical system uses the Hamiltonian formalisms, replaces q_r and p_r by operators and replacing the Poisson brackets by commutators

$$\left\{\cdots\right\} \rightarrow \left[\cdots\right]/i\hbar \tag{1.13}$$

2. Quantum Pictures

2.1 The Dirac or Interaction Picture

In the prerequisites, the are two equivalent pictures of Quantum mechanics: 1) the Schrödinger picture where the wavefunction is time dependant and the operators not and 2) the Heisenberg picture where the wavefunction is time-independant and the time-dependance is carried by the operators. I will introduce a third picture which is called the Dirac picture or, frequently, the interaction picture. First we set the scene. Take a typical situation where the Hamiltonian of a system is described as a "solvable piece" H_0 and a "small perturbation piece" H_I.

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{2.1}$$

Actually the interaction picture doesn't care whether H_I is small or not but is really only usefull when it is. One of the depressing/hopeful features of physics is how few problems have been solved exactly in quantum mechanics. There are actually only two. The first is the simple harmonic oscillator, the second is the hydrogen atom. (a third should or should not be added to this according to taste - it is the two dimensional Ising model.) All other cases which have been solved exactly are equivalent to these two cases. Free Field theory (non-interacting particles) is, as we will see, solvable because it can be related to a sum of independant harmonic oscilators. It is also amazing how far we have taken physics with just these few examples! Perhaps someday someone will solve a further model and physics will advance enormously.

Since there is so little we can solve exactly a great deal of effort has gone into developing approximate methods to calculate. The methods I will develop here are for calculating matrix elements and will be perturbative in the (assumed) small perturbation H_I. These have proved enormously sucessful (but don't answer all questions..) For a given operator $\hat{O}$, we can define the interaction picture operator $\hat{O}_I$ in terms of the Schrödinger operator by

$$\begin{aligned}\hat{O}_I =& e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t} \\ =& e^{i\hat{H}_0 t} e^{-i\hat{H} t}\hat{O}_H e^{i\hat{H} t} e^{-i\hat{H}_0 t} \\ =& \hat{U}(t)\hat{O}_H \hat{U}^{-1}(t)\end{aligned} \tag{2.2}$$

(We set $\hbar = 1$ unless explicitly stated otherwise - it is always a useful exercise to reinsert $\hbar$ in equations.) The operator

$$\hat{U}(t) \equiv e^{i\hat{H}_0 t} e^{-i\hat{H} t} \tag{2.3}$$

will be critical in what follows. In the case where $H_I = 0$ the interaction picture reduces to the Heisenberg picture and $U(t) = 1$. We must make a similar definition for the states in the Dirac picture

$$|a,t\rangle_I = e^{i\hat{H}_0 t}|a,t\rangle_S = \hat{U}(t)|a\rangle_H \tag{2.4}$$

Note that the Dirac picture states contain a time dependance. Since the operators are transformed as if in the Heisenberg picture for H_0 we have

$$i\frac{\partial}{\partial t}\hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0] \tag{2.5}$$

To calculate in the interaction picture we need to evaluate $\hat{U}(t)$. *It is this object which will be the focus of perturbation theory.* We have

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{U}(t) =& - \hat{H}_0 e^{i\hat{H}_0 t}e^{-i\hat{H}t} + e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{H} \\ =& e^{i\hat{H}_0 t}\hat{H}_I e^{-i\hat{H}t} \\ =& (\hat{H}_I)_I \hat{U}(t) \end{aligned} \tag{2.6}$$

where the confusing notation $(\hat{H}_I)_I$ denotes that the *operator* H_I has been transformed into the interaction picture. Clearly if H_I is a function of operators, $H_I(\hat{O}^j)$, then $(\hat{H}_I)_I = H_I(\hat{O}^j_I)$.

We are now in a position to solve this equation perturbatively, always assuming that H_I forms a small perturbation. Expanding $U(t)$ as a series,

$$U(t) = 1 + U_1 + U_2 + U_3 + \cdots \tag{2.7}$$

We can then substitute this into the equation for $U(t)$ and solve order by order. We find for U_1,

$$i\frac{\partial}{\partial t}U_1 = \hat{H}_I(t) \tag{2.8}$$

which can be solved to give

$$U_1 = -i\int_0^t \hat{H}_I(t_1)dt_1 \tag{2.9}$$

and for U_2

$$i\frac{\partial}{\partial t}U_2 = \hat{H}_I(t)U_1(t) \tag{2.10}$$

giving

$$U_2 = (-i)^2\int_0^t dt_2 \int_0^{t_2} dt_1 \hat{H}_I(t_2)\hat{H}_I(t_1) \tag{2.11}$$

From this we can guess the rest (or prove recursivly)

$$U_n = (-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1 \hat{H}_I(t_n)\hat{H}_I(t_{n-1})\cdots\hat{H}_I(t_2)\hat{H}_I(t_1) \tag{2.12}$$

Notice that in the above $t_n > t_{n-1} > \ldots t_2 > t_1$. This can all be massaged into a more standard form. We define the *time ordered product* of any two operators by

$$\begin{aligned} T(\hat{A}(t_1),\hat{B}(t_2)) =& \hat{A}(t_1)\hat{B}(t_2);\;\; t_1 > t_2 \\ =& \hat{B}(t_2)\hat{A}(t_1);\;\; t_2 > t_1 \end{aligned} \tag{2.13}$$

Note that within a time ordered product we can commute two operators as we like. Now the expresion for U_2 may be written

$$(-i)^2\int_0^t dt_2\int_0^{t_2} dt_1\hat{H}_I(t_2)\hat{H}_I(t_1) = \frac{(-i)^2}{2}\int_0^t dt_2\int_0^t dt_1 T(\hat{H}_I(t_2),\hat{H}_I(t_1)) \tag{2.14}$$

where the integrations now both run from 0 to t. The times ordered product ensures that the ordering of operators is as before and the factor of 1/2 comes because the integral now "overcounts". Similarly we obtain,

$$U_n = \frac{(-i)^n}{n!} \int_0^t \prod_i dt_i T(\hat{H}_I(t_n), \hat{H}_I(t_{n-1}), \cdots, \hat{H}_I(t_2), \hat{H}_I(t_1)) \tag{2.15}$$

We are now in a position to formally sum the contributions into an exponential,

$$U(t) = T(exp(-i \int_0^t \hat{H}_I(t)dt)) \tag{2.16}$$

This is in many senses a formal solution. As we will see later the perturbative evaluation typically involves finding U_1, U_2 themselves. We will spend a considerable effort in evaluating the U_i operators later.

2.2 Lagrangian Quantum Mechanics and the Path Integral

We now turn to the second distinct part of this section on Quantum mechanics. This will involve a formulation of quantum mechanics which involves the Lagrangian rather than the Hamiltonian. We will present this for a single coordinate q and momentum p. We will take two steps later: firstly to consider q as a vector of coordinates and secondly to take it as a field. We will initially work with a simplified Hamiltonian,

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{2.17}$$

Recall that we can consider eigenstates or *either* position $|q\rangle$ satisfying $\hat{q}|q\rangle = q|q\rangle$ or momentum $|p\rangle$ satisfying $\hat{p}|p\rangle = p|p\rangle$ but we cannot have simultaneous eigenstates. In fact the momentum and position eigenstates can be expressed in terms of each other via

$$|q\rangle = \int \frac{dp}{2\pi} e^{-ipq} |p\rangle \,, \quad |p\rangle = \int \frac{dq}{2\pi} e^{ipq} |q\rangle \tag{2.18}$$

We consider the amplitude for a particle to start at initial point q_i at time $t = t_i$ and end up at point q_f at $t = t_f$. In the Schrödinger picture this is

$$A = \langle q_f | e^{-i\hat{H}t} | q_i \rangle \tag{2.19}$$

where $|q\rangle$ are the time independant eigenstates of $\hat{q}$ and we take $t_i = 0, t_f = t$. The following manipulation of this amplitude is due to Feynman originaly. We split up the time interval t into a large number, n, of small steps of length $\Delta = (t_f - t_i)/n$. Then, trivially,

$$e^{-i\hat{H}t} = e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdots e^{-i\hat{H}\Delta} \tag{2.20}$$

and

$$A = \langle q_f | e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdot e^{-i\hat{H}\Delta} \cdots e^{-i\hat{H}\Delta} | q_i \rangle \tag{2.21}$$

In between the terms we now insert representations of one (quantum mechanically)

$$\begin{aligned}\int dq|q\rangle\langle q| &=1\\ \int dp|p\rangle\langle p| &=1\end{aligned} \tag{2.22}$$

to obtain the following expression for A,

$$\begin{aligned}\int_{q_i,p_i} \langle q_f|p_n\rangle\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle\langle q_{n-1}|p_{n-1}\rangle\langle p_{n-1}|e^{-i\hat{H}\Delta}|q_{n-2}\rangle\\ \times\langle q_{n-2}|p_{n-2}\rangle\cdots|q_1\rangle\langle q_1|p_1\rangle\langle p_1|e^{-i\hat{H}\Delta}|q_i\rangle\end{aligned} \tag{2.23}$$

In the above we may make the replacement

$$\langle q_i|p_i\rangle = e^{iq_ip_i} \tag{2.24}$$

We may also evaluate approximately

$$\begin{aligned}\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle &\sim\langle p_n|(1-i\hat{H}(\hat{p},\hat{q})\Delta)|q_{n-1}\rangle\\ &=\langle p_n|(1-iH(p_n,q_{n-1})\Delta)|q_{n-1}\rangle\\ &=e^{-iH(p_n,q_{n-1})\Delta}e^{-ip_nq_{n-1}}\end{aligned} \tag{2.25}$$

where we are using the fact that Δ is small and the form of $\hat{H}$. Note that we have turned operators into numbers in the above. We can now rewrite the amplitude and take the limit $n\to\infty$,

$$\begin{aligned}A = & \lim_{n\to\infty}\int\prod_{i=1}^{n}dp_i\prod_{i=1}^{n-1}dq_i\left\{\prod_i e^{-iH(p_i,q_{i-1})\Delta}e^{-ip_iq_{i-1}}e^{ip_iq_i}\right\}\\ = & \lim_{n\to\infty}\int\prod_{i=1}^{n}dp_i\prod_{i=1}^{n-1}dq_i\left\{\exp\left(i\sum_i\Delta(\frac{(q_i-q_{i-1})p_i}{\Delta}-H(p_i,q_{i-1})\right)\right\}\\ \equiv & \int[dq][dp]e^{i\int dt[p\dot{q}-H]}\end{aligned} \tag{2.26}$$

The last line is the *Path Integral formulation.* It is an interesting question what the symbols mean in this equation!. In the integrations all intermediate values of p, q contribute. We can interpret this as an integral over all possible paths a particle may take between q_i and q_f. This expresion is commonly used but is not quite the Lagrangian formalism. To obtain this we must evaluate the dp_j integrals at the penultimet step (before $n\to\infty$). The integral is assuming the simplified form for $H = p^2/2m + V(q)$,

$$\begin{aligned}\int dp_ie^{-i\frac{p_i^2}{2m}\Delta}e^{ip_i(q_i-q_{i-1})} &= e^{i\frac{(q_i-q_{i-1})^2m}{2\Delta}}\\ &\sim e^{i\Delta\frac{m\dot{q}^2}{2}}\end{aligned} \tag{2.27}$$

where we approximate $(q_i - q_{i-1})$ by $\dot{q}_i\Delta$. Using this we can again take $n \to \infty$ to obtain the expresion

$$\int [dq] e^{i \int dt L(q,\dot{q})} \tag{2.28}$$

This Formulation of Quantum mechanics is one we will use extensively. A useful object is the *Action*, S, defined as

$$S = \int dt L \tag{2.29}$$

whence the path integral is

$$\int [dq] e^{iS/\hbar} \tag{2.30}$$

(just for fun I reinserted $\hbar$ in this equation.) The classical significance of S is that it may be used to obtain the equations of motion. Lagranges equations arise by demanding the Action is at an extremal value. A common way to express the path integral, is to say that all paths are summed over, weighted by $e^{i \times \text{action}}$. This has a certain appeal. Think about what happens as $\hbar \to 0$. This formulation has strong anologies with statistical mechanics where the partition function is the sum over all configurations weighted by the energy

$$Z \sim \sum_i e^{-E_i/kT} \tag{2.31}$$

however the factor of i should never be forgotten!

3. Field Theory: A Free Boson

3.1 The classical treatment

In this section we will examine our first Field Theory, look at it initially and then quantise and solve. This will only be possible because it is a non-interacting field theory. We will consider a field, $\phi(x)$. That is an object which has a value at every point in space. This is unlike the harmonic oscilator where, although wavefunctions depend on space these are merely the probability of observing a particle at that point. A field configuration is then described by a (continuous) infinity of real numbers as oppossed to the single number describing a harmonic oscillator. This infinity will, of course, complicate the mathematics. We can easily postulate the Kinetic energy of such a term to be

$$T = \int d^3x \frac{1}{2}(\frac{d\phi(x,t)}{dt})^2 \tag{3.1}$$

This gives the field a Kinetic energy at each point. The potential term we take as

$$V = \frac{m^2}{2}\int d^3x \phi^2(x,t) + \int d^3x \frac{c^2}{2}\sum_{i=1}^{3}(\frac{d\phi(x,t)}{dx_i})^2 \tag{3.2}$$

The "mass term" $\phi^2(x,t)$ is easy to understand. The remaining kinetic term $(\frac{d\phi(x,t)}{dx_i})^2$ is necessary by Lorentz invariance. (Or one may consider the model of an electic sheet with potential energy, consider small perturbations and then evaluate the potential energy: a term such as this then appears.) The c should be the speed of light for Lorentz invariance.

From this we may construct the Lagrangian,

$$L = \int d^3x \left[\frac{1}{2}(\frac{d\phi(x,t)}{dt})^2 - \frac{c^2}{2}\sum_{i=1}^{3}(\frac{d\phi(x,t)}{dx_i})^2 - \frac{m^2}{2}\phi^2(x,t)\right] \tag{3.3}$$

which we may apply Lagranges method to. For field's we often speak of the Lagrangian density $\mathcal{L}$ where $L = \int d^3x\mathcal{L}$. Before doing so we will rewrite this form in a more Lorentz covariant manner. Define a four-vector x^μ where $\mu = 0 \cdots 3$ and $x^0 = t$. We henceforth set $c = 1$ (otherwise $\hbar$ would be jealous). Then

$$\begin{aligned} \partial_\mu\phi =& \frac{\partial\phi}{\partial t} : \mu = 0 \\ =& \frac{\partial\phi}{\partial x_i} : \mu = i \end{aligned} \tag{3.4}$$

It is a fundamental fact of relativity that x^μ and $\partial^\mu\phi$ are 4-vectors. I.e. they transform in a well behaved fashion under Lorentz transformations. Four vectors are similar to normal vectors if one remembers the important minus signs. From the vector x^μ one can define a "co-vector" x_μ by $x_0 = x^0, , x_i = -x^i, i = 1,2,3$. In more fancy language $x_\mu = \sum_\nu g_{\mu\nu}x^\nu$ where $g_{\mu\nu}$ are the elements of a 4×4 matrix g. In this case $g = \text{diag}(+1,-1,-1,-1)$. I mention this to introduce *the Einstein summation convention* where we write $x_\mu = g_{\mu\nu}x^\nu$ and the summation is understood. With this convention, $x_\mu x^\mu = t^2 - x^2 - y^2 - z^2$.

The dot product of two four vectors,

$$A \cdot B \equiv A_\mu B^\mu = A_0 B_0 - \sum_{i=1}^{3} A_i B_i \tag{3.5}$$

is invariant under Lorentz transformations. The action S is

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \; - \frac{m^2}{2}\phi^2\right] \tag{3.6}$$

which since the measure $d^4x \equiv dtd^3x$ is invariant under Lorentz transformation. I am actually sliping in a very very important concept here. Namely that symmetries of the theory are *Manifest* in the action or Lagrangian. (By contrast the Hamiltonian formulation also gives Lorentz invariant behaviour but it is not manifestly Lorentz invariant.) Since symmetries are very important, the Lagrangian formalism is a good place to study them. We can define the momenta conjugate to the field ϕ

$$\Pi(\underline{x}, t) = \frac{\partial \mathcal{L}}{\partial_0 \phi(\underline{x}, t)} \; = \partial^0 \phi \tag{3.7}$$

whence the Hamiltonian becomes

$$H = \int d^3x \left[\frac{\Pi^2(\underline{x},t)}{2} + \frac{1}{2}\Big(\frac{\partial\phi(\underline{x},t)}{\partial x_i}\Big)^2 + \frac{m^2}{2}\phi^2(\underline{x},t)\right] \tag{3.8}$$

Notice that this is *not* invariant under Lorentz transformations. let us now solve this system classically now. First we must present Lagranges equations for a field. Because of the space derivates $\partial\phi/\partial x$ the equations become modified. (We could see this by returning to S and examining the conditions that S is extremeised.)

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] + \frac{\partial}{\partial x_i}\left[\frac{\partial \mathcal{L}}{\partial(\frac{\partial\phi}{\partial x_i})}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{3.9}$$

(where the sum over i is implied). For our Lagrangian this yields

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x_i^2} + m^2\phi = 0 \tag{3.10}$$

or

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{3.11}$$

We now find the general solution to this equation. Since the system is linear in ϕ the sum of any two solutions is also a solution. Try a plane wave solution,

$$\phi(\underline{x}, t) = Ae^{i(\underline{k}\cdot\underline{x} - \omega t)} \tag{3.12}$$

then substituting this into eq.(3.10) gives

$$A\left[-\omega^2 + \underline{k}^2 + m^2\right] e^{i(\underline{k}\cdot\underline{x}-\omega t)} = 0 \tag{3.13}$$

so that the trial form will be a solution provided

$$\omega(k) = \pm\sqrt{\underline{k}^2 + m^2} \tag{3.14}$$

Notice that there are two solutions. From now on take $\omega(k)$ to denote the positive one. The general solution will be

$$\phi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left[a(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)} - a^*(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right] \tag{3.15}$$

The $a(k)$ and $a^*(k)$ are constants. We have also imposed the condition $\phi^* = \phi$ which is necessary for a real field. For purely conventional reasons we have chosen the normalisations given. A classical problem would now just degenerate to finding the $a(k)$ and $a^*(k)$ by e.g., examining the boundary conditions. To finish this section on the classical properties note that

$$\Pi(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3 2} \left[-ia(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)} - ia^*(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right] \tag{3.16}$$

3.2 The Quantum theory

We will now quantise the theory. The field variables are $\phi(\underline{x}, t)$ and $\Pi(\underline{x}, t)$. we must decide upon the commutation relations for these objects. That is, we want the appropriate generalisations of (1.12) for the case where the q and p now are a continuous infinite set. These are

$$\begin{aligned}
\left[\hat\phi(\underline{x}, t),\ \hat\phi(\underline{y}, t)\right] &= 0 \\
\left[\hat\Pi(\underline{x}, t),\ \hat\phi(\underline{y}, t)\right] &= -i\delta^3(\underline{x} - \underline{y}) \\
\left[\hat\Pi(\underline{x}, t),\ \hat\Pi(\underline{y}, t)\right] &= 0
\end{aligned} \tag{3.17}$$

This looks reasonable except that the δ_{ij} present for a discrete number of coordinate is replaced by the Dirac-δ function. I'll try to elucidate this in an exercise.

Let us now, in the Heisenberg picture examine the equations of motion for $\hat\phi$ and $\hat\Pi$,

$$\begin{aligned}
i\dot{\hat\phi}(\underline{x}, t) &= \left[\hat\phi(\underline{x}, t), \hat H\right] \\
&= \int d^3y \left[\hat\phi(\underline{x}, t), \frac{\hat\Pi^2(y, t)}{2}\right] \\
&= \int d^3y \frac{1}{2}\left(\left[\hat\phi(\underline{x}, t), \hat\Pi(y, t)\right]\hat\Pi(y, t) + \hat\Pi(y, t)\left[\hat\phi(\underline{x}, t), \hat\Pi(y, t)\right]\right) \\
&= \int d^3y i\delta^3(\underline{x} - \underline{y})\hat\Pi(y, t) \\
&= \hat\Pi(x, t)
\end{aligned} \tag{3.18}$$

and for $\hat{\Pi}$,

$$\begin{aligned}
i\dot{\hat{\Pi}}(\underline{x},t) &= \left[\hat{\Pi}(\underline{x},t),\hat{H}\right] \\
&= \int d^3y \left[\hat{\Pi}(\underline{x},t),\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y}\right]\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y} + \int d^3y m^2 \left[\hat{\Pi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right]\hat{\phi}(\underline{y},t) \\
&= \int d^3y \left\{-i\frac{\partial}{\partial y}\delta^3(\underline{x}-\underline{y})\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y} - im^2\hat{\phi}(\underline{y},t)\delta^3(\underline{x}-\underline{y})\right\} \\
&= i\frac{\partial^2\hat{\phi}(\underline{x},t)}{\partial x^2} - im^2\hat{\phi}(\underline{x},t)
\end{aligned} \tag{3.19}$$

We can combine and rewrite these two equations as

$$\begin{aligned}
\frac{\partial^2\hat{\phi}(\underline{x},t)}{\partial t^2} &= \nabla^2\hat{\phi} - m^2\hat{\phi} \\
\hat{\Pi}(\underline{x},t) &= \dot{\hat{\phi}}(\underline{x},t)
\end{aligned} \tag{3.20}$$

which is just as before. However, now these are operator equations with the solution

$$\hat{\phi}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{a}(k)e^{i(\underline{k}\cdot\underline{x}-\omega t)} - \hat{a}^\dagger(-k)e^{i(\underline{k}\cdot\underline{x}+\omega t)}\right] \tag{3.21}$$

Now the $\hat{a}$ and $\hat{a}^\dagger$ are operators. This can be rewritten using four vectors in the form

$$\hat{\phi}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{a}(k)e^{ik\cdot x} + \hat{a}^\dagger(k)e^{-ik\cdot x}\right] \tag{3.22}$$

Where the four vector k^μ is formed from ω and $\underline{k}$. (It requires a little care and relabelling under the integral sign to show this.) We can deduce the commutation relationships for them from those for $\hat{\phi}$ and $\hat{\Pi}$,

$$\begin{aligned}
\left[\hat{a}(\underline{k}),\hat{a}(\underline{k}')\right] &= 0 \\
\left[\hat{a}(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k}-\underline{k}') \\
\left[\hat{a}^\dagger(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= 0
\end{aligned} \tag{3.23}$$

Thus as promised we find an infinite set of harmonic oscillators labeled by the momenta $\underline{k}$. If we substitute the forms for $\hat{\phi}$ into the Hamiltonian we find (tediously)

$$\hat{H} = \int \frac{d^3k}{2(2\pi)^3}\hat{a}^\dagger(\underline{k})\hat{a}(\underline{k}) + \text{const.} \tag{3.24}$$

So that the Hamiltonian is a sum of independant harmonic oscillators. We can thus apply our knowledge of such objects to this case. If we denote the ground state by $|0\rangle$ then we will form states by applying raising operators to the vacuum. $\hat{a}^\dagger(\underline{k})$ will create a particle of momentum $\underline{k}$ and energy $\hbar\omega(k)$. (try reinserting the $\hbar$s!) We can easily check

$$\hat{H}\hat{a}^\dagger(\underline{k})|0\rangle = \omega(k)a^\dagger(\underline{k})|0\rangle \tag{3.25}$$

Similarly we may create the two particle states

$$a^\dagger(\underline{k}_1)a^\dagger(\underline{k}_2)|0\rangle \tag{3.26}$$

etc, etc. Notice that because of the commutation relationships that the 2-particles states are even under exchange. That means our system is a system of non-interacting bosons.

We have taken ϕ to be a real field. In practise we wish to consider complex fields. Suppose we have two real fields of the same mass,

$$S = \int d^4x \sum_{r=1}^{2}\left(\frac{1}{2}\partial^\mu\phi_r\partial_\mu\phi_r - \frac{m^2}{2}\phi_r^2\right) \tag{3.27}$$

then we may define the complex field

$$\begin{aligned} \chi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \chi^\dagger &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \end{aligned} \tag{3.28}$$

Then we may easily check

$$S = \int d^4x \left[\partial^\mu\chi^\dagger\partial_\mu\chi - m^2\chi^\dagger\chi\right] \tag{3.29}$$

Solving Heisenberg's equations as before we find

$$\hat{\chi}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}\left[\hat{b}(\underline{k})e^{ik\cdot x} + \hat{d}^\dagger(\underline{k}))e^{-ik\cdot x}\right] \tag{3.30}$$

where $\hat{b}$ and $\hat{d}$ are now independant because χ is a complex field. these must have commutation relationships

$$\begin{aligned} \left[\hat{b}(\underline{k}), \hat{b}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k} - \underline{k}') \\ \left[\hat{d}(\underline{k}), \hat{d}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k} - \underline{k}') \end{aligned} \tag{3.31}$$

all others being zero, with the Hamiltonian

$$\hat{H} = \int \frac{d^3k}{2(2\pi)^3}\left(\hat{b}^\dagger(\underline{k})\hat{b}(\underline{k}) + \hat{d}^\dagger(\underline{k})\hat{d}(\underline{k}) + \text{const.}\right) \tag{3.32}$$

This is fairly important. So far no fundamental scalars have not been observed experimentally although the standard models as we know it contains a fundamental scalar - the Higgs boson. The Higgs boson is complex rather than real. (if it exists!).

4. An interacting Boson Theory: Canonical Quantisation and Feynman Diagrams

We are now in a position to consider an interacting theory. As an example consider a theory which contains a real scalar ϕ and a complex scalar χ. The lagrangian density we take to be

$$\mathcal{L}_\phi + \mathcal{L}_\chi + \mathcal{L}_{int} \tag{4.1}$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\chi$ are the lagrangian densities for a free real and complex scalar (see (3.6) and (3.29)). The interaction term we take

$$\mathcal{L}_{int} = -g\hat{\chi}^\dagger\hat{\chi}\hat{\phi} \tag{4.2}$$

We now work with this system. The Heisenberg equations (which we could solve in the non-interacting case) are

$$\begin{aligned}\left(\partial^2 + m_\phi^2\right)\hat{\phi} + g\hat{\chi}^\dagger\hat{\chi} &=0\\ \left(\partial^2 + m_\chi^2\right)\hat{\chi} + g\hat{\phi}\chi &=0\end{aligned} \tag{4.3}$$

where $\partial^2 = \partial_\mu\partial^\mu$. † These non-linear operator equations have no known solution. We must attack them approximately. As we can see our system ***provided g is small*** is suited for analysis in the interaction picture. We can split the Hamiltonian into the non-interacting piece H_0 plus the small additional $H_I = g\hat{\chi}^\dagger\hat{\chi}\hat{\phi}$. This will allow us to evaluate transitions and scattering perturbatively.

Recall that in the interaction picture, the crucial object is the operator $\hat{U}(t)$. In lowest order this is

$$\begin{aligned}\hat{U}(t_i, t_f) =& - i\int_{t_i}^{t_f} \hat{H}_I(t)dt\\ =& - ig\int_{t_i}^{t_f} d^4x\hat{\chi}^\dagger\hat{\chi}\hat{\phi}\end{aligned} \tag{4.4}$$

We shall use this to examine the transition probability from an initial state containing a single ϕ boson and a final state consisting of a $\chi\chi^\dagger$ pair. We will take the initial time t_i to be $-\infty$ and the final times $t_f = \infty$, we have then,

$$\begin{aligned}|t = -\infty\rangle =&\hat{a}^\dagger(\underline{k})|0\rangle\\ |t = \infty\rangle =&\hat{b}^\dagger(\underline{p})\hat{d}^\dagger(\underline{q})|0\rangle\\ \langle t = \infty| =&\langle 0|b(\underline{p})d(\underline{q})\end{aligned} \tag{4.5}$$

The initial ϕ boson has four momenta k and the final pair of $\chi - \chi^\dagger$ particles have momenta p and q. Recall that in the interaction picture the states evolve with time via the $U(t)$ operator, $|a,t\rangle_I = \hat{U}(t)|a\rangle_H$. Thus the initial state $\hat{a}^\dagger(\underline{k})|0\rangle$ at $t = -\infty$ will evolve into

$$\hat{U}(-\infty, \infty)\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.6}$$

---

† I have slipped over the issue of how to deal with complex fields. The correct proceedure turns out to be to treat χ and $\chi^\dagger$ as independant fields. This can be justified be rewritting χ in terms of it's real components.

(Note that if $H_I = 0$ then the state remains fixed.) The probability that this state at $t = \infty$ is a $\chi\chi^\dagger$ pair is the overlap of this with $\hat{b}^\dagger(\underline{p})\hat{d}^\dagger(\underline{q})|0\rangle$. This is the *matrix element*

$$\langle t = \infty|\hat{U}(-\infty, \infty)|t = -\infty\rangle = \langle 0|b(\underline{p})d(\underline{q})\left(g\int d^4x\hat{\chi}^\dagger\hat{\chi}\hat{\phi}\right)\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.7}$$

This probability we now evaluate. Using the expansions for ϕ and χ this is

$$\begin{aligned}=-ig\int d^4x\int \bar{d}^3p'\int \bar{d}^3k'\int \bar{d}^3q'\langle 0|b(\underline{p})d(\underline{q})\left(\hat{d}(\underline{p}')e^{ip'\cdot x}+\hat{b}^\dagger(\underline{p}')e^{-ip'\cdot x}\right)\\ \times\left(\hat{a}(\underline{k}')e^{ik'\cdot x}+\hat{a}^\dagger(\underline{k}')e^{-ik'\cdot x}\right)\left(\hat{b}(\underline{q}')e^{iq'\cdot x}+\hat{d}^\dagger(\underline{q}')e^{iq'\cdot x}\right)\hat{a}^\dagger(\underline{k}|0\rangle\end{aligned} \tag{4.8}$$

where $\bar{d}^3p = d^3p/2(2\pi)^3\omega$. We will evaluate this by commuting the annihilation operators to the right where they vanish when acting on the vacuum and the creation operators to the left where they vanish whwn multiplied by $\langle 0|$. Since, for example $\hat{b}$ commutes with $\hat{a}^\dagger$ we can throw away the $\hat{b}(\underline{q}')$ terms. Similarly the $\hat{a}^\dagger(\underline{k}')$ term dissappears. (and also the $\hat{d}(\underline{p}')$ with a little more thought) leaving

$$-ig\int d^4x\int \bar{d}^3p'\int \bar{d}^3k'\int \bar{d}^3q'\langle 0|b(\underline{p})d(\underline{q})\hat{b}^\dagger(\underline{p}')\hat{a}(\underline{k}')\hat{d}^\dagger(\underline{q}')e^{-ix\cdot(p'+q'-k')}\hat{a}^\dagger(\underline{k})|0\rangle \tag{4.9}$$

We can continue commuting each annihilation operator to the right, obtaining a variety of δ-functions on route. The final result is

$$-i\int d^4x e^{-i(p+q-k)\cdot x}\langle 0|0\rangle = -ig(2\pi)^4\delta^4(p+q-k)\langle 0|0\rangle \tag{4.10}$$

The δ-function imposes conservation of four-momentum. This is in fact a real perturbative calculation. Notice that it doesn't make a lot of sense unless g is small.

In general, to evaluate to a given order, we need to calculate objects of the form

$$\int dt_1 dt_2\cdots dt_n T(\hat{H}_I(t_1)\hat{H}_I(t_2)\cdots\hat{H}_I(t_n)) \tag{4.11}$$

In principle we can carry out the same proceedure as before. This is sandwiching between states and commuting annihilation operators to the right until we obtain some kind of result. There is a very well specified proceedure for doing so in a systematic manner which is known as Wick's theorem. The diagramatic representation of this is more or less the Feynman diagram approach. We will now think a little more generally in terms of operators. Since we wish to have operators with anihilation operators acting on the right we define the *normal ordered* operator to be precisely this. For example consider the composite operator $T(\hat{\phi}(x)\hat{\phi}(y))$ then

$$:\hat{\phi}(x)\hat{\phi}(y): \tag{4.12}$$

is the same operator but with the anihilation operators pushed to the right. $T(\phi(x)\phi(y))$ and $:\phi(x)\phi(y):$ differ by a term which we call the contraction

$$T(\hat{\phi}(x)\hat{\phi}(y)) =: \hat{\phi}(x)\hat{\phi}(y) : + \underbrace{\hat{\phi}(x)\hat{\phi}(y)} \tag{4.13}$$

since ϕ is linear in operators and hence $T(\phi(x)\phi(y))$ quadratic the contraction term will be a pure number (that is no operator). We may evaluate this by sandwiching the above equation between $\langle 0|$ and $|0\rangle$ so that

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \underbrace{\hat{\phi}(x)\hat{\phi}(y)} \tag{4.14}$$

We now present Wick's theorem which tell's us how to evaluate large collection of operators into the normal ordered pieces and the contraction terms. Consider a large class of operators $A, B, C \cdots X, Y, Z$ which are linear in annihilation/creation operators. Then the time ordered product may be expanded,

$$\begin{aligned} T(ABC \cdots XYZ) = &: ABC \cdots XYZ : \\ &+ \underbrace{AB} : CD \cdots XYZ : + \underbrace{AC} : BD \cdots XYZ : + \text{perms.} \\ &+ \underbrace{AB}\,\underbrace{CD} \ : E \cdots XYZ : + \text{perms.} \\ &+ \cdots \\ &+ \underbrace{AB}\,\underbrace{CD} \ \cdots \underbrace{YZ} + \text{perms.} \end{aligned} \tag{4.15}$$

(This needs a little modification for fermions.) Now we apply this to the case we are interested in. Namely the decay of a ϕ particle into a $\chi\chi^{\dagger}$ pair. We need to sandwich the time-ordered products of Hamiltonians

$$\int dt_1 dt_2 \cdots dt_n T(H_I(t_1)H_I(t_2)\cdots H_I(t_n)) \tag{4.16}$$

between the initial and final states to evaluate the matrix element. We have done this for $n = 1$. Let us examine the systematics of $n > 1$. First we define 'initial' and final state operators (also linear in creation operators),

$$|i>= O^i_{\phi}|0\rangle, \ \ |f>= O^f_{\chi}O^f_{\chi^{\dagger}}|0\rangle \tag{4.17}$$

(The operator for creating a ϕ-state is in many ways a "sub-operator" of the $\hat{\phi}$ operator.) The first correction we can take as

$$g\langle 0|T\Big(O^f_{\chi}O^f_{\chi^{\dagger}}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^{\dagger}(x_1)O^i_{\phi}\Big)|0\rangle \tag{4.18}$$

We can evaluate this using Wicks theorem and throwing away all the normal ordered terms since they vanish we sandwiched between $\langle 0|$ and $|0\rangle$. Fortunately a large number of the possible contractions are zero. For example the contraction between a ϕ and a χ field is zero since the operators in ϕ commute with those in χ. Thus we have

$$\underbrace{\phi(x)\chi(y)} = \underbrace{\chi(x)\chi(y)} = \underbrace{\chi^{\dagger}(x)\chi^{\dagger}(y)} = 0 \tag{4.19}$$

and the only non-zero contractions will be between pairs of $\hat{\phi}$ operators and pairs of χ and $\chi^\dagger$ operators. It is a very useful exercise to repeat the previous calculation using Wick's theorem. Note that the contraction between a $\hat{\phi}(x)$ operator and an initial state operator is rather simple $\underbrace{\phi(x)\hat{O}^i_\phi} = e^{ik\cdot x}$. If we consider the next case the correction is

$$g^2\langle 0|T\Big(O^f_\chi O^f_{\chi^\dagger}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^\dagger(x_1)\hat{\phi}(x_2)\hat{\chi}(x_2)\hat{\chi}^\dagger(x_2)O^i_\phi\Big)|0\rangle \tag{4.20}$$

Since we have an odd number of ϕ terms the contractions must leave a single $\hat{\phi}$ operator which will vanish when sandwiched. Thus the second correction will be identically zero. The third is

$$g^3\langle 0|T\Big(O^f_\chi O^f_{\chi^\dagger}\hat{\phi}(x_1)\hat{\chi}(x_1)\hat{\chi}^\dagger(x_1)\hat{\phi}(x_2)\hat{\chi}(x_2)\hat{\chi}^\dagger(x_2)\hat{\phi}(x_3)\hat{\chi}(x_3)\hat{\chi}^\dagger(x_3)O^i_\phi\Big)|0\rangle \tag{4.21}$$

This will be non-zero and by Wick's theorem will produce a whole splurge of terms. Let us try to organise them. A term will be,

$$\underbrace{O^f_\chi\hat{\chi}^\dagger(x_1)}\;\underbrace{O^f_{\chi^\dagger}\hat{\chi}(x_1)}\;\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)}\;\underbrace{\hat{\chi}(x_2)\hat{\chi}^\dagger(x_3)}\;\underbrace{\hat{\chi}^\dagger(x_2)\hat{\chi}(x_3)}\;\underbrace{\hat{\phi}(x_3)O^i_\phi} \tag{4.22}$$

If we draw a diagram with three points x_1, x_2 and x_3 then we can "join the dots" using the contraction terms as labelled lines and obtain a diagram

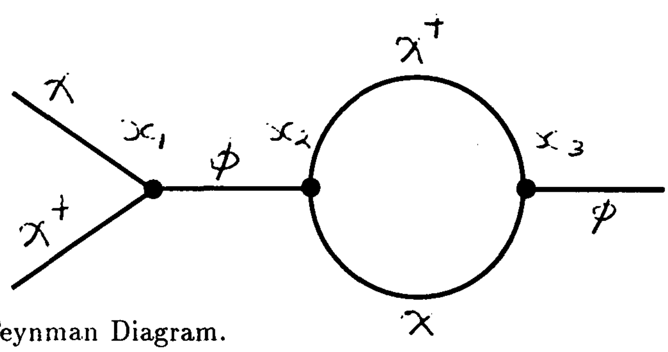

Figure 2. A Feynman Diagram.

Similarly for the other terms we can also draw diagrams. The real trick is, of course, not to do it this way but in reverse. It is much easier to draw diagrams to keep track of contributions than to look after terms. We draw diagrams with the "Feynman rules" which are rules for sewing together vertices with propagators. These may be written down directly from the lagrangian. In our case we have Hamiltonian $\phi\chi\chi^\dagger$ and the rule for vertices is that we have a three point vertex with one ϕ line, one χ line and one $\chi^\dagger$ line. The general case is easy to see (and to understand in terms of what has gone before). For example if we had

$$H_I = \hat{\phi}^n \tag{4.23}$$

then we would have a n-point vertex. The vertices are joined together with lines to form all possibilities. We can then associate with each diagram the approprate contribution. The contributions are given in terms of the contractions of pairs of fields. This contraction is known as the Feynman propagator. Let us now evaluate the Feynman propagator for the ϕ field

$$
\begin{aligned}
i\Delta_F(x,y) &\equiv \hat{\phi}(x)\hat{\phi}(y) = \langle 0|T(\phi(x)\phi(y))|0\rangle \\
&= \langle 0| \int \bar{d}k \int \bar{d}q \Big(\hat{a}(\underline{k})e^{i(\underline{k}\cdot\underline{x}-\omega(k)t_1)}\Big)\Big(\hat{a}^\dagger(\underline{q})e^{-i(\underline{q}\cdot\underline{y}-\omega(q)t_2)}\Big)|0\rangle
\end{aligned}
\tag{4.24}
$$

(we have dropped the terms giving zero trivially) The two operator terms can be commuted past each other to yield

$$
i\Delta_F(x,y) = \int \bar{d}k \int \bar{d}q (2\pi)^3 2\omega \delta^3(\underline{k}-\underline{q}) e^{i(\underline{k}\cdot\underline{x}-\underline{q}\cdot\underline{y})-i(t_1\omega(k)-t_2\omega(q))} \tag{4.25}
$$

The δ-function can now be evaluated. In the above we assumed $t_1 > t_2$ when evaluating. The result in general is

$$
i\Delta_F(x,y) = \int \frac{d^3k}{(2\pi)^3 2\omega}\Big[\theta(t_1-t_2)e^{i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_1-t_2)\omega} + \theta(t_2-t_1)e^{-i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_2-t_1)\omega}\Big] \tag{4.26}
$$

where $\theta(t) = 1, t > 0$ and $\theta(t) = 0, t < 0$. There is a more Lorentz invariant looking expression for the above which is

$$
i\Delta_F(x,y) = \int \frac{d^4k e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \tag{4.27}
$$

where we have slipped into relativistic four vector notation. The proof of the equivalence of these two forms relies upon Cauchy's theorem. For the more mathematically inclined we can prove this by examining the integration in ik_0 and continuing to a complex integration. The poles in the integral occur when

$$
(k^0)^2 - \underline{k}^2 - m^2 + i\epsilon = 0 \tag{4.28}
$$

which happends when $k_0 = \pm w(k) \mp i\epsilon$ The integral in the complex ik_0 plane now lies along the real axis with poles lying at $(-\omega(k), +i\epsilon)$ and $(\omega(k), -i\epsilon)$.

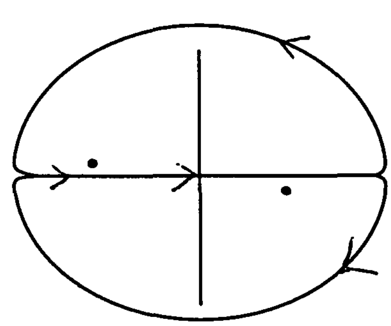

Figure 3. The contour integrations for the Feynman propagator.

We can close the contour with a semi-circle at infinity to obtain a curve which we then apply Cauchy's theorem to. Whether we use the upper or lower hemisphere depends upon whether $t_1 > t_2$ or not. If $t_1 < t_2$ then we close in the upper plane and have to evaluate the residue at $(-\omega(k), +i\epsilon$. The general case can be combined

$$\theta(t_1 - t_2) \int \frac{d^3 k}{(2\pi)^3 2\omega} e^{i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_1-t_2)\omega} + \theta(t_2 - t_1) \int \frac{d^3 k}{(2\pi)^3 2\omega} e^{-i\underline{k}\cdot(\underline{x}-\underline{y})-i(t_2-t_1)\omega} \quad (4.29)$$

which is as before. We now have a form of the propagator which integrates over $d^4 k$ rather than $d^3 k$. We are thus integrating over particles which need not be on mass-shell.

5. Functional Methods

I will now rework some of the results of the previous section but using the path integral approach instead. This is in many ways much slicker. First for a set of discrete coordinates q_i define

$$W[J_i] = \int \prod_i [dq_i] \exp\Big(i \int L(q_i, \dot{q}_i)dt + \sum_j J_j q_j \Big) \tag{5.1}$$

The J_i are dummy variables which will allow us to calculate expectations of q_i etc by derivatives of $W[J]$. For example

$$\frac{\partial W[J_i]}{\partial J_k}\bigg|_{J_i=0} = \int [dq_i] q_k e^{iS} \tag{5.2}$$

We with to extend this concept to a field theory. This means extending $q_i \longrightarrow \phi(x)$. This gives

$$W[J(x)] = \int [d\phi] \exp\Big(i \int d^4x \mathcal{L} + \int d^4x J(x)\phi(x) \Big) \tag{5.3}$$

Now $W[J(x)]$ is a *functional.* That is something which takes a function and produces a number. Before continuing we must define a functional derivative. Consider a functional $F[J(x)]$ then

$$\frac{\delta F}{\delta J(y)} \equiv \lim_{\epsilon \longrightarrow 0} \frac{F[J(x) + \epsilon\delta(x-y)] - F[J(x)]}{\epsilon} \tag{5.4}$$

If we consider a simple example,

$$F[J(x)] = \int d^4x J(x)\phi(x) \tag{5.5}$$

then

$$\begin{aligned} \frac{\delta F}{\delta J(y)} &= \lim_{\epsilon \longrightarrow 0} \int \delta(x-y)\phi(x) \\ &= \phi(y) \end{aligned} \tag{5.6}$$

We now will apply some methods to the theory with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\frac{\partial\phi}{dt})^2 - \sum_{i=1}^{3} \frac{1}{2}(\frac{\partial\phi}{dx_i})^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda\phi^4}{4!} \tag{5.7}$$

This lagrangian has the free part plus an interaction terms ϕ^4. We will consider the free part first. The path integral for the free theory is gaussian and hence calculable by our favourite integrals. However we must carefully take the $q_i \longrightarrow \phi(x)$ transition carefully. Recall that we can carry our gaussian integrals where the exponential contains the term,

$$\sum_{i,j} q_i K_{ij} q_j \tag{5.8}$$

where K is a matrix. The correct generalisation will be the replace K by an operator. We thus wish to transform the exponent in the path integral into the form

$$\int dx \int dy \phi(x) \cdot \text{Operator} \cdot \phi(y) \tag{5.9}$$

By integrating by parts (and neglecting surface terms) the Lagrangian density may be written,

$$\phi(x)\Big[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big]\phi(x) \tag{5.10}$$

whence we may rewrite $W[J]$ as

$$W_0[J] = \int [d\phi] \exp\Big(-\frac{1}{2}\int d^4x \int d^4y \phi(x) K(x,y) \phi(y) - \int d^4x J(x)\phi(x)\Big) \tag{5.11}$$

where

$$K(x,y) = \delta^4(x-y)\Big[-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big] \tag{5.12}$$

We may now evaluate $W_0[J]$ in terms of the inverse operator of K. This is the operator satisfying

$$\int d^4y K(x,y)\Delta(y,z) = \delta(x-z) \tag{5.13}$$

and we find

$$W_0[J] = \exp\Big(-\frac{1}{2}\int d^4x d^4y J(x)\Delta(x,y)J(y)\Big) \tag{5.14}$$

whence

$$\frac{\delta^2 W_0[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0} = \Delta(x,y) \tag{5.15}$$515

Now, the inverse operator Δ is in fact precisely the Feynman propagator encountered in canonical methods (up to the odd normalisation factor of i or -1). To see this

$$\begin{aligned}\int d^4z K(x,z) i\Delta_F(z,y) &= \int d^4z \delta^4(x-z)\Big(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\Big)\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(z-y)}}{k^2-m^2} \\ &= \int d^4z \delta^4(x-z)\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(z-y)} \\ &= \int d^4z \delta^4(x-z)\delta^4(z-y) \\ &= \delta^4(x-y)\end{aligned} \tag{5.16}$$

Now if we wish to evaluate, using functional methods, objects such as

$$\int dt_1 dt_2 \langle 0|T(\phi(x)\phi(y)H_I(t_1)H_I(t_2))|0\rangle \tag{5.17}$$

then we can obtain these by acting upon $W_0[J]$ with

$$\frac{\delta}{J(x)}\frac{\delta}{J(y)}\frac{\delta^4}{J(u)^4}\frac{\delta^4}{J(v)^4} \tag{5.18}$$

and then setting $J = 0$. (together with integrating d^4u and d^4v.) Since, the exponential is quadratic, and we set $J = 0$ finally, every time a propagator is brought down a further functional derivative must act. The end result is that the object is a sum of products of propagators.

As in the canonical case the simplest way to keep track of the terms is by drawing feynman diagrams. This functional approach provides an alternate derivation. In the cases considered up till now we have seen simple vertices (coresponding to just polynomial terms in H_I) this will now be the case for gauge theories but the methods still apply. †

5.2 Momentum space Feynman diagrams

The feynman diagrams I have drawn are not really the conventional ones. These are normally drawn in momentum space rather then x space. The very good reason for this is that the external states are normally momentum eigenstates. The momentum space is really just a Fourier transform of the configuration space rules -and it may be regarded as an exercise to transform these. Just a few points, the rules then require that we draw all diagrams, the momenta now flowing through the legs is now integrated over and each vertex has a δ-function in momenta. Tree level diagrams in momentum space are then merely the product of the propagators $1/(k^2 - m^2)$ however loop diagrams have more integrations over momenta than there are δ- functions and we obtain (the infamously difficult to evaluate) loop momentum integrations. We always obtain (look at our example) a δ-function in our results which imposes total conservation of energy and momentum. From the examples we can easily (?!) see what the general rule for vertices will be - whatever is in $\mathcal{L}_\mathcal{I}$ will be reflected in terms of the rules for the vertex: A $\phi\chi\chi^\dagger$ vertex leads to a vertex with a ϕ a χ and a $\chi^\dagger$ outgoing state: A : $\phi^n(x)$: lagrangian will yield a vertex with n outgoing ϕ states. Constants multipling the vertex (such as g) get reflected in the rules.

† I have cut more corners in this section than I care to think about in an attempt to convey some understanding of the path integral approach. Some of these corners came back to haunt me in tutorials.

6. Gauge Theories 1: Electro-Magnetism

The great sucess in particle physics has been the ability to use gauge theories to describe the fundamental forces. As far as we know, both the strong and electro-weak forces are described by gauge theories. The strong force is beleived to be described by a $SU(3)$ gauge theory known as QCD and the Electro-weak by $SU(2) \times U(1)$. Hopefully these terms will become clearer. I'll take two "bites" at this very important type of field theory. (Graham will also spend a lot of time on gauge theories as will Jonathon). The first bite will be simply electro-magnetism or a U(1) gauge theory - although it might not seem so simple and on the second pass I'll extend to $SU(3)$ and $SU(2)$ (or in fact any gauge group).

The theory of electromagnetism as described by Maxwell's equations is our proto-gauge theory. Maxwell's equations are

$$\begin{aligned} \nabla \cdot \underline{B} &= 0 \\ \nabla \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \nabla \cdot \underline{E} &= \rho \\ \nabla \times \underline{B} &= \underline{j} + \frac{\partial \underline{E}}{\partial t} \end{aligned} \tag{6.1}$$

As might be familiar to you, it is common to express $\underline{E}$ and $\underline{B}$ in terms of the vector and scalar potentials

$$\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \qquad \underline{B} = \nabla \times \underline{A} \tag{6.2}$$

whence the two equations $\nabla \cdot \underline{B} = 0$ and $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ become automatic. Our first task will be to write these equations in manifestly Lorentz covariant form. Firstly we form a 4-vector potential $A_\mu = (\phi, -\underline{A})$ and $j_\mu = (\rho, -\underline{j})$ and define a *field strength* $F_{\mu\nu}$ such that

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.3}$$

This definition is in fact equivalent to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.4}$$

With this definition it is fairly easy to see that the last two of Maxwell's equations (four equations really) can be written (don't forget the Einstein summation convention!)

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{6.5}$$

We now wish to provide a lagrangian formalism for these equations. It turns out that the appropriate Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;\; + j_\mu A^\mu \tag{6.6}$$

whose Lagrange equations are just those of (6.5) . To see this, for example, take the Lagrange equation for A_0,

$$\begin{aligned} &\frac{\partial}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{A}_0}\right] + \frac{\partial}{dx}\left[\frac{\partial \mathcal{L}}{\partial(\partial_x A_0)}\right] + \text{(y and z terms)} - \frac{\partial \mathcal{L}}{\partial A_0} = 0 \\ &0 + \frac{\partial}{dx}\left[-F_{01}\right] + \text{(y and z terms)} + 0 = 0 \\ &\nabla \cdot \underline{E} = 0 \end{aligned} \tag{6.7}$$

There is a difficulty in carrying out a Hamiltonian approach to electro-magnetism. This is because the momentum which is conjugate to A_0 is identically zero,

$$\Pi_{A_0} = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} \equiv 0 \tag{6.8}$$

since the Lagrangian density does not depend upon $\dot{A}_0$.

Although not so obvious a problem in the Lagrangian formalism, this will rear it's ugly head fairly soon. The reason that there is a problem is because, in some ways, we have too many variables A_μ describing the fields. This will lead us into gauge symmetry. Notice that the field strength $F_{\mu\nu}$ is invariant under a transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda(x) \tag{6.9}$$

where $\Lambda(x)$ is an arbitrary function of x. Now, classically, the two choices of A_μ give the same values of $\underline{E}$ and $\underline{B}$ thus since everything can be written in terms of $\underline{E}$ and $\underline{B}$ this symmetry in merely a curiosity. †

Before discussing the quantisation of Electro-magnetism I will consider the theory coupled to Dirac fermion (or scalar) If we consider a Dirac fermion ψ then the Lagrangian

$$\mathcal{L}_\psi = \bar{\psi}\gamma_\mu \partial^\mu \psi \tag{6.10}$$

will be invariant under the transformation,

$$\psi \to \psi' = e^{-ig\alpha}\psi \tag{6.11}$$

where here α is a constant and not a function of x. (We could also consider coupling to the scalar lagrangian $\partial_\mu \chi^\dagger \partial^\mu \chi$.) Suppose we would like to extend our transfomation so that $\alpha(x)$. Then the Lagrangian is not invariant but an extra term

$$-ig\bar{\psi}\gamma_\mu\psi\partial^\mu\alpha \tag{6.12}$$

† An analogy of the problems we are encountering is if we think of the simple pendulum. Suppose I was silly enough to *over* specify my system by decribing it by x, y, and θ. I might be tempted (obviously not but..) because the kinetic term is simple in x and y wheras the potential is simple in terms of θ. If I then chose $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \theta^2$ we would obtain the momentum $p_\theta = 0$. This contraint on (p, q) space is very similar to the electromagnetism case.)

arises. Now we could make the Lagrangian invariant if we add an interaction term

$$\mathcal{L}_{int} = igA^{\mu}\bar{\psi}\gamma_{\mu}\psi \tag{6.13}$$

and the combination

$$\mathcal{L}_A + \mathcal{L}_{\psi} + \mathcal{L}_{int} \tag{6.14}$$

will be invariant under the combined gauge transformation.

$$\psi \to e^{-ig\alpha(x)}\psi \qquad A_{\mu} \to A_{\mu} + \partial_{\mu}\alpha(x) \tag{6.15}$$

In terms of the fermions the transformation act via multiplication by a phase $e^{i\alpha}$. Such phases form a group. A very simple group which is known as $U(1)$- the group of 1×1 unitary matrices. ($U(n)$ will be the group of $n \times n$ unitary matrices). We can include the interaction term with the kinetic term for ψ by defining the ***covariant derivative***

$$D_{\mu}\psi \equiv (\partial_{\mu} + igA_{\mu})\psi \tag{6.16}$$

This is known as the covariant derivative because it transforms in the same way as ψ, namely with just a phase.

$$D_{\mu}\psi \to e^{-ig\alpha(x)}D_{\mu}\psi \tag{6.17}$$

This general trick of gauging symmetries has been enormously useful. It allows us to build models which have proved enormously usefull in describing physics.

There are several conventions for phases in this area. Later I will use a different convention which can be obtained by replaceing α by $-\alpha/g$. Whence the fields transforms as

$$\psi \to e^{i\alpha}\psi, \;\; A_{\mu} \to A_{\mu} - \frac{1}{g}\partial_{\mu}\alpha \tag{6.18}$$

whence

$$D_{\mu} = \partial_{\mu} - igA_{\mu} \tag{6.19}$$

6.2 Quantum Gauge Theories

Our naive attepts to quantise electrodynamics will prove to be sick because we are missing an important point. however, let us see how the sickness developes in the path integral formulation. We attempt to find the propagator. To do so, we must write the quadratic part of the lagrangian as FIELD.OPERATOR.FIELD. The action may be rewritten

$$\begin{aligned}
&\int d^4x(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) \\
&= \int d^4x A_{\nu}\Big(-2\eta^{\nu\nu'}\partial_{\mu}\partial^{\mu} + 2\partial^{\nu}\partial^{\nu'}\Big)A_{\nu'} \\
&= \int d^4x \int d^4x' A_{\nu}(x')\Big(\delta^4(x-x')(-2\eta^{\nu\nu'}\partial_{\mu}\partial^{\mu} + 2\partial^{\nu}\partial^{\nu'})\Big)A_{\nu'}(x)
\end{aligned} \tag{6.20}$$

we thus have the inverse-propagator organised in position space. When we fourier transform the above we obtain the momentum space inverse propagator,

$$P_{\mu\nu} = (k_\mu k_\nu - k^2 \eta_{\mu\nu}) \tag{6.21}$$

This "inverse propagator" has the unfortunate property that it does not have an inverse (so it is not the inverse of anything!). To observe this note that

$$\begin{aligned} P_{\mu\nu} P_{\nu\rho} &= - k^2 (k_\mu k_\rho - k^2 \eta_{\mu\rho}) \\ &= - k^2 P_{\mu\rho} \end{aligned} \tag{6.22}$$

Now any matrix satisfying $M^2 = \lambda.M$ cannot be invertible (unless $M = \lambda.I$ which P clearly is not.) so P is not an invertible operator.

Now we have reached a problem in the path integral formalism (just as we would have in canonical methods.) What is the reason for this? The intrepratation of the "sickness" is that we are actually counting too many states in our path integral. If we have field configurations A_μ and $A_{\mu'}$ related by a gauge transformation, they only represent a single equivalent states so we should only count them once rather than twice. In fact an infinite overcounting occurs in the path integral. Consider the following diagram, where I have "squeezed" the integration of the path integral onto two dimensions. Configurations related to a field configuration lie in the *orbit* of the configuration.

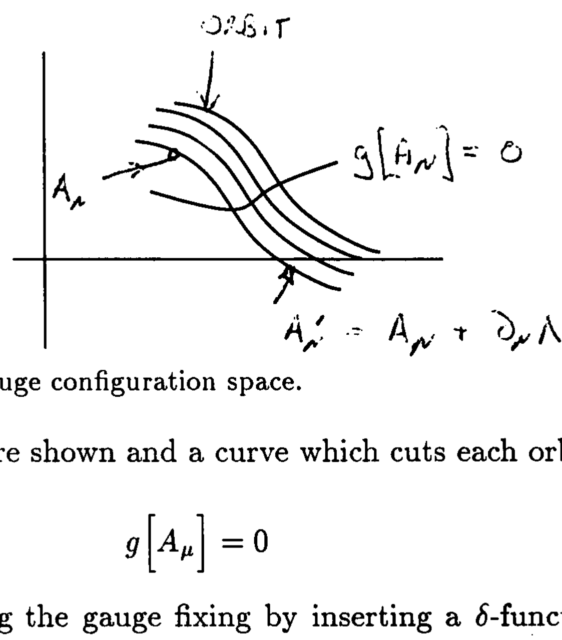

Figure 4. Orbits in gauge configuration space.

In this figure the orbits are shown and a curve which cuts each orbit is shown. Such a curve is given generically by

$$g\left[A_\mu\right] = 0 \tag{6.23}$$

We can think of implementing the gauge fixing by inserting a δ-function into the path integral. (However they are important coefficients!). Such a condition is called a gauge fixing condition. A good function $g[A]$ is clearly one which cuts each orbit once and once only. The implementation of gauge-fixing is important technically in quantising a gauge theory. I will demonstrate (rather than prove) how to implement this. I will try to switch back and forth between a two-dimensional analogy and the real situation.

Consider a two dimensional integral

$$I = \int dxdy f(x,y) \tag{6.24}$$

in analogy with the gauge theories the function f is invariant under rotations thus

$$f(x,y) = F(r,\theta) = F(r) \tag{6.25}$$

by analogy with gauge symmetries let us assume that the different values of θ should not be counted. Thus we wish to evaluate

$$I' = \int drrF(r) \tag{6.26}$$

rather than (6.24) (which differs by a factor of $\int_0^{2\pi} d\theta = 2\pi$. Now we can just implement this by inserting a δ-function within the integral. We define

$$I_\phi = \int d^2r f(x,y)\delta(\theta-\phi) = \int rdrd\theta F(r,\theta)\delta(\theta-\phi) = \int rdrF(r) = I' \tag{6.27}$$

We can define this for any function and by definition

$$I = \int d\phi I_\phi \tag{6.28}$$

however only for rotationally invariant functions will I_ϕ be independant of ϕ. Since I_ϕ is independant of ϕ,

$$I = \int d\phi I_\phi = 2\pi I_{\phi_0} \tag{6.29}$$

where ϕ_0 is any value of ϕ. In many ways I have just cheated! - I "knew" that the curve $\theta = const.$ cut each orbit one and one only (and also smoothly!). In general we want to consider a general curve $g(x,y) = 0$. (analogous to (6.23)). Again I want to insert $\delta(g(x,y))$ into the integral but now we need factors. We can see these from the identity,

$$\left|\frac{\partial g}{\partial \theta}\right|_{g=0} \int d\phi \delta(g(\underline{r}_\phi)) = 1 \tag{6.30}$$

(For intuition on this equation look, for example, at the prerequisites where $\delta(ax) = \delta(x)/|a|$.) It is important that

$$\Delta_g(\underline{r}_\phi) \equiv \left|\frac{\partial g}{\partial \theta}\right|_{g=0} \tag{6.31}$$

is rotation invariant. To see this note

$$\Delta_g^{-1}(\underline{r}_\phi) = \int d\theta \delta(g(r_{\theta+\phi})) = \int d\theta' \delta(g(r_{\theta'})) = \Delta_g^{-1}(\underline{r}) \tag{6.32}$$

We may now insert the factor of one in (6.30) into the integral I

$$I = \int d\theta r dr f(x,y) \int d\phi \Delta_g(\underline{r})\delta(g(\underline{r}_\phi)) = \left(\int d\phi\right) \int d^2 r f(x,y)\Delta_g(\underline{r})\delta(g(\underline{r}_\phi)) \qquad (6.33)$$

So we can obtain

$$I' = \int d^2 r f(x,y)\Delta_g(\underline{r})\delta(g(\underline{r}_\phi)) \qquad (6.34)$$

As expected we have introduced a δ-function but we have a correcting factor Δ_g. In a quite considerable generalisation to gauge theories there is an identity,

$$1 = \Delta_g(A^\mu) \int \prod_x dU(x) \prod_x \delta(g(A^{\mu U}) \qquad (6.35)$$

where

$$\Delta_g(A^\mu) = \det\left(\frac{\delta g}{\delta U}\right) \qquad (6.36)$$

and $U(x) = e^{i\alpha(x)}$ -we are integrating over elements of the $U(1)$ group. Inserting this into the functional integral we obtain,

$$\begin{aligned} &\int d[A^\mu] e^{-Action} \int [dU] \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U})) \\ &\qquad = \int [dU] \int d[A^\mu] e^{-Action} \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U}) \end{aligned} \qquad (6.37)$$

The formal method of quantising is now rather simple - we just throw away the integration of the group variables $\int [dU]$. (analogously to $\int d\phi$) leaving us with a "gauge fixed" path integral which only counts each orbit once.

Great. We however have one more step before this is any use!. (How do we implement a general gauge fixing δ-function?) Obviously, the gauge fixed path integral is independant of g. (It's not easy to show this...) So using the gauge fixing functional

$$g' = g - B \qquad (6.38)$$

where B is just a function of x (just a constant really in functional space!) will give just the same result. Inserting a factor

$$\int [dB] \prod \delta(g(A^{\mu U}) - B) e^{-\frac{1}{2\xi}\int d^4 x B^2(x)} \qquad (6.39)$$

instead of $\prod \delta(g(A^{\mu U}))$ merely changes the path integral by a constant. This is really just averaging (or smearing) over the gauge functions $g - B$ with a factor e^{B^2}. This trivial trick allows us to get rid of the δ-fuctions and the gauge fixed path integral is

$$\begin{aligned} &\int d[A^\mu] \int [dB] e^{-Action} \Delta_g(A^g_\mu) \prod \delta(g(A^{\mu U} - B) e^{-\frac{1}{2\xi}\int d^4 x B^2(x)} \\ &\qquad = \int d[A^\mu] e^{-Action - \frac{1}{2\xi}\int d^4 x g[A]^2} \Delta_g(A^g_\mu) \end{aligned} \qquad (6.40)$$

So we have promoted the δ-function to an extra term in the action - the "gauge-fixing" term plus a determinant in the action (maybe more later). Many choices of "gauge-fixing" exist (and thus much effort to find good gauges - in some sense). I'll try to illustrate one approach via the so-called covariant gauges.

6.3 The Covariant gauges

This gauge choice uses the gauge fixing term,

$$g\left[A^\mu\right] = \partial_\mu A^\mu \tag{6.41}$$

With this gauge choice we find that the gauge fixing term in the action becomes

$$\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{6.42}$$

This will affect the quadratic terms in the action (thankfully!) to be

$$A_\mu \left(k_\mu k_\nu (1 - \frac{1}{\xi}) - k^2 \eta_{\mu\nu}\right) A_\nu \tag{6.43}$$

Now, we *can* invert this operator and obtain a propagator in momentum space

$$\frac{\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2 + i\epsilon} \tag{6.44}$$

Amongst this class of gauge choices two special ones are when $\xi = 0, 1$ These are

$$\begin{aligned} &\text{Feynman Gauge,} \quad \xi = 1, \quad P_{\mu\nu} = \frac{\eta_{\mu\nu}}{k^2} \\ &\text{Landau Gauge,} \quad \xi = 0, \quad P_{\mu\nu} = \frac{\eta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2} \end{aligned} \tag{6.45}$$

So gauge fixing has resolved this (and in fact all other) problems with quantisation of the gauge theory.

In the absence of either scalars or fermions, the quantised theory is a free theory and we may solve as for free scalar theory. (The Lagrangian contains only quadratic terms and, in the Feynman gauge, the propagator is just $\delta_{\mu\nu}/k^2$ which means the A_μ act just like multiple scalar fields.) In the presense of scalar or fermion fields the theory becomes a real live interacting quantum theory - QED for fermions or scalar-QED for scalars. For a fermion the covariant derivative contains an interaction term

$$ig\bar{\psi}\gamma_\mu A^\mu \psi = ig \sum_{a,b} \bar{\psi}_a (\gamma_\mu)_{ab} \psi_b A^\mu \tag{6.46}$$

implying a Feynman vertex

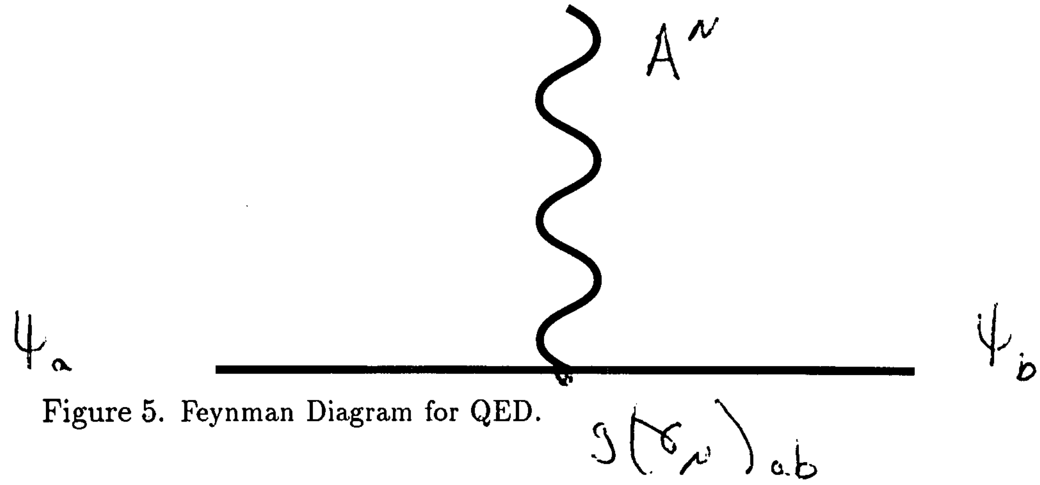

Figure 5. Feynman Diagram for QED.

7. Gauge Theories 2: Non-Abelian gauge theories

In this section we will generalise the concept of a gauge theory to that of a non-abelian gauge theory. Both the strong and weak interactions appear to be described by such theories. Recall that the action of a gauge transformations for electromagnetism act as

$$e^{i\alpha(x)} \tag{7.1}$$

Now complex phases could, if one were perverse, be described as 1×1 unitary matrices. The $U(1)$ such matrices form a group. The basic definition of group's I quickly review here

7.1 basic group theory

A group G, is a set of objects with an action, or multiplication, defined such that the following axioms are satisfied,

$$\begin{aligned} &1 \text{:if } a, b \in G, \text{then } a.b \in G \ \ \text{(closure)} \\ &2 \text{:there exists an identity }, e, \text{s.t.} a.e = e.a = a, \forall a \in G \\ &3 \text{:for all } a \in G, \text{there exists an inverse } a^{-1}, a.a^{-1} = e, a^{-1}.a = e \\ &4 : a.(b.c) = (a.b).c \ \ \forall a, b, c \end{aligned} \tag{7.2}$$

There are many examples of groups. For example,

a) the numbers $\{1, -1\}$ under multiplication

b) the real numbers under addition (but *not* multiplication since zero has no inverse.)

c) the set of $n \times n$ matrices which are unitary ($A^{-1} = A^{\dagger}$) and which have determinant one. This group is known as $SU(N)$.

d) the set of orthogonal matrices ($A^{-1} = A^T$) of determinant one. This is known as $SO(N)$.

Examples c) and d) are examples of *Lie Groups.* Lie groups are groups which depend smoothly (in a well defines mathematical sense) on parameters. For example, a general $SO(2)$ matrix can be written in the form,

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7.3}$$

which we can parameterise by θ. Clearly group multiplication (and inverses etc) depend smoothly upon θ, for example

$$M(\theta)M(\phi) = M(\theta + \phi), \ \ M^{-1}(\theta) = M(-\theta) \tag{7.4}$$

(If you are particulary observant you might notice that there is a lot of similarity between these matrices and $U(1)$. In fact $SO(2)$ and $U(1)$ are essentially the some algebraic structure.) If all elements of a group commute,

$$a.b = b.a \ \ \forall a, b \tag{7.5}$$

then we call the group Abelian.

7.2 Lie Algebras

An important object of interest in a Lie group is it's *algebra.* This is defined in terms of the behaviour of the group elements near the identity. For example consider the group $SU(2)$, ($A^\dagger A = 1$, $\det(A) = 1$). If we have an arbitrary element near the identity, $A = I + iT$ (where T is small) then T must satisfy,

$$T^\dagger = T, \ \ \mathrm{tr}(T) = 0 \tag{7.6}$$

thus T can be parametised as

$$T = \sum_{a=1}^{3} \alpha^a T^a \tag{7.7}$$

where

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.8}$$

The matrices T_i generate an algebra under commutation. That is the commutator of any two T matrices is a sum of T matrices. For example

$$\left[T^1, T^2\right] = iT^3 \tag{7.9}$$

In general for $SU(N)$, if we consider the algebra, then it is generated by hermition traceless matrices of which there are $N^2 - 1$. This is the dimension of the Lie algebra. For $SU(3)$ there are thus eight matrices. A standard basis is

$$\lambda^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \lambda^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{7.10}$$

which are closed under commutation. Elements of the Lie algebra are linear combinations of these. There is a very important relationship between the elements of the algebra and the group itself. Essentially the group elements can be obtained by exponentiating the algebra,

$$U(\alpha) = \exp \sum_a \alpha^a T^a \tag{7.11}$$

where the α are no longer infinitesmal. Similar to the case of $SU(2)$, the T_a obey commutation relations,

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{7.12}$$

where f^{abc} are known as the structure constants of the algebra. For $SU(2)$, $f^{abc} = \epsilon^{abc}$. (We normally normalise the T^a such that $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$.) Although I won't really justify this, the structure constants really contain all the information in the group.

7.3 Representations

The structure of a group is defined abstractly in terms of the multiplication. A concrete realisation of a group is called A representation. A representation has two objects. Firstly, there must be a specific object for each element of the group. Normally we will be interested in matrix representations of a group. So we will have a mapping between the group and our set of matrices,

$$f : f(G) \to M \tag{7.13}$$

which preserves the multiplication structure i.e. $f(G.H) = f(G).f(H)$. For our $SU(2)$ and $SU(3)$ groups we have actually been looking at a representation of the formal mathematical structure. However, it has been a very special representation - the *fundamental*. For a given group there are many representations. For example the is always the trivial representation where every matrix get's mapped to the number 1. Also very importantly, the matrices must have a vector space to act upon. Normally we view this as column vectors. A cultural gap between mathematicians and physicists is that mathematicians focus upon the matrices wheras physicists focus upon the vector space.

7.4 Non Abelian Gauge symmetries

Let us generalise our gauge transformation acting upon a fermion

$$\underline{\psi} \to U(x)\underline{\psi} \tag{7.14}$$

where U is an element of a group G such as $SU(2)$ and $\underline{\psi}$ lies in a representation of G. For example for $SU(2)$ we could take $\underline{\psi}$ to be a doublet of fermions

$$\underline{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{7.15}$$

If U did not vary with x then the Lagrangian

$$\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + \bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 = \underline{\bar{\psi}}^T\gamma^\mu\partial_\mu\underline{\psi} \tag{7.16}$$

is invariant, however for a gauge symmetry we wish the gauge transformation to vary with x. The technique will be to construct a covariant derivative D_μ such that

$$D_\mu\underline{\psi} \to U(x)D_\mu\underline{\psi} \tag{7.17}$$

which will require

$$U(x)D_\mu U^{-1}(x) = D'_\mu \tag{7.18}$$

We will postulate a form for D^μ analogously to the $U(1)$ case,

$$D_\mu = \partial_\mu + ig\sum_a T^a W^a_\mu$$

of the propagator, which in momentum space will lead to k_μ terms. The precise answer for the three point momentum space Feynman vertex, in the Feynman gauge, is

$$V^{abc}_{\mu\nu\rho}(p,q,r) = g f^{abc}\Big(\delta_{\nu\rho}(q_\mu - r_\mu) + \delta_{\rho\mu}(r_\nu - p_\nu) + \delta_{\mu\nu}(p_\rho - q_\rho)\Big) \tag{7.27}$$

as we show diagramatically,

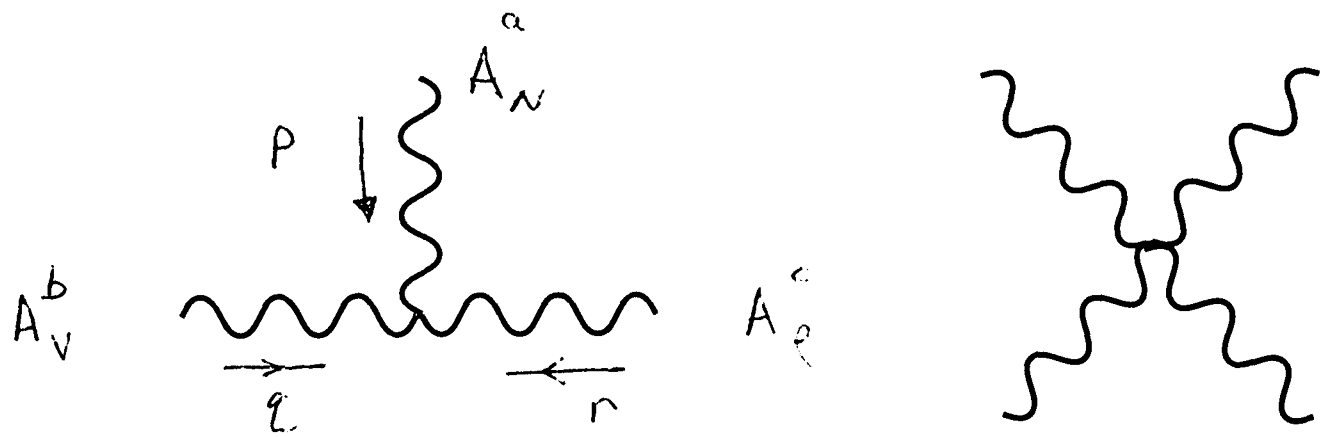

Figure 6. Feynman Diagrams for Non-Abelian Gauge Theory.

Note that is has crossing symmetry under interchange of legs and has one power of momentum in the vertex. The general situation is probably fairly clear from now on. There will also be a 4-point vertex. This contains no momentum (but a factor of g^2 rather than g.)

8. Critique of Perturbation theory

Perturbation theory has been enormously sucessful but it does have limitations. First I'll try to illustrate the "light" and then the "shade"

The Light

Perhaps the most impressive demonstration of perturbative field theory is the evaluation of $g-2$ of the electron in QED. The magnetic moment of a fermion is related to it's spin via

$$\mu = -g\frac{e}{2m}\mathrm{S} \tag{8.1}$$

The *classical* Dirac lagrangian gives a prediction for g to be exactly 2. However, as a purely Quantum mechanical effect, g may not exactly equal 2 but may be anomalous. This is calculable, using Feynman diagrams, perturbatively.

The great success is

$$\begin{aligned}(\frac{g-2}{2}) &= 1159657.7 \pm 3.5 \times 10^{-9} : \text{Experiment} \\ &= 1159655.4 \pm 3.3 \times 10^{-9} : \text{From Theory}\end{aligned} \tag{8.2}$$

The theoretical, prediction includes Feynman diagrams up to three loops. The only sensible conclusion is that

PERTURBATION THEORY WORKS

The Shade

Consider the function

$$\begin{aligned}f(x) &= 0 : x = 0 \\ f(x) &= e^{-\frac{1}{x^2}}\end{aligned} \tag{8.3}$$

This little function has a lot to teach us. It is not a particularly badly behaved function or very exciting to look at. It is continuous differentiable and it isn't very difficult to show that

$$f'(0) = 0 \tag{8.4}$$

If fact, with a little more work we can show that

$$f^{(n)}(0) = 0 \tag{8.5}$$

Thus the Taylor series of $f(x)$ around $x = 0$ is

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n} = 0 \neq f(x) \tag{8.6}$$

Thus it is a fairly simple example where the Taylor series does not equal the function. Now a typical decay amplitude is a function of the coupling constant g

$$R(g)$$

We attempt to evaluate $R(g)$ by perturbation theory - this is essentially just it's Taylor series. So any component of R which takes the form

$$\sim e^{1/g^2} f(g) \tag{8.7}$$

will *never* show up in a perturbative expansion. One might argue that such functions are pathelogical. I.e. that they are really just mathematical and don't effect real problems however I'll try to argue the reverse. Consider $SU(2)$ pure gauge theory. Rescale the potential field

$$W_\mu \to \frac{1}{g} W'_\mu \tag{8.8}$$

whence

$$F_{\mu\nu} \to \frac{1}{g} F'_{\mu\nu}$$

where F' has no explicit dependance on g. Then the Path integral looks a bit like

$$\sim \int [dW] e^{\frac{1}{g^2} \int (F'_{\mu\nu})^2} \tag{8.9}$$

Which definately looks dangerous! Thus we can easily see how contributions not accessible by perturbative results can creep in. This is especially true in any form of classical background

$$A_\mu = A_\mu^B + A'_\mu \tag{8.10}$$

(I.e. looking at transitions in the presence of a non-zero background.)

I present this example (another good example is $1/(1 + g^2\ )$ not to try to destroy Feynman diagram techniques but to point out that they are not *everything*. We must consider the realm of validity. Unfortunately, we have few alternate techniques. One technique is to take the path integral and just evaluate it numerically. To do so we must discretise space-time , the configuration etc etc. It takes a lot of computing effort and still has yet to be enormously fruitful but , at present, we have nothing else other than feynman diagrams (and variations thereof). Despite these concerns, field theory does "produce the goods".

9. Exercises (selected)

1.1 Using Lagrange's Equations solve the double pendulum.

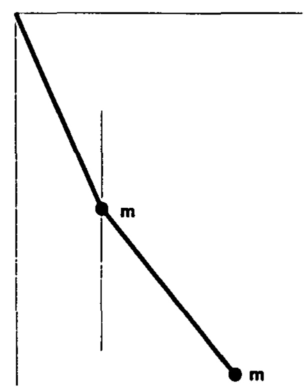

Figure E1. The Double pendulum.

1.2 Calculate the Poisson brackets,

$$\left\{q^2, p\right\}, \quad \left\{q^2, p^2\right\}$$

How do these compare with

$$[\hat{q}^2, \hat{p}], \quad [\hat{q}^2, \hat{p}^2]$$

1.3 Suppose

$$L = \frac{1}{2} v(q)^{-1} \dot{q}^2$$

then what is H?.

1.4 Show that the time dependance of any function $F(p_r, q_r)$ is given by

$$\dot{F} = \left\{F, H\right\} \tag{9.1}$$

2.1 In the low temperature limit of the partition function in statistical mechanics it is the low-energy states whose contributions dominate. In the small-$\hbar$ limit which paths will dominate in the path integral?

3.1 Consider a field $\phi(x)$ we will quantise this theory by discretising in and then let the discretisation go to zero. Suppose we split x-space into N-regions of area δx_i wher ϕ takes on the value $\phi(x_i)$. Then the natural Lagrangian will be

$$\mathcal{L} = \sum_i \delta x_i \frac{1}{2} \dot{\phi}(x_i)^2 + V(\phi(x_i)) \tag{9.2}$$

Define the continuous momentum conjugate to $\phi(x)$ to be $\pi(x)$. and denote it's discretisation by $\pi(x_i)$.

What is the cononical momentum conjugate to $\phi(x_i)$? Suppose we quantise the disctete system. What is implied for

$$\left[\pi(x_i), \phi(x_j)\right]$$

Justify what happens to this in the limit $\delta x_i \to 0$.

5.1 Compute $\frac{\delta}{\delta J(y)}$' and $\frac{\delta^2}{\delta J(y)\delta J(z)}$ of

$$a) \int dx \phi(x) J(x)$$

$$b) \left[\int dx \phi(x) J(x)\right]^2$$

$$c) \int dx \phi(x) J(x)^2$$

6.1 Express both $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F^{\rho\sigma}$ in terms of E and B.

7.1 An alternate Definition of $F_{\mu\nu}$ is

$$F_{\mu\nu} = \left[D_\mu, D_\nu\right]$$

7.2 Find a set of 3×3 matrices which form a representation of $SU(2)$. i.e. matrices satisfying (7.9)

INTRODUCTION TO QUANTUM ELECTRODYNAMICS AND QUANTUM CHROMODYNAMICS

By Dr J Flynn
University of Southampton

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1995

Introduction to Quantum Electrodynamics and Quantum Chromodynamics

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Contents

1 Introduction

The aim of this course is to teach you how to calculate amplitudes, cross-sections and decay rates, particularly for quantum electrodynamics, QED, but in principle also for quantum chromodynamics, QCD. By the end of the course you should be able to go from a Feynman diagram such as the one for $e^+e^- \to \mu^+\mu^-$ in Figure 1.1(a), to a number for the cross-section, for example.

We will restrict ourselves to calculations at *tree level* but will also look qualitatively at higher order *loop* effects which amongst other things are responsible for the running of the QCD coupling constant, where the coupling appears weaker when you measure it at higher enegy scales. This running underlies the useful application of perturbative QCD calculations to high-energy processes. As you can guess, the sort of diagrams which are important here have closed loops of particle lines in them: in Figure 1.1(b) is one example contributing to the running of the strong coupling (the curly lines denote gluons).

In order to do our calculations we will need a certain amount of technology. In particular, we will need to describe particles with spin, especially the spin-1/2 leptons and quarks. We will therefore spend some time looking at the Dirac equation and its free particle solutions. After this will come revision of Fermi's golden rule to find probability amplitudes for transitions, followed by some general results on normalisation, flux factors and phase space, which will allow us to obtain formulas for cross sections and decay rates.

With these tools in hand, we will look at some examples of tree level QED processes. Here you will get hands-on experience of calculating transition amplitudes and getting from them to cross sections. We then move on to QCD. This will entail a brief introduction to renormalisation in both QED and QCD. We will introduce the idea of the running coupling constant and look at asymptotic freedom in QCD.

In reference [1] you will find a list of textbooks which may be useful.

1.1 Units and Conventions

I will use natural units, $c = 1$, $\hbar = 1$, so mass, energy, inverse length and inverse time all have the same dimensions.

$$\begin{array}{lll} \text{4-vector} & a^\mu \quad \mu = 0,1,2,3 & a = (a^0, \mathbf{a}) \\ \text{scalar product} & a{\cdot}b = a^0b^0 - \mathbf{a}{\cdot}\mathbf{b} = g_{\mu\nu}a^\mu b^\nu & \end{array} \tag{1.1}$$

From the scalar product you see that the metric is:

$$g = \mathrm{diag}(1,-1,-1,-1), \qquad g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{1.2}$$

For $c = 1$, $g^{\mu\nu}$ and $g_{\mu\nu}$ are numerically the same.

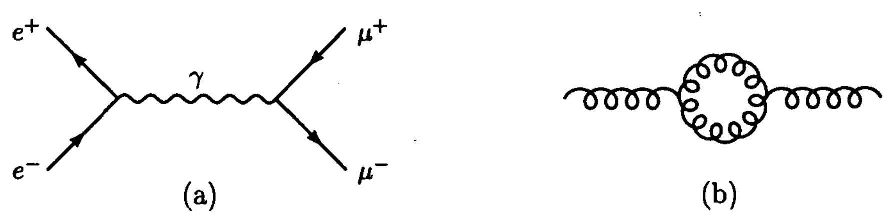


Figure 1.1 Examples of Feynman diagrams contributing to (a) $e^+e^- \to \mu^+\mu^-$ and (b) the running of the strong coupling constant.

From the above, you would think it natural to write the space components of a 4-vector as a^i for $i = 1, 2, 3$. However, for 3-vectors I will normally write the components as a_i. This is confusing only when you convert between ordinary vector equations and their covariant forms, when you have to remember the sign difference between a^i and a_i.

Note that ∂_μ is a covector,

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_\mu x^\nu = \delta_\mu^\nu, \tag{1.3}$$

so $\nabla^i = -\partial^i$ and $\partial^\mu = (\partial^0, -\boldsymbol{\nabla})$

My convention for the totally antisymmetric Levi-Civita tensor is:

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \lambda, \sigma\} \text{ an even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

Note that $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon_{\mu\nu\lambda\sigma}$, and $\epsilon^{\mu\nu\lambda\sigma} p_\mu q_\nu r_\lambda s_\sigma \to (\det \Lambda)\, \epsilon^{\mu\nu\lambda\sigma} p_\mu q_\nu r_\lambda s_\sigma$ for Λ in the Lorentz group.

1.2 Relativistic Wave Equations?

Imagine you are working in the 1920's. You already know quantum mechanics based on Schrödinger's equation and you know relativity. You might ask if you can come up with some relativistic version of a quantum mechanical wave equation. If you do this, you encounter difficulties arising from the one-particle viewpoint, thinking of the equations describing a wave function. These difficulties are solved by ditching the wave function in favour of a *quantum field*, the subject of your quantum field theory course.

What is the problem with the one particle interpretation? Trouble arises from combining the uncertainty principle with the relativistic equivalence of mass and energy-momentum. If you try to localise a particle in a region with dimensions of order L, the particle's momentum and (in the relativistic regime) energy are uncertain by $\sim 1/L$. As the dimension L becomes smaller than the particle's inverse mass, $1/m$, states with more than one particle become energetically accessible. The more you try to localise a particle, the more you become uncertain whether you have one or any number of particles. Relativistic causality is inconsistent with a single particle theory and the real world evades the conflict through pair creation. Quantum field theory is the tool allowing you to reconcile quantum mechanics and special relativity.

What happens in quantum field theory is that field *operators*, which can create or destroy multiparticle states, satisfy Heisenberg equations of motion. If there are no interactions, then the relevant equations are the Klein-Gordon equation for scalar fields or the Dirac equation for spin-half fields (such as the electron). The free quantum fields are expanded as linear combinations of plane wave solutions of these equations, but with operator valued coefficients which can create and destroy single particles. Thus we need to know the properties of the plane wave solutions. This is trivial for the scalar field, but is more interesting for the Dirac field. All the problems with "negative energy solutions" in the wave function approach are non-problems in quantum field theory: the negative energy parts multiply operators which destroy particles.

In fairness I should mention that you can get quite far with the one particle interpretation if you consider external forces which vary slowly on scales of order $1/m$, and thereby don't have enough energy to create new particle pairs. Notably, you can use the Dirac equation, which we'll meet below, in the presence of an electromagnetic field, to calculate fine structure in the spectra of hydrogen-like atoms (see textbooks such as Itzykson and Zuber [1] section 2.3 for example).

1.3 The Klein-Gordon Equation

In your quantum field theory course, you will show that the Heisenberg equations of motion for a free scalar field and its canonical conjugate give the *Klein-Gordon* equation

$$(\Box + m^2)\,\phi(x) = 0 \tag{1.5}$$

where

$$\Box = \partial_\mu \partial^\mu = \partial^2/\partial t^2 - \nabla^2 \tag{1.6}$$

and x is the 4-vector $(t, \mathbf{x})$. Using the substitutions,

$$E \to i\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\boldsymbol{\nabla}, \tag{1.7}$$

you can see that the objects created or destroyed by ϕ satisfy the relativistic energy-momentum relation

$$E^2 = \mathbf{p}^2 + m^2. \tag{1.8}$$

The operator $\Box$ is Lorentz invariant, so the Klein-Gordon equation is relativistically covariant (that is, transforms into an equation of the same form) if ϕ is a scalar function. That is to say, under a Lorentz transformation $(t, \mathbf{x}) \to (t', \mathbf{x}')$,

$$\phi(t, \mathbf{x}) \to \phi'(t', \mathbf{x}') = \phi(t, \mathbf{x})$$

so ϕ is invariant. In particular ϕ is then invariant under spatial rotations so it represents a spin-zero particle (more on spin when we come to the Dirac equation), there being no preferred direction which could carry information on a spin orientation.

The Klein-Gordon equation has plane wave solutions

$$\phi(x) = Ne^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{1.9}$$

where N is a normalisation constant and $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. Thus, there are both positive and negative energy solutions. In the quantum field ϕ, these are just associated with operators which create or destroy particles. However, they are a severe problem if you try to interpret ϕ as a wavefunction. The spectrum is no longer bounded below, and you can extract arbitrarily large amounts of energy from the system by driving it into ever more negative energy states. Any external perturbation capable of pushing a particle across the energy gap of $2m$ between the positive and negative energy continuum of states can uncover this difficulty.

A second problem with the wavefunction interpretation arises when you try to find a probability density. Since ϕ is Lorentz invariant, $|\phi|^2$ doesn't transform like a density. To search for a candidate we derive a continuity equation, rather as you did for the Schrödinger equation in the pre-school problems. Defining ρ and $\mathbf{J}$ by

$$\begin{aligned} \rho &\equiv i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) \\ \mathbf{J} &\equiv -i\left(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*\right) \end{aligned} \tag{1.10}$$

you obtain (see problem) a covariant conservation equation

$$\partial_\mu J^\mu = 0 \tag{1.11}$$

where J is the 4-vector $(\rho, \mathbf{J})$. It is natural to interpret ρ as a probability density and $\mathbf{J}$ as a probability current. However, for a plane wave solution (1.9), $\rho = 2|N|^2 E$, so ρ is not positive definite since we've already found E can be negative.

▷ **Exercise 1.1**
Derive the continuity equation (1.11). Start with the Klein-Gordon equation multiplied by ϕ^* and subtract the complex conjugate of the K-G equation multiplied by ϕ.

Thus, ρ may well be considered as the density of a conserved quantity (such as electric charge), but we cannot use it for a probability density. To Dirac, this and the existence of negative energy solutions seemed so overwhelming that he was led to introduce another equation, first order in time derivatives but still Lorentz covariant, hoping that the similarity to Schrödinger's equation would allow a probability interpretation. In fact, with the interpretation of ϕ as a quantum field, these problems are not problems at all: the negative energy solutions will find an explanation in terms of antiparticles and ρ will indeed be a charge density as hinted above. Moreover, Dirac's hopes were unfounded because his new equation also turns out to admit negative energy solutions. Fortunately it is just what we need to describe particles with half a unit of spin angular momentum, so we will now turn to it.

2 The Dirac Equation

Dirac wanted an equation first order in time derivatives and Lorentz covariant, so it had to be first order in spatial derivatives too. His starting point was

$$i\frac{\partial\psi}{\partial t} = -i\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \beta m\psi \tag{2.1}$$

Remember that in field theory, the Dirac equation is the equation of motion for the field operator describing spin-1/2 fermions. In order for this equation to be Lorentz covariant, it will turn out that ψ cannot be a scalar under Lorentz transformations. In fact this will be precisely how the equation turns out to describe spin-1/2 particles. We will return to this below.

If ψ is to describe a free particle it is natural that it should satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among the $\boldsymbol{\alpha}$ and β. To see these, apply the operator on each side of equation (2.1) twice,

$$-\frac{\partial^2\psi}{\partial t^2} = -\alpha^i\alpha^j\nabla^i\nabla^j - i\,(\beta\alpha^i + \alpha^i\beta)m\nabla^i\psi + \beta^2 m^2\psi$$

The Klein-Gordon equation will be satisfied if

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij} \\ \beta\alpha_i + \alpha_i\beta &= 0 \\ \beta^2 &= 1 \end{aligned} \tag{2.2}$$

for $i, j = 1, 2, 3$. It is clear that the α_i and β cannot be ordinary numbers, but it is natural to give them a realisation as matrices. In this case, ψ must be a multi-component *spinor* on which these matrices act.

▷**Exercise 2.1**

Prove that any matrices $\boldsymbol{\alpha}$ and β satisfying equation (2.2) are traceless with eigenvalues ±1. Hence argue that they must be even dimensional.

In two dimensions a natural set of matrices for the $\boldsymbol{\alpha}$ would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

However, there is no other independent 2×2 matrix with the right properties for β, so the smallest dimension for which the Dirac matrices can be realised is four. One choice is the *Dirac representation*

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix.

There is a theorem due to Pauli which states that all sets of matrices obeying the relations in (2.2) are equivalent. Since the Hermitian conjugates $\boldsymbol{\alpha}^\dagger$ and $\beta^\dagger$ clearly obey the relations, you can, by a change of basis if necessary, assume that $\boldsymbol{\alpha}$ and β are Hermitian. All the common choices of basis have this property. Furthermore, we would like α_i and β to be Hermitian so that the Dirac Hamiltonian (2.14) is Hermitian.

▷ **Exercise 2.2**
Derive the continuity equation $\partial_\mu J^\mu = 0$ for the Dirac equation with

$$\rho = J^0 = \psi^\dagger(x)\psi(x), \qquad \mathbf{J} = \psi^\dagger(x)\boldsymbol{\alpha}\psi(x). \tag{2.5}$$

We will see in section 2.6 that $(\rho, \mathbf{J})$ does indeed transform as a four-vector.

2.1 Free Particle Solutions I: Interpretation

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}$$

where ϕ and χ are two-component spinors, independent of x. Using the Dirac representation, the Dirac equation gives

$$E\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} m & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

so that

$$\chi = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\phi, \qquad \phi = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}\chi.$$

For $E \neq -m$ there are solutions,

$$\psi(x) = \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\phi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{2.6}$$

while for $E \neq m$ there are solutions,

$$\psi(x) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}\chi \\ \chi \end{pmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}, \tag{2.7}$$

for arbitrary constant ϕ and χ. Now, since $E^2 = \mathbf{p}^2 + m^2$ by construction, we find, just as we did for the Klein-Gordon equation (1.5), that there exist positive and negative energy solutions given by equations (2.6) and (2.7) respectively. Once again, the existence of negative energy solutions vitiates the interpretation of ψ as a wavefunction.

Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. By promoting one of these negative energy states to a positive energy one, by supplying energy, you create a pair: a positive energy electron and a hole in the negative energy sea corresponding to a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics. Positrons were discovered in cosmic rays by Carl Anderson in 1932.

The problem with Dirac's hole theory is that it doesn't work for bosons, such as particles governed by the Klein Gordon equation, for example. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy. We need a new interpretation and turn to Feynman for our answer.

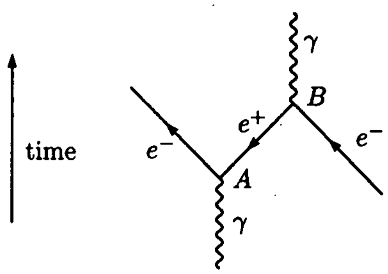


Figure 2.1 Feynman interpretation of a process in which a negative energy electron is absorbed. Time increases moving upwards.

According to Feynman and quantum field theory, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$. So, in Figure 2.1, for example, an electron–positron pair is created at point A. The positron propagates to point B where it is annihilated by another electron.

Thus Feynman tells us to keep both types of free particle solution. One is to be used for particles and the other for the accompanying antiparticles. Let's return to our spinor solutions and write them in a conventional form. Take the positive energy solution of equation (2.6) and write,

$$\sqrt{E+m}\begin{pmatrix} \chi_r \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\chi_r \end{pmatrix} e^{-ip\cdot x} \equiv u_p^r e^{-ip\cdot x}. \tag{2.8}$$

For the former negative energy solution of equation (2.7), change the sign of the energy, $E \to -E$, and the three-momentum, $\mathbf{p} \to -\mathbf{p}$, to obtain,

$$\sqrt{E+m}\begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{E+m}\chi_r \\ \chi_r \end{pmatrix} e^{ip\cdot x} \equiv v_p^r e^{ip\cdot x}. \tag{2.9}$$

In these two solutions E is now (and for the rest of the course) always positive and given by $E = (\mathbf{p}^2 + m^2)^{1/2}$. The subscript r takes the values $1, 2$, with

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

At this point I would like to introduce another notation, and define

$$\omega_p \equiv \sqrt{\mathbf{p}^2 + m^2}, \tag{2.11}$$

so that, ω_p is the energy (positive) of a particle or anti-particle with three-momentum $\mathbf{p}$ (I write the subscript p instead of $\mathbf{p}$, but you should remember it really means the three-momentum). I will tend to use E or ω_p interchangeably.

The u-spinor solutions will correspond to particles and the v-spinor solutions to antiparticles. The role of the two χ's will become clear in the following section, where it will be shown that the two choices of r are spin labels. Note that each spinor solution depends on the three-momentum $\mathbf{p}$, so it is implicit that $p^0 = \omega_p$. In the expansion of the Dirac quantum field operator in terms of plane waves,

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} \sum_{r=1,2} \left[\hat{b}(p,r) u_p^r e^{-ip\cdot x} + \hat{d}^\dagger(p,r) v_p^r e^{ip\cdot x}\right] \tag{2.12}$$

the operator $\hat{b}$ annihilates a fermion of momentum $(\omega_p, \mathbf{p})$ and spin r, whilst $\hat{d}^\dagger$ creates an antifermion of momentum $(\omega_p, \mathbf{p})$ and spin r. The Hermitian conjugate Dirac field contains operators which do the opposite. This discussion should be clearer after your quantum field theory lectures.

The vacuum state $|0\rangle$ is defined by,

$$b(p,r)\,|0\rangle = d(p,r)\,|0\rangle = 0, \tag{2.13}$$

for every momentum $p = (\omega_p, \mathbf{p})$ and spin label r. This ensures the interpretation above: particles are created by the "daggered" operators and destroyed by the undaggered ones.

2.2 Free Particle Solutions II: Spin

Now it's time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. The Dirac Hamiltonian in momentum space is

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \tag{2.14}$$

and the orbital angular momentum operator is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Normally you have to worry about operator ordering ambiguities when going from classical objects to quantum mechanical ones. For $\mathbf{L}$ the problem does not arise — why not?

Evaluating the commutator of $\mathbf{L}$ with H,

$$\begin{aligned} [\mathbf{L}, H] &= [\mathbf{r}\times\mathbf{p}, \boldsymbol{\alpha}\cdot\mathbf{p}] \\ &= [\mathbf{r}, \boldsymbol{\alpha}\cdot\mathbf{p}]\times\mathbf{p} \\ &= i\boldsymbol{\alpha}\times\mathbf{p}, \end{aligned} \tag{2.15}$$

we see that the orbital angular momentum is not conserved. We'd like to find a *total* angular momentum $\mathbf{J}$ which *is* conserved, by adding an additional operator $\mathbf{S}$ to $\mathbf{L}$,

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

To this end, consider the three matrices,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\boldsymbol{\alpha}. \tag{2.16}$$

The $\boldsymbol{\Sigma}/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with $\boldsymbol{\alpha}$ and β are,

$$[\boldsymbol{\Sigma}, \beta] = 0, \qquad [\Sigma_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k. \tag{2.17}$$

▷**Exercise 2.3**
Verify the commutation relations in equation (2.17).

From the relations in (2.17) we find that

$$[\mathbf{\Sigma}, H] = -2i\boldsymbol{\alpha} \times \mathbf{p}.$$

Comparing this with the commutator of $\mathbf{L}$ with H in equation (2.15), you readily see that

$$\left[\mathbf{L} + \tfrac{1}{2}\mathbf{\Sigma}, H\right] = 0,$$

and we can set

$$\mathbf{S} = \frac{1}{2}\mathbf{\Sigma}.$$

We interpret $\mathbf{S}$ as an angular momentum *intrinsic* to the particle. Now

$$\mathbf{S}^2 = \frac{1}{4}\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and recalling that the eigenvalue of $\mathbf{J}^2$ for spin j is $j(j+1)$, we conclude that $\mathbf{S}$ represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised.

We worked in the Dirac representation for convenience, but the result is of course independent of the representation.

Now consider the u-spinor solutions u_p^r of equation (2.8). Choose $\mathbf{p} = (0, 0, p_z)$ and write

$$u_\uparrow = u_{p_z}^1 = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \qquad u_\downarrow = u_{p_z}^2 = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \tag{2.18}$$

It is easy to see that,

$$S_z u_\uparrow = \frac{1}{2}u_\uparrow, \qquad S_z u_\downarrow = -\frac{1}{2}u_\downarrow.$$

So, these two spinors represent spin up and spin down along the z-axis respectively. For the v-spinors, with the same choice for $\mathbf{p}$, write,

$$v_\downarrow = v_{p_z}^1 = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \qquad v_\uparrow = v_{p_z}^2 = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \tag{2.19}$$

where now,

$$S_z v_\downarrow = \frac{1}{2}v_\downarrow, \qquad S_z v_\uparrow = -\frac{1}{2}v_\uparrow.$$

This apparently perverse choice of up and down for the v's is because, as you see in equation (2.12) for the quantum Dirac field, $u_\uparrow$ multiplies an annihilation operator which *destroys* a particle with momentum p_z and spin up, whereas $v_\downarrow$ multiplies an operator which *creates* an antiparticle with momentum p_z and spin up.

2.3 Normalisation, Gamma Matrices

We have included a normalisation factor $\sqrt{E+m}$ in our spinors. With this factor,

$$u_p^{r\dagger}u_p^s = v_p^{r\dagger}v_p^s = 2\omega_p\delta^{rs}. \tag{2.20}$$

This corresponds to the standard relativistic normalisation of $2\omega_p$ particles per unit volume. It also means that $u^\dagger u$ transforms like the time component of a 4-vector under Lorentz transformations as we will see in section 2.6.

▷ **Exercise 2.4**
Check the normalisation condition for the spinors in equation (2.20).

I will now introduce (yet) more standard notation. Define the *gamma matrices*,

$$\gamma^0 = \beta, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}. \tag{2.21}$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{2.22}$$

In terms of these, the relations between the $\boldsymbol{\alpha}$ and β in equation (2.2) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{2.23}$$

Combinations like $a_\mu\gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu\gamma^\mu = a^\mu\gamma_\mu,$$

pronounced "a slash." Note that γ^μ is not, despite appearances, a 4-vector — it just denotes a set of four matrices. However, the notation is deliberately suggestive, for when combined with Dirac fields you can construct quantities which transform like vectors and other Lorentz tensors (see the next section).

Let's close this section by observing that using the gamma matrices the Dirac equation (2.1) becomes

$$(i\not{\partial} - m)\psi = 0, \tag{2.24}$$

or in momentum space,

$$(\not{p} - m)\psi = 0. \tag{2.25}$$

The spinors u and v satisfy

$$\begin{aligned} (\not{p} - m)u_p^r &= 0 \\ (\not{p} + m)v_p^r &= 0 \end{aligned} \tag{2.26}$$

▷ **Exercise 2.5**
Derive the momentum space equations satisfied by u_p^r and v_p^r.

2.4 Lorentz Covariance

We want the Dirac equation (2.24) to preserve its form under Lorentz transformations (LT's). Let $\Lambda^\mu{}_\nu$ represent an LT,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.27}$$

The requirement is,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad \longrightarrow \quad (i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0,$$

where $\partial_\mu = \Lambda^\sigma{}_\mu\partial'_\sigma$. We know that 4-vectors get their components mixed up by LT's, so we expect that the components of ψ might get mixed up also,

$$\psi(x) \to \psi'(x') \equiv S(\Lambda)\psi(\Lambda^{-1}x') \tag{2.28}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the spinor index of ψ. Note that the argument $\Lambda^{-1}x'$ is just a fancy way of writing x, so each component of $\psi(x)$ is transformed into a linear combination of components of $\psi(x)$.

To determine S we rewrite the Dirac equation in terms of the primed variables,

$$(i\gamma^\mu \Lambda^\sigma{}_\mu \partial'_\sigma - m)\psi(\Lambda^{-1}x') = 0. \tag{2.29}$$

The matrices $\gamma'^\sigma \equiv \gamma^\mu \Lambda^\sigma{}_\mu$ satisfy the same anticommutation relations as the γ^μ's in equation (2.23),

$$\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}. \tag{2.30}$$

▷ **Exercise 2.6**

Check relation (2.30).

Now we invoke the theorem (Pauli's theorem) which states that any two representations of the gamma matrices are equivalent. This means that there is a matrix $S(\Lambda)$ such that

$$\gamma'^\mu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda). \tag{2.31}$$

This allows us to rewrite equation (2.29) as

$$(i\gamma^\mu \partial'_\mu - m)S(\Lambda)\psi(\Lambda^{-1}x') = 0,$$

so that the Dirac equation does indeed preserve its form. To construct S explicitly for an infinitesimal LT, let,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \epsilon(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu) \tag{2.32}$$

where ϵ is an infinitesimal parameter and ρ and σ are fixed. Since this expression is antisymmetric in ρ and σ there are six choices for the pair (ρ, σ) corresponding to three rotations and three boosts. Writing,

$$S(\Lambda) = 1 + i\epsilon s^{\rho\sigma} \tag{2.33}$$

where $s^{\rho\sigma}$ is a matrix to be determined, we find that equation (2.31) for γ' is satisfied by,

$$s^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] \equiv \frac{1}{2}\sigma^{\rho\sigma}. \tag{2.34}$$

Here, I have taken the opportunity to define the matrix $\sigma^{\rho\sigma}$.

▷ **Exercise 2.7**

Verify that equation (2.31) relating γ' and γ is satisfied by $s^{\rho\sigma}$ defined through equations (2.33) and (2.34).

We have thus determined how ψ transforms under LT's. To find quantities which are Lorentz invariant, or transform as vectors or tensors, we need to introduce the Pauli and Dirac adjoints. The Pauli adjoint $\overline{\psi}$ of a spinor ψ is defined by

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 = \psi^\dagger \beta. \tag{2.35}$$

The Dirac adjoint is defined by

$$(\overline{\psi} A \phi)^* = \overline{\phi}\,\overline{A}\psi. \tag{2.36}$$

For Hermitian γ^0 it is easy to show that

$$\overline{A} = \gamma^0 A^\dagger \gamma^0. \tag{2.37}$$

Some properties of the Pauli and Dirac adjoints are:

$$\begin{aligned}\overline{(\lambda A + \mu B)} &= \lambda^* \overline{A} + \mu^* \overline{B}, \\ \overline{AB} &= \overline{B}\,\overline{A}, \\ \overline{A\psi} &= \overline{\psi}\,\overline{A}.\end{aligned}$$

With these definitions, $\overline{\psi}$ transforms as follows under LT's:

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi} S^{-1}(\Lambda) \tag{2.38}$$

▷ **Exercise 2.8**

(1) Verify that $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. This says that $\overline{\gamma}^\mu = \gamma^\mu$.

(2) Using (2.33) and (2.34) verify that $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, i.e. $\overline{S} = S^{-1}$. So S is not unitary in general, although it *is* unitary for rotations (when ρ and σ are spatial indices). This is because the rotations are in the unitary $O(3)$ subgroup of the nonunitary Lorentz group. Here you show the result for an infinitesimal LT, but it is true for finite LT's.

(3) Show that $\overline{\psi}$ satisfies the equation

$$\overline{\psi}\,(-i\overleftarrow{\not{\partial}} - m) = 0$$

where the arrow over $\not{\partial}$ implies the derivative acts on $\overline{\psi}$.

(4) Hence prove that $\overline{\psi}$ transforms as in equation (2.38).

Note that result (2) of the problem above can be rewritten as $\overline{S}(\Lambda) = S^{-1}(\Lambda)$, and equation (2.31) for the similarity transformation of γ^μ to γ'^μ takes the form,

$$\overline{S}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu. \tag{2.39}$$

Combining the transformation properties of ψ and $\overline{\psi}$ from equations (2.28) and (2.38) we see that the bilinear $\overline{\psi}\psi$ is Lorentz invariant. In section 2.6 we'll consider the transformation properties of general bilinears.

Let me close this section by recasting the spinor normalisation equations (2.20) in terms of "Dirac inner products." The conditions become,

$$\begin{aligned}\overline{u}^r_p u^s_p &= 2m\delta^{rs} \\ \overline{u}^r_p v^s_p &= 0 \qquad = \overline{v}^r_p u^s_p \\ \overline{v}^r_p v^s_p &= -2m\delta^{rs}\end{aligned} \tag{2.40}$$

▷ **Exercise 2.9**

Verify the normalisation properties in the above equations (2.40).

2.5 Parity

In the next section we are going to construct quantities bilinear in ψ and $\overline{\psi}$, and classify them according to their transformation properties under LT's. We normally use LT's which are in the connected Lorentz Group, $SO(3,1)$, meaning they can be obtained by a continuous deformation of the identity transformation. Indeed in the last section we considered LT's very close to the identity in equation (2.32). The full Lorentz group has four components generated by combining the $SO(3,1)$ transformations with the discrete operations of parity or space inversion, P, and time reversal, T,

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LT's satisfy $\Lambda^T g \Lambda = g$ (see the preschool problems), so taking determinants shows that $\det \Lambda = \pm 1$. LT's in $SO(3,1)$ have determinant 1, since the identity does, but the P and T operations have determinant -1.

Let's now find the action of parity on the Dirac wavefunction and determine the wavefunction ψ_P in the parity-reversed system. According to the discussion of the previous section, and using the result of equation (2.39), we need to find a matrix S satisfying

$$\overline{S}\gamma^0 S = \gamma^0, \qquad \overline{S}\gamma^i S = -\gamma^i.$$

It's not hard to see that $S = \overline{S} = \gamma^0$ is an acceptable solution, from which it follows that the wavefunction ψ_P is

$$\psi_P(t, \mathbf{x}) = \gamma^0 \psi(t, -\mathbf{x}). \tag{2.41}$$

In fact you could multiply γ^0 by a phase and still have an acceptable definition for the parity transformation.

In the nonrelativistic limit, the wavefunction ψ approaches an eigenstate of parity. Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the u-spinors and v-spinors at rest have opposite eigenvalues, corresponding to particle and antiparticle having opposite *intrinsic* parities.

2.6 Bilinear Covariants

Now, as promised, we will construct and classify the bilinears. To begin, observe that by forming products of the gamma matrices it is possible to construct 16 linearly independent quantities. In equation (2.34) we have defined

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

and now it is convenient to define

$$\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.42}$$

with the properties,

$$\gamma_5^\dagger = \gamma_5, \qquad \{\gamma_5, \gamma^\mu\} = 0.$$

Then the set of 16 matrices

$$\Gamma : \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

form a basis for gamma matrix products.

Using the transformations of ψ and $\overline{\psi}$ from equations (2.28) and (2.38), together with the similarity transformation of γ^μ in equation (2.39), construct the 16 fermion bilinears and their transformation properties as follows:

$$\begin{array}{rcll} \overline{\psi}\psi & \rightarrow & \overline{\psi}\psi & \text{S scalar} \\ \overline{\psi}\gamma_5\psi & \rightarrow & \det(\Lambda)\,\overline{\psi}\gamma_5\psi & \text{P pseudoscalar} \\ \overline{\psi}\gamma^\mu\psi & \rightarrow & \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi & \text{V vector} \\ \overline{\psi}\gamma^\mu\gamma_5\psi & \rightarrow & \det(\Lambda)\,\Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\gamma_5\psi & \text{A axial vector} \\ \overline{\psi}\sigma^{\mu\nu}\psi & \rightarrow & \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\overline{\psi}\sigma^{\lambda\sigma}\psi & \text{T tensor} \end{array} \tag{2.43}$$

▷ **Exercise 2.10**

Verify the transformation properties of the bilinears in equation (2.43).

Observe that $\overline{\psi}\gamma^\mu\psi = (\rho, \mathbf{J})$ is just the current we found earlier in equation (2.5). Classically ρ is positive definite, but for the quantum Dirac field you find that the space integral of ρ is the charge operator, which counts the number of electrons minus the number of positrons,

$$Q \sim \int d^3x\, \psi^\dagger\psi \sim \int d^3p\, [b^\dagger b - d^\dagger d].$$

The continuity equation $\partial_\mu J^\mu = 0$ expresses conservation of electric charge.

2.7 Charge Conjugation

There is one more discrete invariance of the Dirac equation in addition to parity. It is charge conjugation, which takes you from particle to antiparticle and vice versa. For scalar fields the symmetry is just complex conjugation, but in order for the charge conjugate Dirac field to remain a solution of the Dirac equation, you have to mix its components as well:

$$\psi \rightarrow \psi_C = C\overline{\psi}^T.$$

Here $\overline{\psi}^T = \gamma^{0T}\psi^*$ and C is a matrix satisfying the condition

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu.$$

In the Dirac representation,

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}.$$

I refer you to textbooks such as [1] for details.

When Dirac wrote down his equation everybody thought parity and charge conjugation were exact symmetries of nature, so invariance under these transformations was essential. Now we know that neither of them, nor the combination *CP*, are respected by the standard electroweak model.

2.8 Neutrinos

In the particle data book [2] you will find only upper limits for the masses of the three neutrinos, and in the standard model they are massless. Let's look therefore at solutions of the Dirac equation with $m = 0$. Specialising from equation (2.1), we have, in momentum space,

$$|\mathbf{p}|\psi = \boldsymbol{\alpha}\cdot\mathbf{p}\,\psi.$$

For such a solution,

$$\gamma_5\psi = \gamma_5\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{|\mathbf{p}|}\psi = 2\frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|}\psi,$$

using the spin operator $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\gamma_5\boldsymbol{\alpha}$, with $\boldsymbol{\Sigma}$ defined in equation (2.16). But $\mathbf{S}\cdot\mathbf{p}/|\mathbf{p}|$ is the projection of spin onto the direction of motion, known as the *helicity*, and is equal to $\pm 1/2$. Thus $(1+\gamma_5)/2$ projects out the neutrino with helicity $1/2$ (right handed) and $(1-\gamma_5)/2$ projects out the neutrino with helicity $-1/2$ (left handed). To date, only left handed neutrinos have been observed, and only left handed neutrinos appear in the standard model. Since

$$\gamma^0\frac{1}{2}(1-\gamma_5)\psi = \frac{1}{2}(1+\gamma_5)\gamma^0\psi,$$

any theory involving only left handed neutrinos necessarily violates parity.

The standard model contains only left handed massless neutrinos. It is really the electroweak symmetry which prevents them having masses, not the fact that they are left handed only. It would be possible to doctor the standard model to contain so-called Majorana neutrinos which can be massive. However, this would entail relinquishing lepton number conservation and break the electroweak symmetry (or involve the introduction of new particles).

2.9 Dirac Lagrangian

In the spirit of the field theory course, we could have started out by looking for objects, transforming in the right way under Lorentz boosts and rotations, to represent spin-1/2 particles. This would have led us to Dirac spinors, for which we would have shown that

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$

is a Lorentz invariant Lagrangian.

Then Lagrange's equations immediately give the Dirac equation, as you can see simply from $\partial\mathcal{L}/\partial\overline{\psi} = 0$ (observing that $\mathcal{L}$ is independent of $\partial\overline{\psi}/\partial t$). Now you could quantise by Hamiltonian or path integral methods. A new feature that appears is that, for consistency, you must impose canonical *anti*commutation relations in the Hamiltonian form, or use *anti*commuting (Grassman) variables in the path integral. Thus, the connection between spin and statistics appears. For example, if $\hat{b}^\dagger(p,r)$ is the creation operator for an electron of momentum p and spin label r, then $\hat{b}^\dagger(p,r)\,|0\rangle$ is a Fock state with one electron, but

$$\hat{b}^\dagger(p,r)\hat{b}^\dagger(p,r)\,|0\rangle = 0,$$

so you can never put more than one electron into the same state. This contrasts with the behaviour of bosons.

3 Cross Sections and Decay Rates

In section 4 we will learn how to calculate quantum mechanical amplitudes for electromagnetic scattering and decay processes. These amplitudes are obtained from the Lagrangian of QED, and contain information about the dynamics underlying the scattering or decay process. This section is a brief review of how to get from the quantum mechanical amplitude to a cross section or decay rate which can be measured. We will commence by recalling Fermi's golden rule for transition probabilities.

3.1 Fermi's Golden Rule

Consider a system with Hamiltonian H which can be written

$$H = H_0 + V \tag{3.1}$$

We assume that the eigenstates and eigenvalues of H_0 are known and that V is a small, possibly time-dependent, perturbation. The equation of motion of the system is,

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = (H_0 + V)|\psi(t)\rangle . \tag{3.2}$$

If V vanished, we could calculate the time evolution of $|\psi(t)\rangle$ by expanding it as a linear combination of energy eigenstates. When V does not vanish, the eigenstates of H_0 are no longer eigenstates of the full Hamiltonian so when we expand in terms of H_0 eigenstates, the coefficients of the expansion become time dependent. To develop a perturbation theory in V we will change our basis of states from the Schrödinger picture to the *interaction* or Dirac picture, where we hide the time evolution due to H_0 and concentrate on that due to V. Thus we define the interaction picture states and operators by,

$$|\psi_I(t)\rangle \equiv e^{iH_0t}|\psi(t)\rangle , \qquad \mathcal{O}_I(t) \equiv e^{iH_0t}\mathcal{O}(t)e^{-iH_0t}, \tag{3.3}$$

so that the interaction picture and Schrödinger picture states agree at time $t = 0$, $|\psi_I(0)\rangle = |\psi(0)\rangle$, with a similar relation for the operators. In the new basis, the equation of motion becomes,

$$i\frac{\partial}{\partial t}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle , \tag{3.4}$$

which can be integrated formally as an infinite series in V,

$$|\psi_I(t)\rangle = \left[1 + \sum_{n=1}^{\infty}\frac{1}{i^n}\int_{-T/2}^{t} dt_1 \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n\, V_I(t_1)V_I(t_2)\cdots V_I(t_n)\right]|\psi(-T/2)\rangle . \tag{3.5}$$

Here, we have chosen to start with some (known) state $|\psi_I(-T/2)\rangle$, at time $-T/2$, and have evolved it to $|\psi_I(t)\rangle$ at time t. The evolution is done by the operator, U, that you've seen in the field theory course:

$$|\psi_I(t)\rangle = U(t, -T/2)|\psi_I(-T/2)\rangle .$$

Now consider the calculation of the probability of a transition to an eigenstate $|b\rangle$ at time t. The amplitude is,

$$\begin{aligned}\langle b|\psi(t)\rangle &= \langle b_I|\psi_I(t)\rangle \\ &= \langle b|\, e^{-iH_0t}|\psi_I(t)\rangle \\ &= e^{-iE_bt}\langle b|\psi_I(t)\rangle,\end{aligned}$$

so $\left|\langle b|\psi(t)\rangle\right|^2 = \left|\langle b|\psi_I(t)\rangle\right|^2$. We let V be time independent and consider the amplitude for a transition from an eigenstate $|a\rangle$ of H_0 at $t = -T/2$ to an orthogonal eigenstate $|b\rangle$ at $t = T/2$. The idea is that at very early or very late times H_0 describes some set of free particles. We allow some of these particles to approach each other and scatter under the influence of V, then look again a long time later when the outgoing particles are propagating freely under H_0 again. To first order in V,

$$\langle b\,|\psi_I(T/2)\rangle = -i\int_{-T/2}^{T/2}\langle b|\,V_I(t)\,|a\rangle\,dt = -i\langle b|V|a\rangle\int_{-T/2}^{T/2} e^{i\omega_{ba}t}dt,$$

where $\omega_{ba} = E_b - E_a$.

▷Exercise 3.1

Show that for $T \to \infty$ the first order transition amplitude for general V can be written in the covariant form

$$\langle b\,|\psi_I(\infty)\rangle = -i\int d^4x\,\phi_b^*(x)V\phi_a(x),$$

where $\phi_i(x) \equiv \phi_i(\mathbf{x})e^{-E_it}$ and $\phi_i(\mathbf{x})$ is the usual Schrödinger wavefunction for a stationary state of H_0, with energy E_i.

The transition rate for time independent V is,

$$\frac{\left|\langle b\,|\psi_I(T/2)\rangle\,\right|^2}{T} = \left|\langle b|V|a\rangle\right|^2\frac{4\sin^2(\omega_{ba}T/2)}{\omega_{ba}^2T}.$$

If $E_b \neq E_a$, this probability tends to zero as $T \to \infty$. However, for $E_b = E_a$ we use the result,

$$\frac{1}{2\pi T}\frac{\sin^2(\omega_{ba}T/2)}{(\omega_{ba}/2)^2} \xrightarrow{T\to\infty} \delta(\omega_{ba}). \tag{3.6}$$

For long times the transition rate becomes,

$$R_{ba} = 2\pi\left|\langle b|V|a\rangle\right|^2\delta(E_b - E_a). \tag{3.7}$$

We need V small for the first order result to be useful and T large so that the delta-function approximation is good. However, T cannot be too large since the transition probability grows with time and we don't want probabilities larger than one.

If we allow for a number of final states $|b\rangle$, with density $\rho(E_b)$ around energy E_b, the transition rate becomes,

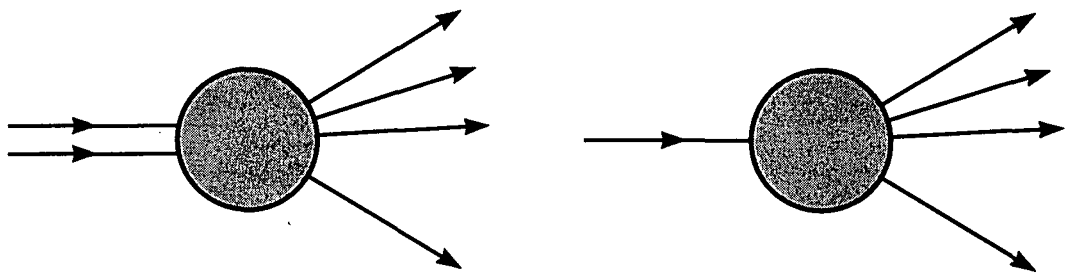
$$\int 2\pi\left|\langle b|V|a\rangle\right|^2\delta(E_b - E_a)\rho(E_b)dE_b = 2\pi\rho(E_a)\left|\langle b|V|a\rangle\right|^2. \tag{3.8}$$

This is *Fermi's golden rule.*

▷Exercise 3.2

Justify the result of equation (3.6) and hence verify Fermi's golden rule in equation (3.8).

I'll stop at first order in V. The answer you get from the formal solution in equation (3.5) depends on the form of V and the initial conditions. Your field theory course gives you a systematic way to perform perturbative calculations of transition amplitudes in field theories by the use of Feynman diagrams. In particular, you've seen the operator method of generating these diagrams, which I've mirrored in deriving the Golden Rule. Let's now move on to see how to get from these amplitudes to cross-sections and decay rates. This corresponds to finding the density of states factor in the Golden Rule.



(a) (b)

Figure 3.1 Scattering (a) and decay (b) processes.

3.2 Phase Space

We saw in the previous section that $\langle b\,|\psi_I(\infty)\rangle$ gives the probability amplitude to go from state $|a\rangle$ in the far past to state $|b\rangle$ in the far future. In quantum field theory you calculate the amplitude to go from state $|i\rangle$ to state $|f\rangle$ to be,

$$i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i), \tag{3.9}$$

where $i\mathcal{M}_{fi}$ is the result obtained from a Feynman diagram calculation, and the overall energy-momentum delta function has been factored out (so when you draw your Feynman diagrams you conserve energy-momentum at every vertex). We have in mind processes where two particles scatter, or one particle decays, as shown in Figure 3.1.

Attempting to take the squared modulus of this amplitude produces a meaningless square of a delta function. This is a technical problem because our amplitude is expressed between non-normalisable plane wave states. These states extend throughout space-time so the scattering process occurs everywhere all the time. To deal with this properly you can construct normalised wavepacket states which do become well separated in the far past and the far future. We will be low-budget and put our system in a box of volume V. We also imagine that the interaction is restricted to act only over a time of order T. The final answers come out independent of V and T, reproducing the luxury wavepacket ones. We are in good company here: Nobel Laureate Steven Weinberg says [3], when discussing cross sections and decay rates, "...(as far as I know) no interesting open problems in physics hinge on getting the fine points right regarding these matters."

Relativistically normalised one particle states satisfy,

$$\langle k|k'\rangle = (2\pi)^3 2\omega_k \delta^3(\mathbf{k} - \mathbf{k}'), \tag{3.10}$$

but the discrete nonrelativistically normalised box states satisfy,

$$\langle \mathbf{k}|\mathbf{k}'\rangle = \delta_{\mathbf{k}\mathbf{k}'}. \tag{3.11}$$

We want to know the transition probability from an initial state of one or two particles to a set of final states occupying some region of $\mathbf{k}$-space, where the density of states in the box normalisation is,

$$\text{box state density} = \frac{d^3\mathbf{k}}{(2\pi)^3}\,V, \tag{3.12}$$

recalling that the spacing of allowed momenta is $2\pi/L$. A particular final state is labelled, $|f\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle$, and the initial state is,

$$|i\rangle = \begin{cases} |\mathbf{k}\rangle & \text{one particle} \\ |\mathbf{k}_1, \mathbf{k}_2\rangle \sqrt{V} & \text{two particles} \end{cases} \tag{3.13}$$

Note the factor of $\sqrt{V}$ in the two particle case. Without this, as V becomes large the probability that the two particles are anywhere near each other goes to zero. From the viewpoint of one particle hitting another, the one particle state is normalised to one (probability 1 of being somewhere in the box), and the two particle state is normalised as a density (think of one particle having probability 1 of being in any unit volume and the second having probability 1 of being somewhere in the box).

The transition probability from i to f is given by (3.9). We want to convert this to the box normalisation. One ingredient of the conversion is the delta function of momentum conservation, arising from,

$$(2\pi)^4\delta^4(P_f - P_i) = \int d^4x\, e^{i(P_f-P_i)\cdot x} = \int_{VT} d^4x\, e^{i(P_f-P_i)\cdot x} = (2\pi)^4\delta^4_{VT}(P_f - P_i),$$

using the box normalisation. Now,

$$\int \frac{d^4p}{(2\pi)^4} \left|(2\pi)^4\delta^4_{VT}(p)\right|^2 = \int_{VT} d^4x\, = VT,$$

so we will say,

$$\left|(2\pi)^4\delta^4_{VT}(p)\right|^2 \simeq VT\,(2\pi)^4\delta^4(p).$$

The second ingredient is a factor of $1/(2E_iV)^{1/2}$ for every particle in the initial or final state (here I am using E_i synonymously with ω_{k_i}). This comes from converting between relativistic and box normalisations for the states.

To see where this arises from we write here the expression for a free field expanded in terms of annihilation and creation operators using three different normalisations: nonrelativistic, $\langle \mathbf{k}|\mathbf{k}'\rangle = \delta^3(\mathbf{k}-\mathbf{k}')$; relativistic, $\langle k|k'\rangle = (2\pi)^3 2\omega_k \delta^3(\mathbf{k}-\mathbf{k}')$; box, $\langle \mathbf{k}|\mathbf{k}'\rangle = \delta_{\mathbf{k}\mathbf{k}'}$.

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[a_{\mathbf{k}} e^{-ik\cdot x} + a^\dagger_{\mathbf{k}} e^{ik\cdot x}\right] && \text{nonrelativistic} \\ &= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \left[a(k) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x}\right] && \text{relativistic} \\ &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_k}\sqrt{V}} \left[a_{\mathbf{k}} e^{-ik\cdot x} + a^\dagger_{\mathbf{k}} e^{ik\cdot x}\right] && \text{box} \end{aligned}$$

Since the discrete sum on $\mathbf{k}$ in the box case corresponds to $\int d^3\mathbf{k}\, V/(2\pi)^3$, we see that,

$$|k\rangle_{\text{rel}} \longleftrightarrow \sqrt{2\omega_k}\sqrt{V}|\mathbf{k}\rangle_{\text{box}}.$$

The box states are normalised to one particle in volume V and the relativistic states have $2\omega_k$ particles per unit volume.

So in the box normalisation, with one or two particles in the initial state and any number in the final state,

$$\text{box amp} = i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i)\prod_{\text{out}}\left[\frac{1}{\sqrt{2E_f V}}\right]\prod_{\text{in}}\left[\frac{1}{\sqrt{2E_i}}\right]\frac{1}{\sqrt{V}},$$

where the initial state energy product depends on the choice of normalisation in equation 3.13 above. The squared matrix element is thus:

$$|\text{box amp}|^2 = |\mathcal{M}_{fi}|^2 T(2\pi)^4\delta^4(P_f - P_i)\prod_{\text{out}}\left[\frac{1}{2E_f V}\right]\prod_{\text{in}}\left[\frac{1}{2E_i}\right],$$

and the differential transition probability into a region of phase space becomes,

$$\frac{\text{differential prob}}{\text{unit time}} = S\,|\mathcal{M}_{fi}|^2\prod_{\text{in}}\left[\frac{1}{2E_i}\right]\times\begin{pmatrix}\text{relativistic density}\\ \text{of final states}\end{pmatrix}, \tag{3.14}$$

where the *relativistic density of final states*, or rdfs, is,

$$\text{rdfs} = D \equiv (2\pi)^4\delta^4(P_f - P_i)\prod_{\text{out}}\frac{d^3\mathbf{k}_f}{(2\pi)^3\,2E_f}. \tag{3.15}$$

You also sometimes hear the name LIPS, standing for Lorentz invariant phase space. Observe that everything in the transition probability is Lorentz invariant save for the initial energy factor (using $d^3k\,/2E = d^4k\,\delta^4(k^2 - m^2)\theta(k^0)$, which is manifestly Lorentz invariant, where $E = (\mathbf{k}^2 + m^2)^{1/2}$). I have smuggled in one extra factor, S, in equation (3.14) for the transition probability. If there are some identical particles in the final state, we will overcount them when integrating over all momentum configurations. The symmetry factor S takes care of this. If there n_i identical particles of type i in the final state, then

$$S = \prod_i \frac{1}{n_i!}. \tag{3.16}$$

▷ **Exercise 3.3**
Show that the expression for two-body phase space in the centre of mass frame is given by

$$\frac{d^3k_1}{(2\pi)^3\,2\omega_{k_1}}\frac{d^3k_2}{(2\pi)^3\,2\omega_{k_2}}(2\pi)^4\delta^4(P - k_1 - k_2) = \frac{1}{32\pi^2 s}\lambda^{1/2}(s, m_1^2, m_2^2)d\Omega^*, \tag{3.17}$$

where $s = P^2$ is the centre of mass energy squared, $d\Omega^*$ is the solid angle element for the angle of one of the outgoing particles with respect to some fixed direction, and

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \tag{3.18}$$

3.3 Cross Sections

The cross section for two particles to scatter is a sum of the differential cross sections for scattering into distinct final states:

$$d\sigma = \frac{\text{transition prob}}{\text{unit time}\times\text{unit flux}} = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{4E_1E_2}S|\mathcal{M}_{fi}|^2 D, \tag{3.19}$$

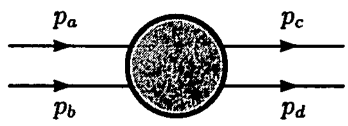


Figure 3.2 $2 \to 2$ scattering.

where the velocities in the flux factor, $1/|\vec{v}_1 - \vec{v}_2|$, are subtracted *nonrelativistically.* I denote them with arrows to remind you that they are ordinary velocities, not the spatial parts of 4-velocities. The amplitude-squared and phase space factors are manifestly Lorentz invariant. What about the initial velocity and energy factors? Observe that

$$E_1 E_2(\vec{v}_1 - \vec{v}_2) = E_2\mathbf{p}_1 - E_1\mathbf{p}_2.$$

In a frame where $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear,

$$|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|^2 = (p_1{\cdot}p_2)^2 - m_1^2 m_2^2,$$

and the last expression is manifestly Lorentz invariant. Hence the differential cross section is Lorentz invariant, as is the total cross section,

$$\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{1}{4E_1E_2} S \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 D. \tag{3.20}$$

3.3.1 Two-body Scattering

An important special case is $2 \to 2$ scattering (see Figure 3.2),

$$a(p_a) + b(p_b) \to c(p_c) + d(p_d).$$

▷ **Exercise 3.4**

Show that in the centre of mass frame the differential cross section is,

$$\frac{d\sigma}{d\Omega^*} = \frac{S\,\lambda^{1/2}(s, m_c^2, m_d^2)}{64\pi^2 s\,\lambda^{1/2}(s, m_a^2, m_b^2)}|\mathcal{M}_{fi}|^2. \tag{3.21}$$

The result of equation (3.21) is valid for any $|\mathcal{M}_{fi}|^2$, but if $|\mathcal{M}_{fi}|^2$ is a constant you can trivially get the total cross section.

Invariant $2 \to 2$ scattering amplitudes are frequently expressed in terms of the *Mandelstam variables*, defined by,

$$\begin{aligned} s &\equiv (p_a + p_b)^2 &= (p_c + p_d)^2 \\ t &\equiv (p_a - p_c)^2 &= (p_b - p_d)^2 \\ u &\equiv (p_a - p_d)^2 &= (p_b - p_c)^2 \end{aligned} \tag{3.22}$$

In fact there are only two independent Lorentz invariant combinations of the available momenta in this case, so there must be some relation between s, t and u.

▷ **Exercise 3.5**
Show that

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2.$$

▷ **Exercise 3.6**
Show that for two body scattering of particles of equal mass m,

$$s \geq 4m^2, \qquad t \leq 0, \qquad u \leq 0.$$

3.4 Decay Rates

With one particle in the initial state,

$$\frac{\text{total decay prob}}{\text{unit time}} = \frac{1}{2E} S \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 D.$$

Only the factor $1/2E$ is not manifestly Lorentz invariant. In the rest frame, for a particle of mass m,

$$\left.\frac{\text{tdp}}{\text{ut}}\right|_{\text{rest frame}} \equiv \Gamma \equiv \frac{1}{2m} \sum \int_{\text{final states}} |\mathcal{M}_{fi}|^2 D. \tag{3.23}$$

This is the "decay rate." In an arbitrary frame we find, $(\text{tdp}/\text{ut}) = (m/E)\Gamma$, which has the expected Lorentz dilatation factor. In the master formula (equation 3.14) this is what the product of $1/2E_i$ factors for the initial particles does.

3.5 Optical Theorem

When discussing the Golden Rule, we encountered the evolution operator $U(t', t)$, which you also met in the field theory course. This takes a state at time t and evolves it to time t'. The scattering amplitudes we calculate in field theory are between states in the far past and the far future: hence they are matrix elements of $U(\infty, -\infty)$, which is known as the S-matrix,

$$S \equiv U(\infty, -\infty) = \text{T} \exp -i \int_{-\infty}^{\infty} dt\, H_I(t).$$

Since the S-matrix is unitary, we can write,

$$(S - I)(S^\dagger - I) = -\Big((S - I) + (S - I)^\dagger\Big). \tag{3.24}$$

Note that $S - I$ is the quantity of interest, since we generally ignore cases where there is no interaction (the "I" piece of S). In terms of the invariant amplitude,

$$\begin{aligned} \langle f|\, S - I\, |i\rangle &= i\mathcal{M}_{fi}(2\pi)^4 \delta^4(P_f - P_i) \\ \langle f|\, (S - I)^\dagger\, |i\rangle &= -i\mathcal{M}^*_{if}(2\pi)^4 \delta^4(P_f - P_i) \end{aligned}$$

Sandwiching the above unitarity relation (equation 3.24) between states $|i\rangle$ and $|f\rangle$, and inserting a complete set of states between the factors on the left hand side,

$$\begin{aligned} &\sum_m \langle f|\, S - I\, |m\rangle \, \langle m|\, S^\dagger - I\, |i\rangle \\ &= \sum_m \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^8 \delta^4(P_f - P_m)\delta^4(P_i - P_m) \prod_{j=1}^{r_m} \frac{d^3\mathbf{k}_j}{(2\pi)^3\, 2E_j} \\ &= \sum_m \int \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^4 \delta^4(P_f - P_i) D_m \end{aligned}$$

where D_m is the phase space factor for the state labelled by m, containing r_m particles, $D_m \equiv D_{r_m}(P_i; k_1, \ldots, k_{r_m})$. Hence,

$$\sum_m \int \mathcal{M}_{fm} \mathcal{M}^*_{im} D_m = i(\mathcal{M}^*_{if} - \mathcal{M}_{fi}).$$

If the intermediate state m contains n_i identical particles of type i, there is an extra symmetry factor S, with,

$$S = \prod_i \frac{1}{n_i!}$$

on the left hand side of the above equation to avoid overcounting. The same factor (see equation 3.16) appears in the cross section formula (equation 3.19) when some of the final state particles are identical.

If $|i\rangle$ and $|f\rangle$ are the same two particle state,

$$4E_T p_i \, \sigma = 2 \, \mathrm{Im} \, \mathcal{M}_{ii}. \tag{3.25}$$

this is the *optical theorem*, relating the forward part of the scattering amplitude to the total cross section. If particles of masses m_1 and m_2 scatter, then $E_T = s^{1/2}$ and $4sp_i^2 = \lambda(s, m_1^2, m_2^2)$, where λ is the function defined in equation (3.18). Then the optical theorem reads, $\mathrm{Im}\, \mathcal{M}_{ii} = \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)\, \sigma$.

4 Quantum Electrodynamics

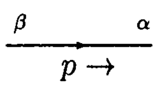
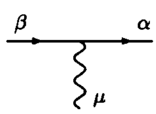
In this section we are going to get some practice calculating cross sections and decay rates in QED. The starting point is the set of Feynman rules derived from the QED Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 + \overline{\psi}(i\not{D} - m)\psi. \tag{4.1}$$

Here, $D_\mu = \partial_\mu + ieA_\mu$ is the electromagnetic covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $(\partial \cdot A)^2/2$ is the gauge fixing term for Feynman gauge. This gives the rules in Table 4.1.

The fermion propagator is (up to factors of i) the inverse of the operator, $\not{p} - m$, which appears in the quadratic term in the fermion fields, as you expect from your field theory course. The derivation of the photon propagator, along with the need for gauge fixing, is also discussed in the field theory course. The external line factors are easily derived by considering simple matrix elements in the operator formalism, where they are left behind from the expansions of fields in terms of annihilation and creation operators, after the operators have all been (anti-)commuted until they annihilate the vacuum. In path integral language the natural objects to compute are Green functions, vacuum expectation values of time ordered products of fields: it takes a little more work to convert them to transition amplitudes and see the external line factors appear.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on

For every ...	draw ...	write ...
Internal photon line		$\dfrac{-ig^{\mu\nu}}{q^2 + i\epsilon}$
Internal fermion line		$\dfrac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$
Vertex		$-ie\gamma^\mu_{\alpha\beta}$
Outgoing electron		$\overline{u}^s_p$
Incoming electron		u^s_p
Outgoing positron		v^s_p
Incoming positron		$\overline{v}^s_p$
Outgoing photon		$\epsilon^{*\mu}$
Incoming photon		ϵ^μ

- Attach a directed momentum to every internal line
- Conserve momentum at every vertex

Table 4.1 Feynman rules for QED. μ, ν are Lorentz indices and α, β are spinor indices.

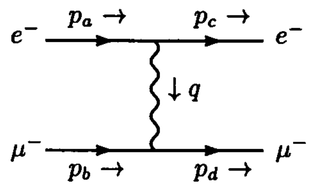


Figure 4.1 Lowest order Feynman diagram for electron–muon scattering.

the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 4.1. This will become clear in the examples which follow. We have already met the Dirac spinors u and v. I will say more about the photon polarisation vector ϵ when we need to use it.

4.1 Electron–Muon Scattering

To lowest order in the electromagnetic coupling, just one diagram contributes to this process. It is shown in Figure 4.1. The amplitude obtained from this diagram is

$$i\mathcal{M}_{fi} = (-ie)\,\overline{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)\,\overline{u}(p_d)\gamma_\nu u(p_b). \tag{4.2}$$

Note that I have changed my notation for the spinors: now I label their momentum as an argument instead of as a subscript, and I drop the spin label unless I need to use it. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication.

The cross-section involves the squared modulus of the amplitude, which is

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\,L^{\mu\nu}_{(e)}L_{(\mu)\,\mu\nu},$$

where the subscripts e and μ refer to the electron and muon respectively and,

$$L^{\mu\nu}_{(e)} = \overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

▷**Exercise 4.1**
Verify the expression for $|\mathcal{M}_{fi}|^2$.

Usually we have an unpolarised beam and target and do not measure the polarisation of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different possibilities are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the following results (I temporarily restore spin labels on spinors):

$$\begin{aligned}\sum_r u^r(p)\,\overline{u}^r(p) &= \not{p} + m \\ \sum_r v^r(p)\,\overline{v}^r(p) &= \not{p} - m\end{aligned} \tag{4.3}$$

▷ **Exercise 4.2**
Derive the spin sum relations in equation (4.3).

Using the spin sums we find,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{e^4}{4q^4}\,\mathrm{tr}\left(\gamma^\mu(\not{p}_a+m_e)\gamma^\nu(\not{p}_c+m_e)\right)\mathrm{tr}\left(\gamma_\mu(\not{p}_b+m_\mu)\gamma_\nu(\not{p}_d+m_\mu)\right).$$

Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of gamma matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook [1]. All these theorems can be derived from the fundamental anticommutation relations of the gamma matrices in equation (2.23) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use,

$$\begin{aligned}\mathrm{tr}(\not{a}\not{b}) &= 4\,a\cdot b \\ \mathrm{tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a\cdot b\,c\cdot d - a\cdot c\,b\cdot d + a\cdot d\,b\cdot c) \\ \mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd}\end{aligned} \tag{4.4}$$

▷ **Exercise 4.3**
Derive the trace results in equation (4.4)

Using these results, and expressing the answer in terms of the Mandelstam variables of equation (3.22), we find,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{2e^4}{t^2}\left(s^2+u^2-4(m_e^2+m_\mu^2)(s+u)+6(m_e^2+m_\mu^2)^2\right).$$

This can now be used in the $2 \to 2$ cross section formula (3.21) to give, in the high energy limit, $s, u \gg m_e^2, m_\mu^2$,

$$\frac{d\sigma}{d\Omega^*} = \frac{e^4}{32\pi^2 s}\,\frac{s^2+u^2}{t^2}. \tag{4.5}$$

for the differential cross section in the centre of mass frame.

▷ **Exercise 4.4**
Derive the result for the electron–muon scattering cross section in equation (4.5).

Other calculations of cross sections or decay rates will follow the same steps we have used above. You draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more wrinkles to be aware of, which we will meet below.

4.2 Electron–Electron Scattering

Since the two scattered particles are now identical, you can't just replace m_μ by m_e in the calculation we did above. If you look at the diagram of Figure 4.1 (with the muons replaced by electrons) you will see that the outgoing legs can be labelled in two ways. Hence we get the two diagrams of Figure 4.2.

The two diagrams give the amplitudes,

$$\begin{aligned} i\mathcal{M}_1 &= \frac{ie^2}{t}\overline{u}(p_c)\gamma^\mu u(p_a)\overline{u}(p_d)\gamma_\mu u(p_b), \\ i\mathcal{M}_2 &= -\frac{ie^2}{u}\overline{u}(p_d)\gamma^\mu u(p_a)\overline{u}(p_c)\gamma_\mu u(p_b). \end{aligned}$$

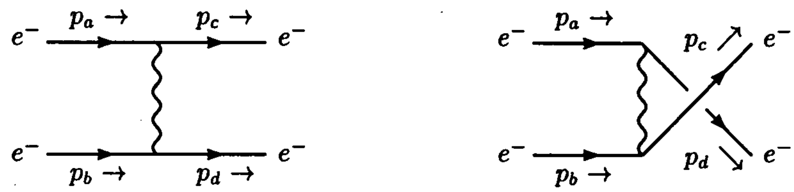


Figure 4.2 Lowest order Feynman diagrams for electron–electron scattering.

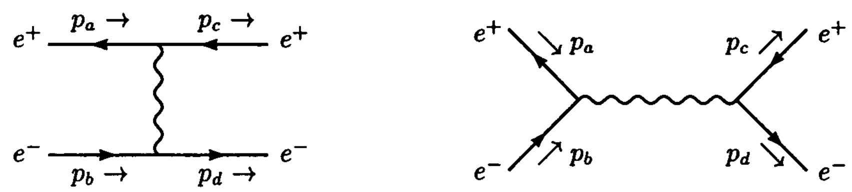


Figure 4.3 Lowest order Feynman diagrams for electron–positron scattering in QED.

Notice the additional minus sign in the second amplitude, which comes from the anti-commuting nature of fermion fields. You should accept as part of the Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$|\mathcal{M}_{fi}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2,$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams.

4.3 Electron–Positron Annihilation

4.3.1 $e^+e^- \to e^+e^-$

For this process the two diagrams are shown in Figure 4.3, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in Figure 4.2 if you twist round the fermion lines. The fact that the diagrams are related this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing.* This is a case where the general results of crossing symmetry can be applied, and our diagrammatic calculations give an explicit realisation. Theorists spent a great deal of time studying such general properties of amplitudes in the 1960's when quantum field theory was unfashionable.

4.3.2 $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons

If electrons and positrons collide and produce muon–antimuon or quark-antiquark pairs, then the annihilation diagram is the only one which contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$,

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{4.6}$$

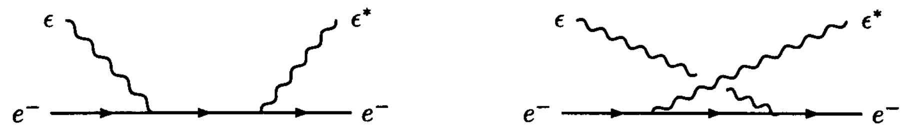


Figure 4.4 Feynman diagrams for Compton scattering.

where the sum is over quark flavours f and Q_f is the quark's charge in units of e. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of R as your e^+e^- collider's CM energy rose through a threshold for producing a new quark flavour. If you didn't know about colour, the height of the step would seem too large. Incidentally, another place the number of colours enters is in the decay of a π^0 to two photons. There is a factor of 3 in the amplitude from summing over colours, without which the predicted decay rate would be one ninth of its real size.

At the energies used today at LEP, of course, you have to remember the diagram with a Z replacing the photon. We will say some more about this later.

▷ **Exercise 4.5**
Show that the cross-section for $e^+e^- \to \mu^+\mu^-$ is equal to $4\pi\alpha^2/(3s)$, neglecting the lepton masses.

4.4 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in Figure 4.4. For unpolarised initial and/or final states, the cross section calculation involves terms of the form

$$\sum_\lambda \epsilon_\lambda^{*\mu}(p)\,\epsilon_\lambda^\nu(p), \tag{4.7}$$

where λ represents the polarisation of the photon of momentum p. Since the photon is massless, the sum is over the two transverse polarisation states, and must vanish when contracted with p_μ or p_ν. In addition, however, since the photon is coupled to the electromagnetic current $J^\mu = \overline{\psi}\gamma^\mu\psi$ of equation (2.5), any term in the polarisation sum (4.7) proportional to p^μ or p^ν does not contribute to the cross section. This is because the current is conserved, $\partial_\mu J^\mu = 0$, so in momentum space $p_\mu J^\mu = 0$. The upshot is that in calculations you can use,

$$\sum_\lambda \epsilon_\lambda^{*\mu}(p)\,\epsilon_\lambda^\nu(p) = -g^{\mu\nu}, \tag{4.8}$$

since the remaining terms on the right hand side do not contribute.

4.5 Form Factors

So far we have considered processes where the strong interactions were absent, or ignored. There are many electroweak processes where a complete computation would require a better understanding of QCD, especially its non-perturbative aspects, than we currently have. However, by using Lorentz and gauge invariance, and any other known symmetries

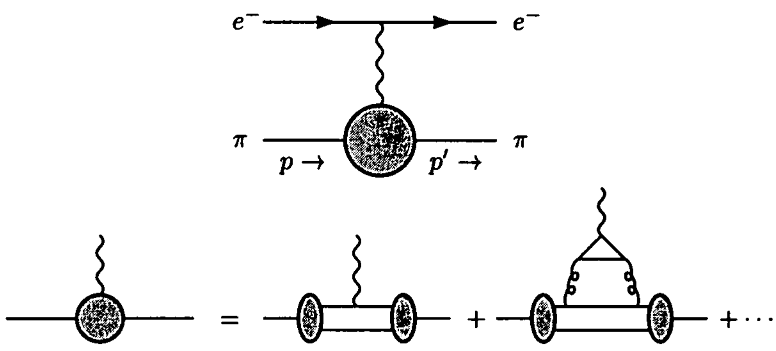


Figure 4.5 Electron–pion scattering (top diagram) and some contributions to the pion electromagnetic form factor (lower diagrams). Wavy lines denote photons and curly lines are gluons. Ordinary lines between the shaded ellipses denote quarks.

of a process, we can parcel up the strong interaction effects in a small number of invariant functions. Let's see how this goes in an example, the electromagnetic form factors of pions and nucleons.

4.5.1 Pion Form Factor

Consider electron–pion scattering as depicted in the top diagram in Figure 4.5. The shaded blob represents all the strong interaction effects in the pion electromagnetic form factor. In the lower part of the figure are represented some contributions to the shaded blob. Note that the blob itself contains more blobs (the shaded ellipses) indicating the unknown wavefunction of the pion in terms of quarks. The electron's coupling to the photon is understood in QED and has been discussed above. Let's see how much we can say about the pion's coupling to photons. This coupling is given by the matrix element $\langle\pi(p')|\, J^\mu(0)\,|\pi(p)\rangle$, where $J^\mu(0)$ is the electromagnetic current at the origin. Using Lorentz covariance we can write,

$$\langle\pi(p')|\, J^\mu(0)\,|\pi(p)\rangle = e\left[F(q^2)(p+p')^\mu + G(q^2)q^\mu\right],$$

where $q = p - p'$. Electromagnetic gauge invariance implies that $q_\mu J^\mu = 0$ so that $G(q^2) = 0$. Hence all the strong interaction effects are contained in $F(q^2)$ and

$$\langle\pi(p')|\, J^\mu(0)\,|\pi(p)\rangle = e\,F(q^2)(p+p')^\mu. \tag{4.9}$$

▷**Exercise 4.6**
Starting from the kinetic term in the Lagrangian for a free charged scalar field, $\partial_\mu\phi^*\partial^\mu\phi$, and introducing the electromagnetic field by minimal substitution, $\partial_\mu \to \partial_\mu - ieA_\mu$, show that, to lowest order in perturbation theory $F(q^2) = 1$ for all q^2. Note that the change of sign in the coupling compared to QED is because QED involves the negatively charged electron, whilst here ϕ is taken as the field which destroys positively charged objects and creates negatively charged ones. You may need to normal order the current.

An additional general piece of information is that $F(0) = 1$ since at $q^2 = 0$ the photon cannot resolve the structure of the pion. This result is a consequence of the conservation of the electromagnetic current, since the space integral of J^0 gives the charge operator. For $q^2 \neq 0$ we expect $F(q^2)$ to fall with q^2 owing to the pion's composite nature.

▷ **Exercise 4.7**
Given that the electric charge operator is defined by

$$eQ = \int d^3x\, J^0(x),$$

show that current conservation implies Q is time independent, and that $F(0) = 1$ for a positively charged pion.

4.5.2 Nucleon Form Factor

For nucleons there are two form factors consistent with Lorentz covariance, current conservation and parity conservation (which holds for electromagnetic and strong interactions). They are defined as follows (again we are working to first order in electromagnetism):

$$\langle N(p', s')|\, J^\mu \,|N(p, s)\rangle = e\,\overline{u}^{s'}(p') \left[\gamma^\mu F_1(q^2) + \frac{i\kappa}{2M} F_2(q^2)\sigma^{\mu\nu} q_\nu\right] u^s(p), \qquad (4.10)$$

where u and $\overline{u}$ are the nucleon spinors, and M the nucleon mass. At zero momentum transfer only the first term contributes and $F_1(0) = 1[0]$ for the proton[neutron]. The factor κ is chosen so that $F_2(0) = 1$: κ is 1.79 for the proton and -1.91 for the neutron. In writing the expression (4.10), use is made of the *Gordon identity*,

$$\overline{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\overline{u}(p')\left[(p + p')^\mu + i\sigma^{\mu\nu}(p' - p)_\nu\right]u(p),$$

to replace a term in $(p + p')^\mu$ with terms of the form given. Given the form factor expression you can compute the angular distribution of electrons in electron–nucleon scattering in terms of F_1 and F_2.

▷ **Exercise 4.8**
Use Lorentz covariance, current conservation and parity invariance to show that there are two electromagnetic form factors for the nucleon in (4.10).

5 Quantum Chromodynamics

In the 1960's most theorists lost interest in quantum field theory. They were discouraged by the apparent non renormalisability of massive vector boson theories which precluded a field theory description of weak interactions. For the strong interactions, their strength and the menagerie of hadrons seemed also to preclude a field theory description. The renaissance of field theory came with the realisation that spontaneous symmetry breaking, the Higgs mechanism and the property of asymptotic freedom made renormalisable gauge theories viable candidates to describe the electroweak and strong interactions.

Our discussion in this section will lead to the property of *asymptotic freedom* which enables us to make phenomenological predictions using perturbation theory for QCD. Since perturbative calculations beyond tree level are not in the scope of this course, the discussion will necessarily be somewhat qualitative. We'll proceed by going back to QED to introduce the idea of renormalisation then work up to the running coupling in QCD and thence to asymptotic freedom.

QCD is a theory of interactions between spin-1/2 quarks and spin-1 gluons. It is a nonabelian gauge theory based on the group $SU(3)$, with Lagrangian,

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f + \begin{array}{l}\text{gauge fixing and}\\ \text{ghost terms}\end{array} \tag{5.1}$$

Here, a is a colour label, taking values from 1 to 8 for $SU(3)$, and f runs over the quark flavours. The covariant derivative and field strength tensor are given by,

$$\begin{aligned} D_\mu &= \partial_\mu - igA^a_\mu T^a, \\ G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \end{aligned} \tag{5.2}$$

where the f^{abc} are the structure constants of $SU(3)$ and the T^a are a set of eight independent Hermitian traceless 3×3 matrix generators in the fundamental or defining representation (see the pre school problems and the quantum field theory course).

As in QED gauge fixing terms are needed to define the propagator and ensure that only physical degrees of freedom propagate. The gauge fixing procedure is more complicated in the nonabelian case and necessitates, for certain gauge choices, the appearance of Faddeev–Popov ghosts to cancel the contributions from unphysical polarisation states in gluon propagators. However, the ghosts first appear in loop diagrams, which we will not compute in this course.

There are no Higgs bosons in pure QCD. The only relic of them is in the masses for the fermions which are generated via the Higgs mechanism, but in the electroweak sector of the standard model.

A fundamental difference between QCD and QED is the appearance in the nonabelian case of interaction terms (vertices) containing gluons alone. These arise from the nonvanishing commutator term in the field strength of the nonabelian theory in equation (5.2). The photon is electrically neutral, but the gluons carry the colour charge of QCD (specifically, they transform in the adjoint representation). Since the force carriers couple to the corresponding charge, there are no multi photon vertices in QED but there are multi gluon couplings in QCD. This difference is crucial: it is what underlies the decreasing strength of the strong coupling with increasing energy scale.

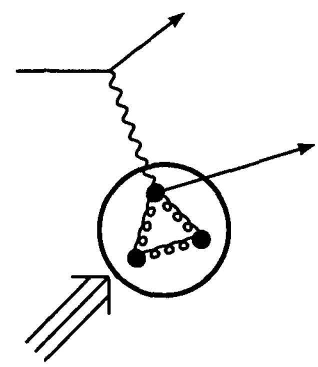


Figure 5.1 Schematic depiction of deep inelastic scattering. An incident lepton radiates a photon which knocks a quark out of a proton. The struck quark is detected indirectly only after hadronisation into observable particles.

In QCD, hadrons are made from quarks. Colour interactions bind the quarks, producing states with no net colour: three quarks combine to make baryons and quark–antiquark pairs give mesons. It is generally believed that the binding energy of a quark in a hadron is infinite. This property, called *confinement*, means that there is no such thing as a free quark. Because of asymptotic freedom, however, if you hit a quark with a high energy projectile it will behave in many ways as a free (almost) particle. For example, in deep inelastic scattering, or DIS, a photon strikes a quark in a proton, say, imparting a large momentum to it. Some strong interaction corrections to this part of the process can be calculated perturbatively. As the quark heads off out of the proton, however, the brown muck of myriad low energy strong interactions cuts in again and "hadronises" the quark into the particles you actually detect. This is illustrated schematically in Figure 5.1.

5.1 Renormalisation: An Introduction

5.1.1 Renormalisation in Quantum Electrodynamics

Let's start by going back to QED and considering how the electric charge is defined and measured. This will bring up the question of what happens when you try to compute higher loop corrections. In fact, the expansion in the number of loops is an expansion in Planck's constant $\hbar$, as you can show if you put back the factors of $\hbar$ for once.

The electric charge $\hat{e}$ is usually defined as the coupling between an on-shell electron and an on-shell photon: that is, as the vertex on the left hand side of Figure 5.2 with $p_1^2 = p_2^2 = m^2$, where m is the electron mass, and $q^2 = 0$. It is $\hat{e}$ and not the Lagrangian parameter e which we measure. That is,

$$\frac{\hat{e}^2}{4\pi} = \frac{1}{137}.$$

We call $\hat{e}$ the renormalised coupling constant of QED. We can calculate $\hat{e}$ in terms of e in perturbation theory. To one loop, the relevant diagrams are shown on the right hand side of Figure 5.2, and the result takes the form,

$$\hat{e} = e + e^3\Big[a_1 \ln\frac{M^2}{m^2} + b_1\Big] + \cdots \tag{5.3}$$

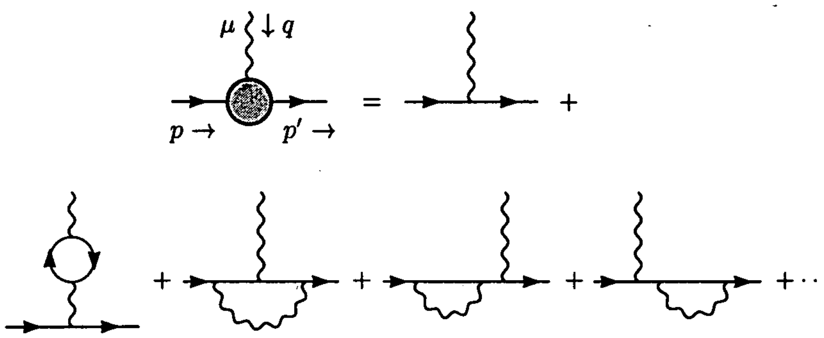


Figure 5.2 Diagrams for vertex renormalisation in QED up to one loop.

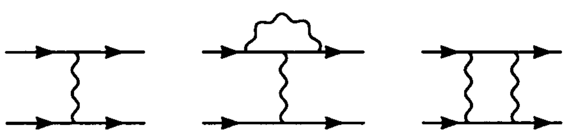


Figure 5.3 Some diagrams for electron–electron scattering in QED up to one loop.

where a_1 and b_1 are constants obtained from the calculation. The e^3 term is divergent, so we have introduced a cutoff M to regulate it. This is called an ultraviolet divergence since it arises from the propagation of high momentum modes in the loops. The cutoff amounts to selecting only those modes where each component of momentum is less than M in magnitude. Despite the divergence in (5.3), it still relates the measurable quantity $\hat{e}$ to the coupling e we introduced in our theory. This implies that e itself must be divergent. The property of renormalisability ensures that in any relation between physical quantities the ultraviolet divergences cancel: the relation is actually independent of the method used to regulate divergences.

As an example, consider the amplitude for electron–electron scattering, which we considered at tree level in section 4.2. Some of the contributing diagrams are shown in Figure 5.3, where the crossed diagrams are understood (we showed the crossed tree level diagram explicitly in Figure 4.2). Ultraviolet divergences are again encountered when the diagrams are evaluated, and the result is of the form,

$$i\mathcal{M}_{fi} = c_0 e^2 + e^4\Big[c_1 \ln\frac{M^2}{m^2} + d_1\Big] + \cdots \tag{5.4}$$

where c_0, c_1 and d_1 are constants, determined by the calculation. In order to evaluate $\mathcal{M}_{fi}$ numerically, however, we must express it in terms of the known parameter $\hat{e}$. Combining (5.3) and (5.4) yields,

$$i\mathcal{M}_{fi} = c_0\hat{e}^2 + \hat{e}^4\Big[(c_1 - 2a_1c_0)\ln\frac{M^2}{m^2} + d_1 - 2b_1c_0\Big] + \cdots \tag{5.5}$$

where the ellipsis denotes terms of order $\hat{e}^6$ and above. Since $|\mathcal{M}_{fi}|^2$ is measurable, consistency (renormalisability) requires,

$$c_1 = 2a_1c_0.$$

This result is indeed borne out by the actual calculations, and the relation between $\mathcal{M}_{fi}$ and $\hat{e}$ contains no divergences:

$$i\mathcal{M}_{fi} = c_0\hat{e}^2 + \hat{e}^4(d_1 - 2b_1c_0) + \mathcal{O}(\hat{e}^6). \tag{5.6}$$

To understand how this cancellation of divergences happened we can study the convergence properties of loop diagrams (although we shall not evaluate them). Consider the third diagram on the right hand side in Figure 5.2 and the middle diagram in Figure 5.3. These both contain a loop with one photon propagator, behaving like $1/k^2$ at large momentum k, and two electron propagators, each behaving like $1/k$. To evaluate the diagram we have to integrate over all momenta, leading to an integral,

$$I \sim \int_{\text{large } k} \frac{d^4k}{k^4}, \tag{5.7}$$

which diverges logarithmically, leading to the $\ln M^2$ terms in (5.3) and (5.4). Notice, however, that the divergent terms in these two diagrams must be the same, since the divergence is by its nature independent of the finite external momenta (the factor of two in equation (5.5) arises because there is a divergence associated with the coupling of each electron in the scattering process). In this way we can understand that at least some of the divergences are common in both (5.3) and (5.4). What about diagrams such as the third box-like one in Figure 5.3? Now we have two photon and two electron propagators, leading to,

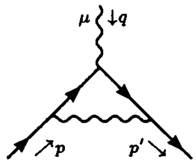
$$I \sim \int_{\text{large } k} \frac{d^4k}{k^6}.$$

This time the integral is convergent.

Detailed study like this reveals that ultraviolet divergences always disappear in relations between physically measurable quantities. We discussed above the definition of the physical electric charge $\hat{e}$. A similar argument applies for the electron mass: the Lagrangian bare mass parameter m is divergent, but we can define a finite physical mass $\hat{m}$.

In fact you find that all ultraviolet divergences in QED stem from graphs of the type shown in Figure 5.4 and known as the *primitive divergences.* Any divergent graph will be found on inspection to contain a divergent subgraph of one of these basic types. For example, Figure 5.5 shows a graph where the divergence comes from the primitive divergent subgraph inside the dashed box. Furthermore, the primitive divergences are always of a type that would be generated by a term in the initial Lagrangian with a divergent coefficient. Hence by rescaling the fields, masses and couplings in the original Lagrangian we can make all physical quantities finite (and independent of the exact details of the adjustment such as how we regulate the divergent integrals). This is what we mean by renormalisability.

This should be made clearer by an example. Consider calculating the vertex correction in QED to one loop,



$$= \overline{u}(p')\Big[A\gamma^\mu + B\sigma^{\mu\nu}q_\nu + Cq^2\gamma^\mu + \cdots\Big]u(p).$$

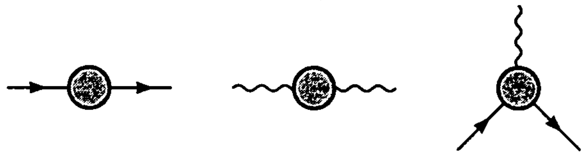


Figure 5.4 Primitive divergences of QED.

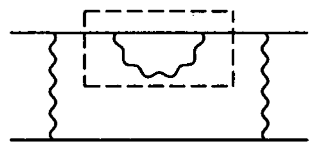


Figure 5.5 Diagram containing a primitive divergence.

The calculation shows that A is divergent. However, we can absorb this by adding a cancelling divergent coefficient to the $\overline{\psi}\not{A}\psi$ term in the QED Lagrangian (4.1). The B and C terms are finite and unambiguous. This is just as well, since an infinite part of B, for example, would need to be cancelled by an infinite coefficient of a term of the form,

$$\overline{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi,$$

which is not available in (4.1).

In fact, the B term gives the QED correction to the magnetic dipole moment, g, of the electron or muon (see page 160 of the textbook by Itzykson and Zuber [1]). These are predicted to be 2 at tree level. You can do the one-loop calculation (it was first done by Schwinger between September and November 1947 [4]) with a few pages of algebra to find,

$$g = 2\left(1 + \frac{\alpha}{2\pi}\right).$$

This gives $g/2 = 1.001161$, which is already impressive compared to the experimental values [2]:

$$\begin{aligned}(g/2)_{\text{electron}} &= 1.001159652193(10),\\ (g/2)_{\text{muon}} &= 1.001165923(8).\end{aligned}$$

Higher order calculations show that the electron and muon magnetic moments differ at two loops and above. Kinoshita and collaborators have devoted their careers to these calculations and are currently at the four loop level. Theory and experiment agree for the electron up to the 11th decimal place.

The C term gives the splitting between the $2s_{1/2}$ and $2p_{1/2}$ levels of the hydrogen atom, known as the Lamb shift. Bethe's calculation [5] of the Lamb shift, done during a train ride to Schenectady in June 1947, was an early triumph for quantum field theory. Here too, the current agreement between theory and experiment is impressive.

5.1.2 Bare Versus Renormalised

In discussing the vertex correction in QED, we said that the divergent part of the A term could be absorbed by adding a cancelling divergent coefficient to the $\overline{\psi}\not{A}\psi$ term in the

QED Lagrangian (4.1). When a theory is renormalisable, *all* divergences can be removed in this way. Thus, for QED, if the original Lagrangian is (ignoring the gauge-fixing term),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\partial\!\!\!/\psi - e\overline{\psi}A\!\!\!/\psi - m\overline{\psi}\psi,$$

then redefine everything by:

$$\psi = Z_2^{1/2}\psi_R, \qquad A^\mu = Z_3^{1/2}A_R^\mu,$$
$$e = Z_e\hat{e} = \frac{Z_1}{Z_2 Z_3^{1/2}}\,\hat{e}, \qquad m = Z_m\hat{m},$$

where the subscript R stands for "renormalised." In terms of the renormalised fields,

$$\mathcal{L} = -\frac{1}{4}Z_3 F_{R\,\mu\nu}F_R^{\mu\nu} + iZ_2\overline{\psi}_R\partial\!\!\!/\psi_R - Z_1\hat{e}\overline{\psi}_R A\!\!\!/_R\psi_R - Z_m Z_2\hat{m}\overline{\psi}_R\psi_R.$$

Writing each Z as $Z = 1 + \delta Z$, reexpress the Lagrangian one more time as,

$$\mathcal{L} = -\frac{1}{4}F_{R\,\mu\nu}F_R^{\mu\nu} + i\overline{\psi}_R\partial\!\!\!/\psi_R - \hat{e}\overline{\psi}_R A\!\!\!/_R\psi_R - \hat{m}\overline{\psi}_R\psi_R + (\delta Z \text{ terms}).$$

Now it looks like the old lagrangian, but written in terms of the renormalised fields, with the addition of the δZ *counterterms.* Now when you calculate, the counterterms give you new vertices to include in your diagrams. The divergences contained in the counterterms cancel the infinities produced by the loop integrations, leaving a finite answer.

The old A and ψ are called the *bare* fields, and e and m are the bare coupling and mass.

Note that to maintain the original form of $\mathcal{L}$, you want $Z_1 = Z_2$, so that the $\partial\!\!\!/$ and $\hat{e}A\!\!\!/$ terms combine into a covariant derivative term. This relation does hold, and is a consequence of the electromagnetic gauge symmetry: it is known as the *Ward identity.*

5.2 Renormalisation in Quantum Chromodynamics

We now try to repeat the procedure we used for the coupling in QED, but this time in QCD, which is also a renormalisable theory. If we define the renormalised coupling $\hat{g}$ as the strength of the quark–gluon coupling, then in addition to the diagrams of Figure 5.2, with the photons replaced by gluons, there are more diagrams at one loop, shown in Figure 5.6. Looking at the second of these new diagrams, it is ultraviolet divergent (containing a $\ln M^2$ term), but also infrared divergent, since there is no mass to regulate the low momentum modes. In QED all the loop diagrams contain at least one electron propagator and the electron mass provides an infrared cutoff (you still have to worry when the electron is on-shell, but this is not our concern here). In the second diagram of Figure 5.6 there is no quark in the loop. Now suppose we choose to define the renormalised coupling off-shell at some non-zero q^2. The finite value of q^2 provides the infrared regulator and the diagram has a term proportional to $\ln(M^2/q^2)$.

Thus in QCD we can't define a physical coupling constant from an on-shell vertex. This is not really a serious restriction since the QCD coupling is not directly measurable anyway. Now the renormalised coupling depends on how we define it and therefore on

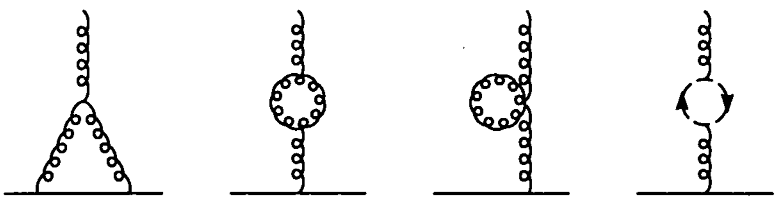


Figure 5.6 Additional diagrams for vertex renormalisation in QCD up to one loop. The dashed line denotes a ghost. For some gauge choices and some regularisation methods not all of these are required.

at least one momentum scale (in almost all practical cases, only one momentum scale). The renormalised strong coupling is thus written,

$$\hat{g}(q^2).$$

When physical quantities are expressed in terms of $\hat{g}(q^2)$ the coefficients of the perturbation series are finite.

It would of course be possible to define the renormalised QED coupling to depend on some momentum scale. However, the on-shell definition used above is a natural one to pick.

You can define counterterms for QCD in the same way as was demonstrated for QED. Now the gauge coupling g enters in many terms where it could get renormalised in different ways. In fact, the gauge symmetry imposes a set of relations between the renormalisation constants, known as the *Slavnov–Taylor* identities, which generalise the Ward identity of QED.

5.3 Asymptotic Freedom

We have just seen that the renormalised coupling in QCD, $\hat{g}(q^2)$, depends on the momentum at which it is defined. We say it depends on the *renormalisation scale*, and commonly refer to $\hat{g}$ as the "running coupling constant." We would clearly like to know just how $\hat{g}$ depends on q^2, so we calculate the diagrams in Figures 5.2 and 5.6, to get the first terms in a perturbation theory expansion:

$$\hat{g}(\mu) = g + g^3\Big[a_1 \ln \frac{M^2}{\mu^2} + b_1\Big] + \cdots \tag{5.8}$$

where a_1 and b_1 are constants and g is the "bare" coupling from the Lagrangian (5.1). I have switched to using μ^2 in place of q^2, and have written $\hat{g}$ as a function of μ for convenience. From this equation it follows that,

$$\mu \frac{\partial \hat{g}}{\partial \mu} \equiv \beta(\hat{g}) = -2a_1\, \hat{g}^3 + \cdots \tag{5.9}$$

The discovery by Politzer and by Gross and Wilczek, in 1973, that $a_1 > 0$ led to the possibility of using perturbation theory for strong interaction processes, since it implies that the strong interactions get weaker at high momentum scales — $\hat{g}(\infty) = 0$ is a stable

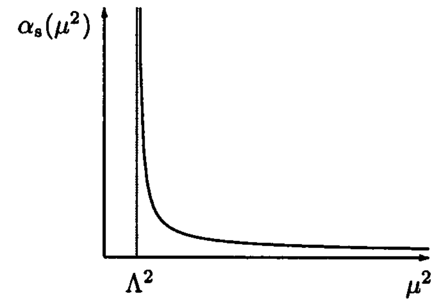


Figure 5.7 Running of the strong coupling constant with renormalisation scale.

solution of the differential equation (5.9). Keeping just the $\hat{g}^3$ term, we can solve (5.9) to find,

$$\alpha_s(\mu) \equiv \frac{\hat{g}^2(\mu)}{4\pi} = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)}, \tag{5.10}$$

where Λ is a constant of integration and $\beta_0 = 32\pi^2 a_1$. Thus $\alpha_s(\mu)$ decreases logarithmically with the scale at which it is renormalised, as shown in Figure 5.7. If for some process the natural renormalisation scale is large, there is a chance that perturbation theory will be applicable. The value of β_0 is,

$$\beta_0 = 11 - \frac{2}{3}n_f, \tag{5.11}$$

where n_f is the number of quark flavours. The crucial discovery when this was first calculated was the appearance of the "11" coming from the self-interactions of the gluons via the extra diagrams of Figure 5.6. Quarks, and other non-gauge particles, always contribute negatively to β_0. Nonabelian gauge theories are the only ones we know where you can have asymptotic freedom (providing you don't have too much "matter" — providing the number of flavours is less than or equal to 16 for QCD).

What is the significance of the integration constant Λ? The original QCD Lagrangian (5.1) contained only a dimensionless bare coupling g (the quark masses don't matter here, since the phenomenon occurs for a pure glue theory), but now we have a dimensionful parameter. The real answer is that the radiative corrections (in all field theories except finite ones) break the scale invariance of the original Lagrangian. In QED there was an implicit choice of scale in the on-shell definition of $\hat{e}$. Lacking such a canonical choice for QCD, you have to say "measure α_s at $\mu = M_Z$" or "find the scale where $\alpha_s = 0.2$," so that a scale is necessarily involved. The phenomenon was called *dimensional transmutation* by Coleman. Λ is given by,

$$\Lambda = \mu \exp\left(- \int^{\hat{g}(\mu)} \frac{dg}{\beta(g)} \right), \tag{5.12}$$

and is μ-independent. The explicit μ dependence is cancelled by the implicit μ dependence of the coupling constant. Today it has become popular to specify the coupling by giving the value of Λ itself.

We've seen that the coupling depends on the scale at which it is renormalised. Moreover, there are many ways of defining the renormalised coupling at a given scale, depending on just how you have regulated the infinities in your calculations and which momentum scales you set equal to μ. The value of $\hat{g}(\mu)$ thus depends on the *renormalisation scheme* you pick, and with it, Λ. In practice, the most popular scheme today is called modified minimal subtraction, $\overline{\text{MS}}$, in which integrals are evaluated in $4-\epsilon$ dimensions and divergences show up as poles of the form ϵ^{-n} for positive integer n. In the particle data book [2] you will find values quoted for $\Lambda_{\overline{\text{MS}}}$ around 200 MeV (it also depends on the number of quark flavours). Don't buy a value of Λ unless you know which renormalisation scheme was used to define it.

In Figure 5.7 you see that the coupling blows up at $\mu = \Lambda$. This is an artifact of using perturbation theory. We can't trust our calculations if $\alpha_s(\mu) > 1$. In practice, you can perhaps use scales for μ down to about 1 GeV, but not much lower, and 2 GeV is probably safer. This region is a murky area where people try to match perturbative calculations onto results obtained from a variety of more or less kosher techniques.

▷ **Exercise 5.1**

Extending the expansion of $\hat{g}$ in terms of g in (5.8) to two loops gives

$$\hat{g}(\mu) = g + g^3\Big[a_1 \ln\frac{M^2}{\mu^2} + b_1\Big] + g^5\Big[a_2 \ln^2\frac{M^2}{\mu^2} + b_2 \ln\frac{M^2}{\mu^2} + c_2\Big],$$

with a similar equation for $\hat{g}(\mu_0)$ in terms of g. Renormalisability implies that $\hat{g}(\mu)$ can be expanded in terms of $\hat{g}(\mu_0)$,

$$\hat{g}(\mu) = \sum_{n=0}^{\infty} \hat{g}^{2n+1}(\mu_0) X_n,$$

where the X_n are finite coefficients. Show that this implies that a_2 is determined once the one loop coefficient a_1 is known. In fact a_1 determines all the terms $(\alpha_s \ln\mu)^n$, called the leading logarithms: from a one loop calculation, you can sum up all the leading logarithms.

For QED there is no positive contribution to the beta function, so the electromagnetic coupling has a logarithmic increase with renormalisation scale. However the effect is small even going up to LEP energies: α goes from 1/137 to about 1/128. The so called Landau pole, where α blows up, is safely hidden at an enormous energy scale.

5.4 Applications

In this section we will briefly consider some places where perturbative QCD can be applied.

5.4.1 $e^+e^- \to$ hadrons

In section 4.3.2 we considered the ratio R of the annihilation cross section for e^+e^- into hadrons to that into $\mu^+\mu^-$. The result we found from the lowest order annihilation diagram proceeding via an intermediate virtual photon was,

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{5.13}$$

where I remind you that sum is over quarks f with Q_f the quark's charge in units of e. Now I would like to extend the discussion in two ways: QED and QCD corrections, and contributions of intermediate Z bosons.

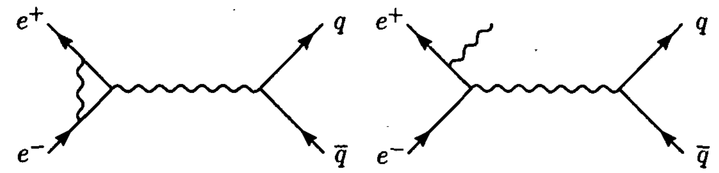


Figure 5.8 QED radiative corrections in e^+e^- annihilation.

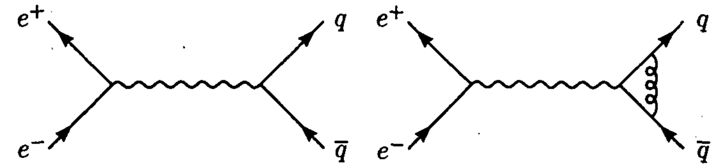


Figure 5.9 QCD radiative corrections in e^+e^- annihilation.

Turning first to QED corrections, consider the two diagrams in Figure 5.8 illustrating two possibilities. The graph on the left contributes to the order α correction to the amplitude. It is ultraviolet divergent, but we have discussed above how to deal with this. However, it is also infrared divergent when the momentum of the photon in the loop goes to zero. The treatment of this problem involves a cancellation of divergences between this graph and the bremsstrahlung diagram on the right of Figure 5.8. Physically, limited detector resolution means you can't tell if the final state you detect is accompanied by one (or infinitely many) very soft photons. So, the rate you calculate should also include these undetected photons, and in summing all the terms, the infrared divergences disappear. Since quarks have electric charge, we can also, of course, have QED corrections where the photon lines connect to the quark legs of the annihilation diagram

For the strong interactions, if α_s is not too large and we aren't near a hadronic resonance, then we expect that calculating the diagrams in Figure 5.9 will give the leading QCD corrections. The gluon is exchanged only between the quarks since the incoming e^+e^- don't feel the strong force. The result of the computation is

$$R = 3\sum_f Q_f^2\Big(1 + \frac{\alpha_s(\mu)}{\pi} + \cdots\Big).$$

What value should we choose for μ in this expression? To answer this you need to know that higher order terms in the perturbation series contain powers of $\ln(s/\mu^2)$, where s is the square of the centre of mass energy. So, to avoid large coefficients in the higher order terms, the preferred choice is $\mu^2 \sim s$. Observe that the leading order graph predicts a back-to-back $q\bar{q}$ pair. Owing to hadronisation, what we actually see is a pair of back-to-back jets. Experimentally, the jets follow the angular distribution predicted for the underlying $q\bar{q}$ process, that is, a $(1 + \cos^2\theta)$ distribution, where θ is the scattering angle in the centre of mass frame. Three jet events can arise from QCD bremsstrahlung where a "hard" (high momentum) gluon radiates from one of the quark legs (see Figure 5.10). The observation of such three jet events at DESY in the 1980's was hailed as the "discovery of the gluon."

At present day e^+e^- colliders, the most important contributions to e^+e^- annihilation come from other diagrams in the standard model. In Figure 5.11 we show two diagrams where the e^+e^- can annihilate into a neutral Z boson or a neutral Higgs scalar, H^0. The Z and Higgs propagators contain factors $1/(q^2 - m^2)$ where $q^2 = s$ and m refers to the

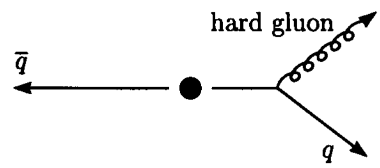


Figure 5.10 QCD bremsstahlung producing a three jet event.

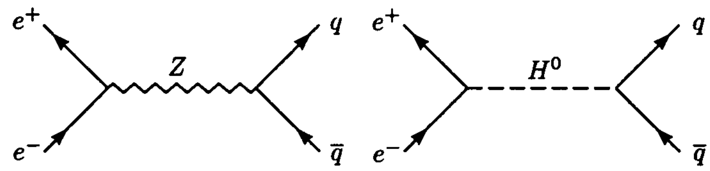


Figure 5.11 Z bosons and Higgs particles in e^+e^- annihilation.

Z or Higgs mass respectively. For the Z graph, the ratio of its amplitude to the QED amplitude is,

$$\frac{\mathcal{M}_Z}{\mathcal{M}_{\text{QED}}} \sim \frac{q^2}{q^2 - m_Z^2}.$$

In the Higgs case the ratio is,

$$\frac{\mathcal{M}_{H^0}}{\mathcal{M}_{\text{QED}}} \sim \frac{q^2}{q^2 - m_Z^2} \frac{m_e m_q}{m_W^2}.$$

The extra factors of the electron and quark masses for the Higgs contribution arise because of the standard model mass generation mechanism (see your standard model lectures), and the factor of m_e means that the Z contribution is most important. These amplitude ratios make it clear that as the centre of mass energy approaches m_Z, the Z process will dominate the pure QED one. This, of course, is exactly the situation at LEP.

I will not go further with this subject, but in closing I note that the agreement between the LEP results and the standard model depends on the inclusion of radiative corrections. This agreement provides compelling evidence for the quantum field theoretic aspects of the standard model.

5.4.2 Deep Inelastic Lepton Hadron Scattering

The process of interest is

$$\text{lepton} + \text{hadron} \to \text{lepton} + X,$$

where X denotes "anything" and the momentum transfer q between the initial and final leptons is large. The initial state lepton may be an electron, muon or neutrino, and the interaction can proceed via the exchange of a photon, W or Z. In Figure 5.12 we illustrate this for electron-proton deep inelastic scattering (DIS), mediated by a photon. The photon couples to one of the qurks in the proton, and since the interaction of the photon and lepton is understood, the strong interaction physics resides in the virtual-photon–proton scattering amplitude.

Choose a Lorentz frame in which the proton is highly relativistic and let the struck quark carry a fraction ξ of the proton's momentum p. Neglecting the struck quark's

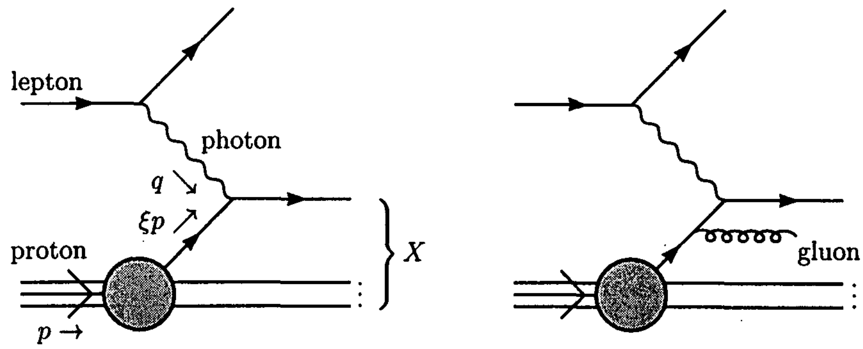


Figure 5.12 DIS process and a QCD correction.

transverse momentum, since the transverse momentum of secondary particles in hadronic experiments is generally small, we have,

$$(\xi p + q)^2 \approx 0,$$

where we assume the struck quark is nearly on shell and has negligible mass. This leads to,

$$\xi = \frac{-q^2}{2p{\cdot}q} \equiv x. \tag{5.14}$$

The fraction of the proton's momentum carried by the struck quark is given by the kinematic variable x (known as Bjorken's x variable). Measurements of the differential DIS cross section thus provide information about the momentum distribution of quarks inside hadrons.

What can we say about this process in perturbation theory? In calculating higher order contributions such as that from gluon radiation in the right hand diagram in Figure 5.12, there is an important difference from the calculation of the R ratio for e^+e^- annihilation in (5.13). The region of phase space where the struck quark is nearly on shell is important, as was anticipated above in the identification of ξ with x in (5.14). This manifests itself in the appearance of terms of the form $\alpha_s^n \ln^n(q^2/\lambda^2)$, where λ is some lower cutoff on the quark's momentum. The choice of λ depends on details of the proton wavefunction and hence these terms can't be calculated in perturbation theory. In other words, the relevant momenta are small, and DIS cross sections are not calculable in perturbation theory. However, for large q^2, it *is* possible to to compute the evolution of these cross sections with q^2, since these effects depend on the region of phase space where the quark is far off shell ($q^2 \gg \Lambda^2$). So, in summary, although DIS cross sections are not themselves calculable, their dependence on q^2 is. This is sufficient for a considerable amount of phenomenology.

DIS cross sections, and hence the momentum distribution of quarks in a proton, depend on q^2. As q^2 increases, theory predicts that there should be fewer quarks at large x and more at small x. This result has a physical interpretation. Imagine probing a proton with a virtual photon and seeing a quark carrying fraction y of the proton's momentum. If you increase the photon energy, you may see that what you thought was a quark with momentum yp is actually a quark with momentum xp together with a gluon of momentum $(y - x)p$. Thus the total momentum of the quark and gluon is yp and the quantum numbers of the pair are those of a single quark. In the first case, the pair was

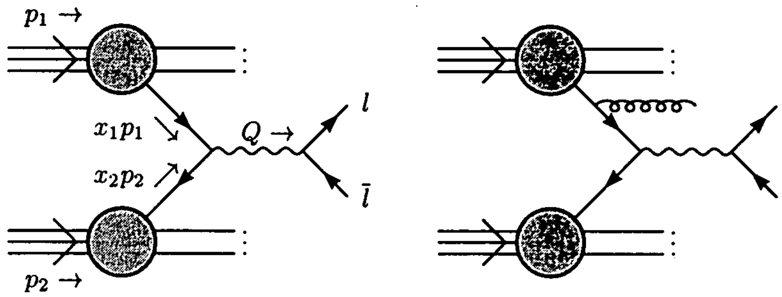


Figure 5.13 Drell–Yan process and a QCD correction.

not resolved, but in the second case we see that since $x < y$ the quark's momentum is less now than when we looked with a lower energy photon.

There is currently great interest in DIS processes at HERA, which is allowing us to explore smaller values of x, giving a new testing ground for theoretical ideas.

5.4.3 Drell–Yan and Related Processes

Now consider a process with two initial state hadrons. For illustration, consider the Drell–Yan process,

$$\text{hadron} + \text{hadron} \rightarrow e^+(\mu^+) + e^-(\mu^-) + X,$$

where the centre of mass energy of the hadrons and the invariant mass of the lepton pair are large and comparable. A parton model for this process, proposed by Drell and Yan is illustrated in Figure 5.13. A quark from one of the initial hadrons, labelled with subscript 1 in the figure, annihilates an antiquark from the other hadron, producing a virtual photon which in turn decays into a lepton–antilepton pair.

The momentum distribution of the quarks in the initial state hadrons can be determined from DIS experiments, so the process is calculable in terms of those distributions:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^4} \sum_f Q_f^2 \int_0^1 dx_1\, dx_2 \\ \times \delta(x_1x_2 - Q^2/s)x_1x_2\Big[q_{1f}(x_1)\overline{q}_{2f}(x_2) + q_{2f}(x_2)\overline{q}_{1f}(x_1)\Big], \tag{5.15}$$

where $s = (p_1 + p_2)^2$ and $q_{if}(x)$ is the probability density for finding a quark of flavour f in hadron i carrying a fraction x of its momentum (similarly for $\overline{q}_{if}$). Now consider some higher order correction such as the gluon radiation graph on the right of Figure 5.13. Just as for DIS there are important contributions from the "long-distance" region of phase space, where the quark and antiquark are almost on-shell. However, close study reveals that these long-distance contributions are precisely the same as in DIS, so can be absorbed into the quark distribution functions. Thus the Drell–Yan and DIS cross sections can be related in perturbation theory. The relation is just equation (5.15) with the $q_{if}(x_i)$ replaced by $q_{if}(x_i, M^2)$, which is the probability density determined from DIS experiments with $q^2 = M^2$. There are further perturbative corrections to (5.15), but the large logarithms coming from long-distance physics can always be absorbed into the distribution functions.

The factorisation of long distance effects into the distribution functions is a common feature of hard inclusive processes, including, besides Drell–Yan production, the production of particles or jets with large transverse momenta. In each case the cross section is a convolution of the partonic distribution functions with the cross section for the quark or gluon hard scattering process. Thus hadrons can be viewed as broad band beams of quarks and gluons, with a known (experimentally determined) momentum distribution. These beams are what we use to search for the Higgs scalar, or signals of new physics such as technicolour or supersymmetry — but that is all material for another course.

Acknowledgements

I would like to thank Tim Jones and Chris Sachrajda for lending me copies of their notes from previous YHEP schools from which I stole shamelessly. These notes are based heavily on those, and some sections are copied almost verbatim.

It is a pleasure to thank Ken Peach for organising and cheerleading the school and Ann Roberts for keeping everything running smoothly. I would also like to thank my fellow lecturers, the tutors and the students for making the school so entertaining.

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A Pre School Problems

The main aim of this course will be to teach the techniques required for performing simple calculations of amplitudes, cross sections and decay rates, particularly in Quantum Electrodynamics but also in Quantum Chromodynamics. Some aspects of quantum mechanics, special relativity and electrodynamics will be assumed during the lectures at the school. The following problems should be helpful in consolidating your knowledge in these areas. The solutions can be found in many standard textbooks.

Probability Density and Current Density

Starting from the Schrödinger equation for the wave function $\psi(\mathbf{x}, t)$, show that the probability density $\rho = \psi^* \psi$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

where

$$\mathbf{J} = \frac{\hbar}{2im}\left[\psi^*(\boldsymbol{\nabla}\psi) - (\boldsymbol{\nabla}\psi^*)\psi\right]$$

What is the interpretation of $\mathbf{J}$?

Rotations and the Pauli Matrices

Show that a 3-dimensional rotation can be represented by a 3×3 orthogonal matrix R with determinant $+1$ (Start with $\mathbf{x}' = R\mathbf{x}$, and impose $\mathbf{x}'\cdot\mathbf{x}' = \mathbf{x}\cdot\mathbf{x}$). Such rotations form the special orthogonal group, $SO(3)$.

For an *infinitesimal* rotation, write $R = \mathbb{1} + iA$ where $\mathbb{1}$ is the identity matrix and A is a matrix with infinitesimal entries. Show that A is antisymmetric (the i is there to make A hermitian).

Parameterise A as

$$A = \begin{pmatrix} 0 & -ia_3 & ia_2 \\ ia_3 & 0 & -ia_1 \\ -ia_2 & ia_1 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} a_i L_i$$

where the a_i are infinitesimal and verify that the L_i satisfy the angular momentum commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

Note that the Einstein summation convention is used here. In general, I will switch around between different notational conventions without warning. You should be able to tell from the context what is meant: notation should be your slave, not your master.

The Pauli matrices σ_i are,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Verify that $\frac{1}{2}\sigma_i$ satisfy the same algebra as L_i. If the two-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

transforms into $(\mathbb{1} + i\mathbf{a}\cdot\boldsymbol{\sigma}/2)\psi$ under an infinitesimal rotation, check that $\psi^\dagger\psi$ is invariant under rotations.

Raising and Lowering Operators

From the angular momentum commutation relations,

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

show that the operators

$$L_\pm = L_1 \pm iL_2$$

satisfy

$$[L_+, L_-] = 2L_3$$
$$[L_\pm, L_3] = \mp L_\pm$$

and show that

$$[L^2, L_3] = 0$$

where $L^2 = L_1^2 + L_2^2 + L_3^2$. From the last commutator it follows that there are simultaneous eigenstates of L^2 and L_3. Let ψ_{lm} be such an eigenvector of L^2 and L_3 with eigenvalues $l(l+1)$ and m respectively. Show that each of $L_\pm\psi_{lm}$ either vanishes or is an eigenstate of L^2 with eigenvalue $l(l+1)$ and of L_3 with eigenvalue $m \pm 1$.

Four Vectors

A Lorentz transformation on the coordinates $x^\mu = (ct, \mathbf{x})$ can be represented by a 4×4 matrix Λ as follows:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

For a boost along the x-axis to velocity v, show that

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.1}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$ as usual.

By imposing the condition

$$g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}x^\mu x^\nu \tag{A.2}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

show that

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g\Lambda = g$$

This is the analogue of the orthogonality relation for rotations. Check that it works for the Λ given by equation (A.1) above.

Now introduce

$$x_\mu = g_{\mu\nu}x^\nu$$

and show, by reconsidering equation (A.2) using $x^\mu x_\mu$, or otherwise, that

$$x'_\mu = x_\nu (\Lambda^{-1})^\nu{}_\mu$$

Vectors A^μ and B_μ that transform like x^μ and x_μ are sometimes called *contravariant* and *covariant* respectively. A simpler pair of names is *vector* and *covector.* A particularly important covector is obtained by letting $\partial/\partial x^\mu$ act on a scalar ϕ:

$$\frac{\partial \phi}{\partial x^\mu} \equiv \partial_\mu \phi$$

Show that ∂_μ does transform like x_μ and not x^μ.

Electromagnetism

The four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Which physical laws are represented by each of these equations? Show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

and explain the significance of this equation. Verify that it can be written in manifestly covariant form

$$\partial_\mu J^\mu = 0$$

where $J^\mu = (c\rho, \mathbf{J})$.

Introduce scalar and vector potentials ϕ and $\mathbf{A}$ by defining $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial \mathbf{A}/\partial t$, and recall the gauge invariance of electrodynamics which says that $\mathbf{E}$ and $\mathbf{B}$ are unchanged when

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \quad \text{and} \quad \phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

for any scalar function Λ. Using this gauge freedom we can set

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

Assuming that ϕ and $\mathbf{A}$ can be combined into a four vector $A^\mu = (\phi/c, \mathbf{A})$, this can be written as $\partial_\mu A^\mu = 0$, which is known as the *Lorentz gauge* condition. Defining $\Box \equiv \partial_\mu \partial^\mu$, show that with this condition Maxwell's equations are equivalent to

$$\Box A^\mu = \mu_0 J^\mu$$

The tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

How many independent components does $F_{\mu\nu}$ have? Rewrite $F_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. Show that,

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2\right)$$

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{8}{c}\mathbf{E}\cdot\mathbf{B}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

This gives the relativistic invariants which can be constructed from $\mathbf{E}$ and $\mathbf{B}$.

Group Theory: in Particular $SU(N)$

Unitary matrices U satisfy $U^\dagger U = \mathbb{1}$. Verify that they form a group by showing that $W = UV$ is unitary if U and V are. In general, you should also show that there is an identity element and that every U has an inverse, but these are both obvious. $U(N)$ is the group of complex unitary $N \times N$ matrices and $SU(N)$ is the subgroup of matrices with determinant +1.

Let U be a $U(N)$ matrix close to the identity. Write

$$U = \mathbb{1} + iG$$

where G has infinitesimal entries. Show that G is hermitian. If, in addition, U has determinant 1, so $U \in SU(N)$, show that G is traceless.

Any $N \times N$ traceless hermitian matrix can be written as a linear combination of a chosen *basis set.* So, for any G we can choose infinitesimal numbers ϵ_i such that

$$G = \sum_{i=1}^{N^2-1} \epsilon_i T_i$$

where the T_i are our basis. Explain why the summation runs from 1 to $N^2 - 1$.

Show that $[T_i, T_j]$ is antihermitian and traceless, and hence can be written

$$[T_i, T_j] = i f_{ijk} T_k \tag{A.3}$$

for some constants f_{ijk}. The commutation relations between the different T_i define the *Lie algebra* of $SU(N)$. The T_i are called the *generators* and the f_{ijk} are called the *structure constants.*

Find a set of 3 independent 2×2 matrices which are generators for $SU(2)$ and a set of 8 independent 3×3 generators for $SU(3)$.

Verify the *Jacobi identity,*

$$\left[T_i, [T_j, T_k]\right] + \left[T_j, [T_k, T_i]\right] + \left[T_k, [T_i, T_j]\right] = 0$$

and hence show that

$$f_{jkl}f_{ilm} + f_{kil}f_{jlm} + f_{ijl}f_{klm} = 0$$

Define a new set of $(N^2 - 1) \times (N^2 - 1)$ matrices

$$(T^i_{\text{adj}})_{jk} = -i f_{ijk}$$

and show that they obey the same commutation relations as the T_i in equation (A.3). The T^i_{adj} define the *adjoint representation.* The W's of the weak interactions and the gluons of the strong interactions belong to the adjoint representations of $SU(2)_L$, left-handed weak $SU(2)$, and $SU(3)$, the strong interaction colour algebra, respectively.

The generators, and hence the algebra, were found by looking at group elements near the identity. Other group elements can be recovered by combining lots of these infinitesimal "rotations"

$$U = \lim_{N\to\infty}(\mathbb{1} + i\theta_i T_i/N)^N = e^{i\theta_i T_i}$$

where the θ_i are finite. This construction generates what mathematicians call a simply connected group. There is a theorem stating that every Lie algebra comes from exactly one simply connected group: $SU(N)$ and its algebra give us one example.

However, we have seen that both $SU(2)$ and the rotation group $SO(3)$ have the same, angular momentum, algebra. What is going on? It must be that $SO(3)$ is not simply connected. In fact, there is a mapping, called a *covering*, from $SU(2)$ to $SO(3)$ which preserves the group property: that is if $U \in SU(2)$ is mapped to $f(U) \in SO(3)$, then $f(UV) = f(U)f(V)$. In the $SU(2) \to SO(3)$ case, two elements of $SU(2)$ are mapped on to every element of $SO(3)$. Whenever a group G has the same Lie algebra as a simply connected group S there must be such a covering $S \to G$.

The double covering of $SO(3)$ by $SU(2)$ underlies the behaviour of spin-1/2 and other half-odd-integer spin particles under rotations: they really transform under $SU(2)$, and rotating them by 2π only gets you half way around $SU(2)$, so you pick up a minus sign. A second 2π rotation gets you back to where you started.

THE STANDARD MODEL

By Dr G M Shore
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Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1995

The Standard Model

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1 Introduction

The standard model is a beautifully crafted and brilliantly predictive theory of all known phenomena in elementary particle physics. It was conceived in the decade from the mid 1960s to the mid 1970s when quantum field theory made a spectacular revival and non-abelian gauge theories were shown to provide a quantitative understanding of particle physics. Those were (I am told) heady times for theorists. Writing in 1984, Sidney Coleman remembers them nostalgically:

"This was a great time to be a high-energy theorist, the period of the famous triumph of quantum field theory. And what a triumph it was, in the old sense of the word: a glorious victory parade, full of wonderful things brought back from far places to make the spectator gasp with awe and laugh with joy."

Since then, the $SU(3)_C \times SU(2)_L \times U(1)_Y$ standard model, the fusion of quantum chromodynamics with the electroweak theory of Glashow, Salam and Weinberg, has successfully described (or at least not contradicted) all experimental data.

These lectures describe the construction of the standard model, with particular reference to the symmetry structure and tree-level dynamics of the electroweak interactions. I have tried to adopt a 'constructive' point of view, emphasising how the phenomenological structure of the fermion currents is incorporated into a gauge field theory. The complete standard model Lagrangian therefore appears as the culmination of the lecture course, rather than the starting point. These notes are complementary to the other lecture courses in this volume, which describe in more depth the quantum dynamics of gauge theories.

Some sections of these notes assume rather more familiarity with quantum field theory than the rest, particularly those associated with anomalies and chiral symmetry. These are marked in the text with an asterisk and may be disregarded. The importance of anomaly freedom in ensuring unitarity and constraining the fermion spectrum of the standard model cannot, however, be overemphasised.

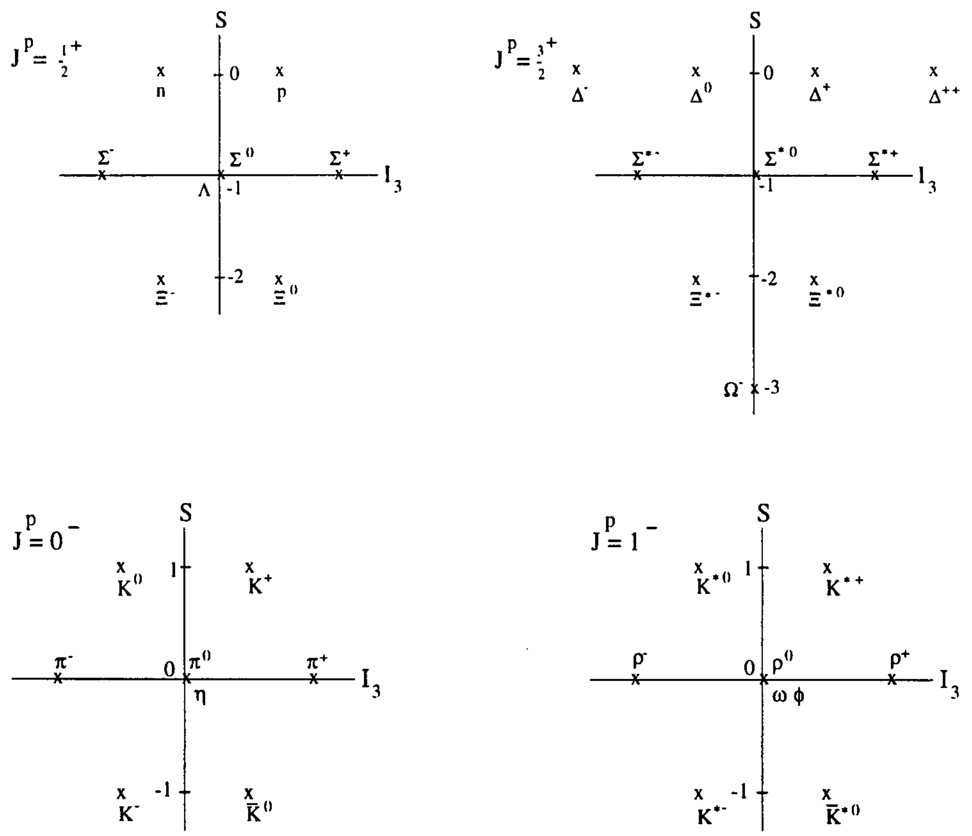
There are many excellent books on gauge theories and the standard model. The description given in these lectures follows quite closely the presentation in the book by Halzen and Martin, 'Quarks and Leptons'. This would provide a good source of supplementary reading and further examples.

2 Elementary particles, QED and QCD

We begin by listing the elementary particles which are currently known to exist in nature. These are the leptons ℓ, quarks q and the gauge bosons which mediate the fundamental forces.

Leptons	$\ell=$	e	μ	τ	ν_e	ν_μ	ν_τ
	mass(MeV)	0.51	105.6	1784	$\leq$46eV	$\leq$0.25	$\leq$70
Quarks	q=	u	d	s	c	b	t
	mass	7MeV	15MeV	200MeV	1.3GeV	4.8GeV	175 GeV
	charge	$\frac{2}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$

The quarks do not exist as free particles, but are permanently bound into hadrons. This is confinement. If we consider just the three quarks u, d, s, we form the baryon and meson octets and decuplets of 'flavour' $SU(3)$:



With the discovery of charm, bottom, ... the picture can be extended. New hadrons exist and fit into multiplets of higher flavour symmetries $SU(4)$, ... For example, there are the charmed mesons such as $D^+ = c\bar{d}$ with $m = 1.86GeV$ which decays by $D^+ \to K^-\pi^+\pi^+$. Of course, because of the mass differences between the quarks, these flavour symmetries are only approximate. All this phenomenology establishes the quarks as the elementary particles; mesons and baryons are bound states.

The next category of elementary particles are the gauge bosons:

$$\gamma \qquad g \qquad W^{\pm} \quad Z$$

The photon γ mediates the electromagnetic interaction, described by quantum electrodynamics (QED). It is massless. The strong (inter-quark, not inter-nuclear) force is mediated by a 'colour' octet of massless gluons and described by another gauge theory, quantum chromodynamics (QCD). Finally, the gauge bosons corresponding to the weak interactions are the charged $W^{\pm}$ and neutral Z, with masses of 80.2 and $91.2GeV$ respectively. These were discovered in 1983 by the UA1 and UA2 collaborations at CERN.

Finally, as we shall see, a further ingredient is required to make the picture work. The minimal standard model also predicts the existence of a scalar particle H^0, the famous Higgs boson.

In the standard quantum field theory model, all these elementary particles are considered to be the quanta of elementary fields.

The simplest example of a gauge theory of this type is QED, describing the interaction of electrons and photons. The action is

$$S = \int dx \left[\bar{\psi}\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2\bar{\psi}\psi\right] \tag{1}$$

where ψ is the electron field and A_{μ} is the photon field. Green functions (and hence S-matrix elements, etc.) are constructed from the path integral,

$$Z = e^{W[J,K,\bar{K}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\; e^{i\int dx\; \mathcal{L}+J^{\mu}A_{\mu}+\bar{K}\psi+\bar{\psi}K} \tag{2}$$

usually using perturbation theory, Feynman diagrams, etc.

In the early 1970s, it was realised that the strong interaction could be described by a non-abelian gauge theory, quantum chromodynamics. Each quark is assigned a colour quantum number, corresponding to the gauge group $SU(3)_C$. QCD is 'flavour blind', i.e. independent of the type of quark.The action is

$$S = \int dx\mathcal{L} = \sum_{flavour}\int dx \left[\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igT^aA^a_{\mu})\psi - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + m^2\bar{\psi}\psi\right] \tag{3}$$

where ψ is the colour triplet quark-field, A^a_{μ} is the colour octet gluon field and T^a is the matrix specifying the quark representation (for quarks, the fundamental representation of $SU(3)_C$).

The physics of non-abelian gauge theories is very different from QED. In particular QCD exhibits asymptotic freedom – the effective coupling $\rightarrow 0$ at small distances. This implies simple (quasi-free) behaviour of quarks in deep inelastic scattering experiments probing the structure of the proton. The inverse effect (infrared slavery), viz. the increase in the effective coupling at long distances, is related to confinement.

At this point, with QED and QCD, we have a theory of the strong and the electromagnetic interactions:

Gauge group	$SU(3)_c \times U(1)_{em}$				
Elementary fields	g	γ			
	e	μ	τ		
	u	d	s	c	$\dots$

$SU(3)_C$ acts only on the colour degree of freedom of the quarks. $U(1)_{em}$ acts on all charged particles. The theory is parametrised by two coupling constants e and g, the latter being traded for Λ_{QCD} according to dimensional transmutation, plus masses. There are no constraints on the masses, mass terms in $\mathcal{L}$ being gauge invariant.

This leaves the weak interactions to be incorporated. These are much more complicated – they act on the flavour degrees of freedom of the quarks and between ν, e, etc. The following are examples of weak interaction processes:

$$
\begin{aligned}
n &\rightarrow p\, e^-\, \bar{\nu}_e \\
(d &\rightarrow u\, e^-\, \bar{\nu}_e) \\
\pi^- &\rightarrow \mu^-\, \bar{\nu}_\mu \\
\mu^- &\rightarrow e^-\, \bar{\nu}_e \nu_\mu \\
\nu_\mu e^- &\rightarrow \mu^-\, \nu_e \\
\nu_\mu N &\rightarrow \mu^-\, X
\end{aligned}
$$

If the weak interactions were really distinct from the other two, we would simply have to enlarge the gauge group to include a new 'quantum flavourdynamics' group G_W acting on the quark flavours and lepton types. However, the picture which will emerge from the following discussion is more subtle. The weak interactions mix with electromagnetism and weave together the intricate tapestry that is the standard model.

3 Weak Interactions

3.1 Effective current-current interaction

The weak interactions were originally described by a phenomenological current-current interaction. From a modern viewpoint, we understand this interaction in terms of the effective low-energy Lagrangian implied by a gauge theory of massive vector bosons.

To motivate this, consider integrating out the gauge field in the QED Lagrangian

$$S = \int dx \left[\frac{1}{2} A^\mu D_{\mu\nu} A^\nu \; - \; e A^\mu J_\mu^{\rm em} \right] \tag{4}$$

where $D_{\mu\nu} = \partial^2 g_{\mu\nu} - (1-\alpha)\partial_\mu \partial_\nu$, where α is the gauge-fixing parameter. Completing the square, we find

$$\int \mathcal{D}A e^{iS} \;=\; \int \mathcal{D}A e^{i\int \frac{1}{2}(A - eJ/D)D(A-eJ/D) - e^2 J^2/2D} \tag{5}$$

$$= \; e^{i\int \mathcal{L}_{eff}} \tag{6}$$

with

$$\mathcal{L}_{eff} = -\frac{i}{2} e^2 J_\mu^{\rm em} \Delta^{\mu\nu} J_\nu^{\rm em} \tag{7}$$

where Δ is the photon propagator, $D\Delta = i\delta$.

The QED interaction is therefore of current-current type, but mediated by a propagator $\sim \frac{1}{q^2}$. It is therefore a long-range interaction. The electromagnetic current is

$$J_\mu^{\rm em} = -\bar{e}\gamma_\mu e - \bar{\mu}\gamma_\mu \mu + \ldots \tag{8}$$

where e, μ, ... are Dirac fields for the electron, muon, etc.

At low energies ($q \ll m_W$), the weak interactions can be well described by an effective theory comprising a current-current interaction. Since the weak interactions are short range, a good approximation is to replace the propagator by a constant, which is equivalent to a point interaction, i.e.

$$\mathcal{L}_{eff}^{\rm weak} = G^{\rm AB} J_\mu^{\rm A} J^{\mu \rm B} \tag{9}$$

Notice that G has dimensions of $mass^{-2}$ which implies that this is a non-renormalisable interaction. It violates unitarity (cross sections $\sigma \sim s$ for large energy). This means that the current-current interaction cannot be fundamental. Nevertheless, it gives an excellent description of weak interaction processes for momenta below m_W.

Our aim is to build a renormalisable gauge theory of the weak interactions. The next step, therefore, is to extract the form of the weak currents $J_\mu^{\rm A}$ from the phenomenology of weak interactions.

3.2 Lorentz structure of currents

The general Lorentz structure for a bilinear fermion current is

$$J = \bar{\psi}\Gamma\psi \tag{10}$$

where $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$ (total=16)

Now, if the current-current interaction is derived from a gauge theory with vector bosons, we will have either $\Gamma = \gamma_\mu$ or $\gamma_\mu\gamma_5$ (the so-called V or A currents). Extensive studies of weak interaction phenomenology in the 1950s showed that this is indeed true – the other forms (S, P and T) are excluded by experiment.

The original assumption was that Γ must be γ_μ, based on the analogy with the electromagnetic current. This was the basis of the 1932 Fermi theory of β decay. The $\gamma_\mu\gamma_5$, or A, interaction would violate parity.

However, in 1956, Lee and Yang surveyed weak interaction data and concluded that parity may not be conserved (e.g. $K^+ \to \pi\pi$ and $\pi\pi\pi$ both occur). The experimental confirmation of parity violation by Wu ($^{60}\mathrm{C} \to {}^{60}\mathrm{Ni}\, e^- \,\bar{\nu}_e$, polarised beta decay), Ledermann ($\pi^- \to \mu^-\bar{\nu}_\mu$ followed by $\mu^- \to e^-\,\bar{\nu}_e\,\nu_\mu$) and others followed shortly after.

The cumulative experimental evidence led to the identification (by Feynman and Gell-Mann and Marshak et al.) of the Lorentz structure of the charged weak current as V$-$A, i.e. $\Gamma = (1-\gamma_5)\gamma_\mu$. Also, only the left-handed (helicity $-\frac{1}{2}$, $\nu_L = \frac{1}{2}(1-\gamma_5)\nu$) neutrino seems to occur in nature, together with the right-handed antineutrino. There is no ν_R state.

Because left and right handed states enter differently in weak interaction theory, it is convenient to use the left and right handed projections for all particles. So, e.g.

$$e_L = \frac{1}{2}(1-\gamma_5)e \tag{11}$$

$$e_R = \frac{1}{2}(1+\gamma_5)e \tag{12}$$

are the helicity $-\frac{1}{2}$ and $+\frac{1}{2}$ components of the electron.

Under parity $(P^{-1}\gamma_5 P = -\gamma_5)$

$$e_L \overset{P}{\longleftrightarrow} e_R \tag{13}$$

Under charge conjugation $(\psi_C = C\gamma^0\psi^* = C\bar{\psi}^T)$

$$e \overset{C}{\longleftrightarrow} \bar{e}^T \tag{14}$$

We can show that

$$
\begin{array}{lll}
J_0^V \xrightarrow{P} J_0^V & J_0^A \xrightarrow{P} -J_0^A & \text{(time cpt.)} \qquad (15)\\
J_i^V \xrightarrow{P} -J_i^V & J_i^A \xrightarrow{P} J_i^A & \text{(space cpts.)} \qquad (16)\\
J_\mu^V \xrightarrow{C} -J_\mu^V & J_\mu^A \xrightarrow{C} J_\mu^A & \qquad (17)
\end{array}
$$

so that $\mathcal{L}_{eff}^{\rm V-A}$ violates P and C individually, but is CP invariant.

This discussion of CP invariance needs to be re-assessed when we have the full weak currents, where ψ is a multiplet of fields and we must allow for flavour mixing. It turns out that CP violation is generic in the physical three generation model. (See the discussion of the CKM matrix in section 5.5)

In terms of left and right-handed fields, the electron mass term in the Lagrangian is written as

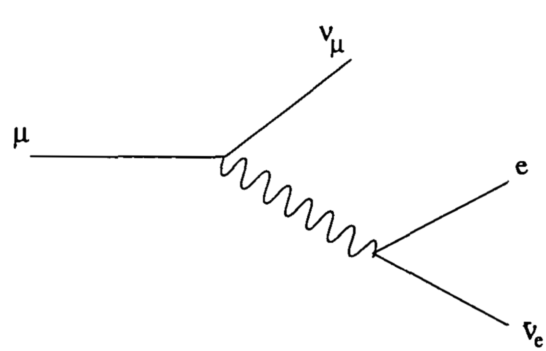
$$\mathcal{L} = m(\bar{e}_L\, e_R + \bar{e}_R\, e_L) \tag{18}$$

Since there is no ν_R state, we cannot construct a similar Dirac mass term for neutrinos.

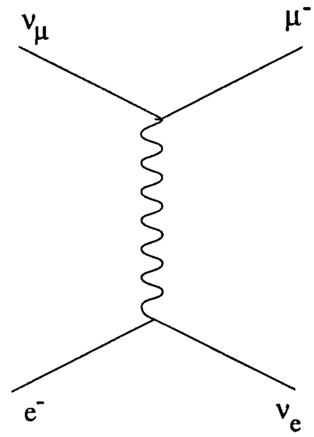
3.3 Charged weak current (leptons)

The individual lepton numbers L_e, L_μ and L_τ are separately conserved (e.g. $\mu \to e\,\gamma$ is forbidden). This implies that we should construct separate lepton currents for the 3 generations. To deduce the structure of these currents, consider the following weak processes:

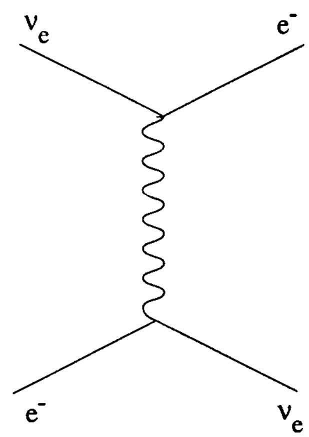
1. $\mu^- \to e^-\,\bar{\nu}_e\,\nu_\mu$



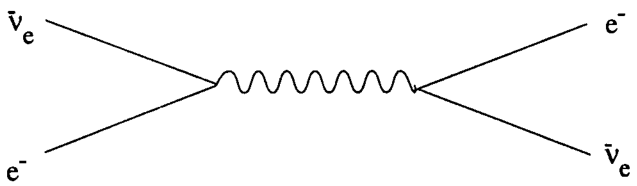
2. $\nu_\mu\, e^- \to \nu_e\, \mu^-$



3. $\nu_e \, e^- \longrightarrow \nu_e \, e^-$



4. $\bar{\nu}_e \, e^- \longrightarrow \bar{\nu}_e \, e^-$



The crossed diagrams for (3) and (4) do not occur. These would require gauge bosons carrying lepton number.

Elastic $\nu_e \, e^-$ and $\bar{\nu}_e \, e^-$ scattering also have neutral current contributions. However, only the charged current contributes to (2).

All these interactions are of the current-current form:

$$\mathcal{L}_{eff} = \frac{4\,G}{\sqrt{2}}\, J_\mu^{CC}\, J^{CC\,\mu\dagger} \tag{19}$$

where the lepton charged current is

$$\begin{aligned} J_\mu^{CC} &= \bar{\nu}_e\,\gamma_\mu\,\tfrac{1}{2}\,(1-\gamma_5)\,e + \bar{\nu}_\mu\,\gamma_\mu\,\tfrac{1}{2}\,(1-\gamma_5)\,\mu + \bar{\nu}_\tau\,\gamma_\mu\,\tfrac{1}{2}\,(1-\gamma_5)\,\tau & (20)\\ &= \bar{\nu}_{eL}\,\gamma_\mu\,e_L + \bar{\nu}_{\mu L}\,\gamma_\mu\,\mu_L + \bar{\nu}_{\tau L}\,\gamma_\mu\,\tau_L & (21)\end{aligned}$$

and

$$J_\mu^{CC\,\dagger} = \bar{e}_L\,\gamma_\mu\,\nu_{eL} + \bar{\mu}_L\,\gamma_\mu\,\nu_{\mu L} + \bar{\tau}_L\,\gamma_\mu\,\nu_{\tau L} \tag{22}$$

Notice that cross terms linking, for example, electron and muon type currents are possible – the gauge bosons are independent of the generation. This is known as "universality" of the weak interactions.

The coupling strength G (the Fermi constant) is the same for all these processes. This indicates a single underlying explanation. If we postulate that the interaction is due to the exchange of a massive vector boson $W^\pm$ with propagator

$$\Delta_{\mu\nu} = i\,\frac{1}{q^2 - m_W^2}\left(-g_{\mu\nu} + \frac{q_\mu\,q_\nu}{m_W^2}\right) \overset{q\,small}{\sim} i\left(\frac{g_{\mu\nu}}{m_W^2}\right) \tag{23}$$

then the effective Lagrangian becomes

$$\mathcal{L}_{eff} = -i\left(\frac{g}{\sqrt{2}}\right)^2 J_\mu^{CC}\,\Delta^{\mu\nu}\,J_\nu^{CC\,\dagger} \sim \frac{g^2}{2\,m_W^2} J_\mu^{CC}\,J^{CC\,\mu\dagger} \tag{24}$$

So we can identify

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8\,m_W^2} \tag{25}$$

The V−A structure can be verified from $\nu\,e$ scattering. If we assume a Lorentz structure $J_\mu^{CC} = \bar{\nu}_\ell\,\gamma_\mu(a + b\gamma_5)\ell$ for the weak current, then process (2) gives

$$\frac{d\sigma}{d\Omega}(\nu_\mu\,e^- \to \nu_e\mu^-) = \frac{G^2\,s}{32\,\pi^2}\left(A^+ + A^-\,\cos^4\frac{\theta}{2}\right) \tag{26}$$

where $A^\pm = (a^2 + b^2)^2 \pm 4a^2b^2$.
(See Halzen and Martin, sect. 12.7 for cross-sections for charged current $\nu\,e$ and $\bar{\nu}\,e$ scattering.)

3.4 Neutral weak current (leptons)

In 1973, neutral current weak interactions were observed in neutrino interactions in the Gargamelle bubble chamber at CERN, e.g.

$$\begin{array}{c} \bar{\nu}_\mu \, e^- \to \bar{\nu}_\mu \, e^- \\ \nu_\mu \, N \to \nu_\mu \, X \\ \bar{\nu}_\mu \, N \to \bar{\nu}_\mu \, X \end{array} \tag{27}$$

These require a new interaction to be added to the effective Lagrangian, viz.

$$\mathcal{L}_{int} = \frac{4\,G}{\sqrt{2}}\, 2\rho \, J_\mu^{NC} \, J^{NC\,\mu} \tag{28}$$

The factor ρ allows for a different strength of coupling compared with the charged current interaction.

Unlike the charged currents, the neutral current is not V$-$A. In fact, we parametrise

$$\begin{aligned} J_\mu^{NC} &= \tfrac{1}{2}\left(\bar{\nu}_L \gamma_\mu \nu_L \;+\; c_L^e \, \bar{e}_L \gamma_\mu e_L + c_R^e \, \bar{e}_R \gamma_\mu e_R\right) \\ &= \tfrac{1}{2}\Big(\bar{\nu}\,\gamma_\mu \tfrac{1}{2}(1-\gamma_5)\,\nu \;+\; \bar{e}\,\gamma_\mu (c_V^e - c_A^e \gamma_5)\, e\Big) \end{aligned} \tag{29}$$

where $c_L = c_V + c_A$ and $c_R = c_V - c_A$.

For the electron (Halzen and Martin, fig. 13.5), experiment gives

$$\begin{aligned} c_V^e &= 0.06 \pm 0.08 \\ c_A^e &= -0.52 \pm 0.06 \end{aligned} \tag{30}$$

Since there is no right-handed neutrino ν_R, we have $c_V^\nu = c_A^\nu = \frac{1}{2}$.

4 Weinberg-Salam Model (leptons)

The Weinberg-Salam model was proposed in 1967 (see also Glashow, 1961), anticipating the discovery of neutral currents as well as the gauge bosons W and Z.

4.1 Currents, gauge bosons and the electroweak group

If we are to derive the effective current-current interaction

$$\mathcal{L}_{int} = \frac{4G}{\sqrt{2}}\Big(J_\mu^{CC} \, J^{CC\,\mu\dagger} \;+\; 2\rho \, J_\mu^{NC} \, J^{NC\,\mu}\Big) \tag{31}$$

from a non-abelian gauge theory with interaction $J_\mu^A A^{A\mu}$, then the currents must form a representation of the gauge group.

For one lepton generation, these currents are

$$\begin{aligned} J_\mu^{CC} = J_\mu^- &= \bar{\nu}_L \gamma_\mu e_L \\ J_\mu^{CC\dagger} = J_\mu^+ &= \bar{e}_L \gamma_\mu \nu_L \\ J_\mu^{NC} &= \frac{1}{2}\Big(\bar{\nu}_L \gamma_\mu \nu_L \; + \; c_L^e \bar{e}_L \gamma_\mu e_L \; + \; c_R^e \bar{e}_R \gamma_\mu e_R\Big) \end{aligned} \tag{32}$$

The charged currents $J_\mu^\pm$ can form two of the three components of the adjoint representation of $SU(2)$. In the fundamental (2-dimensional) representation of $SU(2)$, the generators are $T^A = \frac{1}{2}\tau^A$ and satisfy the commutation relations

$$[T^A, T^B] = i\epsilon^{ABC} T^C, \qquad A = 1, 2, 3 \tag{33}$$

For the charged components,

$$\tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau^2) \tag{34}$$

$$\Rightarrow \; \tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now construct lepton doublets

$$\chi_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \qquad \text{2-dim } SU(2)_L \text{ representation} \tag{35}$$

The currents $J_\mu^\pm$ can then be written as

$$J_\mu^\pm = \bar{\chi}_L \gamma_\mu \tau^\pm \chi_L = J_\mu^1 \pm i J_\mu^2 \tag{36}$$

i.e. as components of the $SU(2)_L$ current

$$J_\mu^A = \bar{\chi}_L \gamma_\mu T^A \chi_L \tag{37}$$

The remaining component is

$$\begin{aligned} J_\mu^3 &= \frac{1}{2}\bar{\chi}_L \gamma_\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi_L \\ &= \frac{1}{2}\left(\bar{\nu}_L \gamma_\mu \nu_L - \bar{e}_L \gamma_\mu e_L\right) \end{aligned} \tag{38}$$

However, this has no right-handed part and so it obviously cannot be identified with the remaining current J_μ^{NC}.

The solution is to introduce a new current, corresponding to a new $U(1)_Y$ interaction. The proposal is to define

$$J_\mu^Y = -\bar{\nu}_L \gamma \nu_L - \bar{e}_L \gamma_\mu e_L - 2\bar{e}_R \gamma_\mu e_R \tag{39}$$

so that $U(1)_Y$ commutes with $SU(2)_L$.

Now try and express J_μ^{NC} and $J_\mu^{\rm em}$ as linear combinations of J_μ^3 and J_μ^Y :

$$\begin{aligned} J_\mu^{\rm em} &= -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R = J_\mu^3 + \frac{1}{2}J_\mu^Y \\ J_\mu^{NC} &= aJ_\mu^3 + bJ_\mu^Y \end{aligned} \tag{40}$$

$$\Rightarrow \left.\begin{aligned} a - 2b &= 1 \\ a + 2b &= -c_L \\ 4b &= -c_R \end{aligned}\right\} \quad \text{solution only if } 1 + c_L - c_R = 0 \tag{41}$$

Then, $a = \frac{1}{2}(1-c_L)$ and $b = -\frac{1}{4}c_R$. So, provided we have $1+c_L-c_R = 0$, we can express

$$J_\mu^{NC} = \frac{1}{2}(1-c_L)J_\mu^3 - \frac{1}{4}c_R J_\mu^Y \tag{42}$$

$$= J_\mu^3 - \frac{1}{2}c_R J_\mu^{em} \tag{43}$$

where recall $c_R = c_V - c_A$

The condition $1 + c_L - c_R = 0$ requires $c_A = -\frac{1}{2}$. This is necessary for this mixing scheme to work.

To incorporate this structure into a gauge theory, choose a gauge group $SU(2)_L \times U(1)_Y$, with gauge bosons W_μ^A and B_μ. The interaction term in the Lagrangian is

$$\mathcal{L}_{int} = -g\, J_\mu^A W^{A\,\mu} - \frac{g'}{2} J_\mu^Y B^\mu \tag{44}$$

In terms of $J_\mu^\pm$, $J_\mu^{\rm em}$ and J_μ^{NC} we have

$$\begin{aligned} \mathcal{L}_{int} = &- \frac{g}{\sqrt{2}}\left(J_\mu^+ W^{+\,\mu} + J_\mu^- W^{-\,\mu}\right) \\ &- J_\mu^{\rm em}\left(g\,\tfrac{1}{2}\,c_R W^{3\,\mu} + g'(1 - \frac{1}{2}c_R)B^\mu\right) \\ &- J_\mu^{NC}\left(gW^{3\,\mu} - g'B^\mu\right) \end{aligned} \tag{45}$$

where $W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right)$.

In the Weinberg-Salam model, the mixing between W_μ^3 and B_μ to give A_μ and Z_μ is of the following form:-

$$\begin{aligned} Z_\mu &= W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \\ A_\mu &= W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \end{aligned} \tag{46}$$

Then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{\sqrt{2}}\frac{e}{\sin\theta_W}\left(J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu}\right) \tag{47}$$

$$- e J^{\rm em}_\mu A_\mu - \frac{e}{\sin\theta_W\cos\theta_W} J^{NC}_\mu Z^\mu \tag{48}$$

where we identify

$$e = g\sin\theta_W = g'\cos\theta_W \tag{49}$$

and

$$\frac{1}{2}c_R = \sin^2\theta_W \tag{50}$$

This last result implies that

$$J^{NC}_\mu = J^3_\mu - \sin^2\theta_W J^{\rm em}_\mu \tag{51}$$

The resulting current-current effective interaction is

$$\mathcal{L} = \left(\frac{g}{\sqrt{2}}\right)^2 J^{CC}_\mu \frac{1}{m_W^2} J^{CC\,\mu\dagger} + \left(\frac{g}{\cos\theta_W}\right)^2 J^{NC}_\mu \frac{1}{m_Z^2} J^{NC\,\mu} \tag{52}$$

Comparing with

$$\mathcal{L} = \frac{4\,G}{\sqrt{2}}\left(J^{CC}_\mu J^{CC\,\mu\dagger} + 2\rho J^{NC}_\mu J^{NC\,\mu}\right) \tag{53}$$

we identify

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8\,m_W^2} = \frac{e^2}{8\,m_W^2\sin^2\theta_W} \tag{54}$$

and

$$\rho = \frac{m_W^2}{m_Z^2\cos^2\theta_W} \tag{55}$$

4.2 Weinberg-Salam Lagrangian (leptons)

The $SU(2)_L \times U(1)_Y$ Lagrangian is therefore

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu\,A} - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} \\ & + \sum_{i=e,\mu,\tau} i\bar{\chi}^i_L\left(\partial_\mu + igT^A W^A_\mu + i\frac{g'}{2}YB_\mu\right)\chi^i_L \\ & + \sum_{i=e,\mu,\tau} i\bar{\psi}^i_R\left(\partial_\mu + i\frac{g'}{2}YB_\mu\right)\psi^i_R\end{aligned} \tag{56}$$

where $T^A = \frac{1}{2}\tau^A$ determines the $SU(2)_L$ representation of χ_L

$$\text{and} \qquad Y = -1 \qquad \text{for } \chi_L = \begin{pmatrix} \nu_{e\,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu\,L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau\,L} \\ \tau_L \end{pmatrix} \tag{57}$$

$$Y = -2 \qquad \text{for } \psi_R = e_R,\ \mu_R,\ \tau_R \tag{58}$$

The parameters are

$$g,\ g',\ m_W,\ m_Z$$

or equivalently

$$e,\ \sin\theta_W,\ m_W,\ m_Z$$

The interaction terms are

$$\mathcal{L} = -e\,J_\mu^{\text{em}} A^\mu - \frac{e}{\sin\theta_W \cos\theta_W} J_\mu^{NC} Z^\mu \tag{59}$$

where

$$\begin{aligned} Z_\mu &= W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \\ A_\mu &= W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \end{aligned} \tag{60}$$

and

$$\begin{aligned} J_\mu^{\text{em}} &= J_\mu^3 + \frac{1}{2} J_\mu^Y \\ J_\mu^{NC} &= J_\mu^3 - \sin^2\theta_W J_\mu^{\text{em}} \end{aligned} \tag{61}$$

The effective interaction is

$$\mathcal{L}_{int} = \frac{4G}{\sqrt{2}} \left(J_\mu^{CC} J^{CC\,\mu\dagger} \ + \ 2\rho J_\mu^{NC} J^{NC\,\mu} \right) \tag{62}$$

together with electromagnetism. This phenomenological description has the parameters e, G, ρ, c_V^e, c_A^e.

The Weinberg-Salam model requires $c_A^e = -\frac{1}{2}$. The other equivalences are

$$\frac{G}{\sqrt{2}} = \frac{e^2}{8 m_W^2 \sin^2\theta_W} \tag{63}$$

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} \tag{64}$$

$$c_V^e = -\tfrac{1}{2} + 2\sin^2\theta_w \tag{65}$$

Universality implies that $c_{V,A}^e = c_{V,A}^\mu = c_{V,A}^\tau$ as well as a single G, ρ.

In fact, the full Weinberg-Salam model including the Higgs mechanism also implies $\rho = 1$ because of an additional (custodial $SU(2)$) symmetry which is built into the model. (See section 9.)

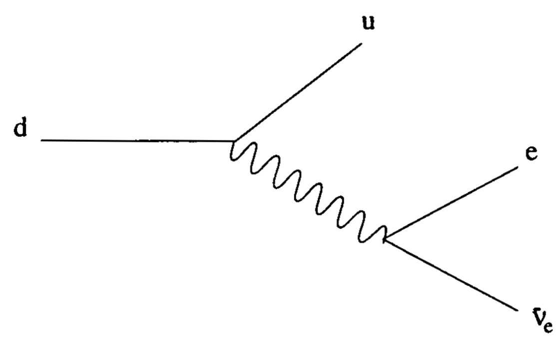
5 Quarks in the Electroweak Model

An analysis of weak interactions involving hadrons leads to a very similar structure for the quark sector of the electroweak model.

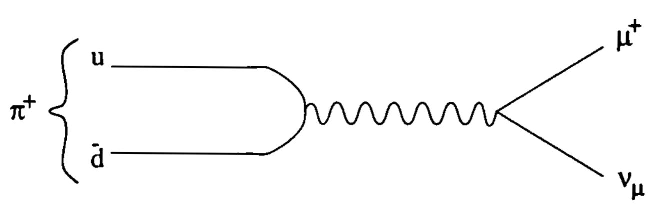
5.1 Charged weak current (quarks)

A selection of key processes includes the following:

1. β decay $\quad n \to p\, e^-\, \bar{\nu}_e \quad$ i.e. $d \to u\, e^-\, \bar{\nu}_e$

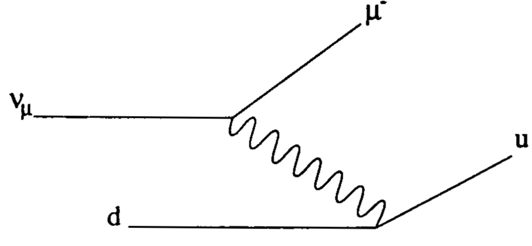


2. π decay $\quad \pi^+ \to \mu^+\, \nu_\mu$

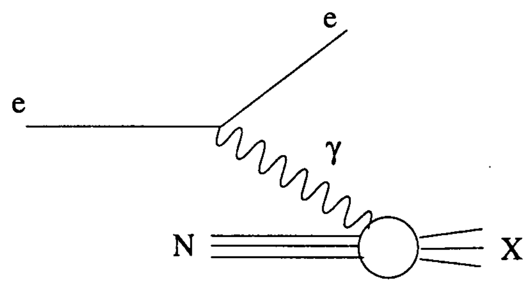


(see Halzen and Martin, sect. 12.6 for a discussion of the "hadronisation" of $u\bar{d}$ into π^+.)

3. $\nu_\mu\, N \to \mu^-\, X \quad$ e.g. $\nu_\mu\, d \to \mu^-\, u$



This is realised in deep inelastic scattering. It is the weak interaction analogue of $e^- N \to e^- X$



All these processes can be described at low energies, $q \ll m_W$, by an effective current-current interaction, also of Lorentz structure $V - A$:

$$\mathcal{L}_{int} = \frac{4G}{\sqrt{2}} J_\mu^{CC} J^{CC\,\mu\dagger} \tag{66}$$

with $J_\mu^{CC} \simeq \bar{u}_L \gamma_\mu d_L$. $\mathcal{L}_{int}$ uses the *same* G as before. This extends electron-muon universality to lepton-quark universality.

In fact, this is too simple. Consider the next generation, with strange and charm quarks. These almost obey

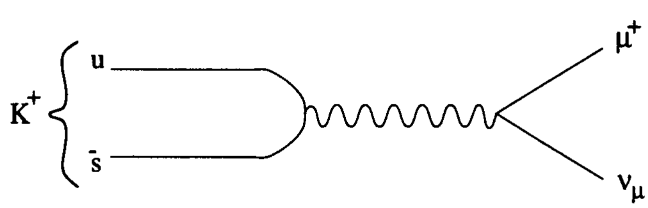
$$J_\mu^{CC} \simeq \bar{u}_L \gamma_\mu d_L + \bar{c}_L \gamma_\mu s_L \tag{67}$$

This corresponds to a structure like the leptons, with families of $SU(2)_L$ doublets

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}.$$

However, processes such as $K^+ \to \mu^+ \nu$ also occur, involving a $u - s$ transition:

$K^+ \to \mu^+ \nu_\mu$



(cf. $\pi^+ \to \mu^+ \nu_\mu$)

To incorporate this flavour mixing, we add an ad-hoc quark mixing angle (the Cabibbo angle) and define

$$\begin{aligned} d' &= d\cos\theta_c + s\sin\theta_c \\ s' &= -d\sin\theta_c + s\cos\theta_c \end{aligned} \qquad (68)$$

so that the $SU(2)_L$ eigenstates are

$$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \qquad \begin{pmatrix} c_L \\ s'_L \end{pmatrix}.$$

Then

$$\frac{\Gamma(K^+ \to \mu^+\nu_\mu)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} \sim \sin^2\theta_c \qquad \text{(up to kinematic factors)}$$

The Cabibbo angle is small: $\theta_c = 13^o$, $\sin\theta_c = 0.23$ (see also the discussion of the CKM matrix in sect. 5.4).

5.2 Neutral current (quarks)

Processes such as $\nu_\mu N \to \nu_\mu X$ were observed at CERN (Gargamelle) in 1973 with strength

$$\frac{\sigma(\nu N \to \nu X)}{\sigma(\nu N \to \mu X)} \sim 0.3 \qquad (69)$$

They can be described by

$$\mathcal{L}_{int} = \frac{4G}{\sqrt{2}}\, 2\,\rho J^{NC}_\mu J^{NC\,\mu} \qquad (70)$$

with

$$J^{NC}_\mu = \frac{1}{2}\bar{q}\,\gamma_\mu\,(c^q_V - c^q_A\gamma_5)\; q \qquad (71)$$

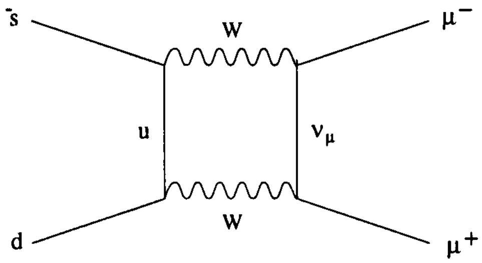
This is the same as for the leptons, except for the different c_V, c_A parameters (see sect. 5.6).

5.3 Charm and flavour changing neutral currents

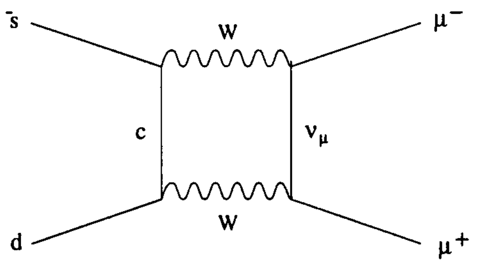
Suppose there was no c quark. With any $u \leftrightarrow d'$ transitions, we would have the flavour changing neutral current ($\Delta S = 1$) decay $K^0 \to \mu^+\mu^-$ from the top diagram overleaf.

The Cabibbo factors from the (du) (us) vertices give $\cos\theta_c\sin\theta_c$. However, experimentally, flavour changing neutral current (FCNC) decays are found to be strongly suppressed, e.g.

$$\frac{\Gamma(K^0_L \to \mu^+\mu^-)}{\Gamma(K^0_L \to anything)} \sim 10^{-8} \qquad (72)$$



$\sim \cos\theta_C \ \sin\theta_C$



$\sim -\cos\theta_C \ \sin\theta_C$

With charm, there is another diagram, shown above. The Cabibbo factors have the opposite sign, giving a cancellation. So FCNCs are strongly suppressed. This is the famous GIM (Glashow, Iliopoulos, Maiani) mechanism.

This was one of the main motivations for the proposal of charm by GIM in 1970. Another was anomalies.

Theoretical Interlude – Anomalies

*Theory

It can happen that a symmetry which holds in the classical theory is no longer a good symmetry in the corresponding quantum theory. This is known as an anomaly. This phenomenon is particularly associated with chiral symmetries (i.e. involving γ_5) such as occur in the electroweak model.

As the simplest example, consider massless QED with the action

$$S = \int dx \mathcal{L} = \int dx \ \left[i\bar{\psi}(\partial_\mu - ieA_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \tag{73}$$

This is invariant under the (global) chiral transformation,

$$\psi \longrightarrow e^{i\alpha\gamma_5}\,\psi \tag{74}$$

By Noether's theorem, there is a conserved current $J_{\mu 5} = \bar{\psi}\gamma_\mu\gamma_5\psi$ corresponding to this symmetry. It satisfies the equation of motion (conservation law)

$$\partial^\mu J_{\mu 5} = 0 \tag{75}$$

Does this remain true in the quantum theory? The equivalent statement would be the chiral Ward identity for, e.g. the two-point Green function

$$< 0|T^*\partial^\mu J_{\mu 5} \;\; \Phi|0 > \stackrel{?}{=} \langle\delta\Phi\rangle \tag{76}$$

where $\langle\delta\Phi\rangle$ is the vacuum expectation value of the chiral variation of some arbitrary (elementary or composite) field Φ.

To compute Green functions in the quantum theory, we need the generating functional,

$$W = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A \; e^{i\int dx\mathcal{L}} \tag{77}$$

Now consider the behaviour of W under a change of integration variable, $\psi \rightarrow e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \rightarrow e^{-i\alpha\gamma_5}\bar{\psi}$. Since this is only a change of variable, W does not change. So (taking $\alpha = \alpha(x)$ as a technical device), we get

$$0 = \frac{\delta W}{\delta\alpha(x)} \stackrel{?}{=} -i\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A \; \partial^\mu J_{\mu 5} \; e^{i\int dx\mathcal{L}} \tag{78}$$

since $\frac{\delta S}{\delta\alpha(x)} = -\partial^\mu J_{\mu 5}$. Since the variation is a total derivative, the global transformation is a symmetry. This gives the naive Ward identity.

However, the integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$, which is the key ingredient in taking us from the classical to the quantum theory, is *not* invariant under chiral transformations. In fact,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \rightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int dx\alpha(x)\frac{e^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}} \tag{79}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. The derivation of this is subtle and difficult. However, the final result for the Ward identity is simple:

$$0 = -i\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A \; (\partial^\mu J_{\mu 5} - \frac{e^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu})e^{i\int dx\mathcal{L}} \tag{80}$$

that is,

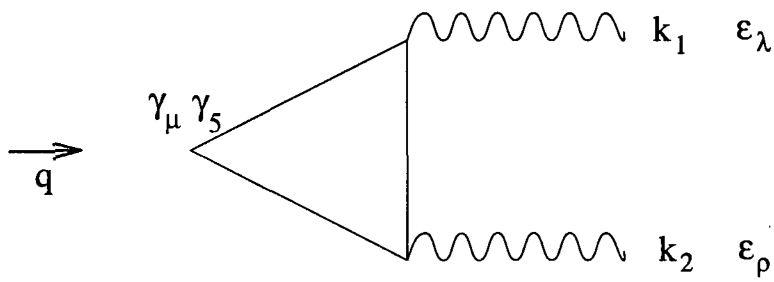
$$< 0|T^*\partial^\mu J_{\mu 5} \;\; \Phi|0 > - < 0|T^*\frac{e^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \;\; \Phi|0 > = \langle\delta\Phi\rangle \tag{81}$$

This result is exact and non-perturbative. In fact, with an appropriate choice of renormalisation for the composite operators $J_{\mu 5}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$, it holds in the same form to all orders (Adler-Bardeen theorem). This is the anomalous chiral Ward identity.

In perturbation theory, the anomaly is manifested in the 1-loop triangle diagram shown overleaf.

Naively, we expect this amplitude to satisfy $q^\mu M_{\mu\lambda\rho} \stackrel{?}{=} 0$ because of the classical current conservation. However a careful treatment of the divergent integrals involved in its calculation actually gives

$$q^\mu M_{\mu\lambda\rho} = \frac{ie^2}{4\pi^2}\epsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu \neq 0 \tag{82}$$



in accordance with the anomalous Ward identity.

The result for non-abelian currents is similar. In this case, the currents at the vertices of the triangle diagram (or equivalently the external gauge fields) include group generators T^a , T^b and T^c. The anomaly is then proportional to

$$\mathcal{A} = Tr\{T^a, T^b\}T^c \tag{83}$$

We have described the 'AVV' anomaly. In theories such as the electroweak model which also has axial gauge bosons there are also 'AAA' and higher-point anomalies.

The physical significance of anomalies depends entirely on whether or not the axial current is coupled to gauge fields.

***Global currents:**

This is the case where the current is not coupled to a gauge field. Here, there is no problem. The quantum theory (anomalous Ward identity) does not look like the classical theory (conserved current), but this does not damage the consistency of the theory. In fact, the existence of these anomalies is an essential and experimentally verified part of the standard model.

For example, the anomaly is essential for the neutral pion decay $\pi^0 \to \gamma\gamma$. The pion couples to the axial current $J_{\mu 5}$ according to $< 0|J_{\mu 5}|\pi >= ik_\mu f_\pi$ where f_π is the pion decay constant, $93MeV$ (see section 7). This allows us to calculate the $\pi^0 \to \gamma\gamma$ decay amplitude from the matrix element $< 0|J_{\mu 5}|\gamma\gamma >$. The divergence of this would vanish if the naive Ward identity was true, predicting $\pi^0 \not\to \gamma\gamma$. In fact, because of the anomaly,

$$\begin{aligned} < 0|\partial^\mu J_{\mu 5}|\gamma\gamma > &= \frac{e^2}{16\pi^2}\sum_f Q_f^2 < 0|F_{\mu\nu}\tilde{F}^{\mu\nu}|\gamma\gamma > \\ &\neq 0 \end{aligned} \tag{84}$$

and this permits a non-zero decay amplitude $\pi^0 \to \gamma\gamma$ in QED and QCD.

The constant multiplying the anomaly, $\sum_f Q_f^2$, measures the sum of the squares of the charges for all the fermions which make up $J_{\mu 5}$ (i.e. which go round the loop in the triangle diagram). Initial calculations with quarks gave a result for the decay amplitude 3 times smaller than experiment. This is resolved if we take the number of colours into account. So, experiment and the anomaly explanation of $\pi^0 \to \gamma\gamma$ implies that QCD must have $N_C = 3$.

Gauged currents:

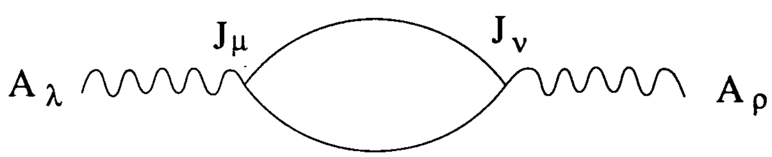
The situation is quite different if we couple a dynamical gauge field to the anomalous current (i.e. promote the anomalous symmetry to a local transformation). Here, the anomaly completely destroys the consistency of the quantum theory. The gauge symmetry is broken (since the current is not conserved) and the quantum theory is non-unitary (the unphysical and Faddeev-Popov ghost degrees of freedom do not decouple).

To see why this is so, consider again the QED action. This can be written as

$$S = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + eJ_{\mu}A^{\mu} \tag{85}$$

where $J_{\mu} = -\bar{\psi}\gamma_{\mu}\psi$ is the electromagnetic current. This action is $U(1)$ gauge invariant provided the current is conserved, i.e. $\partial^{\mu}J_{\mu} = 0$ using the equations of motion.

Now consider the photon self-energy diagram below:

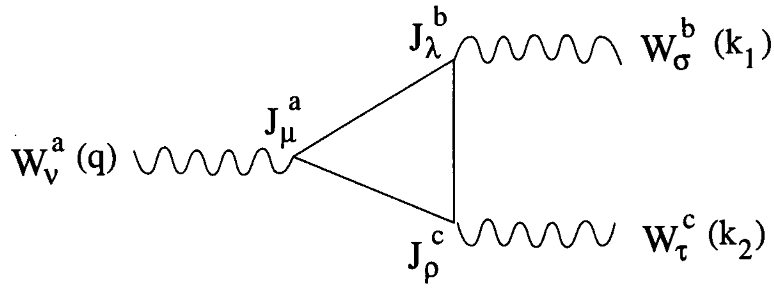


This can be expressed as $\Delta_{\lambda\mu}\Pi^{\mu\nu}\Delta_{\nu\rho}$, where $\Pi_{\mu\nu}$ is the two current Green function $\langle 0|J_{\mu}\ J_{\nu}|0\rangle$ and the photon propagator is $\Delta_{\lambda\mu} = \frac{1}{q^2}\left(g_{\lambda\mu} - (1-\frac{1}{\alpha})\frac{q_{\lambda}q_{\mu}}{q^2}\right)$. The current conservation condition implies that $q^{\mu}\Pi_{\mu\nu} = 0$. In turn, this means that the unphysical, longitudinal degrees of freedom in the photon propagator *decouple.* This must happen for the theory to be unitary. Otherwise, if the longitudinal degrees of freedom are allowed to propagate, the high energy behaviour of the theory is uncontrolled and cross-sections violate unitarity bounds. The conclusion is that the theory is only unitary if the gauged current is conserved. This is true for QED since the gauged current is pure vector, i.e. contains no γ_5 part.

Moving on to the electroweak interactions, we can write the interaction part of the Lagrangian as

$$S = \int J^a_{\mu}W^{\mu a} + J^Y_{\mu}B^{\mu} + \text{kinetic terms} \tag{86}$$

Now consider Feynman diagrams describing the coupling of three gauge bosons, for example:



This can be expressed as $\Delta^a_{\nu\mu} M^{\mu\lambda\rho}_{abc} \Delta^b_{\lambda\sigma} \Delta^c_{\rho\tau}$, where the Δ are the vector boson propagators and $M^{abc}_{\mu\lambda\rho}$ is the three current Green function (to lowest order in perturbation theory, this is just the single fermion loop shown above). Gauge invariance (current conservation) here would imply $q^\mu M^{abc}_{\mu\lambda\rho} = 0$, which just as above is essential if the unphysical degrees of freedom of the vector bosons are to decouple leaving a unitary theory with good high-energy behaviour. But as we have seen, this is not assured. Evaluating the triangle diagram using the Feynman rules, we actually find

$$q^\mu M^{abc}_{\mu\lambda\rho} \simeq \epsilon_{\lambda\rho\sigma\tau} k_1^\sigma k_2^\tau \ \Sigma_{\text{fermions}}\ Tr\Big[\{T^a, T^b\} T^c\Big] \tag{87}$$

where the T^a are the generators in the fermion currents. The theory will therefore only be gauge-invariant and unitary if the fermion spectrum is chosen so that the r.h.s. of this equation ('the anomaly') vanishes.

This is therefore dangerous for a chiral gauge theory such as the electroweak model, since we have gauge fields coupled to axial currents. The theory will only be unitary if all the potential anomalies vanish.

Rewriting in terms of left and right-handed fields, the anomaly coefficient is proportional to

$$\mathcal{A} = \sum_{reps} Tr\Big[\{T_L^a, T_L^b\} T_L^c - \{T_R^a, T_R^b\} T_R^c\Big] \tag{88}$$

There are four possible anomalies to check in the electroweak sector:

(1) a, b, c all $SU(2)_L$ currents:-
All fermions are in doublets, so

$$\begin{aligned} \mathcal{A} &= \sum_{\chi_L^q, \chi_L^\ell} Tr\Big\{\frac{\tau^a}{2}, \frac{\tau^b}{2}\Big\}\frac{\tau^c}{2} \\ &\sim \delta^{ab} \sum Tr T^c = 0 \end{aligned} \tag{89}$$

since the trace of an $SU(2)$ matrix vanishes.

(2) $a = SU(2)_L$ and $b, c = U(1)_Y$:-
In this case,

$$\mathcal{A} = \sum 2Tr\frac{\tau^a}{2} Y_L^2 = 0 \tag{90}$$

for the same reasons.

(3) $a, b = SU(2)_L$ and $c = U(1)_Y$:-
Here,

$$\begin{aligned} \mathcal{A} &= \sum Tr\{\frac{\tau^a}{2}, \frac{\tau^b}{2}\}Y_L \\ &\sim \delta^{ab}\sum TrY_L \sim \sum_{Lreps} TrQ \end{aligned} \tag{91}$$

since for the left-handed representations $\frac{1}{2}Y_L = -T_L^3 + Q$

(4) a, b, c all $U(1)_Y$:-
Here,

$$\begin{aligned} \mathcal{A} &= 2\Big\{\sum_{Lreps} TrY_L^3 - \sum_{Rreps} TrY_R^3\Big\} \\ &\sim \sum_{Lreps} TrQ \end{aligned} \tag{92}$$

since for the right-handed representations $\frac{1}{2}Y_R = Q$.

So, the anomalies of type (1) and (2) necessarily vanish. But the anomalies for type (3) and (4) vanish if and only if

$$\sum_f Q_f = 0 \tag{93}$$

i.e. anomaly cancellation requires the sum of the electric charges of the fermions to vanish.

In the standard model, this is true individually for each generation:-

$$\begin{aligned} \sum_{f=\nu_e,e,u,d} Q_f &= 0 - 1 + N_C(\frac{2}{3} - \frac{1}{3}) = -1 + \frac{1}{3}N_C \\ &= 0 \quad \text{for} \quad N_C = 3 \end{aligned} \tag{94}$$

This theoretical analysis tells us several important things about the standard model

1. The $SU(N_C) \times SU(2)_L \times U(1)_Y$ gauge theory (QCD plus electroweak) with the known quark and lepton spectrum *must* have $N_C = 3$
2. Anomaly cancellation within each generation means that a model with two lepton generations and the quarks u, d, s does not exist. Anomaly freedom implies that charm exists!
3. 3 lepton generations implies that top exists.

4. The condition $\sum_f Q_f = 0$ as described above looks contrived. This suggests that the quarks and leptons may have originally been in some single larger representation. This hints at some form of grand unification.

5.4 Third generation and the CKM matrix

To keep the success of the GIM mechanism when we extend the standard model to three generations, we assign the quarks to $SU(2)_L$ eigenstates $\begin{pmatrix} u_{iL} \\ d'_{iL} \end{pmatrix}$, where $u_{iL} = u_L,\ c_L,\ t_L$ and $d_{iL} = d_L,\ s_L,\ b_L$, with

$$d'_{iL} = V_{ij}\, d_{jL} \tag{95}$$

If V is unitary ($V^\dagger V = 1$), then

$$\bar{d}'_i\, d'_i \;=\; \bar{d}\, V^\dagger\, V\, d \;=\; \bar{d}_i\, d_i \tag{96}$$

This property suppresses the FCNC diagrams discussed in sect. 5.3 and ensures the neutral current is flavour diagonal. V enters the vertices with the $W^\pm$ but not with the Z.

V is the CKM (Cabibbo-Kobayashi-Maskawa) matrix. First, note the parameter counting for an arbitrary number N of generations:

$$\begin{aligned} \text{Unitary } N \times N \text{ matrix} \quad &\to \quad N^2 \text{ parameters.} \\ \text{Orthogonal } N \times N \text{ matrix} \quad &\to \quad \frac{1}{2} N(N-1) \text{ parameters.} \end{aligned}$$

But $(2N-1)$ relative phases for the quarks are irrelevant. So V_{CKM} has $\frac{1}{2}N(N-1)$ real parameters and $N^2 - (2N-1) - \frac{1}{2}N(N-1) = \frac{1}{2}(N-1)(N-2)$ phases.

In the standard model, $N = 3$, so V_{CKM} has 3 angles and 1 phase (important for CP violation). The Kobayashi-Maskawa parametrisation is

$$\begin{aligned} V_{CKM} \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \\ &=\; \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \end{aligned} \tag{97}$$

where we let $c_1 = \cos\,\theta_1$ etc.

The current experimental values are approximately

$$V_{CKM} = \begin{pmatrix} |V_{ud}| = 0.975 & |V_{us}| = 0.222 & |V_{ub}| = 0.005 \\ |V_{cd}| = 0.222 & |V_{cs}| = 0.974 & |V_{cb}| = 0.043 \\ |V_{td}| = 0.010 & |V_{ts}| = 0.041 & |V_{tb}| = 0.999 \end{pmatrix} \tag{98}$$

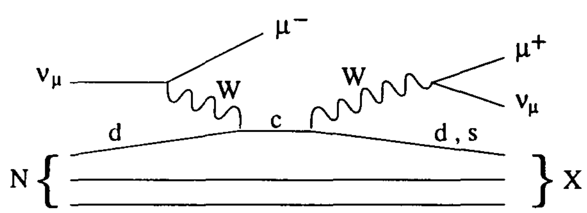
For precise values and errors and a recent review, see Ali and London (Glasgow Conference, 1994).
These arise from:-

V_{ud} : βdecay $n \rightarrow pe^- \bar{\nu}_e$, $\pi^+ \rightarrow \pi^0 e^+ \nu_e$

V_{us} : $K^+ \rightarrow \pi^0 e^+ \nu_e$, $K^0 \rightarrow \pi^- e^+ \nu_e$
semileptonic hyperon decays $\Lambda \rightarrow pe^- \bar{\nu}_e$

V_{ub} : $b \rightarrow ue^- \bar{\nu}_e$, need B decays with no K in final state

V_{cd} : $\nu_\mu d \rightarrow \mu^- e$, as in the diagram

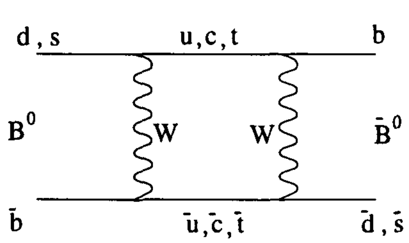


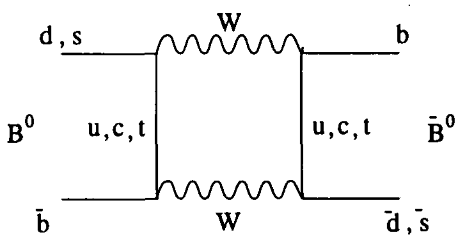
V_{cs} : $\nu_\mu s \rightarrow \mu^- c$, needs estimate of s-content of nucleon
$D^+ \rightarrow \bar{K}^0 e^+ \nu_e$

V_{cb} : $B \rightarrow D^* \ell \bar{\nu}_\ell$, plus heavy quark effective theory

δ : The phase is determined by the ϵ parameter in $K^0 - \bar{K}^0$

V_{td} : $B^0 - \bar{B}^0$ mixing, from diagrams like





Clearly a great deal of experimental work (and theoretical analysis – much depending on models and approximations for heavy quark states) is being done to determine the quark mixing parameters and verify the assumption that V_{CKM} is a unitary matrix. There is at present no accepted theory of these mixing angles – they are all free parameters in the standard model.

5.5 *CP* violation and the CKM matrix

Much of the interest in V_{CKM} is because it is the only source of CP violation in the standard model. We show here why the appearance of a phase in V_{CKM} leads to CP violation. Let

$$\begin{aligned} M &= (\bar{u}_k\gamma^\mu(1-\gamma_5)V_{ki}u_i)(\bar{u}_j\gamma_\mu(1-\gamma_5)V_{jl}u_l)^\dagger \\ &= V_{ki}\,V_{jl}^*(\bar{u}_k\gamma^\mu(1-\gamma_5)u_i)(\bar{u}_l\gamma_\mu(1-\gamma_5)u_j) \end{aligned} \tag{99}$$

be the charged-current induced matrix element for $q_iq_j \rightarrow q_kq_l$. u are the appropriate Dirac spinors. If we can show that the CP transformed matrix element satisfies $M_{CP} = M^\dagger$, then the theory conserves CP. Otherwise, CP is violated.

Under C,

$$\begin{aligned} u &\rightarrow u_C = C\bar{u}^T \\ \bar{u} &\rightarrow \bar{u}_C = -u^TC^{-1} \end{aligned} \tag{100}$$

where $C^{-1}\gamma_\mu C = -\gamma_\mu^T \ , \ C^{-1}\gamma_\mu\gamma_5 C = (\gamma_\mu\gamma_5)^T$.

Under P, $P^{-1}\gamma_\mu(1+\gamma_5)P = \gamma_\mu^\dagger(1-\gamma_5)$ where $\gamma_0^\dagger = \gamma_0\ ,\ \gamma_i^\dagger = -\gamma_i$.

So,

$$\bar{u}_k\gamma^\mu(1-\gamma_5)V_{ki}u_i \xrightarrow{CP} -V_{ki}\bar{u}_i\gamma_\mu^\dagger(1-\gamma_5)u_k \tag{101}$$

and then,

$$M_{CP} = V_{ki}\,V_{jl}^*(\bar{u}_i\gamma^\mu(1-\gamma_5)u_k)(\bar{u}_j\gamma_\mu(1-\gamma_5)u_l) \tag{102}$$

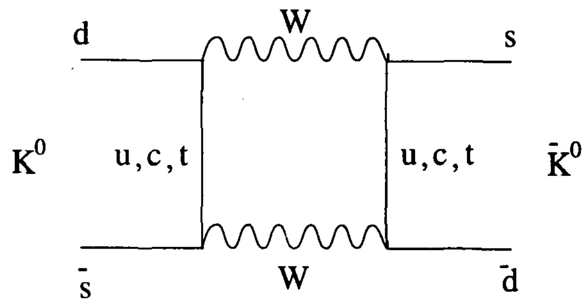
compared with

$$M^\dagger = V_{ki}^*\,V_{jl}(\bar{u}_i\gamma^\mu(1-\gamma_5)u_k)(\bar{u}_j\gamma_\mu(1-\gamma_5)u_l) \tag{103}$$

We find that $M_{CP} = M^\dagger$ provided V_{ij} are *real.*

It follows that in the three generation model where V_{CKM} has a complex parameter, CP is violated. This will show up in $K^0-\bar{K}^0$ or $B^0-\bar{B}^0$ mixing (see figure).

In the two generation model, K^0 and $\bar{K}^0$ are linear combinations of the $CP = +1, -1$ eigenstates, $K_S = \frac{1}{\sqrt{2}}(K^0+\bar{K}^0)$ and $K_L = \frac{1}{\sqrt{2}}(K^0-\bar{K}^0)$, which decay by $K_S \rightarrow 2\pi$ and $K_L \rightarrow 3\pi$. However $K_L \rightarrow 2\pi$ does occur with a small branching ratio of $\sim 10^{-3}$.



5.6 Gauge boson-current interaction (quarks)

The same construction as for the lepton sector now goes through essentially unchanged. In the electroweak Lagrangian, the interaction of the A, W and Z bosons with the quarks is

$$\mathcal{L} = -\frac{g}{\sqrt{2}}\left(J_\mu^+ W^{+\,\mu} + J_\mu^- W^{-\mu}\right) - eJ_\mu^{em}A^\mu - \frac{g}{\cos\theta_W}J_\mu^{NC}Z^\mu \tag{104}$$

where

$$\begin{aligned} J_\mu^+ &= \bar{\chi}_{i\,L}\,\gamma_\mu\tau^+\chi_{i\,L} = \bar{u}_{i\,L}\,\gamma_\mu V_{ij}d_{j\,L} \\ J_\mu^- &= \bar{\chi}_{i\,L}\,\gamma_\mu\tau^-\chi_{i\,L} = \bar{d}_{i\,L}\,\gamma_\mu V_{ij}^\dagger u_{j\,L} \end{aligned} \tag{105}$$

since

$$\chi_{iL} = \begin{pmatrix} u_{i\,L} \\ d'_{i\,L} \end{pmatrix} \qquad \text{with } d'_{i\,L} = V_{ij}\,d_{j\,L} \tag{106}$$

and

$$J_\mu^3 = \bar{\chi}_{i\,L}\,\gamma_\mu\tau^3\chi_{i\,L} = \bar{u}_{i\,L}\,\gamma_\mu u_{i\,L} - \bar{d}_{i\,L}\,\gamma_\mu d_{i\,L} \tag{107}$$

the CKM matrix V dropping out due to its assumed unitarity. The electromagnetic and weak neutral currents are

$$J_\mu^{em} = \bar{q}_f\gamma^\mu Q\,q_f, \qquad f = u,\ c,\ t,\ d,\ s,\ b. \tag{108}$$

and

$$\begin{aligned} J_\mu^{NC} &= J_\mu^3 - \sin^2\theta_W J_\mu^{em} \\ &= \bar{q}_f\gamma^\mu\left(\tfrac{1}{2}(1-\gamma_5)\,t^3 - \sin^2\theta_W Q\right)q_f \\ &= \bar{q}_f\gamma^\mu\tfrac{1}{2}\left(c_V^{q_f} - c_A^{q_f}\gamma_5\right)q_f \end{aligned} \tag{109}$$

where t^3 is the eigenvalue of T_L^3 and Q is the charge.

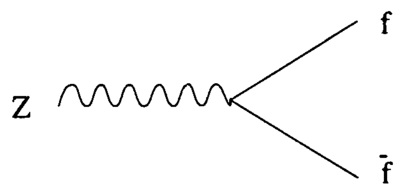
The eigenvalues t^3, Q and parameters c_V and c_A in the neutral current J_μ^{NC} are listed in the table below ($\sin^2\theta_W \simeq 0.234$).
The general formulae are

$$c_A = t^3$$

	t^3	Q	c_A	c_V
u, c, t	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w = 0.19$
d, s, b	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w = -0.34$
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
e, μ, τ	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_w = -0.03$

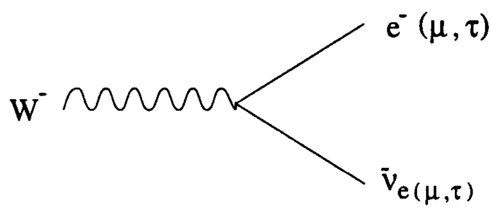
$$c_V = t^3 - 2\sin^2\theta_W Q \tag{110}$$

This determines the $Zf\bar{f}$ vertex:-

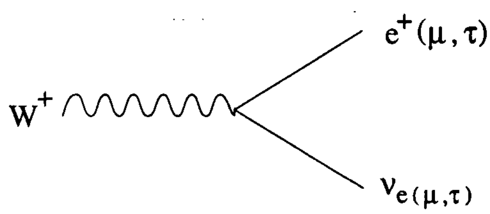


$$\frac{-ie}{\sin\theta_W\cos\theta_W}\gamma_\mu\frac{1}{2}\left(c_V^f - c_A^f\gamma_5\right)$$

along with the $W^\pm$ vertices for leptons

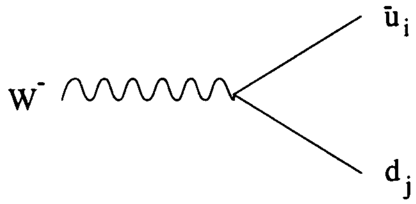


$$-\frac{i}{\sqrt{2}}\frac{e}{\sin\theta_W}\gamma^\mu\frac{1}{2}(1-\gamma_5)$$

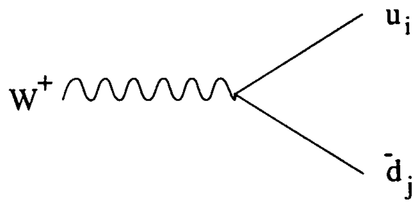


$$-\frac{i}{\sqrt{2}}\frac{e}{\sin\theta_W}\gamma^\mu\frac{1}{2}(1-\gamma_5)$$

and for quarks, including the CKM matrix,



$$-\frac{i}{\sqrt{2}}\frac{e}{\sin\theta_W}\gamma_\mu\frac{1}{2}\left(1-\gamma_5\right)V_{ij}$$



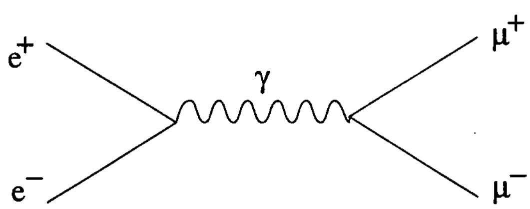
$$-\frac{i}{\sqrt{2}}\frac{e}{\sin\theta_W}\gamma_\mu\frac{1}{2}\left(1-\gamma_5\right)V^\dagger_{ji}$$

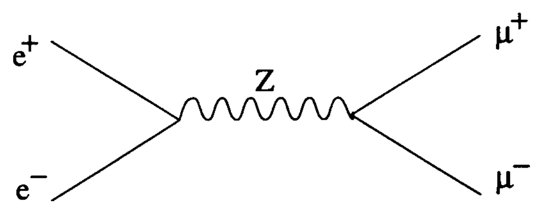
6 Electroweak Processes

In this section, we consider some simple examples of electroweak processes using the structure of the currents described above.

6.1 $\gamma - Z$ interference in $e^+ e^- \;\rightarrow\; \mu^+\mu^-$

Consider the following diagrams, which mediate electron-positron annihilation into leptons:





The amplitudes are

$$M_\gamma = -\frac{e^2}{k^2}\left(\bar{u}_\mu\gamma^\lambda u_\mu\right)\left(\bar{u}_e\gamma_\lambda u_e\right) \tag{111}$$

and

$$\begin{aligned} M_Z \;&=\; -\frac{g^2}{4\,\cos^2\theta_W}\left(\bar{u}_\mu\gamma^\lambda\left(c_V^\mu - c_A^\mu\gamma_5\right)u_\mu\right)\left(\frac{g_{\lambda\rho}-\frac{k_\lambda k_\rho}{m_Z^2}}{k^2-m_Z^2}\right) \\ &\quad\times\left(\bar{u}_e\gamma^\rho\left(c_V^e - c_A^e\gamma_5\right)u_e\right) \end{aligned}$$

$$= -\frac{\sqrt{2}\,G\,m_Z^2}{s-m_Z^2}\left(c_R^{\mu}\bar{u}_{\mu\,R}\gamma^{\lambda}u_{\mu\,R} + c_L^{e}\bar{u}_{e\,L}\gamma^{\lambda}u_{e\,L}\right) \times \left(c_R^{e}\bar{u}_{e\,R}\gamma_{\lambda}u_{e\,R} + c_L^{e}\bar{u}_{e\,L}\gamma_{\lambda}u_{e\,L}\right) \qquad (112)$$

where $s = k^2$ and we neglect lepton masses. Recall $c_{L(R)} = c_V \pm c_A$

To calculate the cross section, we first add these amplitudes then square, i.e. $|M_\gamma + M_Z|^2$. This is electroweak interference.

The unpolarised $e^+e^- \to \mu^+\mu^-$ cross section is found by averaging over the four allowed L, R helicity combinations for e and μ. We find

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left[A_0\left(1+\cos^2\theta\right) + A_1\cos\theta\right] \qquad (113)$$

where

$$\begin{aligned} A_0 &= 1 + 2\Re(r)c_V^2 + |r|^2\left(c_V^2 + c_A^2\right)^2 \\ A_1 &= 4\Re(r)c_A^2 + 8|r|^2 c_V^2 c_A^2 \end{aligned} \qquad (114)$$

with

$$r = \frac{s}{e^2}\,\frac{\sqrt{2}\,G\,m_Z^2}{s - m_Z^2 + i m_Z \Gamma_Z} \qquad (115)$$

r comes from the Z propagator, modified to include the finite resonance width Γ_Z which must be included when $s \simeq m_Z^2$. In pure QED, $A_0 = 1$ and $A_1 = 0$.

This cross section is usually expressed as a forward-backward asymmetry. Define,

$$A_{FB} = \frac{F-B}{F+B}, \qquad F = \int_0^1 \frac{d\sigma}{d\Omega}\,d\Omega\,,\; B = \int_{-1}^0 \frac{d\sigma}{d\Omega}\,d\Omega \qquad (116)$$

Then we have

$$A_{FB} = \frac{3}{8}\frac{A_1}{A_0}, \qquad (s \ll m_Z^2) \qquad (117)$$

and

$$A_{FB} = 3\,\frac{c_V^2\,c_A^2}{(c_V^2 + c_A^2)^2} \qquad (s \simeq m_Z^2) \qquad (118)$$

6.2 Z partial widths

From the $Zf\bar{f}$ vertex,

$$-i\frac{g}{\cos\theta_W}\gamma_\mu \frac{1}{2}\left(c_V^f - c_A^f\gamma_5\right) \qquad (119)$$

we can calculate the decay rate,

$$\Gamma\left(Z \to f\bar{f}\right) = \frac{g^2}{48\pi\,\cos^2\theta_W}\left({c_V^f}^2 + {c_A^f}^2\right) m_Z \qquad (120)$$

This enables us to compute the partial widths for the set of decays:-

$$\begin{aligned}
\Gamma(Z \to \bar{\nu}_e \nu_e) &\equiv \Gamma_Z^0 = \frac{g^2 m_Z}{96\pi \cos^2\theta_W} &= 0.17\,GeV \\
\Gamma\left(Z \to e^+e^-\right) &= \Gamma_Z^0 \left(1 - 4\sin^2\theta_W + 8\sin^4\theta_W\right) &= 0.09\,GeV \\
\Gamma(Z \to \bar{u}\,u) &= 3\,\Gamma_Z^0 \left(1 - \frac{8}{3}\sin^2\theta_W + \frac{32}{9}\sin^4\theta_W\right) &= 0.30\,GeV \\
\Gamma\left(Z \to \bar{d}\,d\right) &= 3\,\Gamma_Z^0 \left(1 - \frac{4}{3}\sin^2\theta_W + \frac{8}{9}\sin^4\theta_W\right) &= 0.39\,GeV
\end{aligned} \quad (121)$$

(The 3 in the last two expressions is the number of colours, $N_C = 3$)

The more light generations, i.e. with mass less than $m_Z/2$, the bigger the Z width. LEP measurements can therefore determine the number of light generations. The experimental value

$$\Gamma_Z\,(total) \sim 2.6GeV \quad (122)$$

confirms $N_\nu = 3$.

Cosmological Interlude – $\mathbf{N_\nu = 3}$ from Big Bang Nucleosynthesis

As well as the LEP measurement of Γ_Z, there is good evidence for $N_\nu = 3$ from measurements of the ^{4}He abundance in the universe. This is based on primordial nucleosynthesis in the big bang model. Very roughly, the argument is as follows:-

1. Most ($\sim 90\%$) of the present day ^{4}He abundance is primordial. ^{4}He production in stars contributes $< 10\%$.

2. At high temperatures ($kT \gg 1MeV$) just after the big bang, neutrons and protons were in equilibrium through the reversible processes

$$\begin{aligned}
n &\to p\,e^-\bar{\nu}_e & (123) \\
n + e^+ &\to p + \bar{\nu}_e & (124) \\
n + \nu_e &\to p + e^- & (125)
\end{aligned}$$

with a neutron to proton ratio (n/p) of

$$\frac{n}{p} \sim e^{-\Delta m/kT} \quad (126)$$

where $\Delta m = m_n - m_p = 1.3MeV$.

3. When kT drops below $1MeV$ (after $t = 10sec$), the rate for $p \to n$ processes becomes much smaller than $n \to p$. When this rate falls below the expansion rate of the universe, the $p \to n$ transitions "freeze-out", fixing the n/p ratio (apart from neutron decay.)
   Now, the expansion rate depends on the square root of the energy density of relativistic particles, so is greater for a larger number of light particles, viz. neutrinos with $m_\nu < 100MeV$.

   $$\begin{aligned} \text{So, a bigger } N_\nu &\Rightarrow \text{ faster expansion rate} \\ &\Rightarrow \text{ earlier freeze-out of } n/p \text{ at higher } T \\ &\Rightarrow \text{ bigger freeze-out } n/p \text{ ratio.} \end{aligned}$$

   In fact, $N_\nu = 3 \Leftrightarrow n/p \sim 1/6$ at freeze-out.

4. Nucleosynthesis begins later, at around $t = 2$ mins, when the temperature is low enough for deuterons to be stable against photodisintegration. By this time, free neutron decay has reduced n/p to $1/7$.

5. Virtually all the neutrons in existence at the start of primordial nucleosynthesis end up as ^{4}He.
   So, the bigger the n/p ratio the greater the abundance of ^{4}He.

The present value for ^{4}He abundance ($\approx 24\%$) rules out $N_\nu = 4$ and is consistent with $N_\nu = 3$. Further evidence comes from a detailed investigation of abundances of ^{4}He, D and ^{7}Li.

6.3 3 and 4 gauge boson vertices

Recalling that

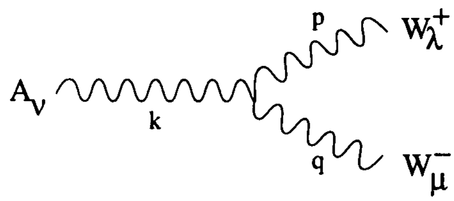
$$F^A_{\mu\nu} = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu - g\,\epsilon^{ABC} W^B_\mu W^C_\nu \tag{127}$$

we see from the field strength terms in the Lagrangian

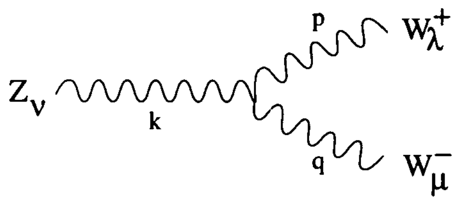
$$\mathcal{L}_{gauge} = \int dx \left[-\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{128}$$

where the first term corresponds to $SU(2)_L$ and the second to $U(1)_Y$, that there will be vertices with 3 and 4 gauge field propagators.

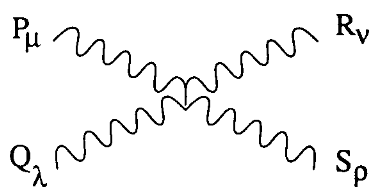
In terms of the A, $W^\pm$ and Z fields, these are:



$$e\,[(k-q)_\lambda\, g_{\mu\nu} \;+\; (q-p)_\nu\, g_{\lambda\mu} \;+\; (p-k)_\mu\, g_{\lambda\nu}]$$



$$g\,\cos\theta_W\,[(k-q)_\lambda\, g_{\mu\nu} \;+\; (q-p)_\nu\, g_{\lambda\mu} \;+\; (p-k)_\mu\, g_{\lambda\nu}]$$



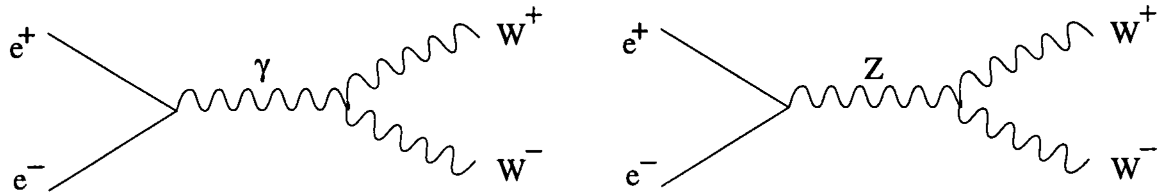
$$K_{PQRS}\,[2\,g_{\mu\nu}g\lambda\rho \;-\; g_{\mu\lambda}g\nu\rho \;-\; g_{\mu\rho}g\nu\lambda]$$

where for the different possible vertices:-

P	Q	R	S	K_{PQRS}
W^+	W^-	W^+	W^-	ig^2
A	W^+	A	W^-	$-ie^2$
Z	W^+	Z	W^-	$-ig^2\cos^2\theta_w$
A	W^+	Z	W^-	$-ie\,g\cos\theta_w$

(recall $e = g\sin\theta_W$)

At LEP 200, with e^+e^- collisions at $100 + 100GeV$, it will soon be possible to pair produce W^+W^- through the diagrams:



This will provide the first direct measurement of the 3 gauge boson coupling.

7 Spontaneous Symmetry Breaking I – Global Symmetries and Goldstone's Theorem

7.1 A global $U(1)$ model

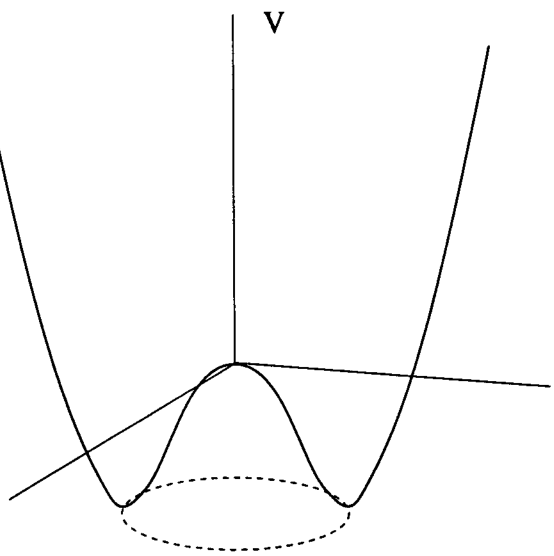
As a toy model, consider a complex scalar field with Lagrangian

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - V(\phi, \phi^*) \tag{129}$$

with

$$V(\phi, \phi^*) = -\mu^2 \phi^* \phi + \lambda(\phi^* \phi)^2 \tag{130}$$

We have chosen the opposite sign from usual for the quadratic term.



Plot of V over the complex ϕ plane.

Rewriting ϕ in modulus-phase form, $\phi = \frac{1}{\sqrt{2}}\rho \, e^{i\chi}$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2(\partial_\mu \chi)^2 - V(\rho) \tag{131}$$

where

$$V(\rho) = -\frac{1}{2}\mu^2\rho^2 + \frac{\lambda}{4}\rho^4 \tag{132}$$

This theory has a global $U(1)$ symmetry, i.e. invariance under $\phi \to \phi\, e^{i\alpha}$, $\alpha =$ constant, i.e. $\chi \to \chi + \alpha$. This is reflected in the potential, which depends only on ρ.

With the sign of μ^2 chosen, the minimum of $V(\rho)$ is not $\rho = 0$, but at the bottom of the rim. The minimum is not unique – there is a family of degenerate minima connected by $U(1)$ transformations.

Select one of these equivalent minima, say $\chi = 0$, $\rho = v$, then write $\rho = v + H$. In the quantum theory v is the vacuum expectation value (VEV) of ϕ, i.e.

$$\langle 0|\phi|0\rangle = v \neq 0. \tag{133}$$

Then,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^2 + \frac{1}{2}v^2(\partial_\mu\chi)^2 + \left(v\, H + \tfrac{1}{2}H^2\right)(\partial_\mu\chi)^2 - V(H) \tag{134}$$

where

$$V(H) = -\frac{1}{4}\lambda v^4 + \lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4 \tag{135}$$

At the minimum, $v^2 = \mu^2/\lambda$.

In perturbation theory about this minimum, the H field describes a massive scalar particle with $m_H^2 = 2\lambda v^2$.

Rewriting in terms of λ and m_H, and rescaling $\tilde{\chi} = v\chi$ so that, as usual for a scalar field, $\tilde{\chi}$ has dimension 1, the Lagrangian is

$$\begin{aligned}\mathcal{L} = \; & \tfrac{1}{2}(\partial_\mu\tilde{\chi})^2 + \tfrac{1}{2}(\partial_\mu H)^2 - \tfrac{1}{2}m_H^2 H^2 + \tfrac{\sqrt{2\lambda}}{m_H}H(\partial_\mu\tilde{\chi})^2 \\ & + \tfrac{\lambda}{m_H^2}H^2(\partial_\mu\tilde{\chi})^2 - \sqrt{\tfrac{\lambda}{2}}\,m_H H^3 - \tfrac{\lambda}{4}H^4 + \tfrac{1}{16}\tfrac{1}{\lambda}m_H^4\end{aligned} \tag{136}$$

We can read off the spectrum of the quantum theory from $\mathcal{L}$. The theory has one massive scalar H – this corresponds to fluctuations up the side of the walls in the potential. Crucially, it also has one *massless* scalar $\tilde{\chi}$, corresponding to fluctuations around the circle of degenerate minima. This is known as a Goldstone boson.

There are also interaction terms, and a constant non-zero vacuum energy density. (This could be a problem if we think of including gravity in a theory with SSB.)

7.2 Goldstone's theorem

This model illustrates a general theorem. We say that a symmetry is spontaneously broken if the vacuum is not invariant under the symmetry, i.e. if a field which varies under the symmetry acquires a VEV. This field is said to be an "order parameter" in the language of statistical mechanics.

In the model above, $\mathcal{L}$ is invariant under $U(1)$, but the vacuum state has no residual invariance. $U(1)$ is broken to the identity.

In general, $\mathcal{L}$ will have a symmetry G and the vacuum will have a residual invariance under a subgroup G_0. We say the symmetry is broken from G to G_0. In that case, the space of degenerate minima is the coset manifold G/G_0.

Goldstone's theorem:

This states that corresponding to each broken generator of G (i.e. a generator in G which is not in G_0) there is a massless scalar boson in the spectrum.

The corresponding scalar field $\chi(x)$ takes values in the coset manifold G/G_0.

***Proof:**

We give a general, non-perturbative proof in quantum theory. Corresponding to each symmetry generator in G there is a conserved current. The Ward Identity is

$$\langle 0|T^* \partial^\mu J_\mu^a \ \Phi|0\rangle = \langle 0|\delta^a \Phi|0\rangle \tag{137}$$

where $\delta^a\Phi$ is the variation of Φ under the generator T^a of the group G.

The VEV is equal to zero for the unbroken generators, i.e. T^a in G_0. But for the broken generators, i.e. T^a in G but not in G_0, we have

$$k^\mu \langle 0|J_\mu^a(k) \ \Phi(-k)|0\rangle \neq 0 \tag{138}$$

writing the Green function in momentum space. This is true for all momenta, in particular $k_\mu = 0$.

The only way this can be true is if there exists a massless state $|\chi\rangle$ in the spectrum coupling to the broken current. Then

$$\begin{aligned} k^\mu \langle 0|J_\mu^a|\chi\rangle \ \Delta_{\chi\chi} \ \langle\chi|\Phi|0\rangle &= k^\mu i k_\mu F_\chi \ \frac{1}{k^2}\langle\chi|\Phi|0\rangle \\ &\neq 0. \end{aligned} \tag{139}$$

where $\Delta_{\chi\chi}$ is the χ propagator $\sim \frac{1}{k^2}$ and F_χ is the decay constant. Clearly there is one massless χ state for each broken current.

7.3 *Chiral symmetry breaking in QCD

An important example of global spontaneous symmetry breaking occurs in QCD. Consider QCD with just two flavours u and d and neglect their masses. Since QCD is independent of flavour, there is a rotation symmetry between u and d. Also, since parity is conserved for massless quarks, we can rotate the left and right handed fields separately. So, massless QCD has a global symmetry $SU(2)_L \times SU(2)_R$.

This is spontaneously broken to the $SU(2)_V$ subgroup (the axial generators are all broken) by the appearance of a VEV $\langle 0|\bar{u}u + \bar{d}d|0\rangle$, also called a "condensate".

Since we have SSB with $G = SU(2)_L \times SU(2)_R$ and $G_0 = SU(2)_V$, Goldstone's theorem says there are 3 massless pseudoscalar bosons (since $3 = \dim G/G_0$).

These are the pions, π^+, π^-, π^0, which would be exactly massless in QCD with $m_u = m_d = 0$.

8 Spontaneous Symmetry Breaking II – Gauged Symmetries and the Higgs Mechanism

8.1 A local $U(1)$ model

Now go back to the toy model of section 7.1 and make the $U(1)$ into a local (gauge) symmetry. The Lagrangian is

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - V(\phi,\phi^*) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{140}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$. The potential is the same, with a non-zero vacuum expectation value for ϕ. Making the substitution $\phi = \frac{1}{\sqrt{2}}(v+H)e^{i\chi}$ we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^2 + \frac{1}{2}(v+H)^2(\partial_\mu\chi - eA_\mu)^2 - V(H) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{141}$$

with $V(H)$ as before. Now write

$$W_\mu = A_\mu - \frac{1}{e}\partial_\mu\chi \tag{142}$$

Since this is a gauge transformation, $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$, independent of χ. This leaves

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^2 + \frac{1}{2}e^2v^2 W_\mu W^\mu + e^2(vH + \frac{1}{2}H^2)W_\mu W^\mu - V(H) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{143}$$

In this form we can read off the particle content we expect the quantum theory to have:

The χ field has disappeared! So there are no massless scalar bosons.

The W_μ field is massive, with $m_W = e^2v^2$. It therefore has 3 degrees of freedom (two inherited from A and one from χ).

So, starting from a theory with a $U(1)$ gauge symmetry, we find that the spectrum in the SSB phase has a massive gauge boson. This is the Higgs Mechanism.

Going back to $\mathcal{L}$, in its original form it had 3 parameters, e, λ, μ. The final theory has a massive W and a massive H, so we can write $\mathcal{L}$ in terms of e, m_H, m_W.
(Check: $v^2 = \frac{\mu^2}{\lambda}$, then $m_H^2 = 2\lambda v^2$ and $m_W^2 = e^2 v^2$).
This gives

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + \frac{1}{2}m_W^2 W_\mu W^\mu + \frac{1}{2}(\partial_\mu H)^2 \\ &- \frac{1}{2}m_H^2 H^2 + e m_W H W_\mu W^\mu + \frac{1}{2}e^2 H^2 W_\mu W^\mu \\ &- \frac{1}{2}e\frac{m_H^2}{m_W}H^3 - \frac{1}{8}e^2\frac{m_H^2}{m_W^2}H^4 + \frac{1}{8e^2}m_H^2 m_W^2 \end{aligned} \tag{144}$$

8.2 Quantisation and renormalisation

Notice that the above description of the Higgs Mechanism was entirely at the classical level. Strictly speaking, it is no more than a plausibility argument as to what we expect in the full quantum theory.

Remember that to quantise a gauge theory, we have to start with the functional integral, introduce a gauge-fixing term, and construct the Faddeev-Popov ghosts. To obtain the physical spectrum, we have to prove that these ghosts decouple along with the unphysical components of the gauge field.

All this has to be re-done in a theory with SSB. It works and the spectrum is as described above.

Gauge invariance is essential to the renormalisation of the theory. We have to prove ('t Hooft, 1971) that SSB does not spoil renormalisation, despite the appearance of gauge boson masses.

The beauty of the Higgs Mechanism is that this is true – gauge theories with spontaneous symmetry breaking *are* renormalisable.

9 The Higgs Mechanism and Mass Generation in the $SU(2)_L \times U(1)_Y$ Model

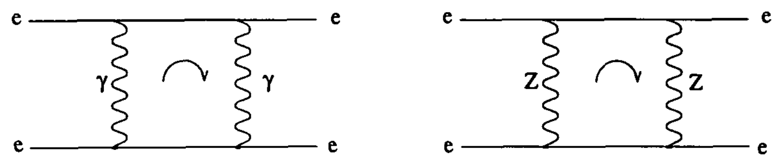
9.1 Mass generation

In the form we have presented so far, the $SU(2)_L \times U(1)_Y$ electroweak model has no mass for either the gauge bosons or fermions.

Gauge bosons:

Mass terms for $W^{\pm}$ or Z simply added to $\mathcal{L}$ violate the gauge symmetry. (This is also true for any gauge theory.) But the gauge symmetry is necessary for the theory to be renormalisable, and therefore predictive.

For example, consider $e^- e^-$ scattering at 1-loop. The Feynman diagrams include



The loop gives

$$\int d^4q \Delta_e \Delta_e \Delta_\gamma \Delta_\gamma \overset{q\to\infty}{\sim} \int d^4q \frac{1}{q}\frac{1}{q}\frac{1}{q^2}\frac{1}{q^2} \sim \int d^4q \frac{1}{q^6} = convergent$$

for the photon diagram, since the photon propagator (in Feynman gauge) is $\Delta_\gamma = -\frac{i}{q^2} g_{\mu\nu}$.

On the other hand, the Z propagator is $\Delta_Z = \frac{i}{q^2 - m_Z^2}(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_Z^2})$ so the loop gives

$$\int d^4q \Delta_e \Delta_e \Delta_Z \Delta_Z \overset{q\to\infty}{\sim} \int d^4q \frac{1}{q^2} = divergent$$

The divergence has to be cancelled by a counterterm

$$\mathcal{L}_{counterterm} = (div)\frac{1}{m_Z^2} \int dx\, e\, e\, e\, e$$

which is a four-Fermi interaction (dim = 6). But this introduces a new parameter. The process continues and an infinite set of higher dimension operators are induced. The theory is non-renormalisable.

We therefore need a dynamical mechanism to generate vector boson masses while keeping gauge invariance. This is achieved by the Higgs mechanism.

Fermions:

In general, we can add fermion mass terms to the Lagrangian in a gauge theory. For example, in QCD we can add quark masses, $\mathcal{L}_{mass} = \int dx\, m(\bar{q}_R q_L + \bar{q}_L q_R)$. $\mathcal{L}_{mass}$ is gauge invariant.

However, in $SU(2)_L \times U(1)_Y$, because $SU(2)_L$ is a chiral gauge theory (the group acts only on the left handed fields) fermion masses violate the gauge symmetry. For example $\mathcal{L}_{mass} = \int dx\, m(\bar{e}_R e_L + \bar{e}_L e_R)$ is *not* invariant under an $SU(2)_L$ transformation.

We therefore need a mechanism to generate fermion masses dynamically in the standard model. Remarkably, the Higgs fields can also achieve this, through Yukawa couplings.

9.2 Higgs mechanism in $SU(2)_L \times U(1)_Y$.

We need to repeat the analysis of the $U(1)$ Higgs mechanism described earlier, generalised to a non-abelian theory. The aim is to find a Higgs sector which will break $SU(2)_L \times U(1)_Y$ to $U(1)_{em}$.

The simplest choice (Weinberg and Salam, 1967) is to take

$$D_\mu \phi = (\partial_\mu + igT^A W_\mu^A + \frac{ig'}{2} Y B_\mu)\phi \tag{145}$$

where ϕ is a complex $SU(2)_L$ doublet with $Y = 1$. Then

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D_\mu \phi) - V(\phi, \phi^\dagger) \tag{146}$$

with the potential $V(\phi, \phi^\dagger) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$. So, remembering the relation $Q = T_L^3 + \frac{1}{2}Y$, the charge assignment is

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{147}$$

If charge conservation is to remain unbroken, only the ϕ^0 should get a vacuum expectation value. This motivates rewriting

$$\phi = \frac{1}{\sqrt{2}} e^{i\frac{1}{v}T^A \chi^A} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{148}$$

where χ^A, H are real fields. The potential is $V(H) = -\frac{1}{2}\mu^2 (v+H)^2 + \frac{\lambda}{4}(v+H)^4$ and we have chosen $v^2 = \frac{\mu^2}{\lambda}$ to give the minimum at $H = 0$.

Now substitute ϕ into $\mathcal{L}_{Higgs}$. We have

$$\phi = \frac{1}{\sqrt{2}} U \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{149}$$

where U is an $SU(2)_L$ gauge transformation. But since $\mathcal{L}$ is gauge invariant, it will not depend on U, which can be absorbed into a trivial redefinition of the gauge fields, just as in section 8.1 for the $U(1)$ transformation $U = e^{i\chi}$.

In this so-called 'unitary gauge' the Lagrangian has the form

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D_\mu \phi) - V(\phi, \phi^\dagger) \tag{150}$$

with

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{151}$$

The Goldstone boson fields χ^A, which parametrise the space of flat directions in the potential, disappear from the spectrum. The vacuum expectation value for the scalar fields is $\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$.

This is not invariant under $SU(2)_L$ transformations or $U(1)_Y$, since we have assigned $Y=1$ to ϕ. However it *is* invariant under $U(1)_{em}$ transformations, since

$$\begin{aligned} Q\langle\phi\rangle &= (T^3+\frac{1}{2}Y)\langle\phi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}0\\ v\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = 0 \end{aligned} \tag{152}$$

So $G = SU(2)_L \times U(1)_Y$ is spontaneously broken to $G_0 = U(1)_{em}$. There are dim $G/G_0 = 3$ broken generators, which implies 3 Goldstone boson χ^A. These are absorbed by the vector bosons $W^{\pm}, Z$, which acquire masses. The remaining vector boson, the photon, is still massless because $U(1)_{em}$ is unbroken.

There is one massive neutral scalar left in the physical spectrum - the Higgs boson H.

To find the masses and couplings, we expand out $\mathcal{L}_{Higgs}$. In the unitary gauge, where

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H\end{pmatrix}$$

we have

$$\begin{aligned} D_\mu\phi &= \begin{pmatrix}\partial_\mu+\frac{ig}{2}W^3_\mu+\frac{ig'}{2}B_\mu & \frac{ig}{2}(W^1_\mu-iW^2_\mu)\\ \frac{ig}{2}(W^1_\mu+iW^2_\mu) & \partial_\mu-\frac{ig}{2}W^3_\mu+\frac{ig'}{2}B_\mu\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{ig}{2}(W^1_\mu-iW^2_\mu)(v+H)\\ \partial_\mu H+(-\frac{ig}{2}W^3_\mu+\frac{ig'}{2}B_\mu)(v+H)\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{ig}{2}W^+_\mu(v+H)\\ \partial_\mu H-\frac{i}{2}(g\cos\theta_W+g'\sin\theta_W)Z_\mu(v+H)\end{pmatrix} \end{aligned} \tag{153}$$

in terms of W^+, W^-, Z. Notice that, as expected, the photon field A_μ does not appear.

Substituting into $\mathcal{L}_{Higgs}$, we find the vector boson masses

$$m_W^2 = \frac{1}{4}g^2v^2 \tag{154}$$

$$m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2 = \frac{1}{4}\frac{g^2}{\cos^2\theta_W}m_W^2 \tag{155}$$

$$m_H^2 = 2\lambda v^2 \tag{156}$$

We therefore predict the ρ parameter, originally introduced as the relative strength of the neutral and charged current interactions:-

$$\rho = \frac{m_W^2}{m_Z^2\cos^2\theta_W} = 1 \tag{157}$$

in the Weinberg-Salam-Higgs model. This is a special property of the particular representation of Higgs field we have chosen to induce the breaking of $SU(2)_L \times U(1)_Y$. Other choices are possible – not all give $\rho = 1$ however.

The deeper reason is that $\rho = 1$ is a prediction of a global $SU(2)$ ('custodial') symmetry implicit in $\mathcal{L}_{Higgs}$. Writing ϕ in terms of real components, $\phi^+ = \phi^1 + i\phi^2$ and $\phi^0 = \phi^3 + i\phi^4$ gives $V(\phi^\dagger\phi) = V(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)$. The potential has an O(4) symmetry, broken to O(3) by the vacuum expectation value. In fact, the Higgs sector is a linear sigma model with coset manifold $O(4)/O(3) \sim SU(2) \times SU(2)/SU(2)$. The unbroken custodial global SU(2) ensures the mass relation between m_W, m_Z that gives $\rho = 1$.

Finally, rewriting $\mathcal{L}_{Higgs}$ in terms of the parameters m_W^2, m_Z^2, m_H^2, g rather than the original set μ, λ, g', g we have

$$\begin{aligned}\mathcal{L}_{Higgs} &= \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}m_H^2 H^2 + m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \\ &+ g m_W H W_\mu^+ W^{-\mu} + \frac{1}{4}g^2 H^2 W_\mu^+ W^{-\mu} + \frac{1}{4}g\frac{m_Z^2}{m_W} H Z_\mu Z^\mu \\ &+ \frac{1}{8}g^2\frac{m_Z^2}{m_W^2} H^2 Z_\mu Z^\mu - \frac{1}{16}g\frac{m_H^2}{m_W}H^3 - \frac{1}{32}g^2\frac{m_H^2}{m_W^2}H^4 + \frac{1}{2g^2}m_H^2 m_W^2 \quad (158)\end{aligned}$$

The corresponding Feynman rules for the vertices are shown in the figures.

9.3 Fermion masses

Yukawa interactions involving 2 fermion fields and the Higgs field can be constructed in such a way as to be $SU(2)_L \times U(1)_Y$ invariant, and so preserve renormalisability. When the Higgs field gets a vacuum expectation value, the interaction terms give rise to mass terms for the fermions.

Leptons:

Choose the $SU(2)_L \times U(1)_Y$ invariant Yukawa terms,

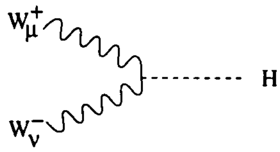
$$\mathcal{L}_{Yukawa} = -G_e\left[(\bar{\nu}_{eL} \;\; e_L)\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} e_R + \bar{e}_R(\phi^- \;\; \phi^{0*})\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix}\right] + e \to \mu + e \to \tau \quad (159)$$

Setting

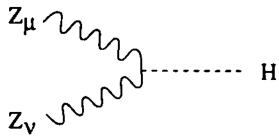
$$\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H\end{pmatrix} \quad (160)$$

as above, we have

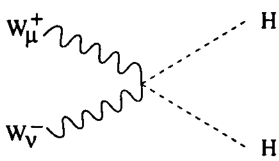
$$\begin{aligned}\mathcal{L}_{Yukawa} &= -\frac{G_e}{\sqrt{2}}v(\bar{e}_L e_R + \bar{e}_R e_L) - \frac{G_e}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L)H + \mu, \tau \text{ terms} \\ &= -m_e\bar{e}e - \frac{g}{2}\frac{m_e}{m_W}\bar{e}eH + \mu, \tau \text{ terms} \quad (161)\end{aligned}$$



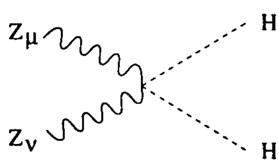
$$i g m_W g_{\mu\nu}$$



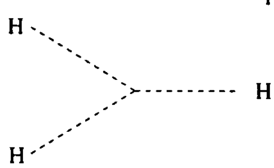
$$i g \frac{m_Z^2}{m_W} g_{\mu\nu}$$



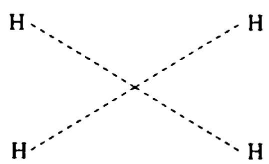
$$\frac{i}{2} g^2 g_{\mu\nu}$$



$$\frac{i}{2} g^2 \frac{m_Z^2}{m_W^2} g_{\mu\nu}$$



$$-i \frac{3}{2} g \frac{m_H^2}{m_W}$$



$$-i \frac{3}{4} g^2 \frac{m_H^2}{m_W^2}$$

The Yukawa coupling (another free parameter) is traded for the lepton mass. There is also a lepton-Higgs boson vertex, proportional to m_{lepton}/m_W. This is a general feature of the model – the Higgs boson couples to particles with a strength proportional to their mass.

Quarks:

This is slightly trickier because we have to arrange masses for the upper components of the $SU(2)_L$ doublets as well.
Define

$$\phi_C = -i\tau_2\phi^* = \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} v+H \\ 0 \end{pmatrix} \qquad (162)$$

in the unitary gauge.

Allowing for quark mixing, remembering that the $SU(2)_L$ eigenstates are $\begin{pmatrix} u_{iL} \\ d'_{iL} \end{pmatrix}$ with $d'_i = V_{ij}^{CKM} d_j$, we can write

$$\begin{aligned}
\mathcal{L}_{Yukawa} &= -(\bar{u}_{iL} \;\; \bar{d}'_{iL})\phi_C\, G^u_{ij}\, u_{jR} - (\bar{u}_{iL} \;\; \bar{d}'_{iL})\phi\, G^d_{ij}\, d_{jR} + h.c. \\
&= -\frac{1}{\sqrt{2}}\bar{u}_{iL} G^u_{ij} u_{jR}(v+H) - \frac{1}{\sqrt{2}}\bar{d}'_{iL} V^\dagger_{ik} G^d_{kj} d_{jR}(v+H) + h.c. \\
&= -m_u^{(i)}\,\bar{u}_i\, u_i - \frac{g}{2}\frac{m_u^{(i)}}{m_W}\bar{u}_i\, u_i H - m_d^{(i)}\,\bar{d}_i\, d_i - \frac{g}{2}\frac{m_d^{(i)}}{m_W}\bar{d}_i\, d_i H \qquad (163)
\end{aligned}$$

where we have chosen G^u_{ij} to be diagonal and G^d_{kj} such that $V^\dagger G$ is diagonal.

10 The Standard Model Lagrangian

This completes the construction of the standard model Lagrangian. The standard model is the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory with quarks, leptons and the Higgs field, with Lagrangian:-

$$\mathcal{L}_{SM} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}$$

$SU(2)_L$ $U(1)_Y$ $SU(3)_C$

$$+ \; i\bar{\chi}_L(\partial_\mu + ig\frac{\tau^A}{2}W^A_\mu + i\frac{g'}{2}YB_\mu)\chi_L$$

$SU(2)_L$ and $U(1)_Y$ fermion-gauge interaction on L fields

$$+ \; i\bar{\psi}_R(\partial_\mu + i\frac{g'}{2}YB_\mu)\psi_R$$

$U(1)_Y$ interaction on R fields

$$+ \; i\bar{q}(ig_{QCD}\lambda^a G^a_\mu)q$$

$SU(3)_C$ interaction on quarks.

λ^a = $SU(3)_C$ triplet representation generators.

G^a_μ = gluon field. q = colour triplet quarks.

$$+ \quad |(\partial_\mu + ig\frac{\tau^A}{2}W_\mu^A + i\frac{g'}{2}YB_\mu)\phi|^2 \quad - V(\phi^\dagger \phi)$$

Higgs sector $\Rightarrow$ W, Z masses and H interactions.

$\phi = SU(2)_L$ doublet Higgs field $(Y = 1)$ V is the Higgs potential

$$- \quad (G^d\, \bar{\chi}_L\, \phi\, d_R + G^u\, \bar{\chi}_L\, \phi_C\, u_R \;\; + \; h.c.)$$

Yukawa interactions $\Rightarrow$ fermion mass

With the construction of this Lagrangian, our task in these lectures comes to a close. This is, however, more of a beginning than an end.

Many questions immediately arise. Going beyond the tree level dynamics and symmetries we used to guide us to the Lagrangian, what does the standard model actually predict and is it true? Here, the evidence for the model is strong and compelling. Perturbative radiative corrections to the tree-level predictions are impressively verified in precision electroweak experiments at LEP, and perturbative QCD, exploiting the power of the renormalisation group, is well established. Non-perturbative phenomena are much harder, but lattice gauge theories and other approaches are beginning to make serious inroads into the physics of QCD bound states. Beyond that, there are predictions, in general yet to be tested, concerning the role played by extended objects such as instantons, monopoles and strings which are implicit in the model.

The least tested and most controversial aspect of the standard model is of course the symmetry breaking, Higgs sector. Here, even the confrontation of the model with precision electroweak data provides little more than circumstantial evidence in favour of the precise mechanism presented here. Experimental confirmation of the Higgs mechanism, or indeed an alternative dynamical symmetry breaking scheme, will probably have to await the LHC.

Finally, we are led to the big questions. Assuming the standard model to be true, why is it the way it is? What determines the symmetries and the representations in which the elementary quark and lepton fields lie? What determines the parameters, nineteen in all? Aesthetic criteria, often so successful in fundamental physics, tempt us to the view that the standard model is just the low energy effective theory of a deeper, more unified theory of the fundamental interactions. But that would be another lecture course.

Acknowledgements

And finally, it is a great pleasure once again to thank everyone responsible for making lecturing at the School such an enjoyable and rewarding experience. In particular, I would like to thank Ken Peach for keeping the show rolling with such good humour, Ann Roberts for all the behind the scenes organisation, my fellow lecturers and tutors. Most of all, I would like to thank the students for their goodwill and enthusiasm.

Problems

1. Check that $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma_\mu\} = 0$.
Show that $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$ are projection operators, i.e.

$$P_L^2 = P_L \quad P_R^2 = P_R \quad P_L P_R = P_R P_L = 0 \quad P_L + P_R = 1$$

Consider a massless fermion with $p_\mu = (E, 0, 0, E)$. Show that $P_L u(p)$ and $P_R u(p)$ are eigenstates of helicity h with eigenvalues $-1/2$ and $1/2$ respectively.

$$h = \frac{1}{2}\frac{\underline{\sigma}.\underline{p}}{|\underline{p}|} = -\frac{1}{2}\frac{\gamma_0\gamma_5\underline{\gamma}.\underline{p}}{E}$$

2. Consider the current $J_\mu = \frac{1}{2}\bar{u}\gamma_\mu(1-\gamma_5)u$. Show that under a combined CP transformation,

$$J_\mu \to -\frac{1}{2}\bar{u}\gamma_\mu^\dagger(1-\gamma_5)u$$

Hence verify that the product $J_\mu J^{\dagger\mu}$ is CP invariant.

What happens if we have different types of fermions u_i and a current $J_\mu = \frac{1}{2}\bar{u}_i\gamma_\mu(1-\gamma_5)V_{ij}u_j$, for some matrix V?

3. Suppose that the weak charged current had the Lorentz structure

$$J_\mu^{CC} = \bar{\nu}_e\gamma^\mu(a + b\gamma_5)e \quad + \quad (\mu \to e)$$

Calculate the cross section for $\nu_\mu e^- \to \mu^- \nu_e$ and show that

$$\frac{d\sigma}{d\Omega} = \frac{G^2 s}{32\pi^2}(A^+ + A^- \cos^4\frac{\theta}{2})$$

where $A^\pm = (a^2+b^2)^2 \pm 4a^2b^2$. Neglect m_e and m_μ and assume

$$\begin{aligned} Tr\gamma.k\gamma^\mu\gamma.k'\gamma^\nu &= 4(k^\mu k'^\nu + k'^\mu k^\nu - k.k' g^{\mu\nu}) \\ Tr\gamma_5\gamma.k\gamma^\mu\gamma.k'\gamma^\nu &= 8i\epsilon^{\mu\nu\lambda\rho}k_\lambda k'_\rho \\ -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\alpha\beta} &= (\delta_\alpha^\lambda\delta_\beta^\rho - \delta_\beta^\lambda\delta_\alpha^\rho) \end{aligned}$$

4. The decay rate for the 2-body decay $Z \to f\bar{f}$ is

$$\Gamma = \frac{1}{2m_Z}\int D\, |M|^2 = \frac{1}{64\pi^2 m_Z}\int d\Omega\, |M|^2$$

where D denotes the phase space measure.

The $Zf\bar{f}$ vertex is $-i\frac{g}{\cos\theta_W}\gamma^\mu \frac{1}{2}(c_V^f - c_A^f\gamma_5)$.

First show that, summing over the fermion and averaging over the boson spins,

$$|M|^2 = \frac{1}{12}\frac{g^2}{\cos^2\theta_W}(c_V^{f2} + c_A^{f2})(-g_{\mu\nu})Tr\gamma^\mu\gamma.k_1\gamma^\nu\gamma.k_2$$

where k_1, k_2 are the fermion momenta and the gauge boson polarisation sum is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_Z^2}$$

Then show that the decay rate is

$$\Gamma = \frac{1}{48\pi}\frac{g^2}{\cos^2\theta_W}(c_V^{f2} + c_A^{f2})m_Z$$

5. Using the explicit forms for c_V and c_A in the electroweak model, derive expressions for the decay rates $Z \to \nu_e\bar{\nu}_e$, $Z \to e^+e^-$, $Z \to \bar{u}u$ and $Z \to \bar{d}d$ in terms of $\sin^2\theta_W$

What is the total width of the Z in the standard model?
$[G_F = 1.2 \times 10^{-5}GeV^2, \ \ \sin^2\theta_W = 0.23, \ \ m_Z = 91GeV\ ]$

6. Consider a Higgs theory for a general gauge group G and Higgs field ϕ. Show that the vector boson mass matrix is

$$(m^2)^{AB} = g^2\langle\phi^\dagger\rangle\{T^A, T^B\}\langle\phi\rangle$$

where $\langle\phi\rangle$ is the vacuum expectation value of ϕ and T^A is the generator of G in the representation to which ϕ belongs.

Specialise the above result to $G = SU(2)_L \times U(1)_Y$, with ϕ in an $SU(2)_L$ doublet representation with $Y = 1$ and assume the breaking conserves $U(1)_{em}$. Show that in the charged sector, $m_{W^\pm}^2 = g^2v^2$, where v is the magnitude of the VEV for ϕ, while in the neutral sector, the mass matrix for W_μ^3 and B_μ is

$$\frac{1}{4}v^2\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}$$

Diagonalise this to find the mass eigenstates. Show that these are the photon A_μ and Z_μ defined as

$$\begin{aligned} Z_\mu &= W_\mu^3\cos\theta_W - B_\mu\sin\theta_W \\ A_\mu &= W_\mu^3\sin\theta_W + B_\mu\cos\theta_W \end{aligned}$$

TOPICS IN STANDARD MODEL PHENOMENOLOGY

By E W N Glover
University of Durham

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1995

Topics in Standard Model Phenomenology

Nigel Glover

University of Durham

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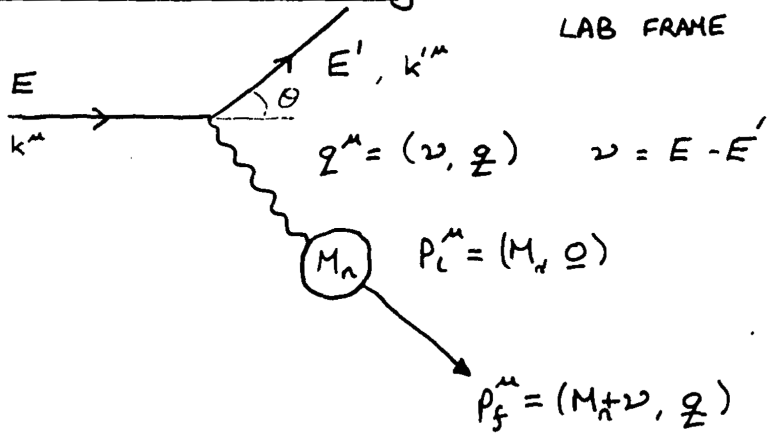
- Deep Inelastic Scattering
- Structure Functions
- Scaling
- Parton Model
- Parton Density Functions
- Neutrino-Proton scattering
- DIS at high Q^2
- Momentum Sum Rule and Scaling Violations

STRUCTURE OF PROTON

Electron - nucleus scattering

LAB FRAME



FOR ELASTIC SCATTERING

$$p_f^\mu p_{f\mu} = M_n^2 = M_n^2 + 2M_n\nu + \underbrace{\nu^2 - \underline{q}^2}_{q_\mu q^\mu = -Q^2}$$

$$\Rightarrow \quad \nu = \frac{Q^2}{2M_n}$$

USUAL VARIABLES

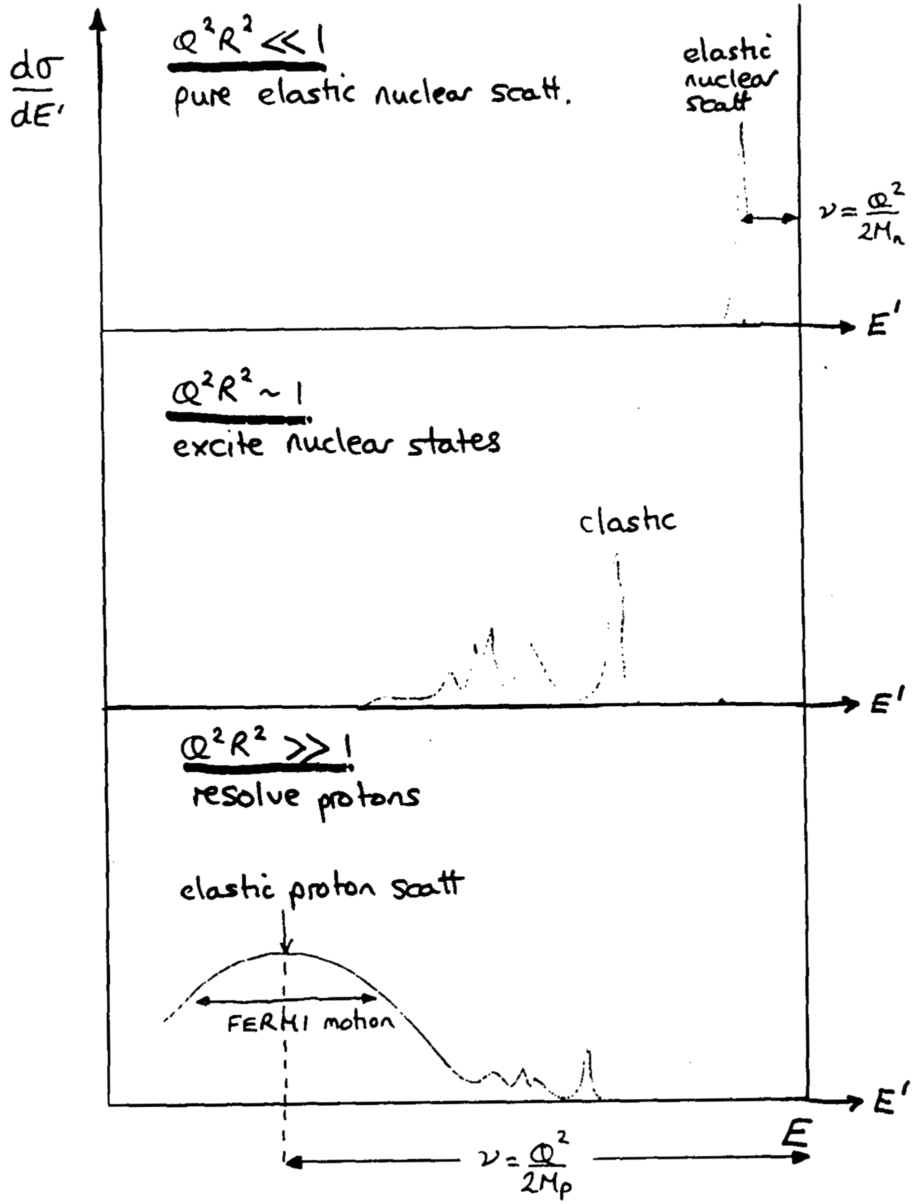
$$Q^2 = -q^2 = -(k-k')^2 = 2k\cdot k' = 2EE'(1-\cos\theta)$$

$$x = \frac{Q^2}{2p_i \cdot q} = \frac{Q^2}{2M(E-E')}$$

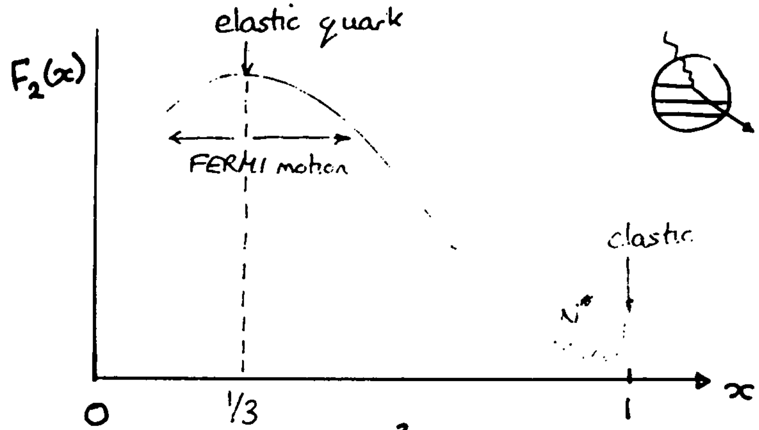
$$s = 2ME = 2k\cdot p_i$$

$$y = E-E'/E = \nu/E = q\cdot p_i/k\cdot p_i = Q^2/xs$$

e - NUCLEUS SCATT.

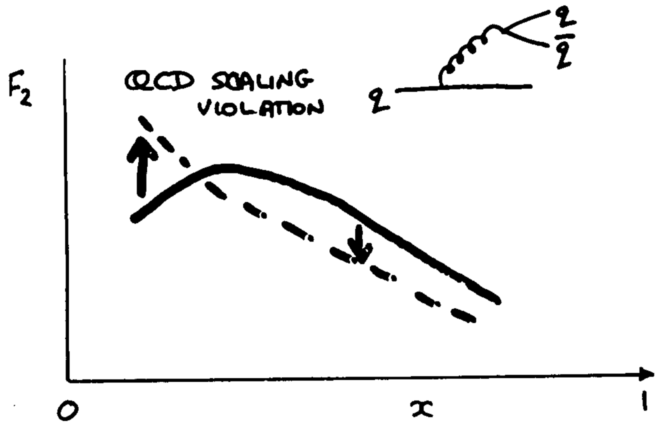


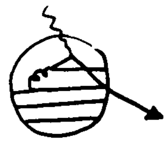
e - PROTON SCATTERING



$$x = \frac{Q^2}{2M\nu}$$

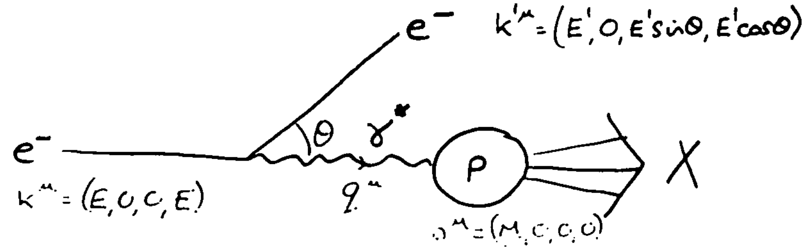
- SAME SORT OF BEHAVIOUR AS e-nucleus SCATTERING
- AS Q^2 INCREASES, START RESOLVING SEA QUARKS





SCATTERING OFF SEA QUARK

DEEP INELASTIC SCATTERING



$Q^2 = -q^2 = 2EE'(1 - \cos\theta) \gg M^2$ "DEEP"

$$x = \frac{Q^2}{2p.q} = \frac{Q^2}{2M(E-E')}$$

$$s = 2ME$$

$W^2 = (p+q)^2 \gg M^2$ "INELASTIC"

$$y = \frac{E-E'}{E} = \frac{Q^2}{xs}$$

$$d\sigma = \frac{1}{2s} \sum_X \int d\Phi \; \frac{1}{4} \sum_{spins} |M|^2_{ep \to eX}$$

($\frac{1}{2s}$: FLUX; $d\Phi$: PHASE SPACE; $\frac{1}{4}\sum_{spins}|M|^2$: SPIN AVERAGED |AMPLITUDE|^2)

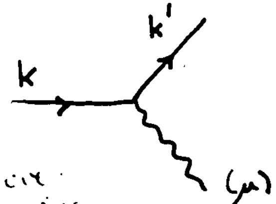
now $$\int d\Phi = \frac{1}{(2\pi)^3} \frac{d^3k'}{2E'} \, d\Phi_X$$

where $$\frac{d^3k'}{2E'} = \pi E' dE' d\cos\theta = \frac{\pi Q^2 dQ^2 dx}{2 s x^2}$$

and $\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{e^4}{Q^4} L^{\mu\nu} h_{X\,\mu\nu}$

($\frac{e^4}{Q^4}$ ↑ photon propagator; $L^{\mu\nu}$ ↑ lepton tensor; $h_{X\,\mu\nu}$ ↑ hadron tensor)

$L^{\mu\nu}$

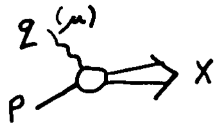


$\bar{u}(k')\gamma_\mu u(k)$

(average over spins)

$$L_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\, \not{k}\,\gamma_\nu\, \not{k}'\,\gamma_\mu = 2(k_\mu k'_\nu + k_\nu k'_\mu - k'.k\, g_{\mu\nu})$$

HADRONIC TENSOR



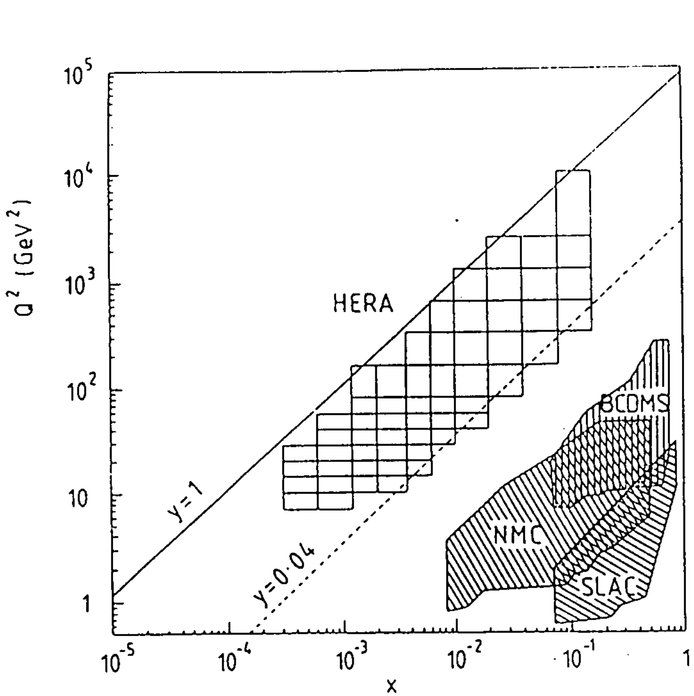
define $H_{\mu\nu} = \sum_X \int d\Phi_X \; h_{X\,\mu\nu}$

which is function of q^2, $q \cdot p$ $(= x, Q^2)$ ONLY

$$\Rightarrow \quad \frac{d^2\sigma}{dx\,dQ^2} = \frac{1}{2s} \cdot \frac{1}{(2\pi)^3} \frac{\pi Q^2}{2 s x^2} \cdot \frac{(4\pi\alpha)^2}{Q^4} L^{\mu\nu} H_{\mu\nu}$$

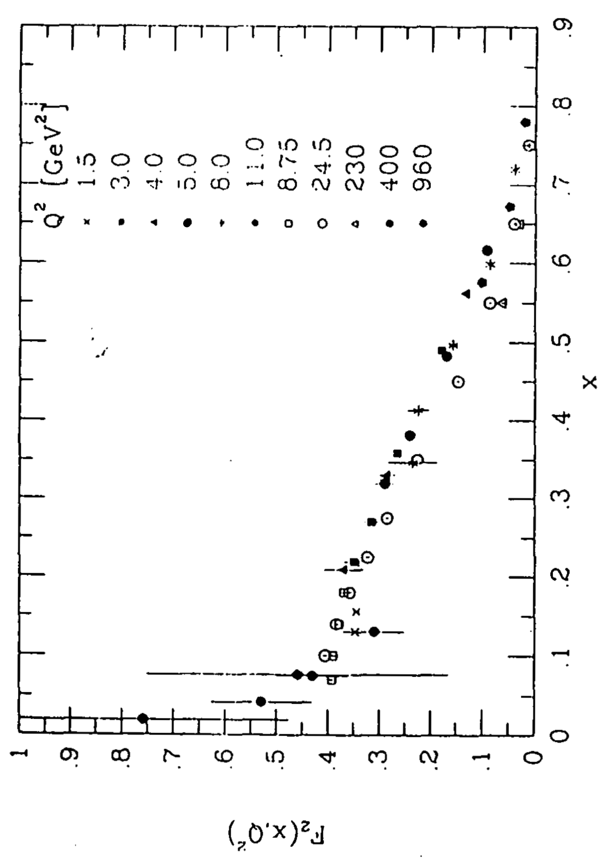
$$= \frac{\alpha^2 y^2}{2Q^6} L^{\mu\nu} H_{\mu\nu}(x, Q^2)$$

(x, Q^2) regions probed by DIS expts.



$Q^2 = s x y$

$\log Q^2 = \log x + \log y + \log s$



BJORKEN SCALING.

F_2 INDEP OF Q^2 $\Rightarrow$ SCATTERING OFF POINTLIKE CONSTITUENTS

— OTHERWISE VARIATION WITH Q/Q_0 WITH SIZE SCALE Q_0^{-1}

Most general form of $H_{\mu\nu}$ (symmetric fn of $g_{\mu\nu}, p_\mu, q_\mu$)

$$H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2 \frac{p_\mu p_\nu}{Q^2} + H_4 \frac{q_\mu q_\nu}{Q^2} + H_5 \frac{(p_\mu q_\nu + q_\mu p_\nu)}{Q^2}$$

(the H_4 and H_5 terms) vanish when contracted with $L^{\mu\nu}$

$$\Rightarrow \quad L^{\mu\nu} H_{\mu\nu} = 4 k.k' \, H_1 + 4 \frac{p.k \, p.k'}{Q^2} H_2$$

$$= 2Q^2 H_1 + \frac{Q^2(1-y)}{x^2 y^2} H_2$$

STANDARD DEFINITIONS

$$H_1 = 4\pi F_1 \qquad H_2 = 8\pi x F_2$$

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left[y^2 x F_1 + (1-y) F_2 \right]}$$

$\sqrt{s} \gg M$

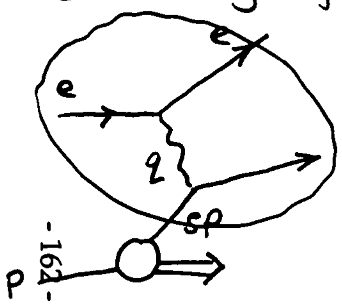
F_1, F_2 contain all information about proton

— no assumption of partons made

NAIVE PARTON MODEL

$\gamma^* p$ interaction at large Q^2 can be expressed as sum of incoherent scatt from point like quarks.

Over short time scale $1/\sqrt{Q^2}$ photon "sees" non-interacting quarks. Final hadronize occurs long after



$f_q(\xi)\,d\xi$ represents prob. that quark carries momentum fraction between ξ and $\xi + d\xi$

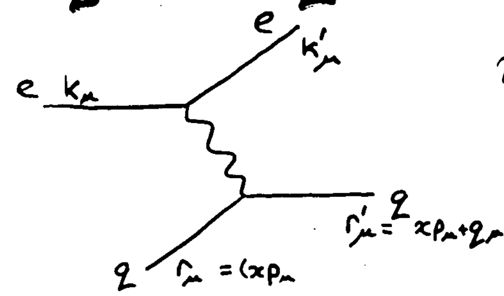
$$\Rightarrow \quad \frac{d^2\sigma}{dx\,dQ^2} = \sum_q \int_0^1 d\xi\, f_q(\xi)\, \frac{d^2\sigma}{dx\,dQ^2}(eq \to eX)$$

$$(q + \xi p)^2 = 2p.q\,\xi - Q^2 = 0$$

ie $\boxed{\xi = x}$ $\qquad Q^2 \gg m_q^2$

fractional mom = Bjorken x

$eq \to eq$



$$\mathcal{M} = \frac{e^2 e_q}{Q^2}\, \bar{u}(k')\gamma_\mu u(k)\; \bar{u}(r')\gamma^\mu u(r)$$

$$\Rightarrow \quad \frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = e^4 e_q^2 \; 2\, \frac{(k.r)^2 + (k'.r)^2}{(k.k')^2}$$

$$k.r = x\,p.k = xs/2$$

$$k'.r = k.r - q.r = xs(1-y)/2$$

$$k.k' = Q^2/2 = xsy/2$$

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4} \sum_q f_q^p(x)\, e_q^2\, \frac{x}{2}\left[1 + (1-y)^2\right]}$$

$\Rightarrow$ COMPARING WITH GENERAL FORMULA

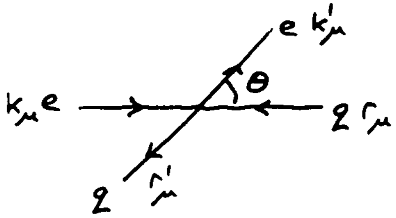
$$\boxed{\begin{aligned} F_2^{ep} &= 2x F_1^{ep} \\ &= \sum_q e_q^2\, x\, f_q^p(x) \end{aligned}}$$

CALLAN GROSS RELATION – SPIN 1/2

SCALING $F_2(x, \cancel{Q^2})$

INSIGHT INTO y DEPENDENCE

eq scattering in CM frame



$k_\mu = E(1,0,0,1)$

$r_\mu = E(1,0,0,-1)$

$k'_\mu = E(1,0,\sin\theta,\cos\theta)$

$$\Rightarrow \quad k.r = \frac{xS}{2} = 2E^2$$

$$k.r' = \frac{xS}{2}(1-y) = E^2(1+\cos\theta)$$

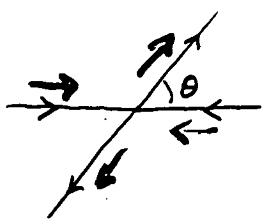
$$k.k' = \frac{xSy}{2} = E^2(1-\cos\theta)$$

$$\Rightarrow \quad y = \frac{1}{2}(1-\cos\theta)$$

$y=0$ forward scatt

$y=1$ backward scatt

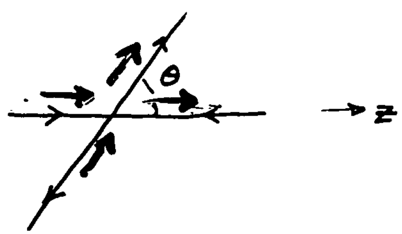
BUT $|\mathcal{M}|^2 \propto (1 + (1-y)^2)$



same helicity

RR $\rightarrow$ RR

$J_z^{in} = 0 = J_z^{out}$



opposite helicity

RL $\rightarrow$ RL

MUST VANISH AS $\theta \rightarrow \pi$

$J_z^{in} = 1$

Parton density functions

proton = uud + $q\bar{q}$ pairs

"valence" "sea"

$$f_u^p(x) = u(x) = u_V(x) + u_{sea}(x)$$

$$f_{\bar{u}}^p(x) = \bar{u}(x) = u_{sea}(x)$$

$$\Rightarrow \quad \int_0^1 (u-\bar{u})\,dx = \int_0^1 u_V\,dx = 2$$

(↑ prob. of finding u)

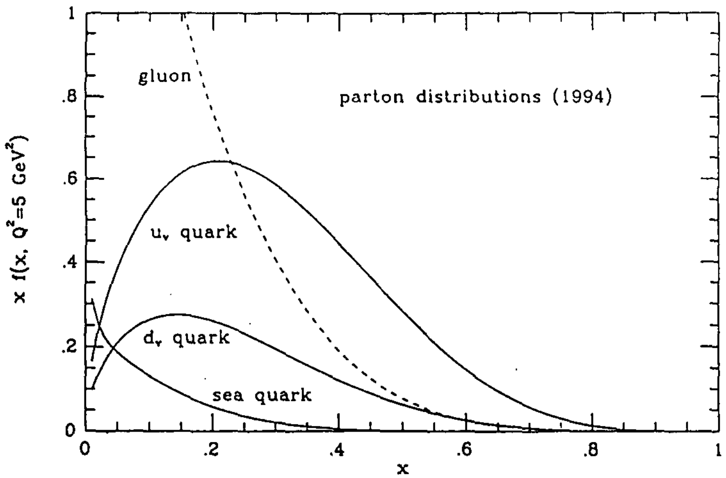
$$F_2^{ep}(x) = x\left(\frac{4}{9}u + \frac{1}{9}d + \frac{1}{9}s + \dots + \frac{4}{9}\bar{u} + \frac{1}{9}\bar{d} + \frac{1}{9}\bar{s} + \dots\right)$$

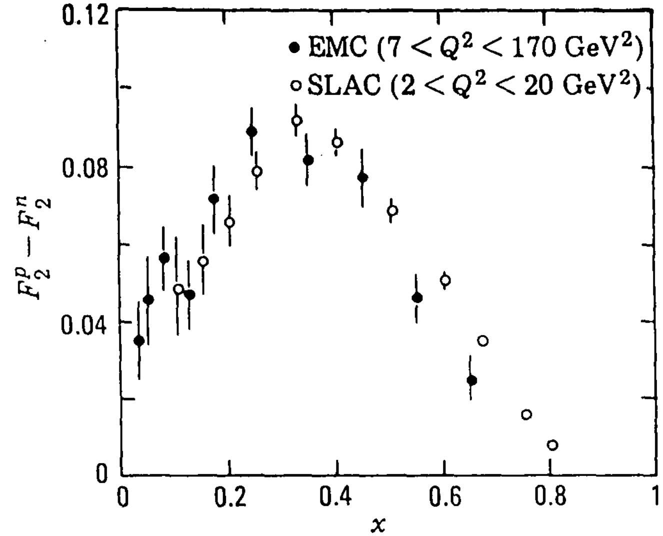
usually write all $f_q^X(x)$ in terms of proton distributions

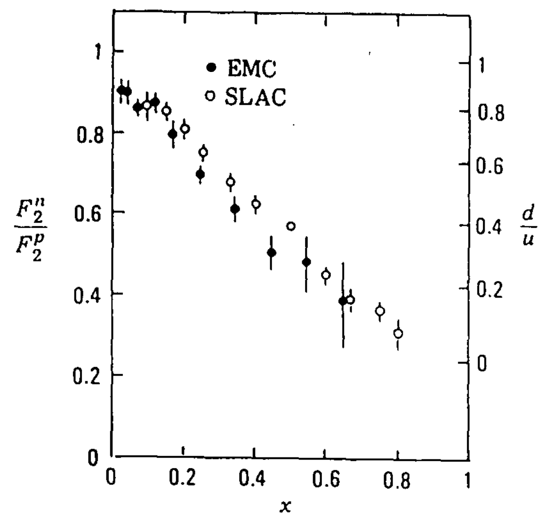
$$F_2^{en}(x) = x\left(\frac{4}{9}d + \frac{1}{9}u + \frac{1}{9}s + \dots + \frac{4}{9}\bar{d} + \frac{1}{9}\bar{u} + \frac{1}{9}\bar{s} + \dots\right)$$

$u \leftrightarrow d$, $\bar{u} \leftrightarrow \bar{d}$ by ISOSPIN

$$\Rightarrow \quad F_2^{ep} - F_2^{en} = \frac{x}{3}\left[u + \bar{u} - d - \bar{d}\right]$$


gluon
parton distributions (1994)
u_v quark
d_v quark
sea quark
x f(x, Q²=5 GeV²)
x
0
.2
.4
.6
.8
1


● EMC ($7 < Q^2 < 170$ GeV^2)
○ SLAC ($2 < Q^2 < 20$ GeV^2)
$F_2^p - F_2^n$
x
0
0.04
0.08
0.12
0.2
0.4
0.6
0.8
1


● EMC
○ SLAC
$\frac{F_2^n}{F_2^p}$
$\frac{d}{u}$
x
0
0.2
0.4
0.6
0.8
1

$\nu q \to \nu q$

$\nu \xrightarrow{L} \xleftarrow{L} d$ $\qquad \dfrac{d\sigma(\nu d \to \mu u)}{dx\, dQ^2} = \dfrac{G_F^2}{\pi}$

$\nu \xrightarrow{L} \xleftarrow{R} \bar{u}$ $\qquad \dfrac{d\sigma(\nu \bar{u} \to \mu \bar{d})}{dx\, dQ^2} = \dfrac{G_F^2}{\pi}(1-y)^2$

so
$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{G_F^2}{\pi x}\left\{ x(d+s) + x(\bar{u}+\bar{c})(1-y)^2 \right\}}$$

OR
$$\boxed{\begin{aligned} F_2^{\nu p} &= 2x\,(d+\bar{u}+s+\bar{c}) \\ xF_3^{\nu p} &= 2x\,(d-\bar{u}+s-\bar{c}) \end{aligned}}$$

for scattering off neutrons $u \leftrightarrow d$, $\bar{u} \leftrightarrow \bar{d}$

N

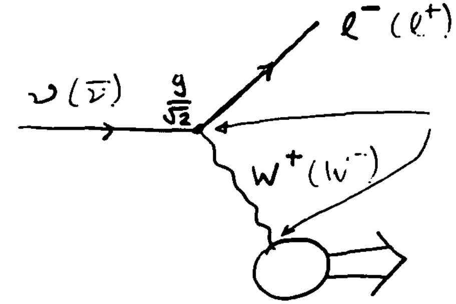
$$F_2^{\nu N} = \frac{1}{2}\left(F_2^{\nu p} + F_2^{\nu n}\right) = x\,(u+\bar{u}+d+\bar{d}+2s+2c)$$

$$xF_3^{\nu N} = x\,(u-\bar{u}+d-\bar{d}+2s-2c)$$

Together with F_2^{ep}, F_2^{ed} "fix"

$u+\bar{u}$, $d+\bar{d}$, $\bar{u}+\bar{d}$, s

$\nu N \to \mu X$



PARITY VIOLATION

$$\bar{u}(k')\gamma_\mu \frac{(1-\gamma_5)}{2} u(k)$$

FROM γ_5 IN TRACE

$$\Rightarrow L_{\mu\nu}^{\nu(\bar{\nu})} = L_{\mu\nu}^{e} \pm 2i\,\varepsilon_{\mu\nu\rho\sigma} k^\rho k'^\sigma$$

$$H^{\mu\nu} = \;\ldots\ldots\; - \frac{i}{Q^2}\varepsilon^{\mu\nu\lambda\kappa} p_\lambda q_\kappa H_3$$

$\Rightarrow$ NEW TERM IN $L_{\mu\nu}^{\nu(\bar{\nu})} H^{\mu\nu}$
$$= \pm \frac{2H_3}{Q^2}\left(p.k\, q.k' - p.k'\, q.k\right)$$
$$= \pm \frac{H_3 Q^2}{xy}\left(1 - y/2\right)$$

$$H_3 = 8\pi x F_3$$

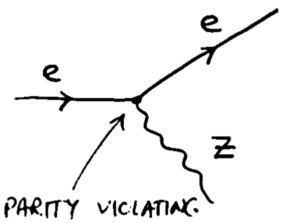
$Q^2 \ll M_W^2$

$$\boxed{\begin{aligned} \frac{d^2\sigma^{\nu(\bar{\nu})}}{dx\,dQ^2} &= \frac{G_F^2}{\pi x}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 \left\{ y^2 x F_1 + (1-y)F_2 \pm y(1-y/2)\,xF_3 \right\} \\ &= \frac{G_F^2}{\pi x}\left\{ \frac{y^2}{2}(2xF_1 - F_2) + \frac{1}{2}(1+(1-y)^2)F_2 \pm \frac{1}{2}(1-(1-y)^2)\,xF_3 \right\} \end{aligned}}$$

STRUCTURE FUNCTIONS @ HERA

- NEUTRAL CURRENT

f	v_f	a_f
e	$-1+4\sin^2\theta$	-1
u	$1-\frac{8}{3}\sin^2\theta$	1
d	$-1+\frac{4}{3}\sin^2\theta$	-1



$$-\frac{ig\gamma_\mu}{\cos\theta_W}(v-a\gamma_5)$$

$$\frac{d^2\sigma^{e^\pm}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left[xy^2F_1 + (1-y)F_2 \mp xy\left(1-\frac{y}{2}\right)F_3\right]$$

$$F_2 = 2xF_1 = \sum_q \left[xq + x\bar{q}\right] A_q$$

$$xF_3 = \sum_q \left[xq - x\bar{q}\right] B_q$$

$$A_q = e_q^2 - 2e_q v_e v_q \chi(Q^2) + (v_e^2+a_e^2)(v_q^2+a_q^2)\chi^2(Q^2)$$

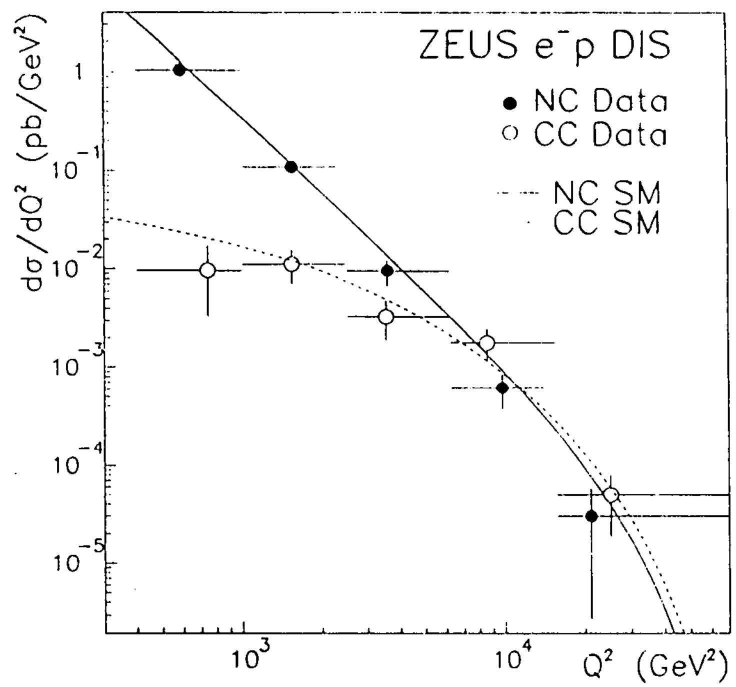
$\gamma-\gamma$ $\qquad$ $\gamma-Z$ $\qquad$ $Z-Z$

$$B_q = -2e_q a_e a_f \chi(Q^2) + 4v_e a_e v_q a_q \chi^2(Q^2)$$

$$\chi(Q^2) = \left(\frac{\sqrt{2}\,G_F M_Z^2}{16\pi\alpha}\right)\left(\frac{Q^2}{M_Z^2+Q^2}\right)$$ — SMALL UNLESS Q^2 LARGE

- CHARGED CURRENT (LIKE νN SCATTERING)

$$\frac{d^2\sigma^{e^-}}{dx\,dQ^2} = \frac{G_F^2}{\pi x}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\left\{x(u+c) + x(\bar{d}+\bar{s})(1-y)^2\right\}$$



MORE NEEDED

1) <u>MOMENTUM SUM RULE</u>

$\int_0^1 x\, q(x)\, dx$ measures momentum carried by q

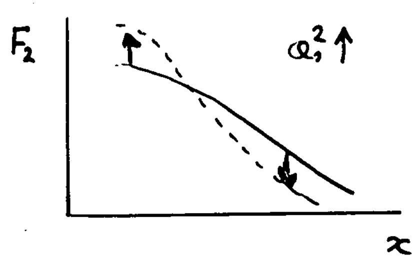
$\Rightarrow$ expect

$$\sum_{q,\bar{q}} \int_0^1 x\, q(x)\, dx = 1$$

~ 0.5

– rest of momentum carried by GLUONS

2) <u>BJORKEN SCALING VIOLATED</u>



see small "logarithmic" violations

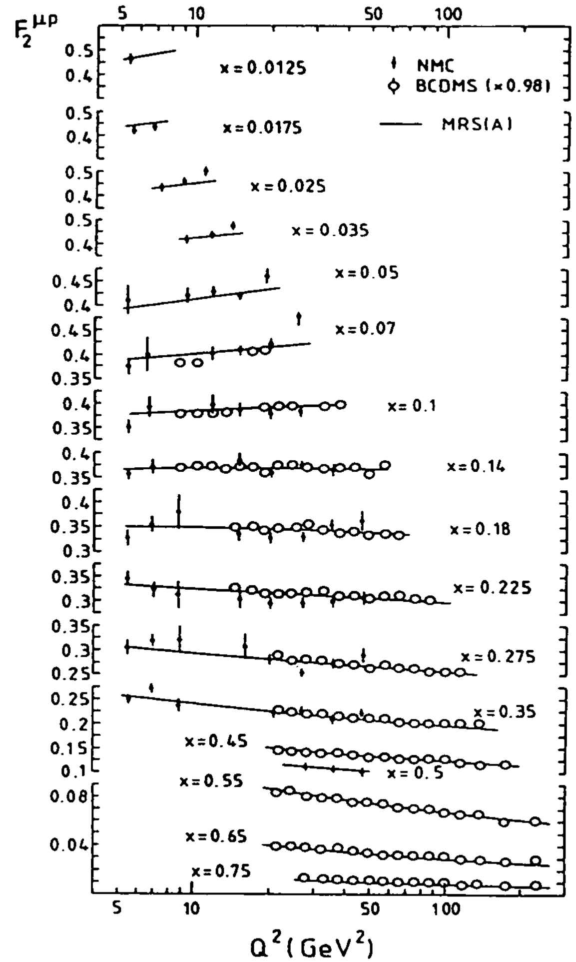


Fig. 6

2. $\sigma(e^+e^- \to \text{hadrons})$

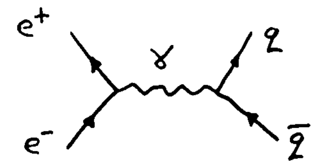
- $e^+e^- \to q\bar{q}$ and $R_{e^+e^-}$
- $R_{e^+e^-}$ at LEP/SLC
- $\mathcal{O}(\alpha_s)$ corrections to $R_{e^+e^-}$
- Real radiation: $e^+e^- \to q\bar{q}$ + gluon
  - Singularities
  - Soft and Collinear Limits
  - Radiation Patterns
- First measurement of α_s
- Running Coupling Constants
- Second measurement of α_s and renormalisation scale
- World average α_s

TOTAL HADRONIC CROSS SECTION e^+e^-

– compute $\sigma(e^+e^- \to \text{hadrons})$ from $e^+e^- \to$ quarks gluons

since Probability (quarks, gluons $\to$ hadrons) = 1

– At LOWEST ORDER $e^+(p_1) + e^-(p_2) \to q(q_1) + \bar{q}(q_2)$



(For Z exchange see later)

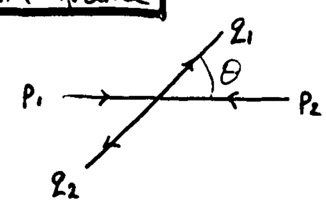
ASSUME $M_q \ll \sqrt{s}$

$$\mathcal{M} = e^2 e_q \delta_{ij} \bar{v}(p_1)\gamma_\mu u(p_2) \frac{-g^{\mu\nu}}{(p_1+p_2)^2} \bar{u}_i(q_1)\gamma_\nu v_j(q_2)$$

colours of $q, \bar{q}$ same

$$\to \frac{1}{4}\sum_{\substack{\text{spins}\\ \text{colours}}} |\mathcal{M}|^2 = e^4 e_q^2 N \; 2\, \frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2}{(p_1\cdot p_2)^2}$$

CM frame



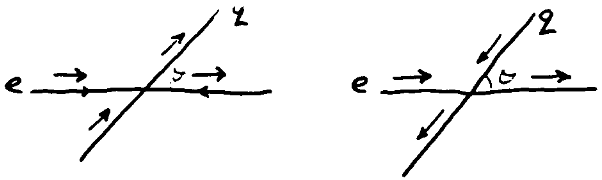
$$p_1^\mu = \frac{\sqrt{s}}{2}(1,0,0,1)$$
$$p_2^\mu = \frac{\sqrt{s}}{2}(1,0,0,-1)$$
$$q_1^\mu = \frac{\sqrt{s}}{2}(1,0,\sin\theta,\cos\theta)$$
$$q_2^\mu = \frac{\sqrt{s}}{2}(1,0,-\sin\theta,-\cos\theta)$$

$$\Rightarrow \quad p_1\cdot q_1 = \frac{s}{4}(1-\cos\theta) \qquad p_1\cdot p_2 = \frac{s}{2} = q_1\cdot q_2$$

$$p_1\cdot q_2 = \frac{s}{4}(1+\cos\theta)$$

$$\Rightarrow \quad \frac{1}{4}\sum |M|^2 = \frac{e^4 e_q^2}{2} N\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right]$$

Helicities



$J_z = +1$ VANISHES $\theta \to \pi$ $\sim (1+\cos\theta)^2$ VANISHES $\theta \to 0$ $\sim (1-\cos\theta)^2$

QED doesn't distinguish LH and RH

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$$\frac{1}{4}\sum |M|^2 = (\underbrace{4\pi\alpha}_{=e^2})^2 e_q^2 \cdot N\,(1+\cos^2\theta)$$

- Phase Space

$$d\Phi = \frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)$$

$$= \frac{1}{16\pi^2}\frac{E_1^2 dE_1\, d\cos\theta\, d\phi}{E_1 E_2}\delta(\sqrt{s}-E_1-E_2)$$

$$= \frac{1}{8\pi} dE_1\, d\cos\theta\, \frac{1}{2}\delta(\sqrt{s}/2 - E_1)$$

$$d\Phi = \frac{d\cos\theta}{16\pi}$$

$$\int d\sigma = \frac{1}{2s}\int \frac{1}{4}\sum|M|^2 d\Phi \qquad \int_{-1}^{1} = \frac{8}{3}$$

$$= \frac{1}{2s} 16\pi^2\alpha^2 e_q^2 N \overbrace{(1+\cos^2\theta)\frac{d\cos\theta}{16\pi}}$$

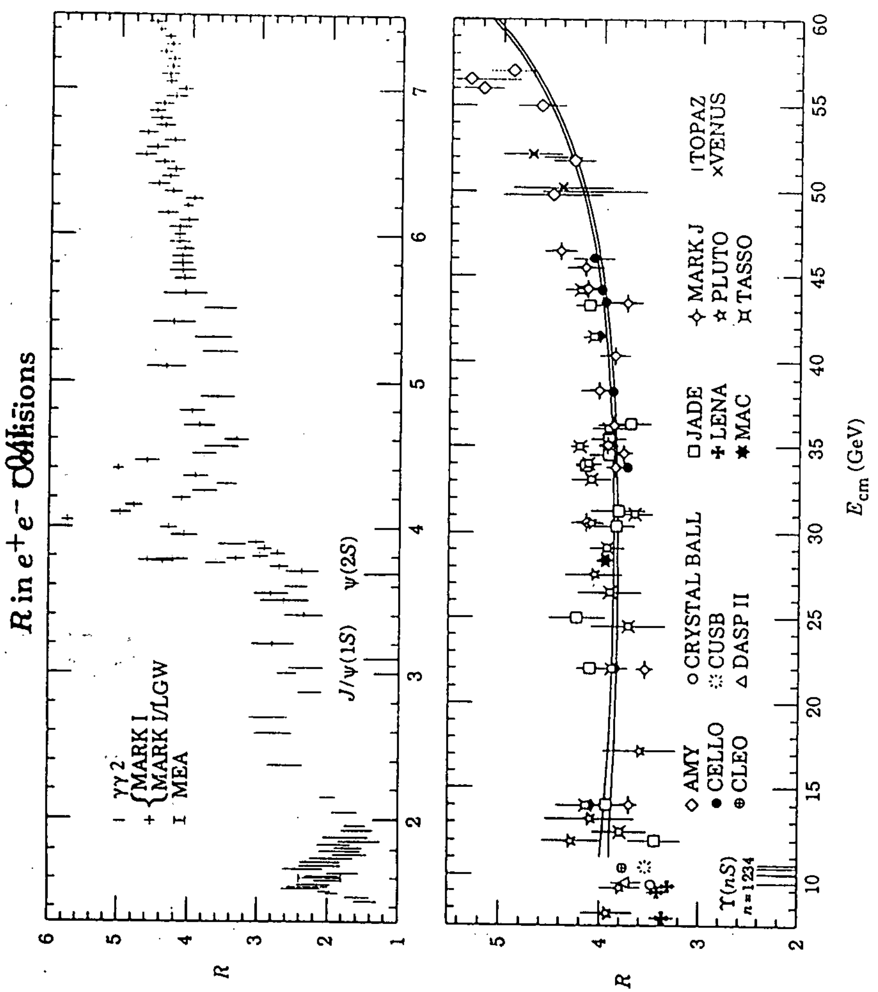
$$\sigma(e^+e^-\to q\bar{q}) = \boxed{\frac{4\pi\alpha^2}{3s}}\, e_q^2 N$$

$= \sigma(e^+e^-\to\mu^+\mu^-)$

$$R = \frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \approx \underbrace{\sum_q e_q^2 N}_{=\frac{11}{3}}$$

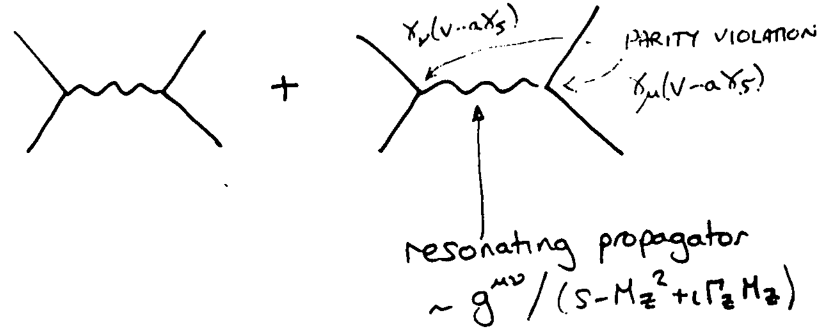
e.g at $\sqrt{s} = 34$ GeV, $q = u,d,s,c,b$

$$R^{EXP} = 3.88 \pm 0.06$$



Ke^+e^- at LEP/SLC

- include effect of Z exchange



$$\Rightarrow \quad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} N \left[(1+\cos^2\theta)\, A_q^{(s)} + 2\cos\theta\, B_q^{(s)} \right]$$

$$A_q(s) = \underset{\gamma-\gamma}{e_q^2} - \underset{\gamma-Z}{2 e_q v_q v_e \chi_1(s)} + \underset{Z-Z}{(v_e^2+a_e^2)(v_q^2+a_q^2)\chi_2(s)}$$

$$B_q(s) = -2 e_q a_q a_e \chi_1(s) + 4 v_e a_e v_q a_q \chi_2(s)$$

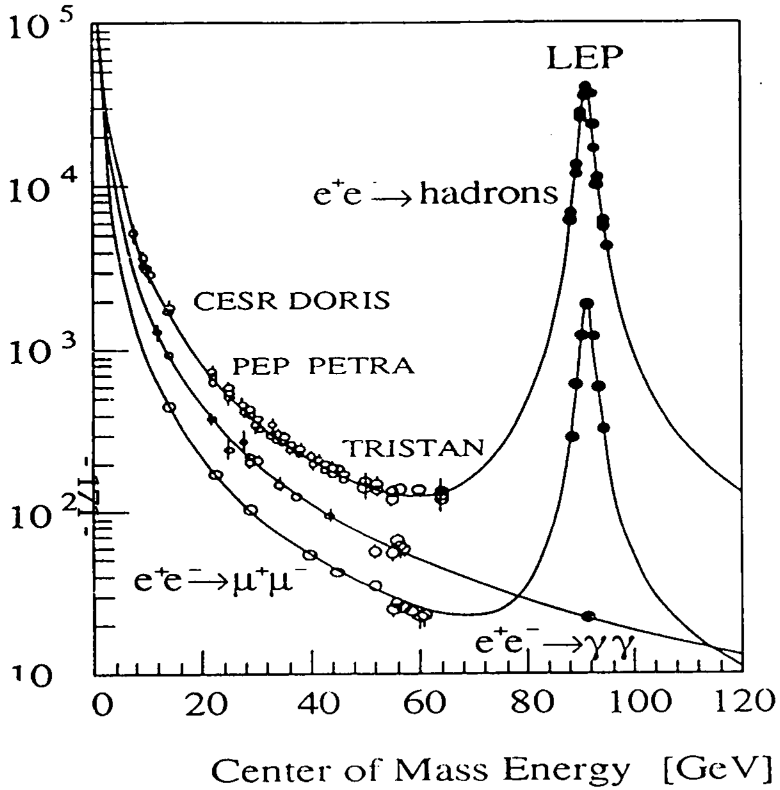
$$\chi_1(s) = \frac{\sqrt{2} G_F M_Z^2}{16\pi\alpha} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$

$$\chi_2(s) = \left(\frac{\sqrt{2} G_F M_Z^2}{16\pi\alpha}\right)^2 \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$

$$\Rightarrow \quad R_0 = N \sum e_q^2 \longrightarrow \boxed{N \frac{\sum A_q(s)}{A_\mu(s)} = R_0(s)}$$

at $\sqrt{s} = 34$ GeV $\qquad R_0 \to \frac{11}{3} + 0.05 = 3.716$

cf $R^{DATA} = 3.88 \pm 0.05$



igure 1: Total cross section for e^+e^- annihilation into hadrons and ıuon pairs as function of the centre-of-mass energy. Also given is ıe two-photon cross section. Experimental data are compared with redictions from the standard model.

QCD LAGRANGIAN SU(3)

quark field ψ_i $\quad i = 1,2,3 \quad (G,R,B)$

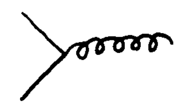
gluon field G_μ^a $\quad a = 1 \ldots 8$

gluon field strength $G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{abc} G_\mu^b G_\nu^c$

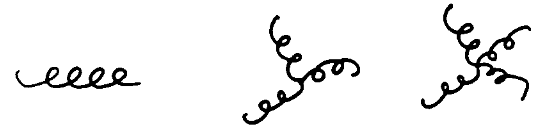
cf QED $F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$

$$\mathcal{L} = i \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - m \bar{\psi}_i \psi_i$$

$$- g_s \bar{\psi}_i T^a_{ij} \gamma^\mu \psi_j G^a_\mu$$



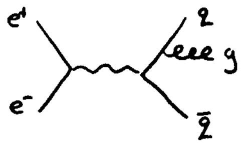
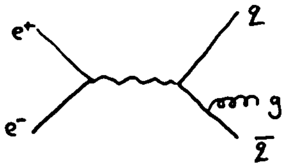
$$- \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a}$$



$$[T^a, T^b] = i f^{abc} T^c \qquad T^a_{ij} T^b_{ji} = \frac{\delta^{ab}}{2}$$

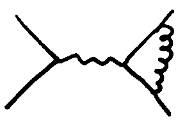
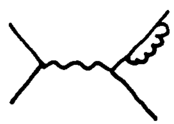
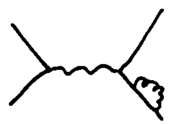
$O(\alpha_s)$ QCD CORRECTIONS

- REAL GLUON EMISSION

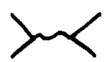
$\mathcal{M}_{gq\bar{q}} \sim$  + 

$$\mathcal{M} \propto g_s \Rightarrow |\mathcal{M}|^2 \propto \alpha_s$$

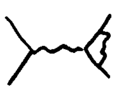
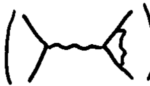
- VIRTUAL GLUON EMISSION

$\mathcal{M} \sim$  +  +  $\sim g_s^2$

$\mathcal{M}_{q\bar{q}} \sim$  +

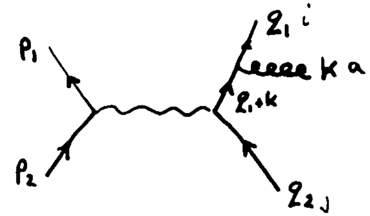
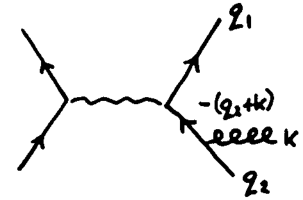
$|\mathcal{M}_{q\bar{q}}|^2 \sim$ |  $|^2$ $O(1)$

$+$  $\times$ ( $)^*$ $O(\alpha_s)$

$+$  $\times$ ( $)^*$ $O(\alpha_s^2)$

- ONLY INTERESTED IN INTERFERENCE OF TREE AND ONE LOOP GRAPHS AT $O(\alpha_s)$

1) $e^+e^- \to q\bar{q}g$

 + 

$$\mathcal{M} = e^2 e_q g_s T^a_{ij} \quad \bar{v}(p_1)\gamma_\mu u(p_2) \frac{g^{\mu\nu}}{(p_1+p_2)^2}$$

$$\bar{u}_i(q_1)\left[\frac{\gamma^\alpha(\not{q}_1+\not{k})\gamma_\nu}{(q_1+k)^2} - \frac{\gamma_\nu(\not{q}_2+\not{k})\gamma^\alpha}{(q_2+k)^2}\right]v_j(q_2)\,\varepsilon^a_\alpha(k)$$

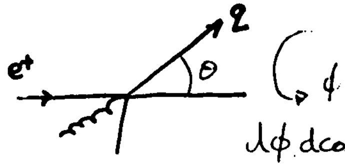
GLUON POLARISATION VECTOR ($\varepsilon^a_\alpha(k)$)

$$\Rightarrow \frac{1}{4}\sum_{\substack{\text{spins}\\ \text{colours}}} |\mathcal{M}|^2 = 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \cdot \frac{\left[(q_1\cdot p_1)^2 + (q_1\cdot p_2)^2 + (q_2\cdot p_1)^2 + (q_2\cdot p_2)^2\right]}{q_1\cdot k \;\; k\cdot q_2 \;\; p_1\cdot p_2}$$

<u>Note</u>

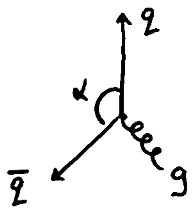
$$\sum_{\substack{a=1,8\\ i,j=1,3}} T^a_{ij}(T^a_{ij})^* = \sum T^a_{ij}T^a_{ji} = \mathrm{Tr}\,T^aT^a = \frac{N^2-1}{2}$$

Phase Space



$d\phi \, d\cos\theta$

- Because of momentum conservation $q\bar{q}g$ lie in plane



$dx_1 \, dx_2 \, d\alpha$

↑ ↑ FRACTIONAL ENERGIES OF QUARKS

$x_1 = \frac{2E_q}{\sqrt{s}} \quad x_2 = \frac{2E_{\bar{q}}}{\sqrt{s}} \quad x_3 = \frac{2E_g}{\sqrt{s}} \qquad x_1 + x_2 + x_3 = 2$

$$d\Phi = \frac{s}{2^{10}\pi^5} \, d\phi \, d\cos\theta \, d\alpha \, dx_1 dx_2$$

$k.q_1 = \frac{s}{2}(1-x_2)$

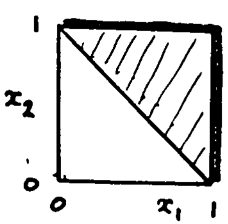
$k.q_2 = \frac{s}{2}(1-x_1)$

⇒ integrating out Euler angles and summing over quark flavours

$$d\sigma_{q\bar{q}g} = \underbrace{\frac{4\pi\alpha^2}{3s} N \sum e_q^2}_{\sigma^0_{q\bar{q}}} \cdot \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right) \int_{\blacktriangledown} dx_1 dx_2 \, \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

integration region

DALITZ



see that integrand diverges as $x_1, x_2 \to 1$

- divergences occur when gluon is COLLINEAR with quark $\theta_{qg} \to 0$ so $1 - \cos\theta_{qg} \to 0$

  OR when gluon is SOFT $E_g \to 0$

- in both limits, processes look like tree level **INFRARED**

COLLINEAR LIMIT OF $\frac{1}{4}\sum |M|^2 \qquad q_1 /\!/ k$

LET $\quad q_1 = (1-z)\, Q_1 \qquad Q_1$ parent momentum

$\qquad k = z\, Q_1 \qquad$ COLLINEAR LIMIT $Q_1^2 = 0$

$\Rightarrow \quad q_1 . p_1 \to (1-z)\, Q_1 . p_1 \qquad q_2 . p_1 \to Q_1 . p_2$

$\qquad q_1 . p_2 \to (1-z)\, Q_1 . p_2 \qquad q_2 . p_2 \to Q_1 . p_1$

ie $\quad [\ldots] \longrightarrow 2\left[1 + (1-z)^2\right]\left[(Q_1.p_1)^2 + (Q_1.p_2)^2\right]$

$\qquad k.q_2 \longrightarrow z\, Q_1.q_2 = z\, p_1.p_2$

ie.

$$\frac{1}{4}\sum |M|^2_{q\bar{q}g} \xrightarrow{q /\!/ g} \frac{g_s^2}{2k.q_1}\left(\frac{N^2-1}{2N}\right) 2\,\frac{\left[1+(1-z)^2\right]}{z} \frac{1}{4}\sum |M|^2_{q\bar{q}}$$

↑ SINGULARITY

SOFT LIMIT OF $\frac{1}{4}\sum |M|^2$

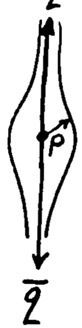
$q_2\cdot p_1 \rightarrow q_1\cdot p_2 \qquad q_2\cdot p_2 \rightarrow q_1\cdot p_1$

ie $[\ldots\ldots] \longrightarrow 2\left[(q_1\cdot p_1)^2 + (q_1\cdot p_2)^2\right]$

and

$$\frac{1}{4}\sum |M|^2_{q\bar{q}g} \xrightarrow{g \text{ soft}} g_s^2 \frac{N^2-1}{2N} \frac{2 q_1\cdot q_2}{q_1\cdot k\, k\cdot q_2} \frac{1}{4}\sum |M|^2$$

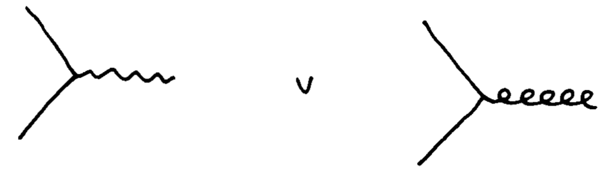
ANTENNA RADIATION PATTERNS $\quad P \sim \dfrac{1-\cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{\bar{q}g})\,E_g^2}$



- more soft particles get radiated in $q\bar{q}$ directions (dipole)
- JUST like QED – photon radiation from charged line
  → gluon radiation from coloured line

less radiation — more radiation

RADIATION IN $q\bar{q}\gamma$ v $q\bar{q}g$



v

≡

single colour antenna — double colour antenna

- if soft gluons related to hadronic particle directions

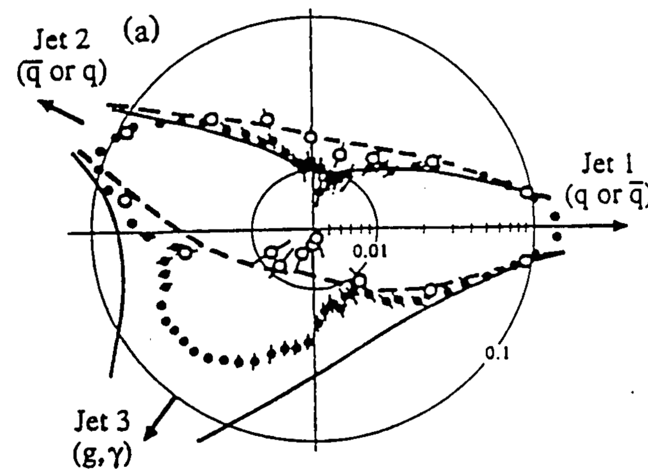


Figure 42: Particle flow on a logarithmic scale as a function of angle in the pl the event for $q\bar{q}\gamma$, (open points) and $q\bar{q}g$, (closed points).

$\sigma_{q\bar{q}g}$ CONTINUED

- TO DO INTEGRAL OVER x_1, x_2 NEED SOME REGULARISATION PROCEDURE
  - DIMENSIONAL REGULARISATION work in $n = 4 - 2\varepsilon$ dimensions and divergences appear as $1/\varepsilon^2$, $1/\varepsilon$
  - BOTH $|M|^2$ and $d\Phi$ change. get integrals like

$$\int_0^1 \frac{dx_1}{(1-x_1)} (1-x_1)^{-\varepsilon} \rightarrow \frac{-1}{\varepsilon} \qquad \left(C^{-\varepsilon} \equiv C \right)$$

$$\Rightarrow \sigma_{q\bar{q}g} = \sigma^0_{q\bar{q}} \frac{\alpha_s}{2\pi} \left(\frac{N^2-1}{2N} \right) H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + O(\varepsilon) \right]$$

$H(\varepsilon) = 1 + C(\varepsilon)$

virtual corrections

$$\sigma_{q\bar{q}(g)} = \sigma^0_{q\bar{q}} \frac{\alpha_s}{2\pi} \left(\frac{N^2-1}{2N} \right) H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + O(\varepsilon) \right]$$

- WHEN ADDED TOGETHER SINGULARITIES IN ε CANCEL AND TOTAL CORRECTION TO CROSS SECTION IS FINITE AS $\varepsilon \rightarrow 0$ (4-D)
- NOT ACCIDENT :- POWERFUL THEOREMS (Bloch, Nordsieck, Kinoshita, Lee, Nauenberg) GUARANTEE THIS FOR SUITABLY DEFINED INCLUSIVE CROSS SECTIONS

e.g. $\boxed{\sigma_{TOT} = \sigma^0_{q\bar{q}} \left(1 + \frac{\alpha_s}{\pi} \right)}$ OK $\qquad \sigma_{q\bar{q}}$ ✗

1st MEASUREMENT OF α_s

$$R = \frac{\sigma_{TOT}}{\sigma_{\mu^+\mu^-}} = R_0(s) \left(1 + \frac{\alpha_s}{\pi} \right) = 3.88 \pm 0.06$$

$R_0(s) = 3.7\,$; $\sqrt{s} = 34$

$$\Rightarrow \boxed{\alpha_s = 0.135 \pm 0.05}$$

RUNNING COUPLING CONSTANTS

- PROBLEM IS FEYNMAN GRAPHS LIKE

CONTAIN ULTRAVIOLET DIVERGENCES WHEN LOOP MOMENTA ARE LARGE

- UNLIKE INFRARED DIVERGENCES, THESE ARE NOT CANCELLED BY REAL RADIATION
- SOLUTION IS TO RENORMALIZE COUPLINGS AT A RENORMALISATION SCALE

$$g_s^2 \longrightarrow g_s^2(\mu) - \infty(\mu)$$

g_s^2 ↑ BARE COUPLING IN $\mathcal{L}$ $\qquad$ $g_s^2(\mu)$ ↑ FINITE COUPLING AT SCALE μ

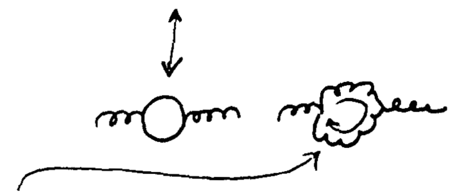
$\Rightarrow$ PHYSICAL QUANTITIES COMPUTED IN TERMS OF $g_s^2(\mu)$ ARE FINITE.

- BECAUSE μ DOES <u>NOT</u> APPEAR IN $\mathcal{L}$, PHYSICAL QUANTITIES COMPUTED TO ALL ORDERS IN PERTURBATION THEORY ARE INDEPENDENT OF μ $\quad$ ie $\dfrac{\partial \sigma}{\partial \mu} = 0$

- HOWEVER, CROSS SECTIONS COMPUTED ONLY TO A FINITE ORDER IN PERTURBATION THEORY ARE NOT INDEPENDENT OF μ.

- COUPLINGS AT DIFFERENT SCALES ARE RELATED BY PERTURBATION THEORY

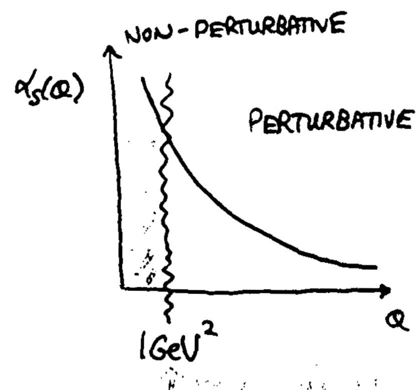
$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \beta(\alpha_s) = -b_0 \alpha_s^2 + \cdots\cdots$$



- IN QCD $\quad b_0 = \dfrac{11N - 2n_f}{12\pi}$

QED $\quad b_0 = -\dfrac{1}{3\pi}$

$\Rightarrow$ RUNNING DIFFERENT (α_{QED} INCREASES WITH Q^2)

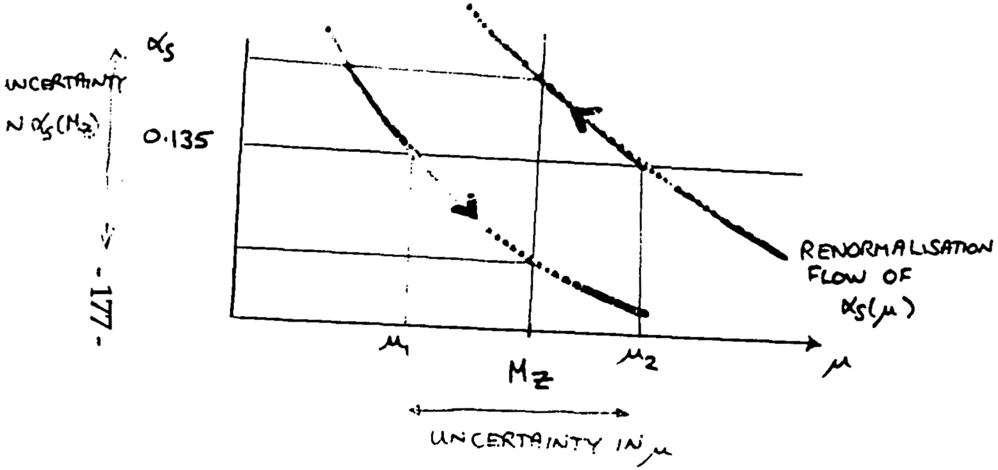


Λ IS SCALE WHERE α_s BECOMES ∞

RETURNING TO MEASUREMENT OF α_S

$$\alpha_S(\mu) = 0.135 \pm 0.05$$

- RELATE TO $\alpha_S(M_Z)$. DEPENDS ENTIRELY ON CHOICE FOR μ

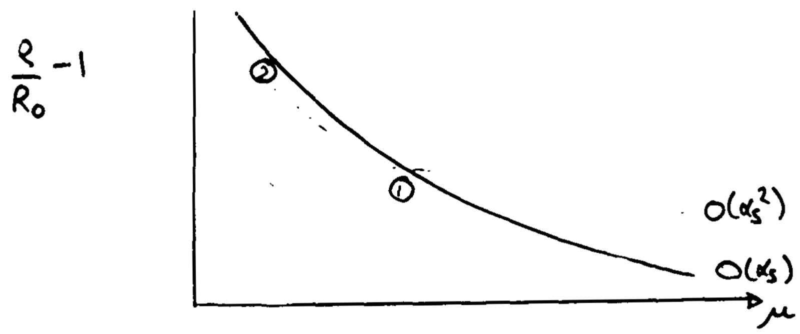


$\Rightarrow$ DEPENDING ON CHOICE OF μ GET ANY VALUE FOR $\alpha_S(M_Z)$

- SOME CHOICES "MORE SENSIBLE" THAN OTHERS, BUT STILL UNCERTAIN
- PROBLEM REDUCED AT HIGHER ORDER IN α_S

eg
$$\frac{R}{R_0} = 1 + \frac{\alpha_S(\mu)}{\pi} + \left(\frac{\alpha_S(\mu)}{\pi}\right)^2 \left[1.41 + b_0 \log \frac{\mu^2}{S}\right]$$

in μ ↑ ↓ ↓ ↑



TYPICAL CHOICES

① FAC ; fastest apparent convergence

② PMS ; principle of minimal sensitivity

③ $\mu \sim \sqrt{S}$ physical scale

– then vary by factor either way.

- VARYING μ IS ATTEMPT TO GUESS HOW BIG HIGHER ORDER CORRECTIONS ARE

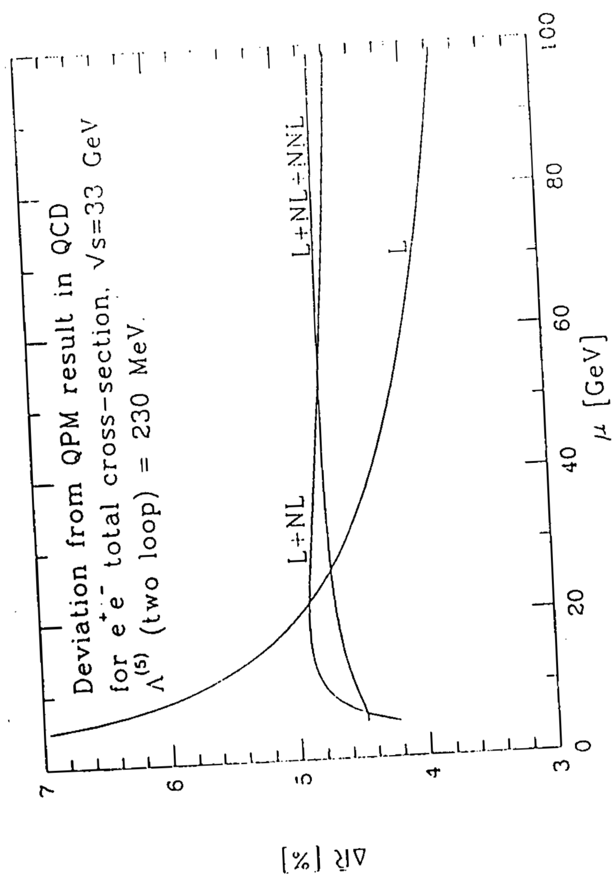


Figure 10: The effect of higher order QCD corrections to R, as a function of the renormalization scale μ.

α_s MEASURED AT $\sqrt{s} \sim M_Z$

$$R_0(M_Z^2) = N \frac{A_q(M_Z^2)}{A_\mu(M_Z^2)} \propto N \frac{\sum_q (v_q^2+a_q^2)}{v_\mu^2+a_\mu^2}$$

$$= 19.937 \quad \text{for } m_t = 174,\ m_H = 300$$

$$R^{LEP} = 20.788 \pm 0.032 \qquad \text{BRUSSELS 95}$$

SOLVE (LET $\mu = M_Z$)

$$R^{LEP} = R_0(M_Z^2)\left(1 + 1.06\frac{\alpha_s}{\pi} + 0.85\left(\frac{\alpha_s}{\pi}\right)^2 - 15\frac{\alpha_s^3}{\pi^3}\right)$$

$\Rightarrow$ at $O(\alpha_s)$ $\quad \alpha_s|^1 = 0.126$

$O(\alpha_s^2)$ $\quad \alpha_s|^2 = 0.123$

$O(\alpha_s^3)$ $\quad \alpha_s|^3 = \underline{0.125}$

DECREASING ERROR FROM μ DEP

$\Delta\alpha_s^{\mu} = 0.002$

$M_Z/4 < \mu < M_Z$

$\Delta\alpha_s^{EXP} = 0.005$

$\Delta\alpha_s^{EW} = 0.002$ $\quad Z \to b\bar{b}$

$\Delta\alpha_s^{H} = 0.002$ $\quad 60 < M_H < 1000$

$\Delta\alpha_s^{t} = 0.001$ $\quad 100 < M_t < 200$

- taking α_s as parameter in EW fits with m_t, M_H

$\Rightarrow$

$$\alpha_s(M_Z) = 0.122 \pm 0.005$$
$$m_t = 171 \pm 11$$
$$m_H = 93 {}^{+189}_{-63}$$

Chankowski + Pokorski hep/ph/9505304

α_s WORLD AVERAGE

- measured in many processes with high precision at many scales
  - all results consistent within errors

- recent compilation

$$\alpha_s(M_Z^2) = 0.117 \pm \Delta$$

$\Delta \approx 0.006$

↑ ESTIMATE, NOT RIGOUROUS ERROR SINCE ERRORS NON GAUSSIAN

Bethke, Moriond 95

Process	Ref.	Q [GeV]	$\alpha_s(Q)$	$\alpha_s(M_{Z^0})$	$\Delta\alpha_s(M_{Z^0})$ exp.	theor.	Theory
GLS (CCFR)	[15]	1.73	0.24 ± 0.047	$0.107 {}^{+0.007}_{-0.009}$	${}^{+0.006}_{-0.007}$	${}^{+0.004}_{-0.006}$	NNLO
R_τ (CLEO)	[16]	1.78	0.302 ± 0.024	0.116 ± 0.003	0.002	0.002	NNLO
R_τ (ALEPH)	[17]	1.78	0.355 ± 0.021	0.122 ± 0.003	0.002	0.002	NNLO
R_τ (OPAL)	[17]	1.78	$0.375 {}^{+0.032}_{-0.025}$	0.123 ± 0.003	0.002	0.002	NNLO
R_τ (Raczka)	[18]	1.78	0.333 ± 0.021	0.120 ± 0.003	0.002	0.002	NNLO
$\eta_c \to \gamma\gamma$ (CLEO)	[16]	2.98	0.187 ± 0.029	0.101 ± 0.010	0.008	0.006	NLO
$Q\overline{Q}$ states	[19]	5.0	0.188 ± 0.018	0.110 ± 0.006	0.000	0.006	q LGT
$b\bar{b}$ states	[19]	5.0	0.203 ± 0.007	0.115 ± 0.002	0.000	0.002	LGT
Υ(1S) (CLEO)	[16]	9.46	0.164 ± 0.013	0.111 ± 0.006	0.001	0.006	NLO
$e^+e^- \to jets$ (CLEO)	[16]	10.53	0.164 ± 0.015	0.113 ± 0.006	0.002	0.006	NLO
$ep \to jets$ (H1)	[20]	5 - 60		0.123 ± 0.018	0.014	0.010	NLO
$p\bar{p} \to W$ jets (D0)	[21]	80.6	0.123 ± 0.015	0.121 ± 0.014	0.012	0.005	NLO
$e^+e^- \to Z^0$:							
scal. viol. (ALEPH)	[17]	91.2		0.127 ± 0.011	-	-	NLO
ev. shapes (SLD)	[22]	91.2		0.120 ± 0.008	0.003	0.008	resum.
$\Gamma(Z^0 \to had.)$ (LEP)	[23]	91.2		0.127 ± 0.006	0.005	${}^{+0.003}_{-0.004}$	NNLO

Table 1. Summary of most recent measurements of α_s, presented at this conference. Abbreviations: GLS-SR = Gross-Llewellyn-Smith sum rules; (N)NLO = (next-)next-to-leading order perturbation theory; LGT = lattice gauge theory (q stands for quenched approximation); resum. = resummed next-to-leading order. Most results are still preliminary.

3. Structure of hadronic events

- Event shapes
- Thrust Distribution
  - Spin of the Gluon
  - $\mathcal{O}(\alpha_s^2)$ Corrections
  - Resumming large logs
- Jet algorithms
- Measuring QCD group parameters
- Jet production, Local Parton Hadron Duality and Hadronisation
- A simple Hadronisation model
- Hadronisation and Thrust

EVENT SHAPES

- global observables characterising structure of hadronic event

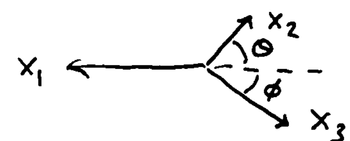
Thrust axis

e.g. THRUST $\quad T = \max \dfrac{\sum |\underline{p}_a \cdot \hat{n}|}{\sum |\underline{p}_a|}$

- for 2 particle (jet) events, thrust axis lies along particle direction

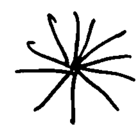
$\Rightarrow \quad T = 1$

- for 3 particle events, thrust axis coincides with most energetic particle



$$T = \frac{x_1 + x_2\cos\theta + x_3\cos\phi}{x_1 + x_2 + x_3} < 1$$

- for spherical events



$T = \frac{1}{2}$

e.g TOTAL JET BROADENING, B_T

$$B_T = \frac{\sum |\underline{p}_a \times \underline{n}|}{2 \sum |\underline{p}_a|}$$

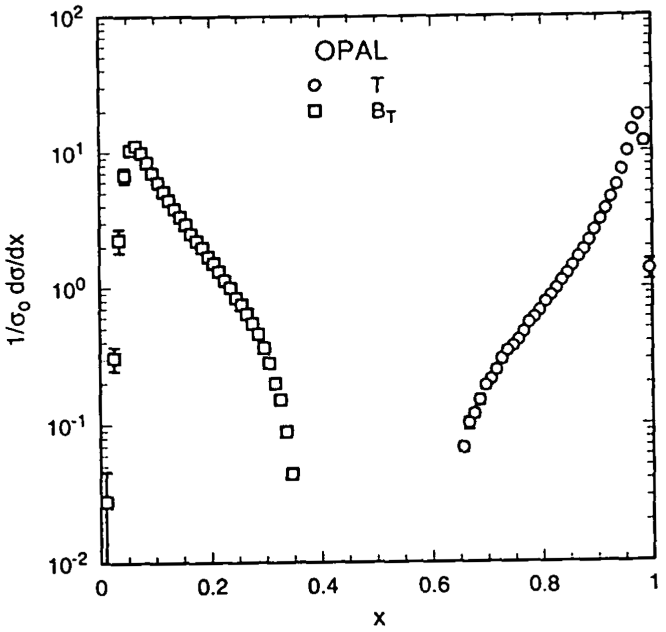
($\underline{n}$: THRUST AXIS)

- for 2 particle events $B_T = 0$

  3 particle events

$$B_T = \frac{x_2 \sin\theta + x_3 \sin\phi}{2(x_1 + x_2 + x_3)} > 0$$

→ FIG.



As $T \to 1$ or $B_T \to 0$ $\quad \frac{d\sigma}{dx} \to 0$

INFRARED SAFENESS

— FOR INFRARED POLES TO CANCEL FOR A PARTICULAR OBSERVABLE, IT MUST BE INFRARED SAFE

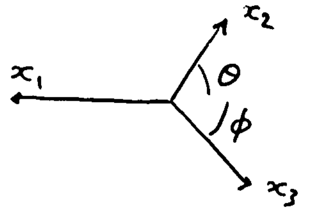
i.e. OBSERVABLE MUST BE INVARIANT UNDER

$$\underline{p}_i \to \underline{p}_j + \underline{p}_k$$

when $\underline{p}_j \parallel \underline{p}_k$ or $|\underline{p}_j| \to 0$

$O(\alpha_s)$ DISTRIBUTION

- $e^+e^- \to q\bar{q}g$



$x_1 > x_2, x_3$

$$\frac{2p_1\cdot p_2}{s} = \frac{2E_1E_2(1+\cos\theta)}{s}$$

$$= \frac{x_1x_2}{2}(1+\cos\theta) = \frac{(p_\gamma - p_3)^2}{s}$$

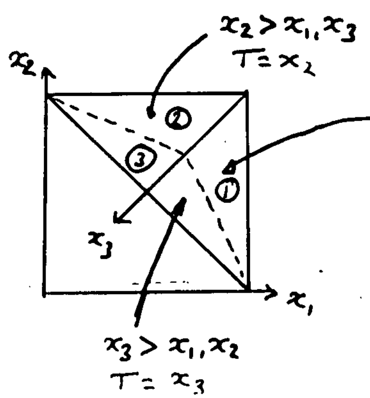
$$= 1 - \frac{2E_3}{\sqrt{s}} = 1 - x_3$$

i.e. $$x_2\cos\theta = \frac{2(1-x_3)}{x_1} - x_2$$

$$x_3\cos\phi = \frac{2(1-x_2)}{x_1} - x_3$$

$$\Rightarrow \quad T = \frac{x_1 + x_2\cos\theta + x_3\cos\phi}{x_1+x_2+x_3}$$

$$\boxed{T = x_1} \quad \text{if } x_1 > x_2, x_3$$



When $x_1 = x_2 = x_3$

$T = 2/3$

Minimum for 3 particle event.

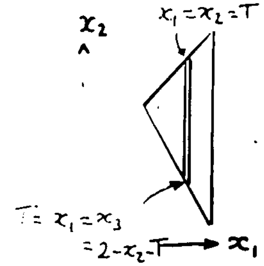
Starting from

$$\frac{1}{\sigma^0_{q\bar{q}}}\frac{d\sigma}{dx_1dx_2} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

obtain $\frac{1}{\sigma^0}\frac{d\sigma}{dT}$ by integrating over the 3 regions

1) $x_1 > x_2, x_3 \Rightarrow x_1 = T$

i.e. $2(1-T) < x_2 < T$

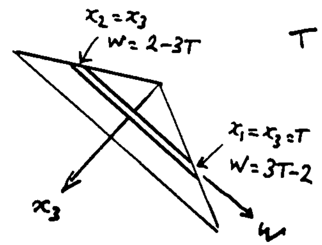


$$\left.\frac{1}{\sigma^0}\frac{d\sigma}{dT}\right)_1 \propto \int_{2(1-T)}^{T} \frac{T^2 + x_2^2}{(1-T)(1-x_2)}\, dx_2$$

$$= \frac{\alpha_s}{2\pi}\frac{N^2-1}{2N}\left[\frac{(1+T^2)}{(1-T)}\ln\left(\frac{2T-1}{1-T}\right) + \frac{1}{1-T}\left(\frac{4-7T+3T^2}{2}\right)\right].$$

2) $x_2 > x_1, x_3$ — get same contribution as 1)

3) $x_3 > x_1, x_2$



rewrite variables

$$T = x_3 = 2 - x_1 - x_2$$

$$\omega = x_1 - x_2$$

$$dT\,d\omega = 2\,dx_1dx_2$$

$$x_1 = 1 + \frac{W-T}{2} \qquad x_2 = 1 - \frac{(W+T)}{2}$$

$$1 - x_1 = \frac{T-W}{2} \qquad 1 - x_2 = \frac{W+T}{2}$$

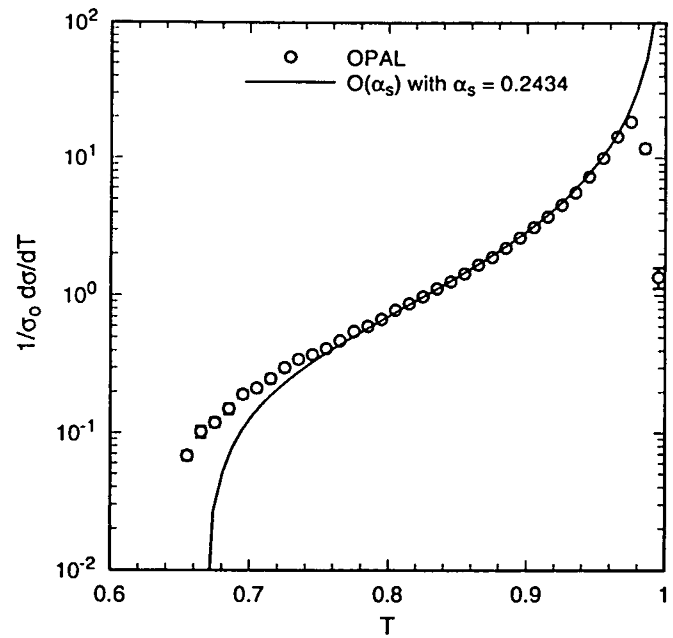
ie $\left.\frac{1}{\sigma^0}\frac{d\sigma}{dT}\right)_3 \propto \frac{1}{2}\int_{2-3T}^{3T-2} \frac{(2-T+w)^2+(2-T-w)^2}{(T+w)(T-w)}\,dw$

$$= \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\left[\frac{4-4T+2T^2}{T}\ln\left(\frac{2T-1}{1-T}\right)-6T+4\right]$$

⇒ adding up contributions from 3 regions

$$\frac{1}{\sigma^0}\frac{d\sigma}{dT} = \frac{\alpha_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\left[2\,\frac{(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - 3\,\frac{(3T-2)(2-T)}{1-T}\right]$$

- $T > 2/3$
- As $T \to 1$ $\quad \frac{1}{\sigma_0}\frac{d\sigma}{dT} \sim \frac{1}{1-T}\log\left(\frac{1}{1-T}\right)$
-  virtual gluon contribution occurs at $T = 1$
- EXPECT LARGE HADRONISATION CORRECTIONS AS $T \to 1$



- DEFICIENCY AT SMALL T DUE TO KINEMATIC BOUND
- SHAPE GOOD $0.75 < T < 0.95$

SPIN OF THE GLUON

if the gluon is a SCALAR

$$-\!\!-\!\!< -ig_s T^a_{ij} \qquad \mathcal{L}_{INT} \sim \bar{g}_s \, \overline{\psi}_i \, T^a_{ij} \, \phi^a \, \psi_j$$

QUARKS: $\overline{\psi}_i$, ψ_j

COUPLING: $\bar{g}_s$

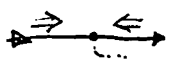
SCALAR GLUON FIELD: ϕ^a

$$\Rightarrow \quad \frac{1}{\sigma^0}\frac{d\sigma}{dx_1 dx_2} = \frac{\bar{\alpha}_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{x_3^2}{2(1-x_1)(1-x_2)}$$

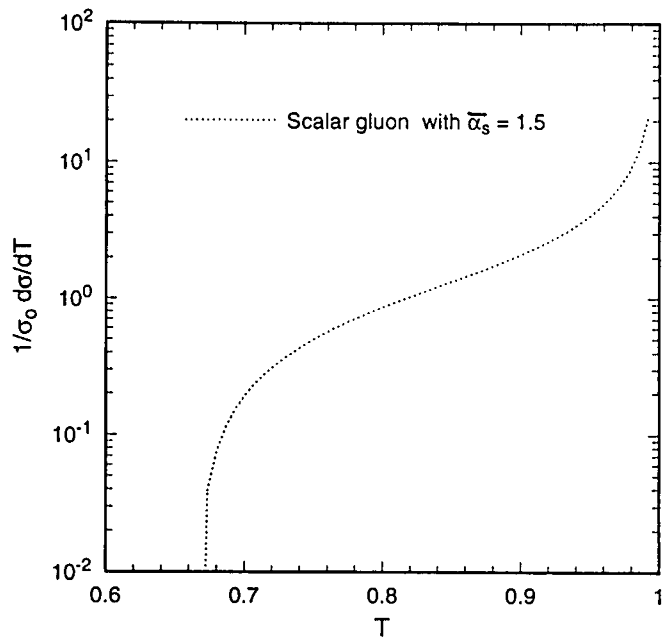
$$\frac{1}{\sigma^0}\frac{d\sigma}{dT} = \frac{\bar{\alpha}_s}{2\pi}\left(\frac{N^2-1}{2N}\right)\frac{1}{2}\left[2\ln\left(\frac{2T-1}{1-T}\right) + \frac{(3T-2)(4-3T)}{(1-T)}\right]$$

<u>NOTE</u> as $E_g \sim x_3 \to 0 \qquad |M|^2 \to 0$

- emission of gluon (scalar) changes helicity



• helicity violating



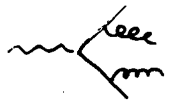
• SHAPE FOR SCALAR GLUON VERY DIFFERENT $\Rightarrow$ EVIDENCE THAT $J_g = 1$

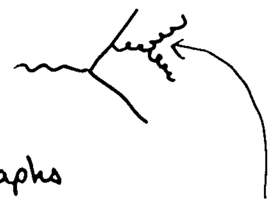
• $\bar{\alpha}_s$ MUST BE LARGE

$O(\alpha_s^2)$ Thrust Distribution

At NLO, get contributions from

double bremstrahlung





and virtual graphs

ggg VERTEX

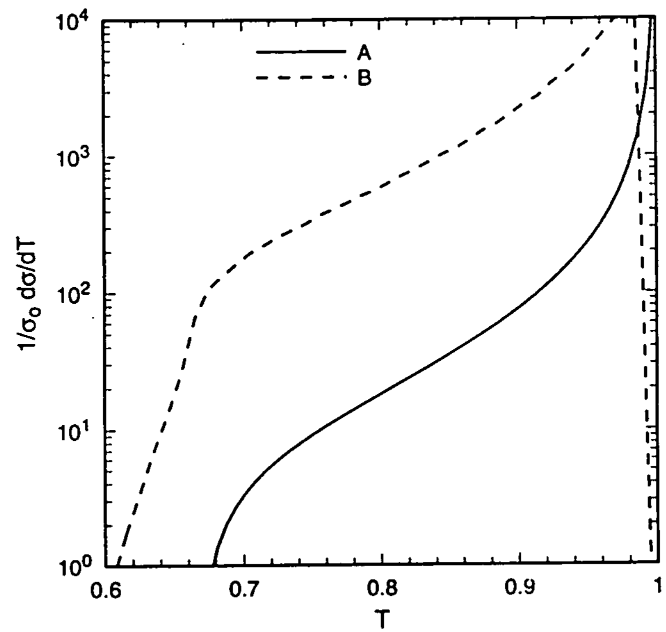
Same structure of NLO corrections as before

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{\alpha_s(\mu)}{2\pi}\,A(T)$$

$$+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left[\underbrace{2\pi A(T)\,b_0\log\frac{\mu^2}{s}}_{\text{renormalisation term}} + B(T)\right]$$

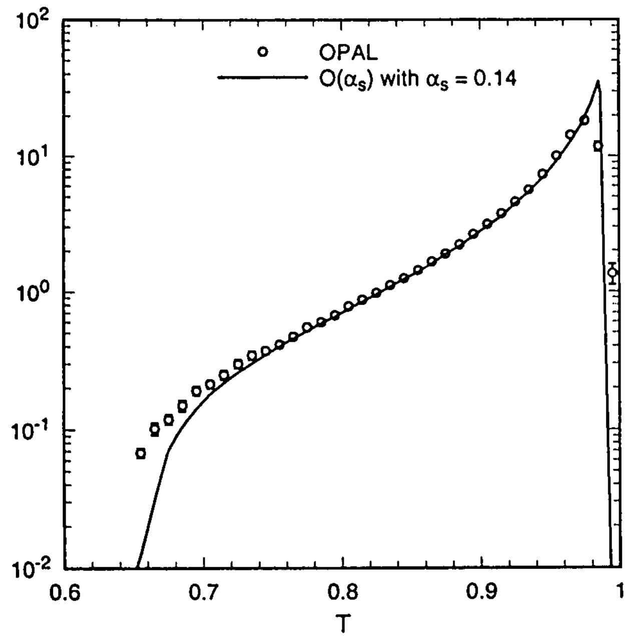
renormalisation term $b_0 = \frac{11N - 2n_f}{12\pi}$

genuine NLO
- too hard to do analytically
- numerically using EVENT program by Kunszt + Nason



- As $T \to 1$ — A diverges ↑, B diverges ↓

- $\frac{1}{\sqrt{3}} < T$ ALLOWED at NLO



- AGREEMENT WITH DATA OVER WIDER RANGE OF T
- SMALLER VALUE OF α_s
- PREDICTION $\to -\infty$ AS $T \to 1$

RESUMMING LARGE LOGS

as $T \to 1$

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} \sim \frac{\alpha_s}{2\pi}\underbrace{\left(\frac{N^2-1}{2N}\right)}_{C_F}\left\{\frac{4}{1-T}\log\left(\frac{1}{1-T}\right) - \frac{3}{1-T}\right\}$$

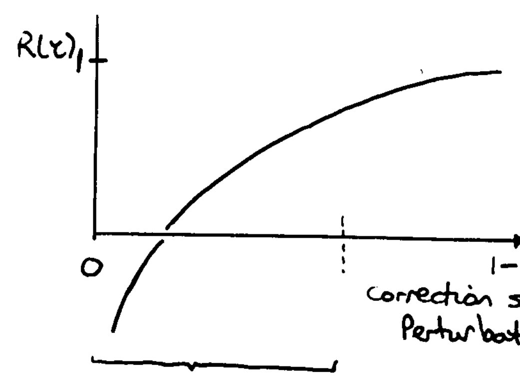
$\Rightarrow$ define cross section $1-\tau < T$

$$R(\tau) = \int_{1-\tau}^{1} dT \, \frac{1}{\sigma_0}\frac{d\sigma}{dT}$$

singularity at $T=1$ cancelled by

$\sim$ fraction of events with $T > 1-\tau$

$$R(\tau) \sim 1 - \underbrace{\frac{C_F \alpha_s}{\pi} \log^2(1-\tau)}_{\text{not small unless } \alpha_s \log^2(1-\tau) \text{ small}}$$



correction small
Perturbation theory believable

very large corrections
≡ don't believe fixed order PT.

in fact

$$R(\tau) = 1 - \frac{C_F\alpha_s}{\pi}\log^2(1-\tau) + \frac{1}{2}\left(\frac{C_F\alpha_s}{\pi}\right)^2 \log^4(1-\tau)$$

. . . .

keeping only leading logs. – CAN RESUM for few variables e.g. T, B_T, EEC, jet crossections in k_T alg.

$$\boxed{R(\tau) \sim \exp -\frac{C_F\alpha_s}{\pi}\log^2(1-\tau)}$$

so that as $\tau \to 1$ $\log^2(1-\tau) \longrightarrow 0$

$$\underline{R(\tau) \longrightarrow 0}$$

- This is SUDAKOV form factor effect
  - for event to have high thrust, must have radiated very few gluons
  - very improbable

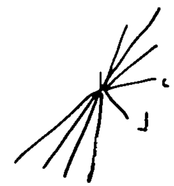
cf data, very improbable to have only 2 particle event.

- can also resum next-to-leading logs (Webber, Dokshitzer, Catani et al)

$\Rightarrow$ calculations believable when $\alpha_s \log(1-\tau)$ small.

WARNING: NOT ALL OBSERVABLES ARE EQUAL

e.g. JETS DEFINED THROUGH CLUSTERING



- FIND min d_{ij}
- IF $d_{ij} < d_{cut}$ COMBINE ij
- REPEAT UNTIL NO MORE CLUSTERING
  – ALL $d_{ij} > d_{cut}$

JADE $\quad d_{ij} = 2E_iE_j(1-\cos\theta_{ij})$

K_T ALGORITHM $\quad d_{ij} = 2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})$

– BOTH INFRARED SAFE

BUT LOGS CAN BE RESUMMED FOR K_T ALGORITHM

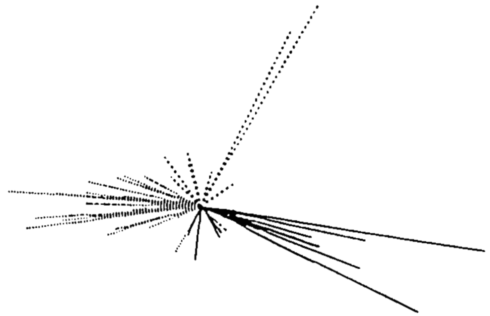
– MORE OBVIOUS ASSIGNMENT OF PARTICLES IN JETS

– HADRONISATION CORRECTIONS SMALLER

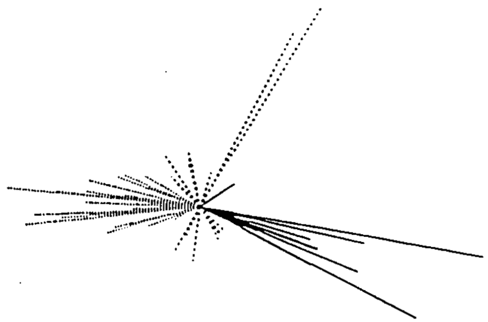
→ FIG

3 JET EVENTS

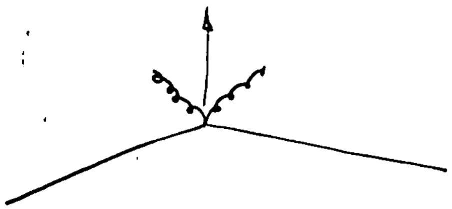
k_T ALGORITHM



JADE



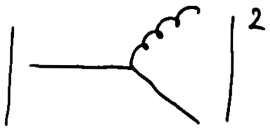
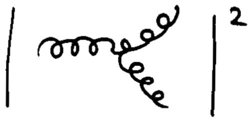
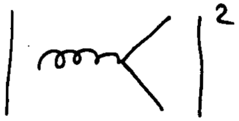
JADE



ADDS TWO SOFT GLUONS TOGETHER TO MAKE JET

MEASURING QCD GROUP PARAMETERS

Probabilities for parton splitting

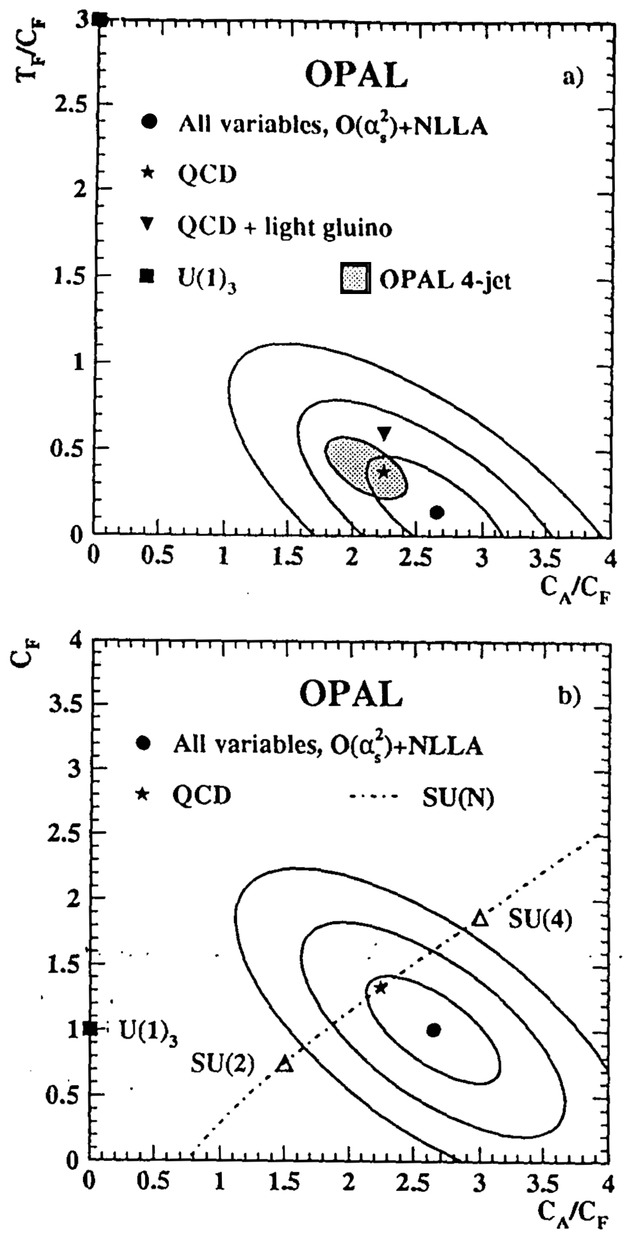
				QCD
$q \to qg$		$\propto$	$C_F \alpha_s$	$\left(\frac{N^2-1}{2N}\right) \alpha_s$
$g \to gg$		$\propto$	$C_A \alpha_s$	$N \cdot \alpha_s$
$g \to q\bar{q}$		$\propto$	$T_F \alpha_s$	$\frac{1}{2} \alpha_s$

All present in $O(\alpha_s^2)$ EVENT shapes and 4 jet rate

i.e. $$B(T) = C_F \left[C_F B_{C_F}(T) + C_A B_{CA}(T) + T_F B_{TF}(T) \right]$$

FIT TO DATA ON THRUST, B_T etc.

→ agreement with QCD.



JET PRODUCTION

THREE DISTINCT STAGES

1) ON MOMENTUM SCALE $\boxed{t \sim Q^2}$ HARD SCATTERING INVOLVING SMALL NUMBER OF PRIMARY PARTONS e.g $e^+e^- \to q\bar{q}$

2) OVER A PERIOD $\boxed{Q_0^2 < t < Q^2}$ PRIMARY PARTONS CASCADE / SHOWER BY MULTIPLE BREMSTRAHLUNG. CASCADES TEND TO FOLLOW DIRECTION OF PRIMARY PARTONS DUE TO SOFT / COLLINEAR BEHAVIOUR OF $|M|^2$

- COHERENCE (ANGULAR ORDERING) AND SUDAKOV BRANCHING — LEADING LOGS RESUMMED

3) $\boxed{\Lambda^2 < t < Q_0^2}$ NON - PERTURBATIVE (LONG DISTANCE + LONG TIMES) WHERE PARTONS BLANCH AND FORM COLOURLESS HADRONS

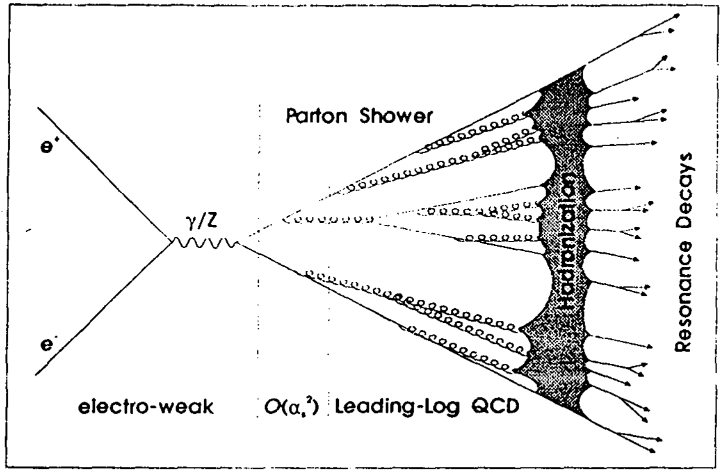


Figure 3: Schematic representation of a parton shower.

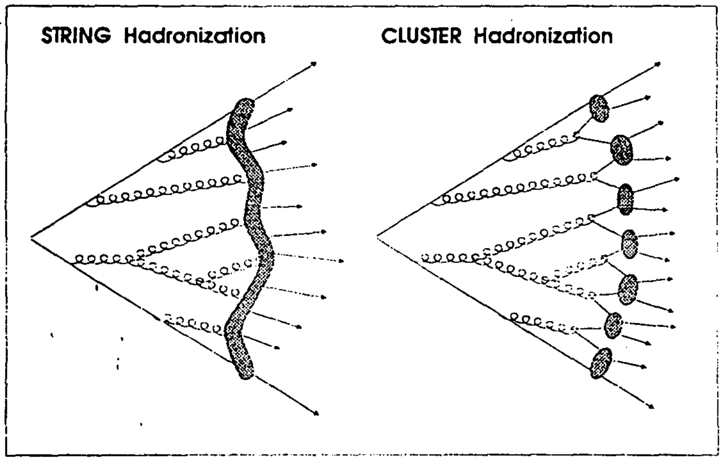


Figure 4: Pictorial presentation of the string and the cluster hadronization model.

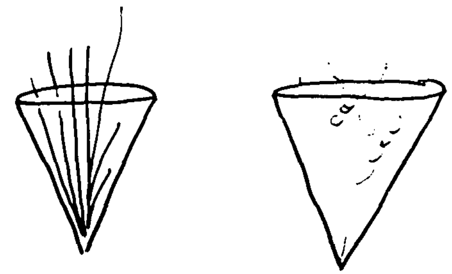
LOCAL PARTON HADRON DUALITY

- FLOW OF MOMENTUM + QUANTUM NUMBERS AT HADRON LEVEL TENDS TO FOLLOW FLOW ESTABLISHED AT PARTON LEVEL

e.g. The flavour of quark initiating jet should be found in hadron near jet axis

The extent the hadron flow deviates from parton flow reflects smearing due to hadronisation – of order λ.

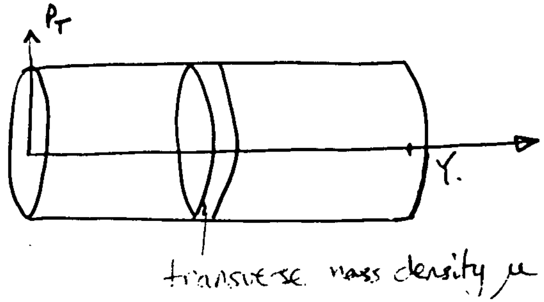
e.g



JET energy + direction determined at parton level ≈ that at hadron level.

SIMPLE HADRONISATION MODEL

(Feynman)



pseudorapidity $= \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$

Parton produces a tube in (y, p_T) space of light hadrons wrt. initial parton direction

$$E_{jet} = \mu \int_0^Y \cosh y \, dy = \mu \sinh Y$$

$$P_{jet} = \mu \int_0^Y \sinh y \, dy = \mu(\cosh Y - 1)$$

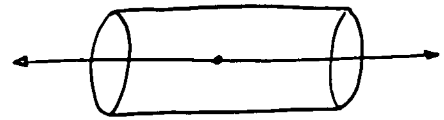
$$\Rightarrow \quad m_{jet}^2 = E_{Jet}^2 - P_{Jet}^2 = 2\mu^2(\cosh Y - 1)$$
$$= 2\mu P_{jet}$$

$$\boxed{E_{jet} = P_{jet} + \mu}$$

$\mu \sim 0.5 - 1$ GeV from expt.

HADRONISATION AND THRUST

$$T = \max \frac{\sum |p_a \cdot n|}{Q}$$



2 jet event $\quad T_{parton} = 1$

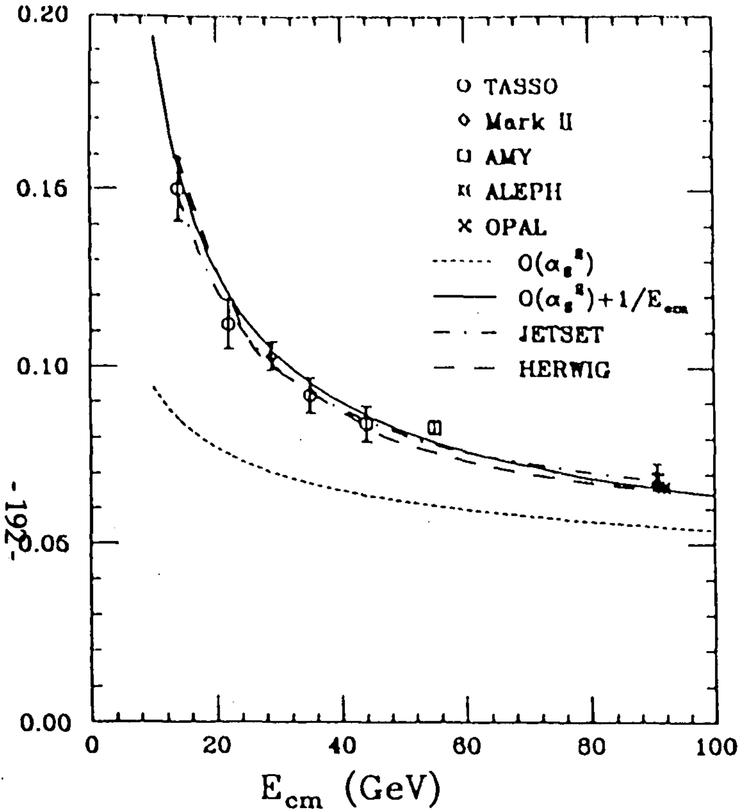
$$T_{hadron} = \frac{2P_{jet}}{Q} = \frac{2E_{jet} - 2\mu}{Q}$$
$$= 1 - \frac{2\mu}{Q}$$

ie. $\boxed{\delta T = -\frac{2\mu}{Q}}$

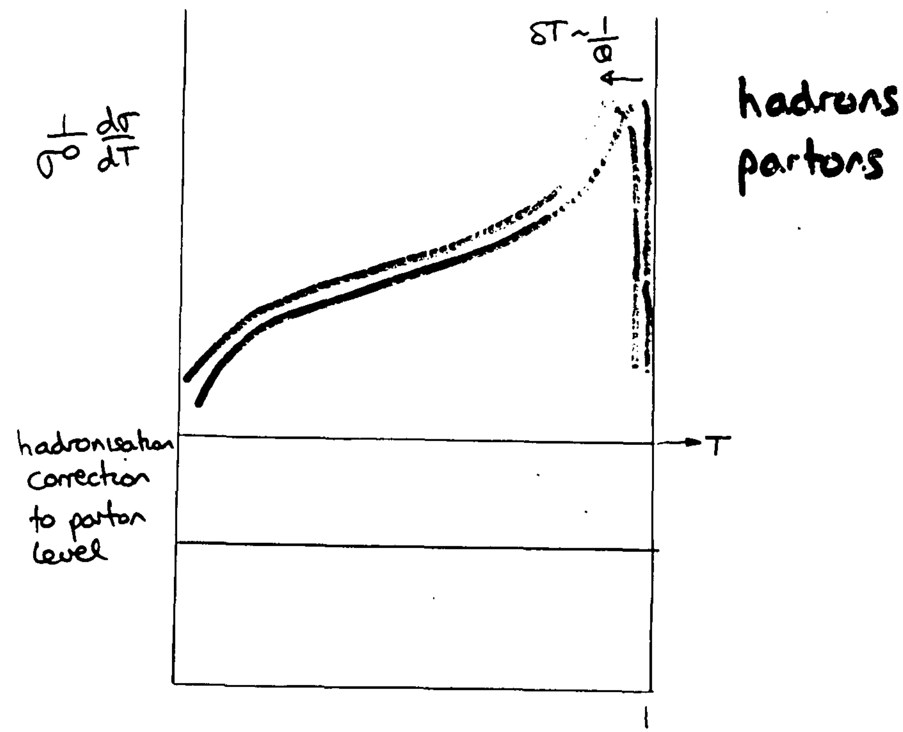
MEAN VALUE OF THRUST

$$\langle 1-T \rangle = \int (1-T) \frac{1}{\sigma_0} \frac{d\sigma}{dT} dT$$

$$= 0.334\,\alpha_s + 1.02\,\alpha_s^2 + \frac{1\,\text{GeV}}{Q}$$



HADRONISATION CORRECTION TO THRUST DISTRIBUTION



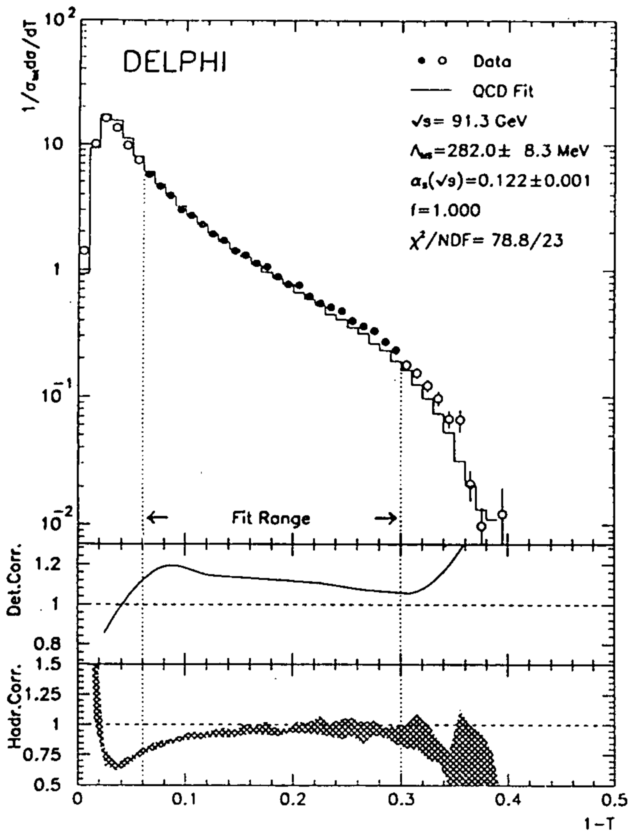


Figure 11: Measurement of the strong coupling constant from the Thrust distribution. The data points used in the fit are indicated by the full dots. Detector and hadronization corrections are indicated below. The theoretical prediction is the second order matrix element plus resummation of leading and next-to-leading logarithms.

4. Deep Inelastic Scattering and QCD

- QCD improved Parton Model
- Mass Factorisation
- Evolution equations
- Callan Gross Relation
- Global fits and PDF's

QCD + PARTON MODEL
– DIS REVISITED

- EXPLAIN LOGARITHMIC SCALING VIOLATIONS SEEN IN DATA
- WILL IGNORE HIGHER TWIST TERMS

$$F(x,Q^2) = F^{(2)}(x,Q^2) + \frac{1}{Q^2} F^{(4)}(x,Q^2) + \dots$$

LEADING TWIST HIGHER TWIST

e.g final state interactions

Milsztajn + Virchaux
- EMPIRICAL FITS AT low Q^2 (SLAC) + high Q^2 (BCDMS)

$\Rightarrow$ negligible for $Q^2 > 5\,GeV^2$, $W^2 = \frac{Q^2(1-x)}{x} \geq 10\,GeV^2$

LEADING ORDER

$$F_2(x) = \sum_q e_q^2 \, x \, f_q(x)$$

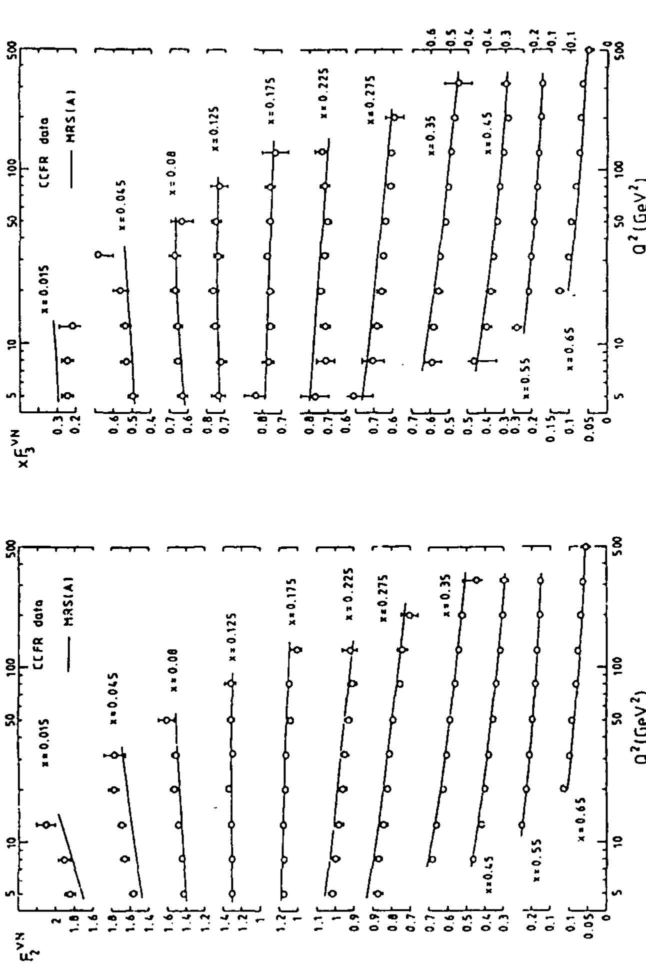
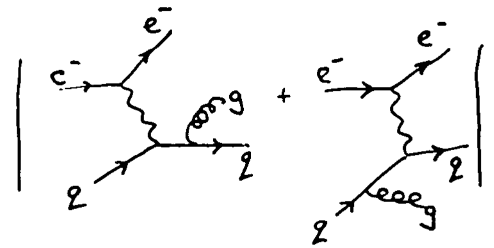


Fig. 8

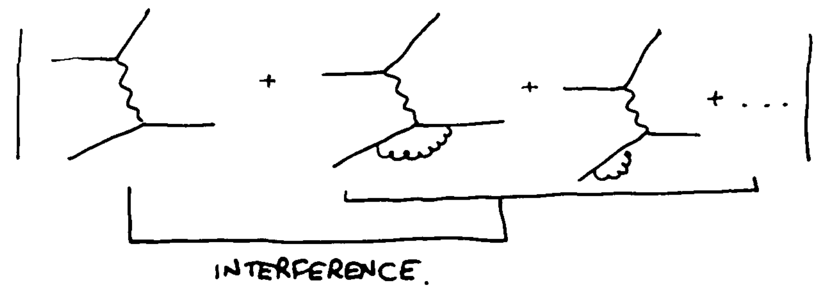
QCD IMPROVED PARTON MODEL

$O(\alpha_s)$ CORRECTIONS

1) BREMSTRAHLUNG



2) VIRTUAL

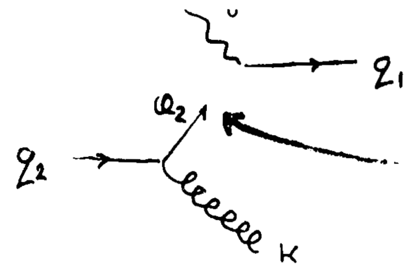


— SAME $|M|^2$ AS $e^+e^- \to q\bar{q}g$ and $e^+e^- \to q\bar{q}$ WITH e^+ CROSSED TO FINAL STATE AND $\bar{q}$ CROSSED TO INITIAL STATE

e.g $\quad e^-(p_2) + q(q_2) \longrightarrow e^-(p_1) + q(q_1) + g(k)$

$$\frac{1}{4}|M|^2 = 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \times \frac{\left[(q_1\cdot p_1)^2 + (q_1\cdot p_2)^2 + (q_2\cdot p_1)^2 + (q_2\cdot p_2)^2\right]}{q_1\cdot K \;\; K\cdot q_2 \;\; p_1\cdot p_2}$$

from before



SINGULAR WHEN $k.q_2 \to 0$
ie. INITIAL STATE
COLLINEAR SINGULARITY

→ IN COLLINEAR LIMIT

$$Q_2 = z q_2 \qquad k = (1-z) q_2 = \frac{(1-z)}{z} Q_2$$

so

$$q_1 \cdot k = \frac{1-z}{z}\, q_1 \cdot Q_2 = \frac{1-z}{z}\, p_1 \cdot p_2$$

$$q_2 \cdot p_1 = \frac{1}{z}\, Q_2 \cdot p_1 = \frac{1}{z}\, q_1 \cdot p_2$$

$$q_2 \cdot p_2 = \frac{1}{z}\, Q_2 \cdot p_2 = \frac{1}{z}\, q_1 \cdot p_1$$

$$\frac{1}{4}\sum |M|^2 \longrightarrow 2e^4 e_q^2 \left(\frac{N^2-1}{2}\right) g_s^2 \,\frac{\left[(q_1\cdot p_1)^2 + (q_1 \cdot p_2)^2\right]}{(p_1\cdot p_2)^2}$$

$$\times \frac{1}{q_2 \cdot k}\left[\frac{1+z^2}{z(1-z)}\right]$$

$$= \frac{g_s^2}{2 q_2 \cdot k}\left(\frac{N^2-1}{2N}\right) \frac{2(1+z^2)}{z(1-z)}\; \frac{1}{4}\sum |M|^2_{eq}$$

$$\text{FLUX} = \frac{1}{2 q_2 \cdot p_2} \longrightarrow \frac{z}{2 Q_2 \cdot p_2}$$

Phase Space

$$d\Phi(p_1, q_1, k) \longrightarrow d\Phi(p_1, q_1)\, \frac{1}{16\pi^2}\, dt\, \frac{dz}{z}$$

(t circled, annotated: $\sim k.q_2$)

Putting pieces together

$$d\sigma(eq \to eqg) = \frac{1}{\text{Flux}}\,\frac{1}{4}\sum |M|^2 d\Phi$$

$$\longrightarrow \frac{\alpha_s}{2\pi}\, \frac{N^2-1}{2N}\, \frac{1+z^2}{1-z}\, \frac{dt}{t}\, \frac{dz}{z}\, d\sigma(eq \to eq)$$

$$= \frac{\alpha_s}{2\pi} \qquad\qquad \frac{dz}{z}\, d\sigma(eq \to eq)$$

SMALL SCALE TO
CUT OFF INTEGRATION

Altogether, have now found $O(\alpha_s)$ CORRECTIONS TO F_2 $\left\{ z = x/y \Rightarrow dz/z = -dy/y \quad \int_{x/y}^1 \to -\int_x^1 \right\}$

$$F_2(x, Q^2) = x \sum_q e_q^2 \int_x^1 \frac{dy}{y} f_q(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_s}{2\pi}\left\{ P_{qq}\left(\frac{x}{y}\right) \log\frac{Q^2}{m^2} + R\left(\frac{x}{y}\right)\right\}\right]$$

VIRTUAL GRAPHS CONTRIBUTE

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+ + 2\delta(1-x)$$

– $R\left(\frac{x}{y}\right)$ IS CALCULABLE
FINITE CORRECTION

- As $t \to 0$, the collinear quark propagates for longer times
  - eventually hadronisation occurs!

DIVIDE LONG DISTANCE / SHORT DISTANCE EFFECTS WITH SCALE μ

FINITE

$Q^2 > t > \mu^2$ — SHORT DISTANCE — Perturbative

$\mu^2 > t > m^2$ — LONG DISTANCE — NON-PERT.

— "RENORMALIZE" INTO PARTON DISTRIBUTIONS

MASS FACTORISATION

$$f_q(x,\mu^2) = f_q(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} f_q(y)\left\{P_{qq}\left(\frac{x}{y}\right)\log\frac{\mu^2}{m^2} + R_2\left(\frac{x}{y}\right)\right\}$$

- $f_q(x)$: "bare" parton density function
- $\log\frac{\mu^2}{m^2}$: divergence as $m \to 0$
- R_2: scheme dependent

$$F_2(x,Q^2) = x\sum e_q^2 \int_x^1 \frac{dy}{y} f_q(x,\mu^2)\left\{\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{y}\right)\log\frac{Q^2}{\mu^2} + R_1(x/y)\right\}\right\}$$

- $f_q(x,\mu^2)$: PROCESS INDEPENDENT → CAN BE USED ELSEWHERE
- R_1: $R - R_2$ ⇕ Factorisation scheme dep.

Just as $\alpha_s(M_Z^2)$ is not directly calculable, neither is $f_q(x,\mu^2)$

— BUT VARIATION WITH μ^2 IS

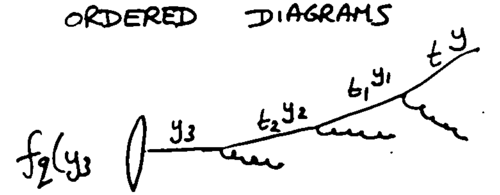
$$\mu^2 \frac{\partial f_q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y} f_q(x,\mu^2) P_{qq}(x/y)$$

— this is DGLAP equation and predicts scaling violations

- expt $\longrightarrow f_q(x,Q_0^2)$	- expt $\Rightarrow \alpha_s(M_Z)$
- running given by QCD	- running α_s
- μ is FACTORISATION scale	- μ is RENORMALISATION SCALE

NOTE — Physical cross sections are independent of μ — FIXED ORDER CALCULATIONS ARE NOT

DGLAP corresponds to resumming STRONGLY ORDERED DIAGRAMS



$t \gg t_1 \gg t_2 \gg t_3$

$y < y_1 < y_2 < y_3$

DGLAP

At $O(\alpha_s)$ ALSO GET CONTRIBUTION FROM GLUON PARTON DENSITY FUNCTION

– ALSO [gluon → quark splitting] [gluon → gluon splitting] SPLITTINGS

$$\mu^2 \frac{\partial f_q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dy}{y} \left[P_q\left(\frac{x}{y}\right) f(y,\mu^2) + P_g\left(\frac{x}{y}\right) f(y,\mu^2) \right]$$

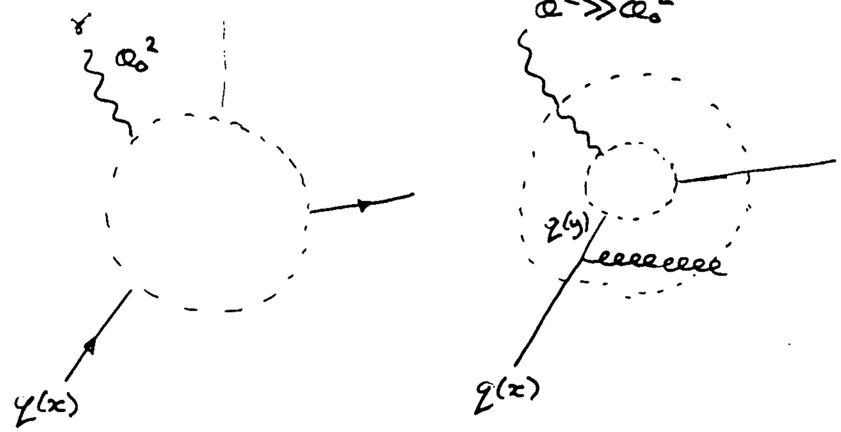
$$\mu^2 \frac{\partial f_g(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dy}{y} \left[P_g\left(\frac{x}{y}\right) f(y,\mu^2) + \sum_q P_g\left(\frac{x}{y}\right) f(y,\mu^2) \right]$$

$\frac{\alpha_s}{2\pi} P_{ab}\left(\frac{x}{y}\right)$ IS PROBABILITY OF FINDING a WITH MOMENTUM FRACTION x INSIDE b WITH FRACTION y

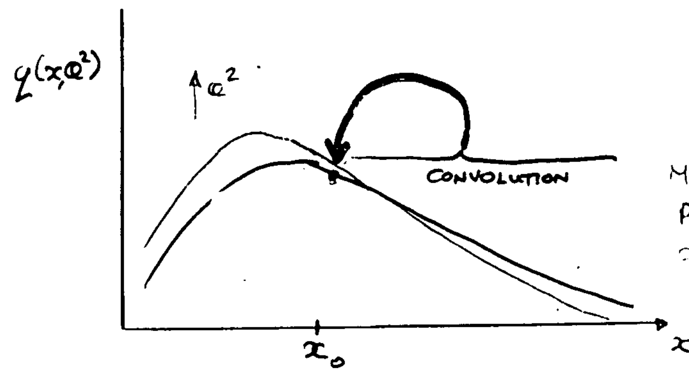
– for some input $f_i(x,\mu_0^2)$ can solve to find $f_i(x,\mu^2)$

– EVOLUTION known to $O(\alpha_s^2)$

SIMPLE PICTURE

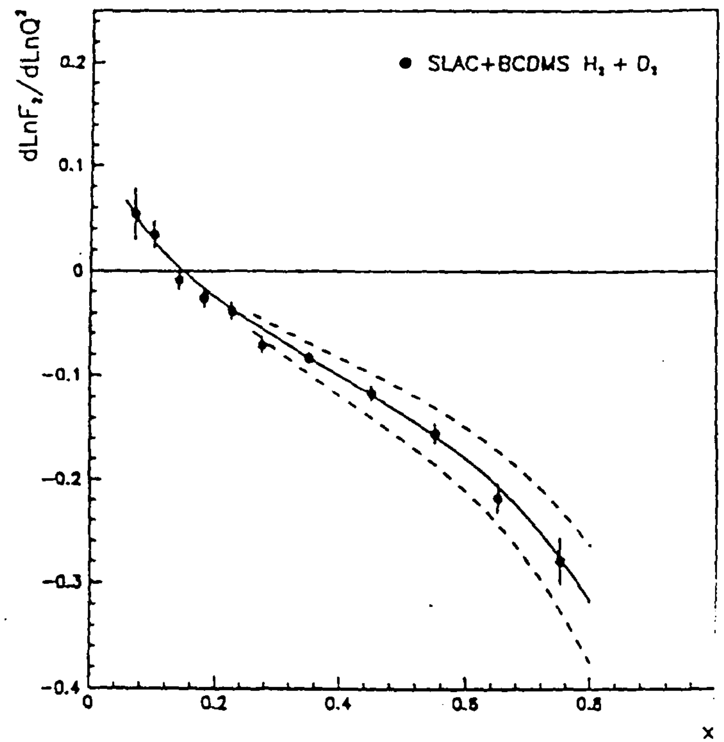


– AS $Q^2 \uparrow$ PHOTON RESOLVES QUARKS WITH LESS MOMENTA DUE TO RADIATION



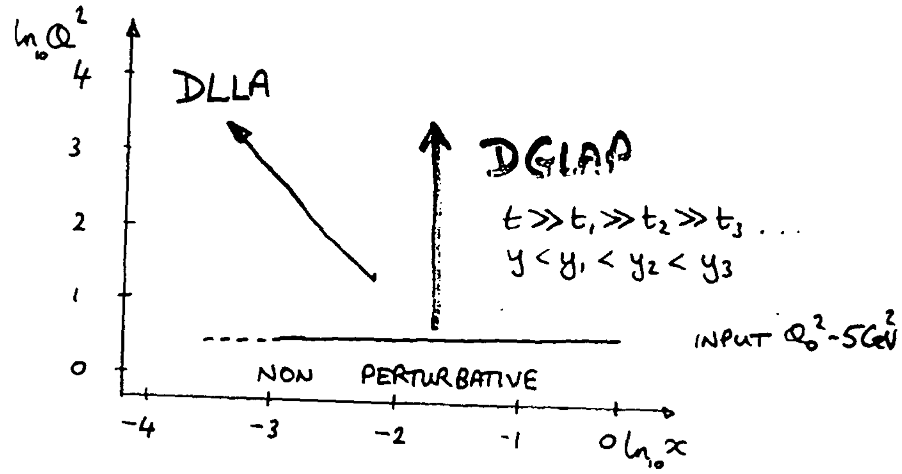
MOVEMENT OF PDF TO LOWER x – DICTATED BY $q(x)$ for $x > x_0$

$$q(x_0, Q^2+\Delta Q^2) = q(x_0, Q^2) + \frac{\alpha_s}{2\pi} \int_{x_0}^1 \frac{dy}{y} P\left(\frac{x_0}{y}\right) q(y, Q^2) \ln\frac{Q^2+\Delta Q^2}{Q^2}$$



-- different values of $\alpha_s(M_Z)$

EVOLUTION IN x, Q^2 SPACE



DLLA – strong ordering in t and y

$$t \gg t_1 \gg t_2 \gg t_3$$
$$y \ll y_1 \ll y_2 \ll y_3$$

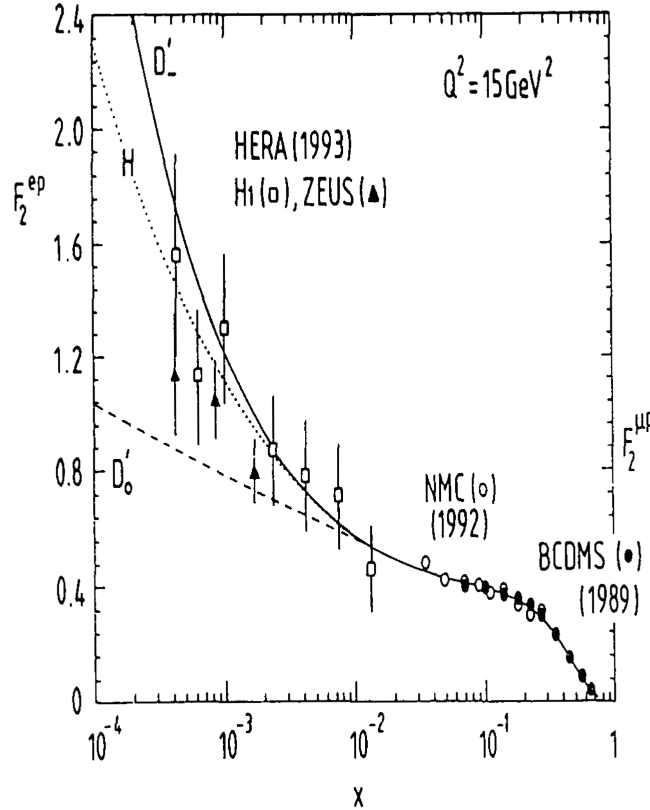
$\Rightarrow$ RESUM $\left[\alpha_s \log\left(\frac{1}{x}\right) \log\left(\frac{Q^2}{Q_0^2}\right)\right]^n$

BFKL – strong ordered ONLY in y

$$y \ll y_1 \ll y_2 \ll y_3$$

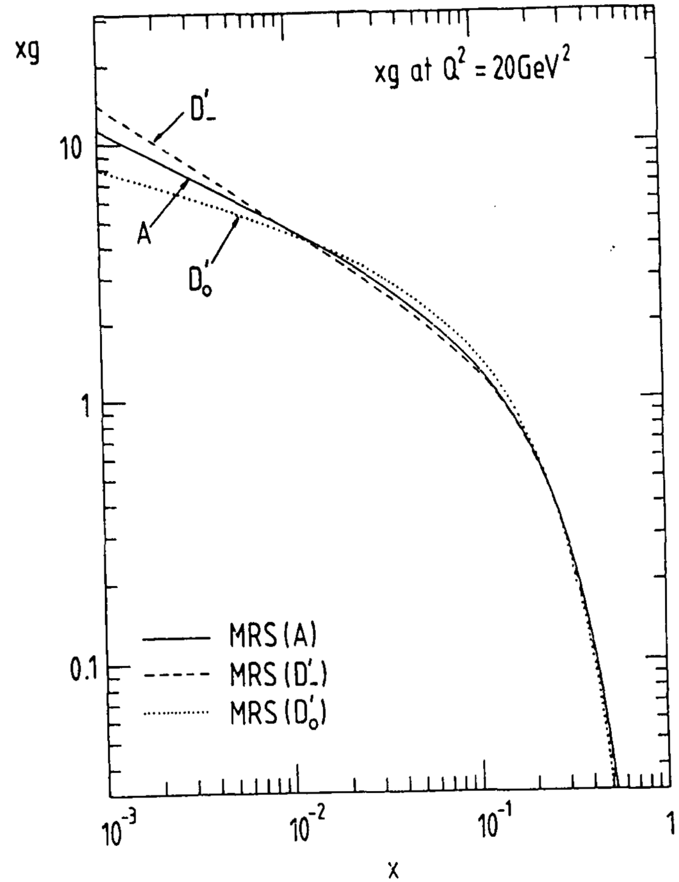
$\Rightarrow$ RESUM $\left[\alpha_s \log\left(\frac{1}{x}\right)\right]^n$

– BOTH PREDICT GROWTH AT SMALL x $[F_2]$



$Q^2 = 15 GeV^2$
HERA (1993)
H1 (□), ZEUS (▲)
D'_-
H
D'_0
NMC (○)
(1992)
BCDMS (•)
(1989)
F_2^{ep}
$F_2^{\mu p}$
2.4
2.0
1.6
1.2
0.8
0.4
0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

Fig.1

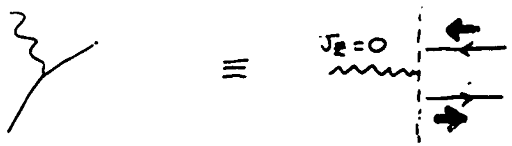


xg
xg at $Q^2 = 20 GeV^2$
D'_-
A
D'_0
MRS (A)
MRS (D'_-)
MRS (D'_0)
10
1
0.1
10^{-3}
10^{-2}
10^{-1}
1
x

Fig. 15

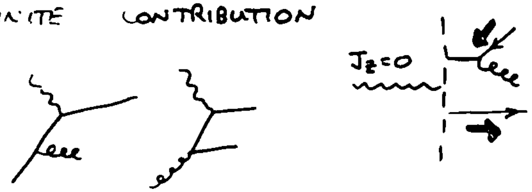
CALLAN - GROSS RELATION / F_L

- AT LOWEST ORDER $\boxed{F_2 - 2xF_1 = 0}$
  - This combination sensitive to helicity zero component of γ^* (LONGITUDINAL)
  - ⇒ Define $\boxed{F_L = F_2 - 2xF_1}$
- IN BREIT / BRICK WALL FRAME



DIRECTION OF QUARKS REVERSED

- helicity conserved by QED, but FORBIDDEN
- AT NLO, RADIATION OF GLUON GIVES FINITE CONTRIBUTION



$F_L \neq 0$

$$\boxed{F_L(x,Q^2) = \frac{\alpha_s}{\pi}\int_x^1 \frac{dy}{y}\left(\frac{x}{y}\right)^2\left[F_2(y,Q^2) + 2\sum_q e_q^2\left(1-\frac{x}{y}\right) y f_g(y,Q^2)\right]}$$

↑ known (under $F_2(y,Q^2)$) ↑ direct measure (under $f_g(y,Q^2)$)

GLOBAL FITS → PDF

- PARAMETERISE $f_q(x,Q_0^2)$, $f_g(x,Q_0^2)$
  + QCD EVOLUTION + COMPARE WITH WIDE RANGE OF DATA (FIT) (MRS) CTEQ. GRV.

e.g.

$$x u_v = A_u x^{\eta_1}(1-x)^{\eta_2}(1+\varepsilon_u\sqrt{x} + \gamma_u x)$$
$$x d_v = A_d x^{\eta_3}(1-x)^{\eta_4}(1+\varepsilon_d\sqrt{x} + \gamma_d x)$$
$$x S = A_s x^{-\lambda}(1-x)^{\eta_5}(1+\varepsilon_s\sqrt{x} + \gamma_s x)$$
$$x g = A_g x^{-\lambda}(1-x)^{\eta_g}(1+\gamma_g x)$$

$$2\bar{u} = 0.4\,S - \Delta$$
$$2\bar{d} = 0.4\,S + \Delta$$
$$2\bar{s} = 0.2\,S$$

$x\Delta = A x^{\eta_\Delta}(1-x)^{\eta_s}$ — GOTTFRIED SUM RULE

→ MANY PARAMETER FITS

→ AS DATA IMPROVE, FITS CHANGE

→ CAN USE IN $p\bar{p}$, pp COLLISIONS TO PREDICT CROSS SECTIONS

MRS input

Process/ Experiment	Leading order subprocess	Parton determination
DIS ($\mu N \to \mu X$) BCDMS, NMC $F_2^{\mu p}, F_2^{\mu n}$	$\gamma^* q \to q$	Four structure functions $\to$ $u+\bar{u}$, $d+\bar{d}$, $\bar{u}+\bar{d}$, s (assumed $= \bar{s}$), but only $\int xg(x)dx \simeq 0.5$ [$\bar{u}-\bar{d}$ is not determined]
DIS ($\nu N \to \mu X$) CCFR (CDHSW) $F_2^{\nu N}, xF_3^{\nu N}$	$W^* q \to q'$	
$\nu N \to \mu^+\mu^- X$ CCFR	$\nu s \to \mu^- c$, $c \to \mu^+$	$s \approx \frac{1}{2}\bar{u}$ (or $\frac{1}{2}\bar{d}$) ⟶ FIG
DIS (HERA) F_2^{ep} (H1,ZEUS)	$\gamma^* q \to q$	λ ($x\bar{q} \sim xg \sim x^{-\lambda}$, via $g \to q\bar{q}$)
$pp \to \gamma X$ WA70 (UA6)	$qg \to \gamma q$	$g(x \approx 0.4)$
$pN \to \mu^+\mu^- X$ E605	$q\bar{q} \to \gamma^*$	$\bar{q} = ...(1-x)^{\eta_S}$
$pp, pn \to \mu^+\mu^- X$ NA51	$u\bar{u}, d\bar{d} \to \gamma^*$; $u\bar{d}, d\bar{u} \to \gamma^*$	$(\bar{u}-\bar{d})$ at $x = 0.18$
$p\bar{p} \to WX(ZX)$ UA2, CDF, D0	$ud \to W$	u,d at $x_1x_2s \simeq M_W^2 \to$ $x \approx 0.13$ CERN, $x \approx 0.05$ FNAL
$\to W^\pm$ asym CDF		slope of u/d at $x \approx 0.05$ ⟶ FIG

RISE OF F_2 AT LOW x

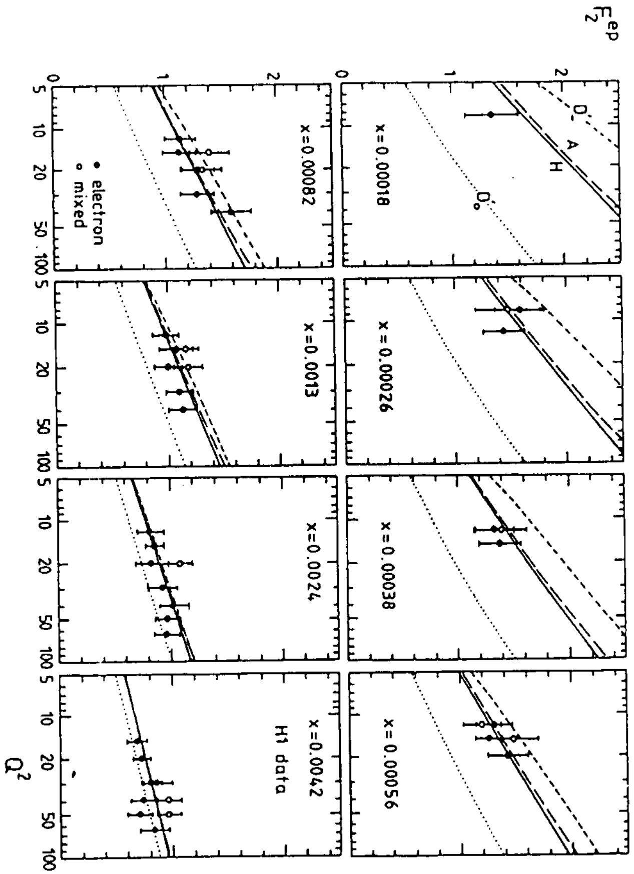
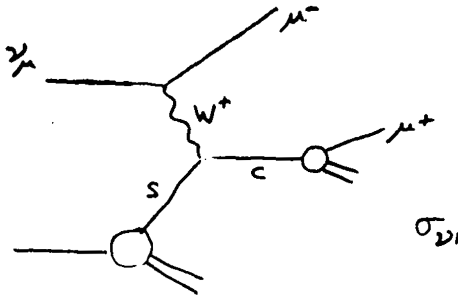


Fig. 11



$$\sigma_{\nu N} \sim |V_{cs}|^2 \, s(x) \, D^{c \to \mu^+}$$

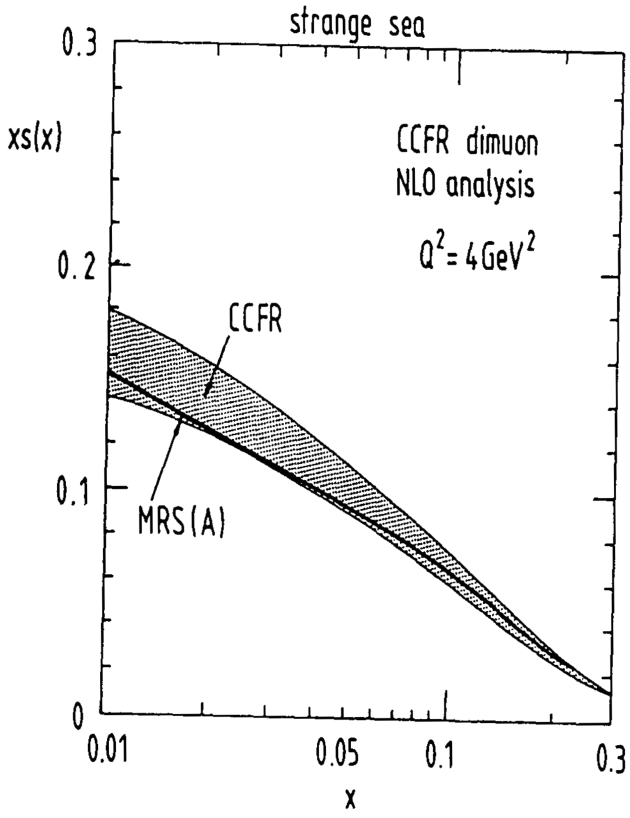
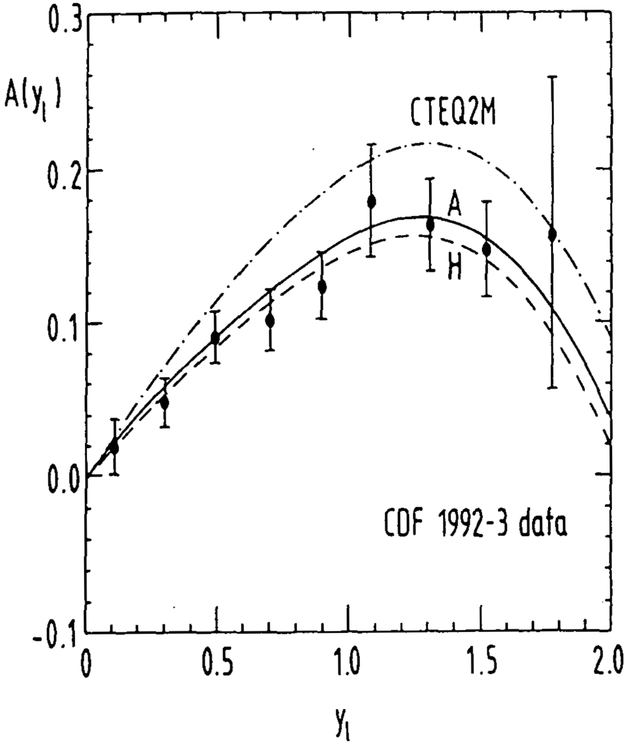


Fig.4

$$\frac{\sigma^+(y) - \sigma^-(+y)}{\sigma^+(y) + \sigma^-(+y)} = A(y_l)$$



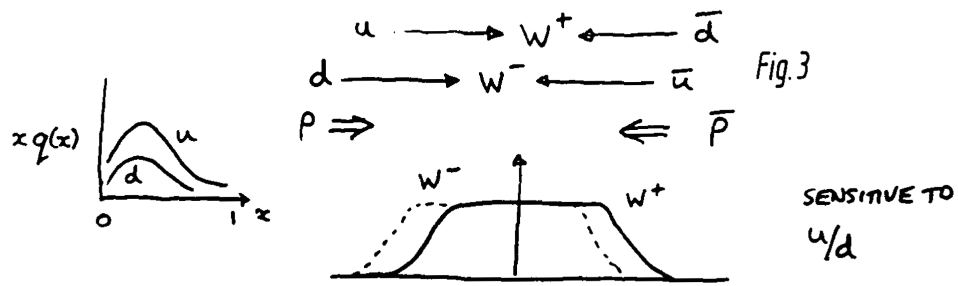
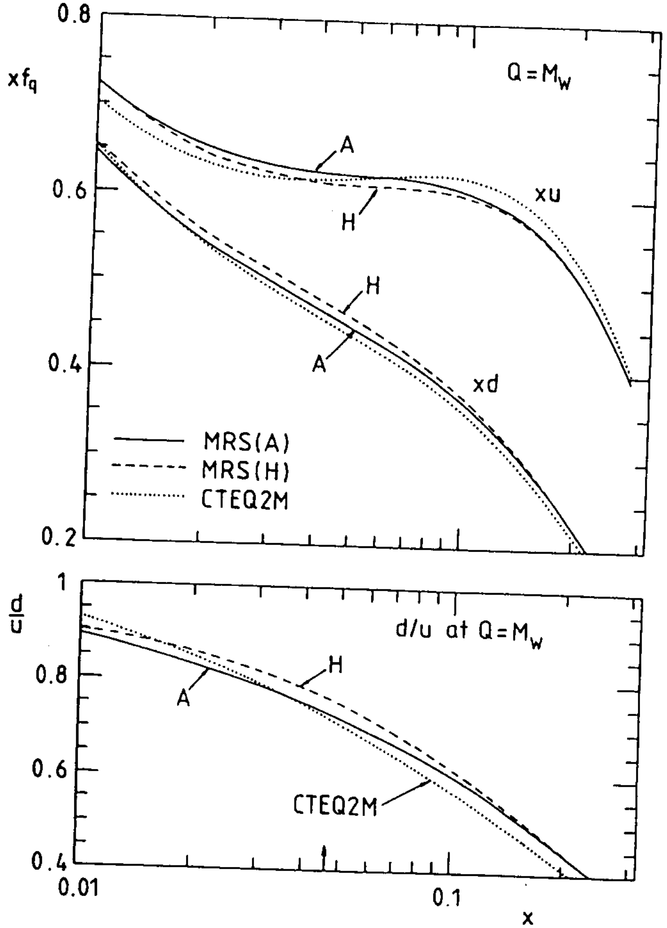
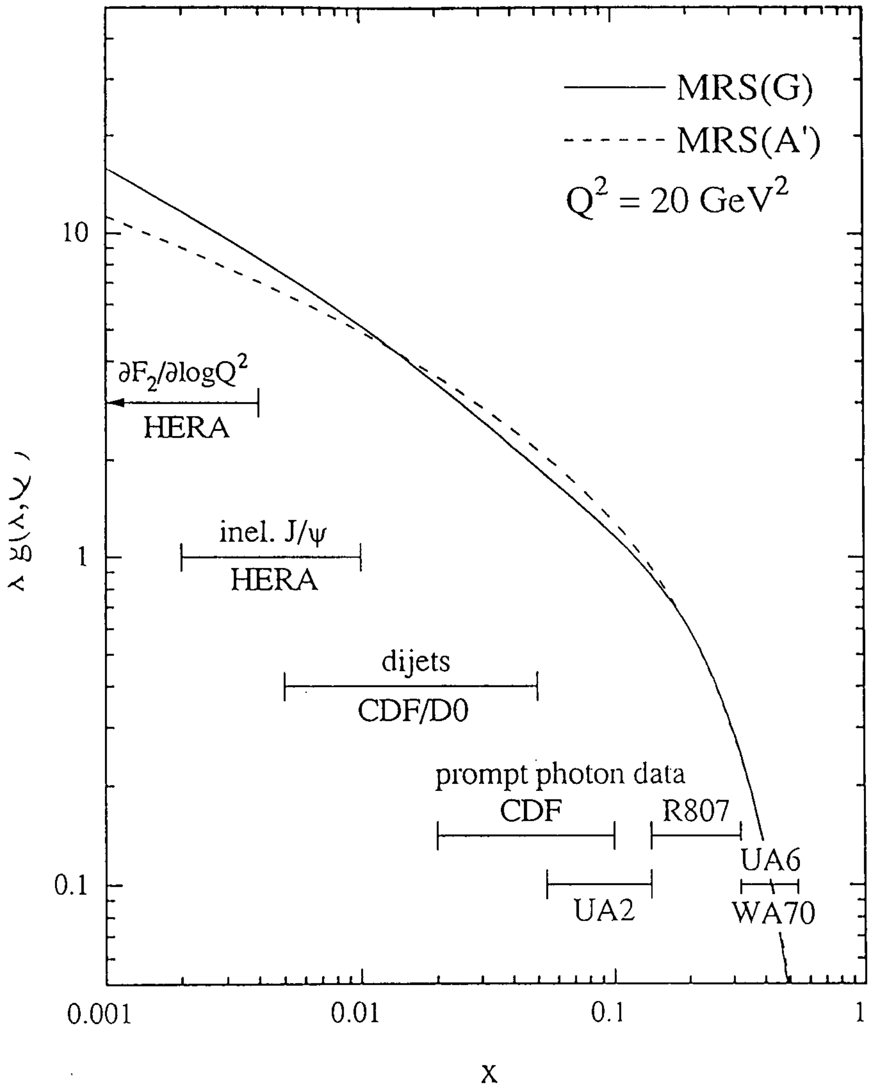


Fig.3

W ASYMMETRY MAINTAINED AFTER $W^\pm \to \ell^\pm \nu$



xf_q
Q = M_W
A
H
xu
xd
MRS(A)
MRS(H)
CTEQ2M
0.8
0.6
0.4
0.2
d/u
d/u at Q = M_W
CTEQ2M
1
0.8
0.6
0.4
0.01
0.1
x

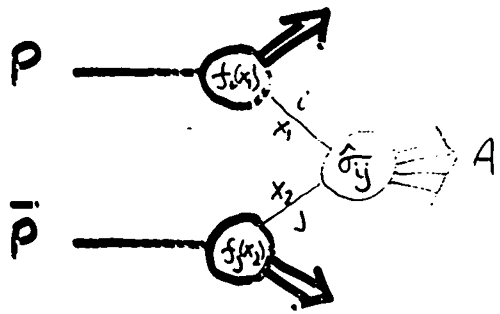


MRS(G)
MRS(A')
Q^2 = 20 GeV^2
x g(x,Q^2)
∂F_2/∂logQ^2
HERA
inel. J/ψ
HERA
dijets
CDF/D0
prompt photon data
CDF
R807
UA6
UA2
WA70
10
1
0.1
0.001
0.01
0.1
1
x

5. Hadron Collider Physics

- Parton Model in Hadron-Hadron Collisions
- Jets with large transverse momentum
- 2 → 2 QCD scattering processes
- Single Jet inclusive tranverse energy distribution
- Theoretical Uncertainties

PARTON MODEL IN HADRON HADRON COLLISIONS



FINAL STATE PRODUCED BY HARD SCATTERING OF TWO PARTONS

$\hat{\sigma}_{ij}$ $ij \rightarrow A$

ie. AT LEAST ONE HARD SCALE

$$\sigma = \sum_{ij} \int dx_1 dx_2 \, f_i(x_1, \mu_F^2) \, f_j(x_2, \mu_F^2) \times \hat{\sigma}_{ij}(x_1 P_1, x_2 P_2, \alpha_S(\mu_R), \mu_R, \mu_F, Q)$$

FACTORISATION

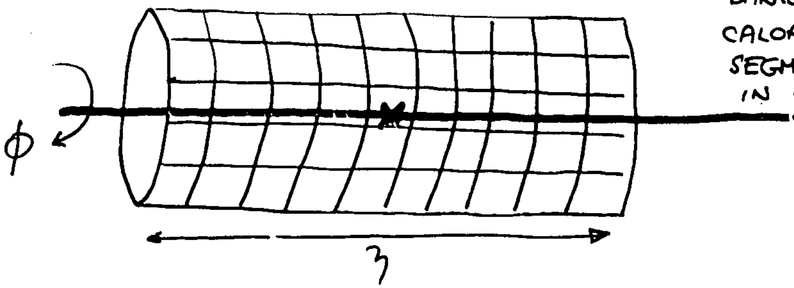
- INITIAL STATE SINGULARITIES CAN BE FACTORIZED INTO $f_i(x, \mu_F^2)$ (LONG DISTANCE)

⇒ SAME $f_i(x, \mu_F^2)$ AS IN DIS

- TWO SCALES
  - μ_F – FACTORIZATION
  - μ_R – RENORMALISATION

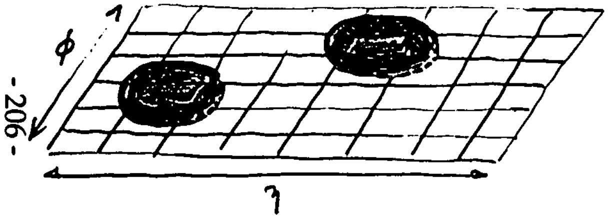
  } SOME DEPENDENCE AT FIXED ORDER – NONE IN PRINCIPLE

LARGE P_T / E_T JET PRODUCTION



"BARREL" CALORIMETERS SEGMENTED IN η, ϕ

$$\eta = \text{Pseudorapidity} = \frac{1}{2}\log\left(\frac{E+p_z}{E-p_z}\right) = \log\tan\theta/2$$

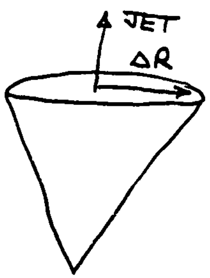


JETS DEFINED USING CONE ALGORITHMS (COULD + WILL USE CLUSTER ALG AS IN e^+e^-)

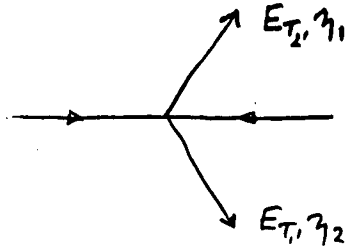
$$-\quad E_T^{JET} = \sum E_{T_i}$$

$$\eta^{JET} = \frac{\sum E_{T_i}\eta_i}{\sum E_{T_i}}$$

"SNOWMASS" SUM RUNS OVER ~~ALL~~ CELLS INSIDE $\Delta R^2 = \Delta\eta^2 + \Delta\phi^2$ OF JET CORE



LAB



AT LOWEST ORDER

$E_{T_1} = E_{T_2}$

(SPOILED BY SOFT GLUONS)

$$d\sigma = \int dx_1 dx_2 \,.\, f_i(x_1,\mu^2) f_j(x_2,\mu^2) \frac{1}{2\hat{s}} \overline{\sum |M|^2_{ij}}\, d\phi_2$$

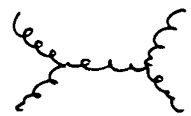
$$= \int d\eta_1 d\eta_2 dE_T \,.\, \frac{2\pi}{\hat{s}^2} E_T x_1 f_i(x_1,\mu^2)\, x_2 f_j(x_2,\mu^2) \overline{\sum |M|^2_{ij}}$$

using $\quad dx_1 dx_2 d\phi = \frac{4\pi}{s} d\eta_1 d\eta_2 E_T dE_T$

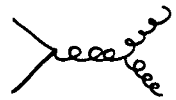
$$\hat{s} = x_1 x_2 s$$

– Many different $2 \to 2$ parton scattering processes

e.g. $\quad gg \to gg$



$q\bar{q} \to gg$



– **ALL** CONTRIBUTE

$\overline{\sum |M|^2} \sim g_s^4 \quad$ ie $\quad \sigma \sim \alpha_s^2 \quad$ at LO.

Process	$\sum\|M\|^2/g^4$	$\theta^* = \pi/2$
$q\,q' \to q\,q'$	$\frac{4}{9}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$	2.22
$q\,q \to q\,q$	$\frac{4}{9}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}+\frac{\hat{s}^2+\hat{t}^2}{\hat{u}^2}\right)-\frac{8}{27}\frac{\hat{s}^2}{\hat{u}\hat{t}}$	3.26
$q\,\bar{q} \to q'\,\bar{q}'$	$\frac{4}{9}\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$	0.22
$q\,\bar{q} \to q\,\bar{q}$	$\frac{4}{9}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}+\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right)-\frac{8}{27}\frac{\hat{u}^2}{\hat{s}\hat{t}}$	2.59
$q\,\bar{q} \to g\,g$	$\frac{32}{27}\frac{\hat{t}^2+\hat{u}^2}{\hat{t}\hat{u}}-\frac{8}{3}\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$	1.04
$g\,g \to q\,\bar{q}$	$\frac{1}{6}\frac{\hat{t}^2+\hat{u}^2}{\hat{t}\hat{u}}-\frac{3}{8}\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}$	0.15
$g\,q \to g\,q$	$-\frac{4}{9}\frac{\hat{s}^2+\hat{u}^2}{\hat{s}\hat{u}}+\frac{\hat{u}^2+\hat{s}^2}{\hat{t}^2}$	6.11
$g\,g \to g\,g$	$\frac{9}{2}\left(3-\frac{\hat{t}\hat{u}}{\hat{s}^2}-\frac{\hat{s}\hat{u}}{\hat{t}^2}-\frac{\hat{s}\hat{t}}{\hat{u}^2}\right)$	30.4

SINGLE JET INCLUSIVE DIST.

$\left.\frac{d\sigma}{dE_T}\right|_{\eta_1}$ — INTEGRATE η_2 AND LOOK AT E_T OF JET FOR FIXED η_1

CDF $0.1 < |\eta_1| < 0.7$

$50 < E_T < 500$ GeV

— PROBES DISTANCES

~ $0.05 \left(\frac{100 \text{ GeV}}{E_T}\right)$ fm

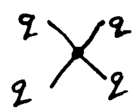
i.e. 0.1 → 0.01 fm

→ FIG.

— EXCESS IN TAIL DUE TO ?

- EXPERIMENTAL PROBLEM (PRELIM DATA)
- THEORETICAL UNCERTAINTY
- NEW PHYSICS

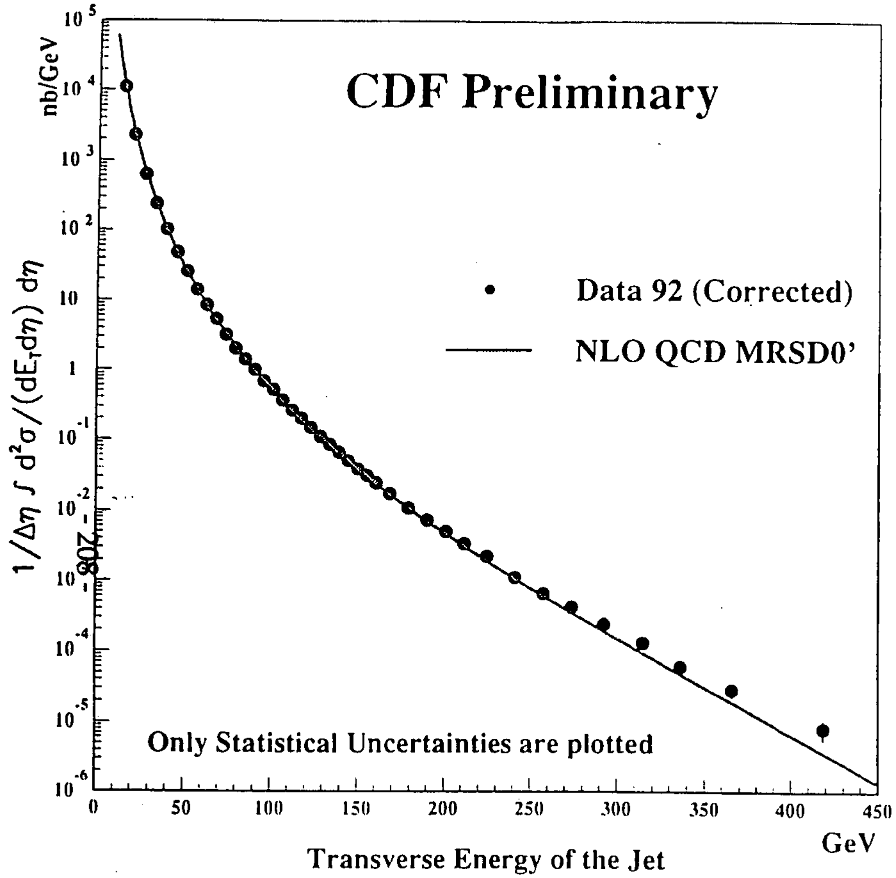
e.g. NEW NON-RENORMALISABLE INTERACTION



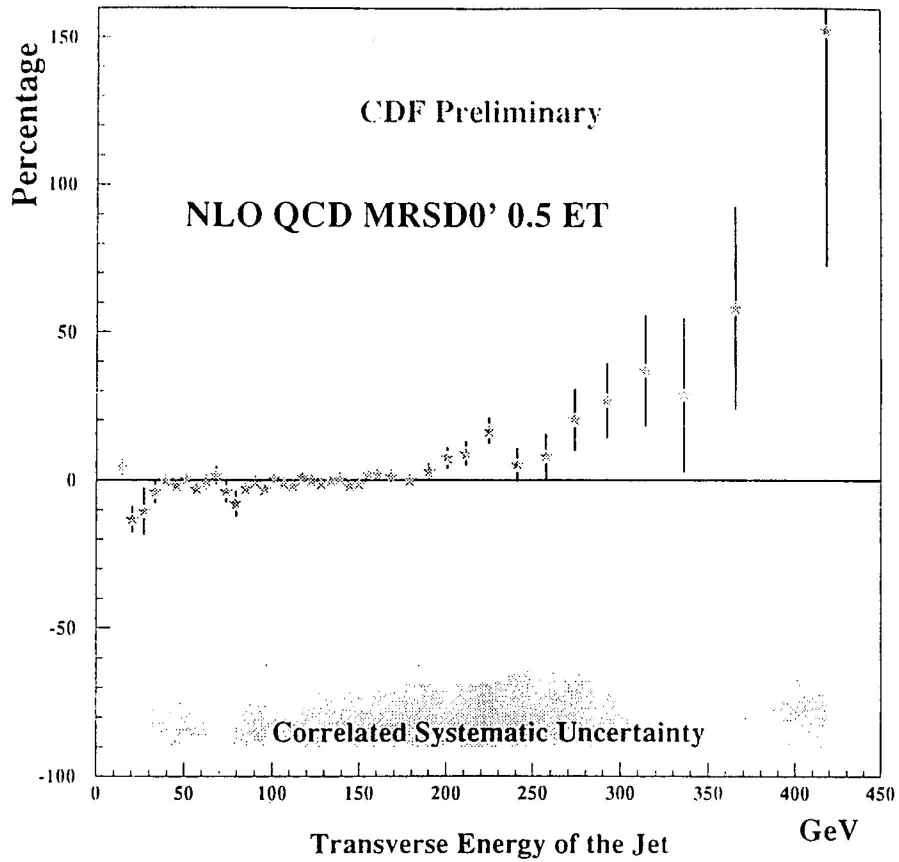
$$\mathcal{L} \sim \frac{g_s^2}{\Lambda^2}\,\bar{\psi}\psi\bar{\psi}\psi$$

$$\Rightarrow \sigma_{22} \sim |\mathrm{I} + \mathrm{X}|^2 \sim |\mathrm{I}|^2 + \frac{1}{\Lambda^2}\mathrm{I}\mathrm{X} + \frac{1}{\Lambda^4}\mathrm{X}\mathrm{X}$$

Inclusive Jet Cross-Section



(Data - Theory) / Theory



ignificant excess of jets with E_T>200 GeV
ompared to NLO QCD prediction

THEORETICAL UNCERTAINTIES

- RENORMALISATION SCALE μ
  - CHANGES NORMALISATION OF $\frac{\text{DATA} - \text{THEORY}}{\text{THEORY}}$
- NLO CORRECTIONS ARE LARGE?

  $K = \frac{\sigma^{NLO}}{\sigma^{LO}}$ – FLAT UNTIL $E_T \sim 600$ GeV

  → FIG
- UNCERTAINTY IN $\alpha_S(M_Z)$?

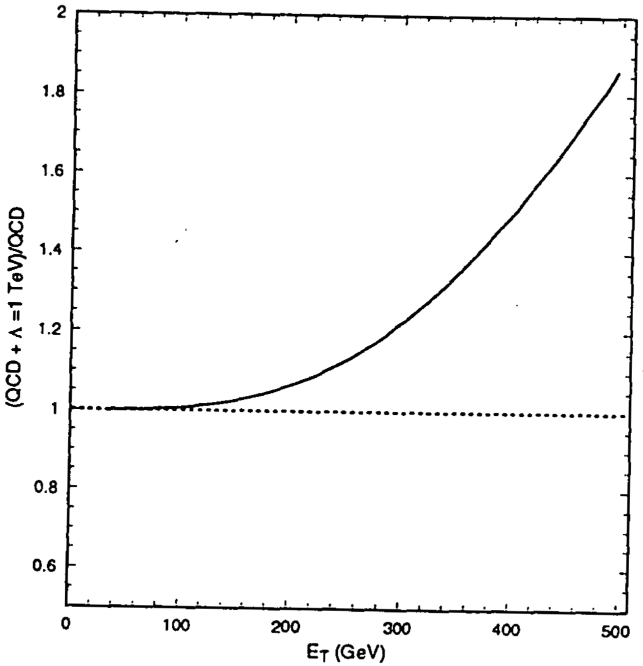
  TILTS THEORY – EXPECT THEORY ↓ AS α_S ↓, BUT MODIFIED BY DGLAP EVOLUTION OF PDF
  - PARTON DENSITIES AT LARGE x LESS DEPLETED IF α_S SMALL

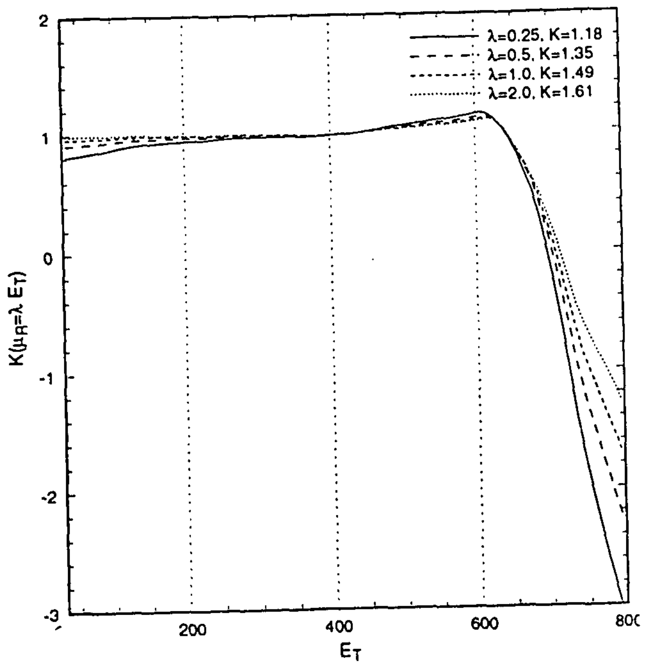
  → FIG
- UNCERTAINTY in $f_q(x, Q^2)$

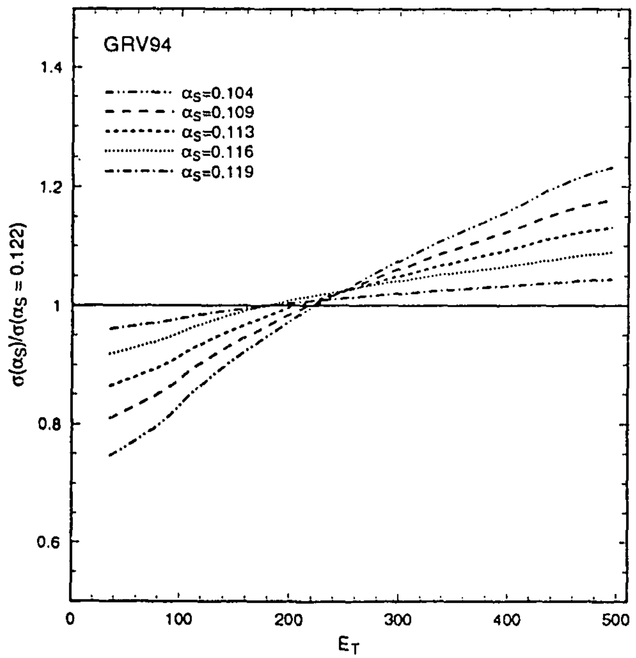
  $x \sim \frac{2E_T}{\sqrt{s}} \sim 0.2 - 0.5$

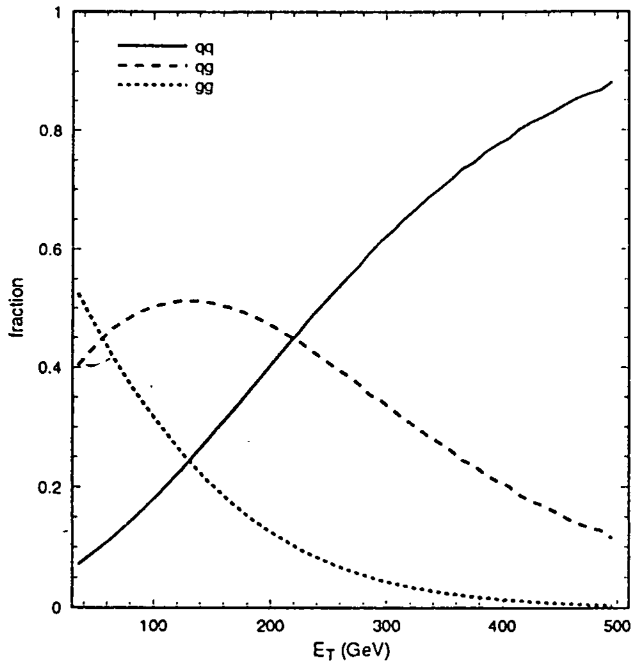
  $Q^2 \sim E_T^2 \sim 4 \times 10^4 - 2 \times 10^5 \text{ GeV}^2$

  - CHANGING INPUT PDF – CTEQ/MRS/GRV gives small change.




λ=0.25, K=1.18
λ=0.5, K=1.35
λ=1.0, K=1.49
λ=2.0, K=1.61
K(μR=λ ET)
ET
2
1
0
-1
-2
-3
200
400
600
800


GRV94
αS=0.104
αS=0.109
αS=0.113
αS=0.116
αS=0.119
σ(αS)/σ(αS = 0.122)
ET
1.4
1.2
1
0.8
0.6
0
100
200
300
400
500


qq
qg
gg
fraction
0
0.2
0.4
0.6
0.8
1
100
200
300
400
500
E_T (GeV)

6. Precision Electroweak Physics at LEP

- Parameters and Observables
- Radiative corrections from Vacuum Polarisation
- ρ parameter
- LEP Physics
  - Lineshape: M_Z, Γ_Z and peak cross section
  - Bhabha scattering and Luminosity
  - Forward Backward Asymmetry
- Radiative Corrections
- R_b
- Bounds on m_H and m_t

ELECTROWEAK PHYSICS

BASIC IDEA

$$\mathcal{L}(g, g', g_s, \lambda, \mu^2, g_f)$$

g, g', g_s: gauge couplings; λ, μ^2: Higgs; g_f: fermion masses

$\Rightarrow$ all observables can be computed in terms of $g, g', g_s, \lambda, \mu^2, g_f$

USUALLY choose 6 PARAMETERS THAT CAN BE DETERMINED DIRECTLY

$$e^2,\ M_Z,\ G_F,\ M_H,\ m_f,\ \alpha_s(M_Z)$$

$$\downarrow$$

$$M_W$$

$\Rightarrow$ compute observable O as

$$O = O_0 \left[1 + \alpha C_1 + \alpha^2 C_2 \ldots\right]$$

O_0: LOWEST ORDER; $[1 + \alpha C_1 + \alpha^2 C_2 \ldots]$: PERTURBATION SERIES

O_0, C_1, C_2 DEPEND ON PARAMETERS

e.g. QED + $(g-2)_e$ parameters m_e, α

$$\left(\frac{g-2}{2}\right)_e = 0.001\ 159\ 652\ 188\ (4) \quad \text{expt}$$
$$0.001\ 159\ 652\ 133\ (29) \quad \text{theory}.$$

- here m_e, α input.
- if, for example, m_e <u>not</u> known, could use this to determine m_e

ELECTROWEAK PHYSICS

<u>inputs</u>

$\alpha^{-1}(m_e^2) = 137.035\,9895\ (61)$ Josephson

$G_F = 1.16637\ (2) \times 10^{-5}\ \text{GeV}^{-2}$ μ decay

$M_Z = 91.1884 \pm 0.0022$ GeV LEP

$\alpha_s(M_Z) = 0.117 \pm 0.006$ WORLD.

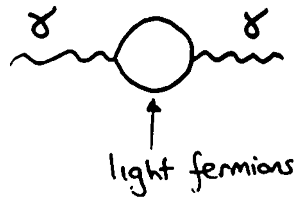
<u>NOT KNOWN</u>

m_t , m_H

- $\sin^2\theta_W$ DERIVED FROM INPUTS

e.g $\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}$ USEFUL QUANTITY.

MAIN (VACUUM POLARISATION) EFFECTS DUE TO

① 

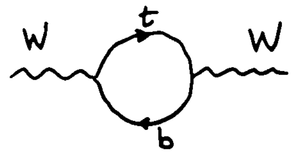
RUNNING α

$\alpha^{-1}(M_Z) = 128.89 \pm 0.09$

$= \alpha^{-1} - \frac{\Delta\alpha}{\alpha}$

② 

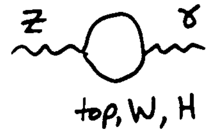
running Z MASS

③ 

running W mass

- actually since M_Z known only difference

appears

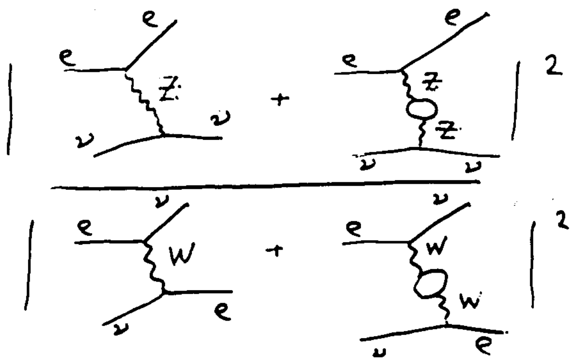
④ 

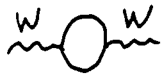
Z γ MIXING

- other diagrams will contribute but main m_t effects from these.

ρ PARAMETER

– Defined as RATIO OF NEUTRAL AND CHARGED CURRENT INTERACTIONS

$\rho \sim$ 

$\approx \rho_0 \Big[1 +$  $-$  $\Big]$

ρ_0 ↑ 1 IN SM; ↑ TOP; ↑ TOP VELTMAN

$$\approx 1 + \frac{3 G_F M_t^2}{8\pi^2\sqrt{2}} + \dots$$

DATA $= 1.0044 \pm 0.0016$

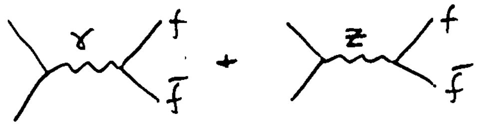
$\Rightarrow \quad m_t = 118 \,^{+20}_{-24}$ GeV

✱  EVALUATED AT LOW q^2.

LEP Physics

- BASIC PROCESS

$$e^+ + e^- \longrightarrow f + \bar{f}$$



$$\Rightarrow \quad \frac{d\sigma_0^q}{d\cos\theta} = N \frac{\pi\alpha^2}{2s}\left[(1+\cos^2\theta)\, A_q(s) + 2\cos\theta\, B_q(s)\right]$$

from before

$$A_q(s) = e_q^2 - 2 e_q v_q v_e \chi_1(s) + (v_e^2+a_e^2)(v_q^2+a_q^2)\chi_2(s)$$

$$B_q(s) = -2 e_q a_q a_e \chi_1(s) + 4 v_e a_e v_q a_q \chi_2(s)$$

$$\chi_1(s) = \frac{\sqrt{2}\, G_F M_Z^2}{16\pi\alpha} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$

$$\chi_2(s) = \left(\frac{\sqrt{2}\, G_F M_Z^2}{16\pi\alpha}\right)^2 \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$

$$\Rightarrow \quad \boxed{\sigma_0^q(s) = \frac{4\pi\alpha^2}{3s} \cdot A_q(s) \cdot N}$$

⟶ LINESHAPE

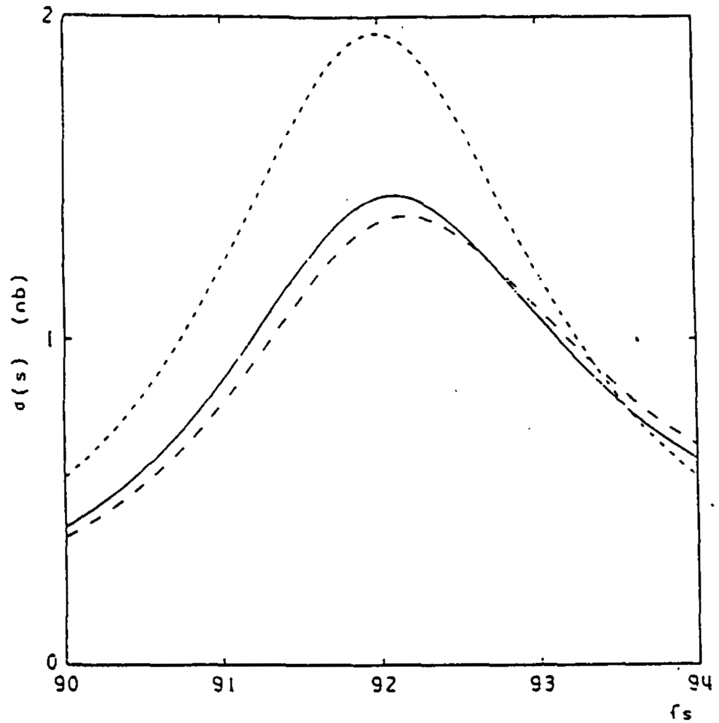


Fig. 2

Muon pair line shape curves including successive stages of corrections: only non-photonic corrections (fine dashed line), first order QED corrections applied to the previous one (dashed line) and second order exponentiated corrections applied to the first curve (solid line). The masses are $M_Z = 92$, $M_H = 100$ and $m_t = 60$ GeV and the minimum s' value is 1 GeV^2.

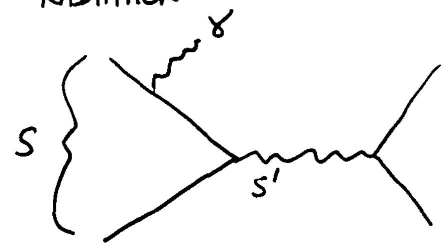
- SEE VERY LARGE QED CORRECTIONS

$$\sigma^f(s) = \int H(s,s') \boxed{\sigma_0^f(s')}\, ds'$$

$H(s,s')$: RADIATOR FUNCTION TO DESCRIBE QED EMISSION

$\sigma_0^f(s')$: CALCULATED ABOVE

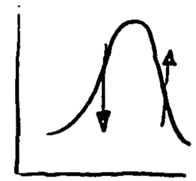
- LARGE EFFECT DUE TO QED INITIAL STATE RADIATION



Photon carries away energy So $S' < S$

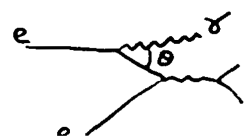
- if $S > M_Z^2$, CAN PUT Z ON SHELL $S' \sim M_Z^2$ $\sigma \uparrow$

  $S \sim M_Z^2$, CAN PUT Z OFF SHELL $\sigma \downarrow$



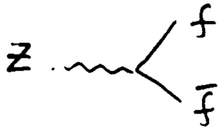
OBSERVED
- PEAK SHIFTS TO HIGHER S $\sim +100$ MeV
- needs to be corrected.

• LARGE EFFECT BECAUSE OF COLLINEAR RADIATION



$\sim \alpha \log \frac{M_Z^2}{M_e^2} \sim 24\alpha$

Z WIDTH



$$d\Gamma = \frac{1}{2M_Z}\frac{1}{3}|M|^2 d\phi$$

↑ AVERAGE OVER Z POLARISATIONS

$$\Gamma_q = \frac{G_F M_Z^3}{24\pi\sqrt{2}} \cdot N \cdot \left[1 + (1 - 4|e_q|\sin^2\theta_W)^2\right]$$

$$\Gamma_Z = \sum_\ell \Gamma_{\ell\ell} + \Gamma_{had} + \Gamma_{INV}$$

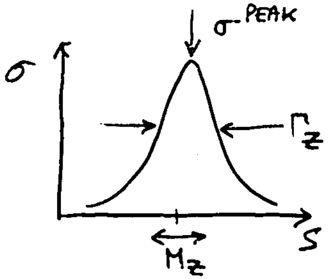
↓

$$N_\nu \Gamma_\nu = N_\nu\, 165.8\,\text{MeV}$$

⇒ EXTRACTION OF $M_Z, \Gamma_Z, \sigma_{PEAK}$ FROM LINESHAPE

IN RESONANCE REGION, IGNORE γ EXCHANGE

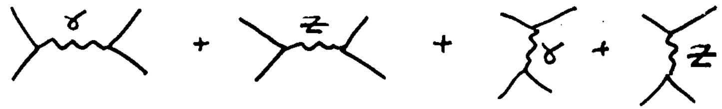
$$\sigma_0^f = \sigma_0^{PEAK} \frac{s\Gamma_Z^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}$$



⇒ 3 PARAMETER FIT

$$\sigma_0^{PEAK} = \frac{12\pi\Gamma_e\Gamma_f}{M_Z^2\Gamma_Z^2} = N_{events}/\mathcal{L}$$

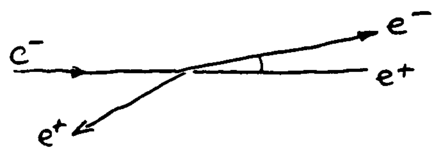
Bhabha scattering $e^+e^- \to e^+e^-$



- MUST INCLUDE ALL GRAPHS
  - LARGE CONTRIBUTIONS FROM S-CHANNEL Z WHEN $s \sim M_Z^2$ AND t-CHANNEL γ EXCHANGE WHEN $t \sim 0$ (i.e. $\theta \sim 0$)

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\left[(1+\cos^2\theta)A_q(s) + 2\cos\theta\, B_q(s) + 2\left(\frac{(1+\cos\theta)^2 + 4}{(1-\cos\theta)^2}\right) + \text{other terms}\right]$$

↳ FIG



- USE LOW ANGLE TAGGER TO MEASURE QED Bhabha ⇒ NORMALISATION OF σ ($\mathcal{L}$)

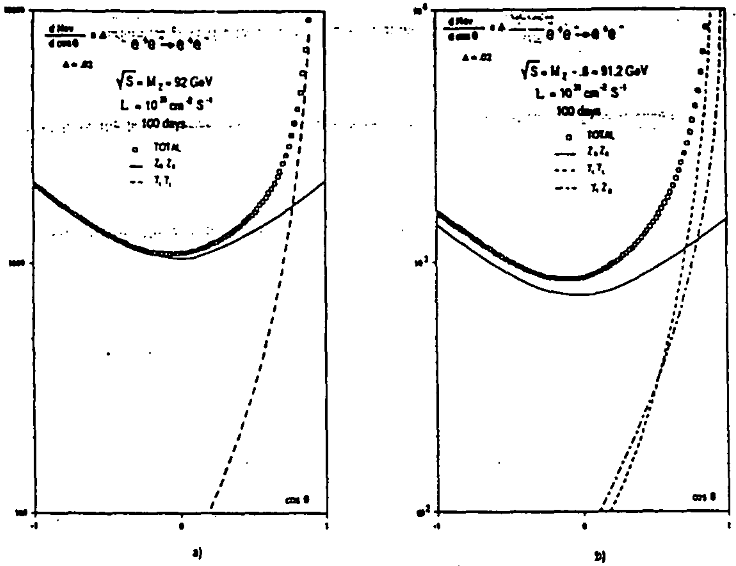


Figure 2 : The differential cross section $d\sigma/d\cos\theta$ for $e^+e^- \rightarrow e^+e^-$ at first order

a) at an energy $\sqrt{s} = M_z$

b) at an energy $\sqrt{s} = M_z - .8$

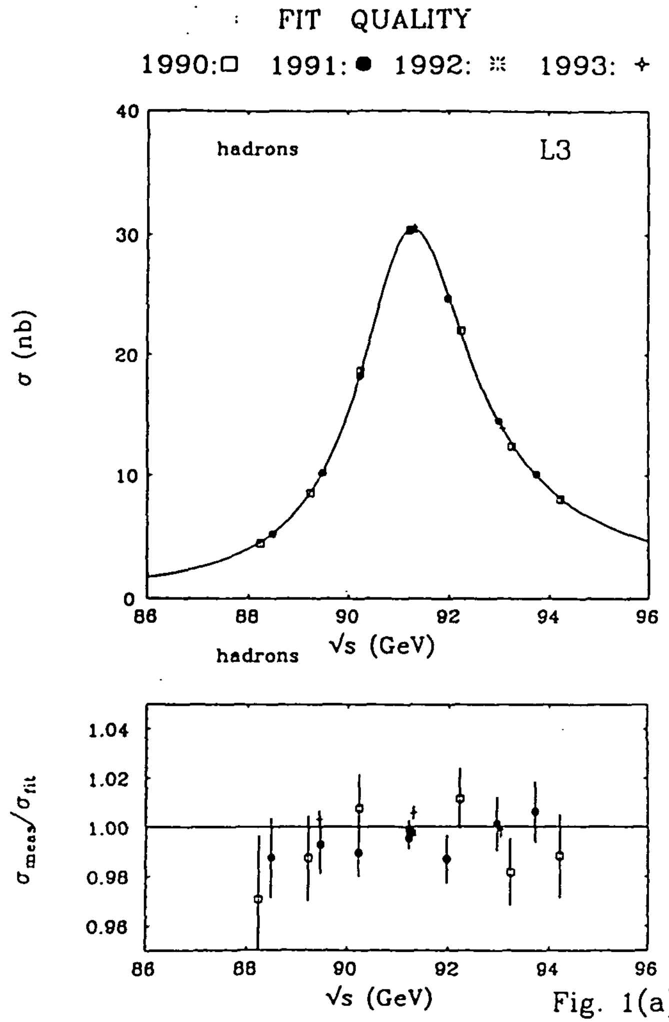


Fig. 1(a)

total error quoted — common syst. subtracted

M_Z [GeV]

ALEPH
91.1915±.0052
24 57

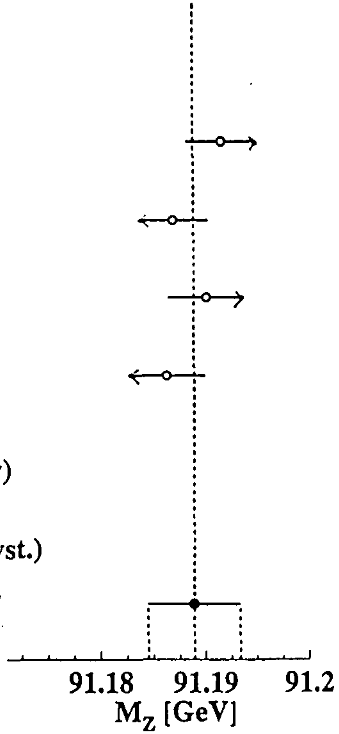
DELPHI
91.1869±.0052
49 34

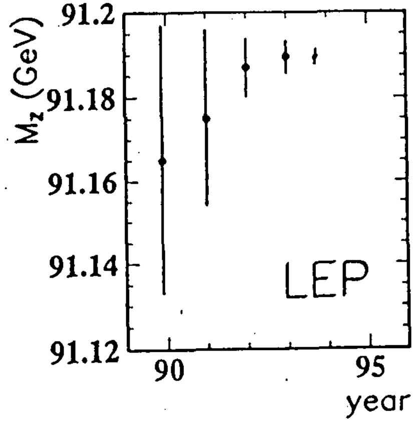
L3
91.1900±.0054
38 36

OPAL
91.1862±.0054
46 35

common syst.:
1.5 **4 MeV (LEP energy)**
χ^2/D.O.F.=0.5

LEP (incl. comm. syst.)
91.1888±.0044
4 22





Olchevski EPS

Γ_Z [GeV]

ALEPH
2.4959±.0061
1 55

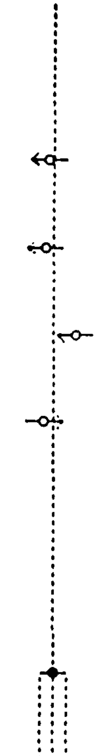
DELPHI
2.4951±.0059
13 53

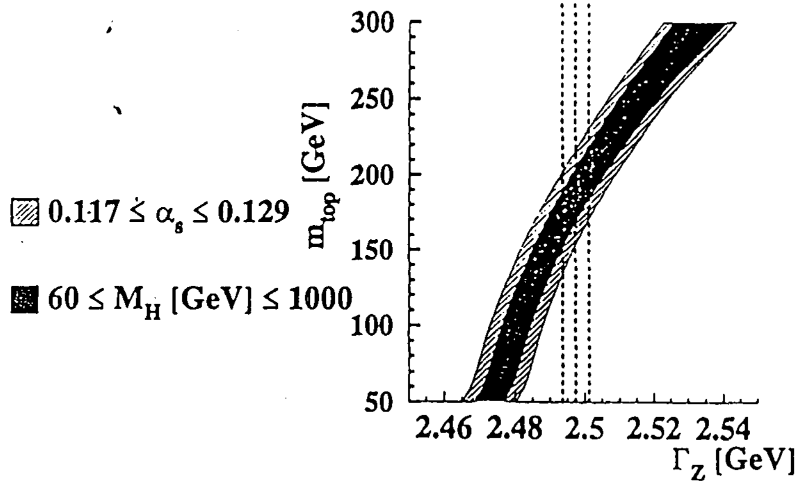
L3
2.5040±.0058
23 54

OPAL
2.4946±.0061
59 52

common syst.:
2.7 MeV (LEP energy)
χ^2/D.O.F.=0.7

LEP (incl. comm. syst.)
2.4974±.0038
63 32





10

Olchevski EPS

FORWARD - BACKWARD ASYMMETRY

$$A^f_{FB} = \frac{\int_0^1 \frac{d\sigma}{d\cos\theta}\, d\cos\theta - \int_{-1}^0 \frac{d\sigma}{d\cos\theta}\, d\cos\theta}{\int_0^1 \frac{d\sigma}{d\cos\theta}\, d\cos\theta + \int_{-1}^0 \frac{d\sigma}{d\cos\theta}\, d\cos\theta}$$

$$= \frac{3}{4} \frac{B_f(s)}{A_f(s)}$$

$s \sim M_Z^2$

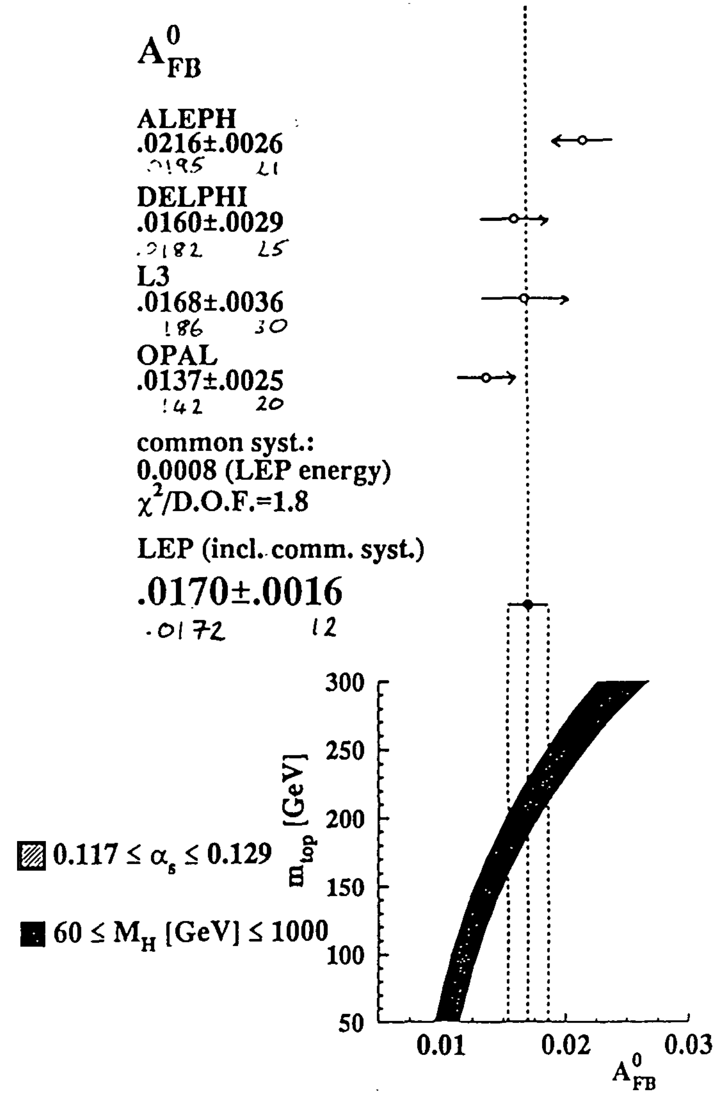
$$A^f_{FB} \approx \frac{3}{4} \cdot \frac{2 v_e a_e}{(v_e^2 + a_e^2)} \cdot \frac{2 v_f a_f}{(v_f^2 + a_f^2)}$$

∴ so that for leptons

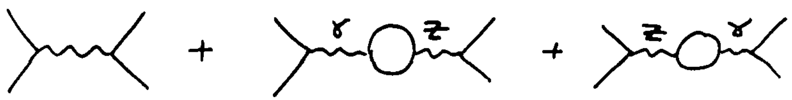
$$A^\ell_{FB} = 3 \frac{v_e^2 a_e^2}{(v_e^2 + a_e^2)^2}$$

cf leptonic width of Z

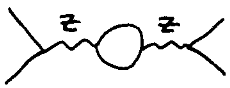
$$\Gamma_\ell = \frac{G_F M_Z^3}{24\pi\sqrt{2}} (v_e^2 + a_e^2)$$



ELECTROWEAK CORRECTIONS

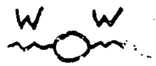


+



+ many more

−



NOT b's

– these universal self energy contributions can be absorbed into definitions

$$\Gamma_\ell = \frac{G_F M_Z^3}{24\pi\sqrt{2}}\,\rho\,\left[1 + (1 - 4\,\kappa\sin^2\theta_W)^2\right]$$

ρ parameter contains 

$$\rho \sim 1 + \frac{3\,G_F M_t^2}{8\pi^2\sqrt{2}} = 1 + \Delta\rho$$

$$\kappa \sim 1 + \frac{\cos^2\theta_W}{\sin^2\theta_W}\,\Delta\rho$$

$$A_{FB}^{\ell} = \frac{3\,[1 - 4\kappa\sin^2\theta_W]^2}{[1 + (1 - 4\kappa\sin^2\theta_W)^2]^2}$$

• ONLY κ

Measurement	$\sin^2\theta_{\rm eff}^{\rm lept}$
LEP:	
$A_{FB}^{0,\ell}$	0.2311 ± 0.0009
$\mathcal{A}_\tau$	0.2320 ± 0.0013
$\mathcal{A}_e$	0.2330 ± 0.0014
$A_{FB}^{0,b}$	0.2327 ± 0.0007
$A_{FB}^{0,c}$	0.2310 ± 0.0021
$\langle Q_{FB}\rangle$	0.2320 ± 0.0016
Average LEP:	0.2321 ± 0.0004
SLC:	
A_{LR} (SLD)	0.2294 ± 0.0010
Average LEP+SLC:	0.2317 ± 0.0004

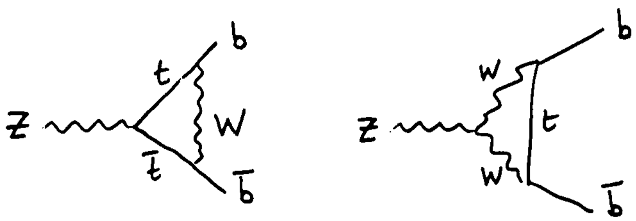
I USED $\sin^2\theta_W = 1 - \dfrac{M_W^2}{M_Z^2}$

OFTEN $\boxed{\sin^2\theta^{EFF} = \kappa\sin^2\theta_W}$

So $$A_{FB}^{\ell} = \frac{3\,[1 - 4\sin^2\theta^{EFF}]^2}{[1 + (1 - 4\sin^2\theta^{EFF})^2]^2}$$

$$R_b = \Gamma_{b\bar{b}} / \Gamma_{had}$$

- ADDITIONAL m_t DEPENDENCE FROM VERTICES



$$\Rightarrow \quad \Gamma_{b\bar{b}} = \frac{N G_F M_Z^3}{24\pi\sqrt{2}} \rho_b \left(1 + \left(1 - \frac{4}{3} K_b \sin^2\theta_W\right)^2\right)$$

$$\rho_b = 1 - \frac{1}{3}\Delta\rho$$

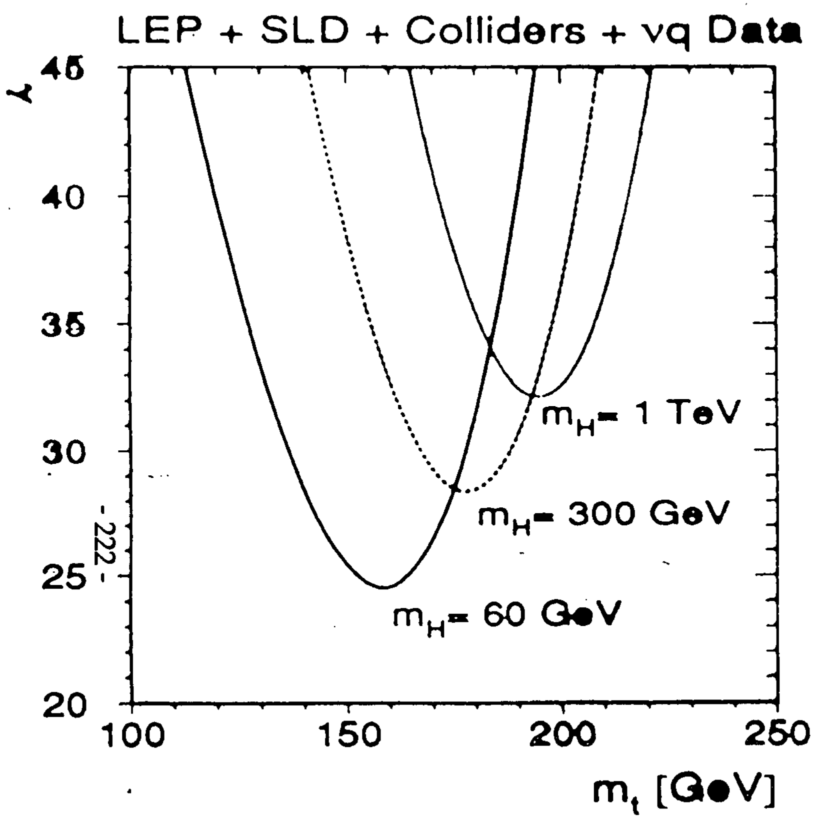
$$K_b = 1 + \left(\frac{\cos^2\theta_W}{\sin^2\theta_W} + \frac{2}{3}\right)\Delta\rho$$

$\longrightarrow$ FIG.

- ALSO m_b mass effects – reduce R_b by ~ 10%

	Measurement	Standard Model Fit	Pull
a) LEP			
line-shape and lepton asymmetries:			
m_Z [GeV]	91.1884 ± 0.0022	91.1882	0.1
Γ_Z [GeV]	2.4963 ± 0.0032	2.4973	−0.3
σ_h^0 [nb]	41.488 ± 0.078	41.450	0.5
R_ℓ	20.788 ± 0.032	20.773	0.5
$A_{FB}^{0,\ell}$	0.0172 ± 0.0012	0.0159	1.1
+ correlation matrix Table 8			
τ polarization:			
$\mathcal{A}_\tau$	0.1418 ± 0.0075	0.1455	−0.5
$\mathcal{A}_e$	0.1390 ± 0.0089	0.1455	−0.7
b and c quark results:			
R_b [(a)]	0.2219 ± 0.0017	0.2156	3.7
R_c [(a)]	0.1543 ± 0.0074	0.1724	−2.5
$A_{FB}^{0,b}$ [(a)]	0.0999 ± 0.0031	0.1020	−0.7
$A_{FB}^{0,c}$ [(a)]	0.0725 ± 0.0058	0.0728	0.0
+ correlation matrix Table 13			
$q\bar{q}$ charge asymmetry:			
$\sin^2\theta_{eff}^{lept}$ ($\langle Q_{FB}\rangle$)	0.2325 ± 0.0013	0.23172	0.6
b) SLC			
$\sin^2\theta_{eff}^{lept}$ ($\mathcal{A}_{LR}$ [11])	0.23049 ± 0.00050	0.23172	−2.5
R_b [46] [(a)]	0.2171 ± 0.0054	0.2156	−0.3
$\mathcal{A}_b$ [52–55]	0.841 ± 0.053	0.935	−1.8
$\mathcal{A}_c$ [52–55]	0.606 ± 0.090	0.667	−0.7
c) $p\bar{p}$ and νN			
m_W [GeV] ($p\bar{p}$ [67])	80.26 ± 0.16	80.35	−0.5
$1 - m_W^2/m_Z^2$ (νN [12–14])	0.2257 ± 0.0047	0.2237	0.4

CORRECTIONS $\sim \; a\, m_t^2 - b\, M_Z^2 \log\left(M_H^2/M_Z^2\right)$



FITS TO ALL DATA

m_t =	171 ± 11	GeV
m_H =	93^{+189}_{-63}	GeV
$\alpha_s(M_Z)$ =	0.122 ± 0.005	

POKORSKI + CHANKOWSKI

$\chi^2 = 25/14$

7. Higgs Physics

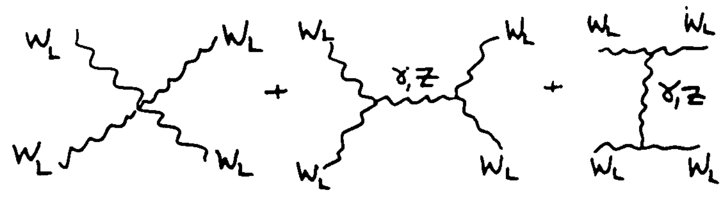
- General arguments
- Experimental bounds on m_H and m_t
- Theoretical bounds on m_H and m_t
- Higgs Decays
- Higgs search at LEP-I
- Higgs search at LEP-II
- Higgs search at LHC

HIGGS PHYSICS

– A "HIGGS" MUST EXIST !

– IF NOT UNITARITY VIOLATED.

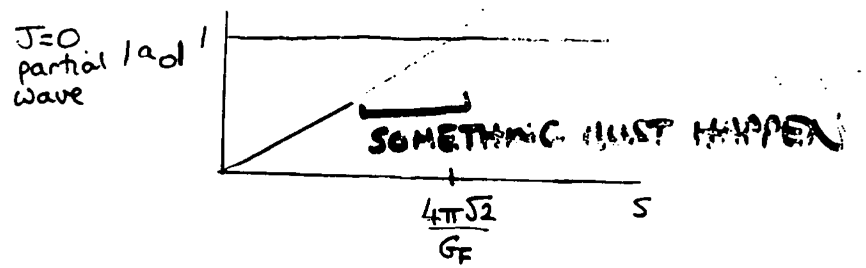
e.g. SCATTERING OF LONGITUDINALLY POLARISED W BOSONS



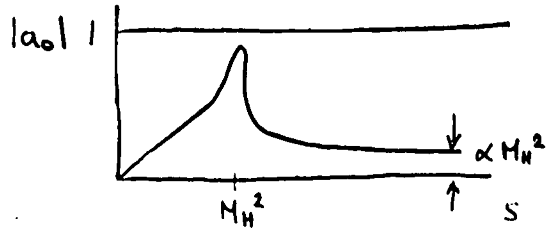
$$\mathcal{M} \sim s$$

ie $$\sigma \sim \frac{1}{2s} \int |\mathcal{M}|^2 d\phi \sim s$$

– this violates UNITARITY (PROBABILITY OF SCATTERING > 1 !)

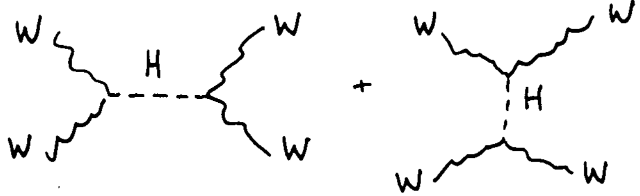


$$s \sim \frac{4\pi\sqrt{2}}{G_F} = (1.2\,\text{TeV})^2$$



Higgs exchange graphs prevent unitarity violation

$M_H^2 \leq \frac{8\pi\sqrt{2}}{3G_F} \sim (1\,\mathrm{TeV})^2.$

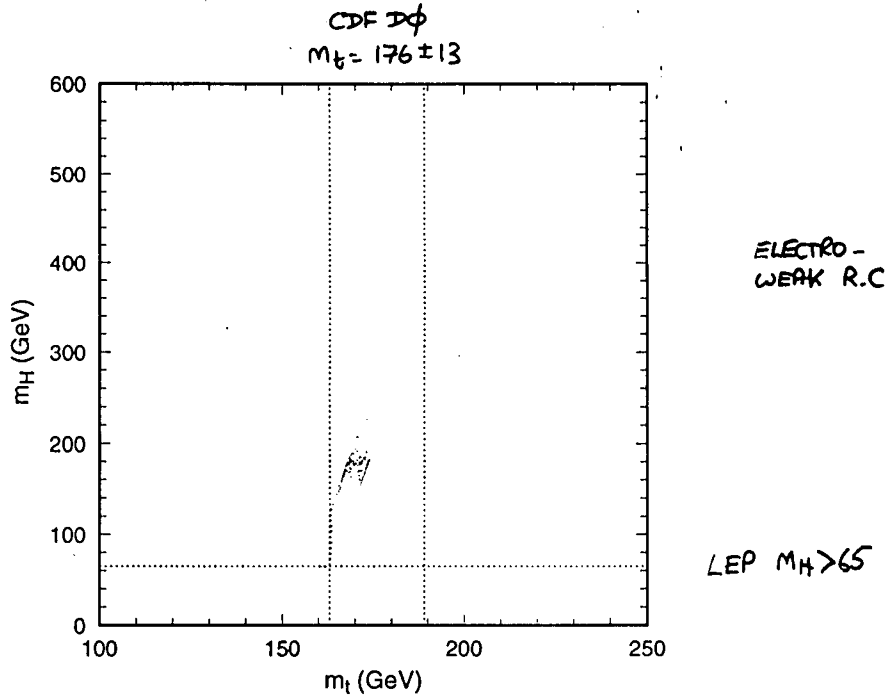


— also seen in $t\bar{t} \to W_L W_L$

⇒ "NEW PHYSICS (HIGGS) COUPLING TO MASSES MUST OCCUR ON A SCALE OF O(1TeV)"

Bounds on $m_H - m_t$

1) EXPERIMENTAL

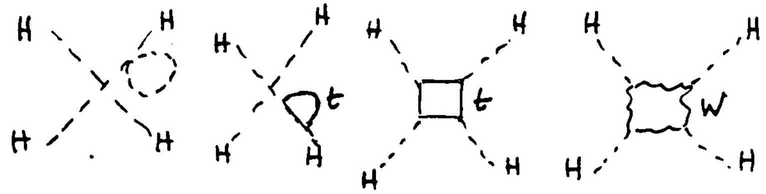


2) THEORETICAL

– JUST AS STRONG COUPLING CONSTANT AND BOSON MASSES

THE TOP QUARK MASS AND HIGGS SELF COUPLING ALSO RUN

- $$\frac{d m_t^2}{d \ln \mu^2} \sim a m_t^4 - b m_t^2 \alpha_s - c m_t^2 \alpha_W \ldots$$



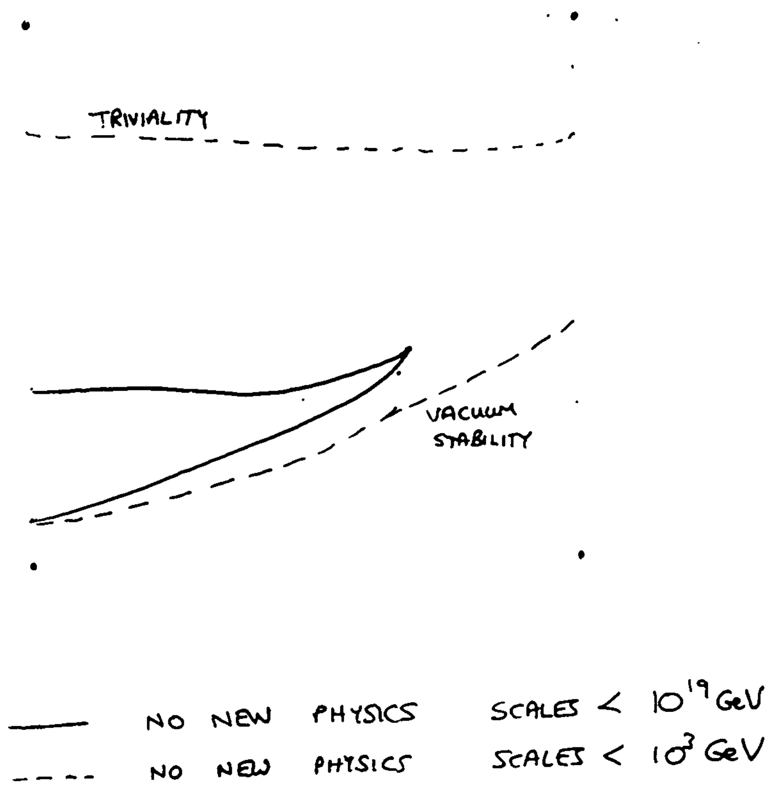
- $$\frac{d\lambda}{d \ln \mu^2} \sim a' \lambda^2 + b' \lambda m_t^2 + c' m_t^4 + \ldots$$

<u>BUT</u> 1) minimum of $V(\phi)$ must still be at $\frac{v}{\sqrt{2}}$ VACUUM STABILITY

2) $\lambda(\mu)$ MUST NOT DIVERGE $\boxed{\mu < \Lambda}$

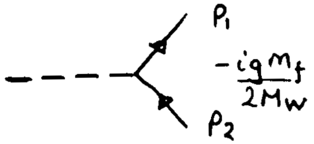
– IF IT DOES THEN $\lambda(v) \to 0$

– NO SPONTANEOUS SYMMETRY BREAKING TRIVIALITY.



HIGGS DECAYS

1) DECAY TO FERMIONS



$$\mathcal{M} = \bar{u}(p_1)\, v(p_2)\, \frac{g m_f}{2M_W}$$

$$\Rightarrow \sum |M|^2 = \frac{g^2 m_f^2}{4M_W^2} \cdot Tr\Big[\underbrace{(\not p_1 + m_f)(\not p_2 - m_f)}_{\begin{subarray}{l}4p_1\cdot p_2 - 4m_f^2 \\ = 2(m_H^2 - 4m_f^2)\end{subarray}}\Big]$$

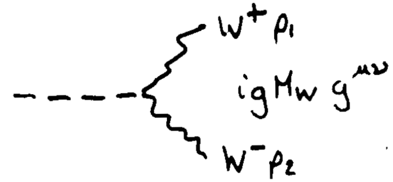
$$\Rightarrow \Gamma_f = \frac{1}{2m_H} \sum |M|^2 d\phi_2$$

$$\Gamma_f = \frac{g^2 m_f^2 m_H}{32\pi M_W^2}\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}$$

- if f is quark, extra factor of N colours

- LARGER m_f, larger decay rate.

2) DECAY TO GAUGE BOSONS



$$\mathcal{M} = g M_W \varepsilon_\mu^+ \varepsilon_\nu^- g^{\mu\nu}$$

$$\Rightarrow \sum |M|^2 = g^2 M_W^2 \underbrace{\sum \varepsilon_\mu^+ (\varepsilon_\nu^+)^*}_{-g_{\mu\nu} + \frac{p_{1\mu} p_{1\nu}}{M_W^2}} \sum \varepsilon_\mu^- (\varepsilon_\nu^-)^*$$

$$= g^2 M_W^2 \left[4 - \frac{p_1^2}{M_W^2} - \frac{p_2^2}{M_W^2} + \underline{\frac{(p_1\cdot p_2)^2}{M_W^4}} \right]$$

$$= \frac{g^2}{4M_W^2}\left[m_H^4 - 4m_H^2 m_W^2 + 3m_W^4 \right]$$

grows with m_H^2

$$\Rightarrow \Gamma_W = \frac{1}{2m_H} \sum |M|^2 d\phi$$

$$= \frac{g^2}{64\pi} \frac{m_H^3}{m_W^2} \sqrt{1 - \frac{4m_W^2}{m_H^2}} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{3m_W^4}{m_H^4}\right)$$

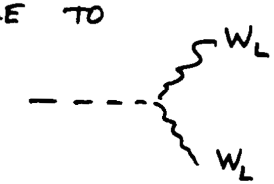
similarly

$$\Gamma_Z = \frac{g^2}{128\pi} \frac{m_H^2}{m_W^2} \sqrt{1 - \frac{4M_Z^2}{m_H^2}} \left(1 - \frac{4M_Z^2}{m_H^2} + \frac{3m_Z^4}{m_H^4}\right)$$

- FACTOR $\frac{1}{2}$ IN Γ_Z DUE TO ZZ IDENTICAL PARTICLES !

- RAPID GROWTH OF $\Gamma_W, \Gamma_Z \propto M_H^3$

DUE TO



"ORIGINALLY GOLDSTONE BOSON BEFORE SSB"

- **RULES OF THUMB**

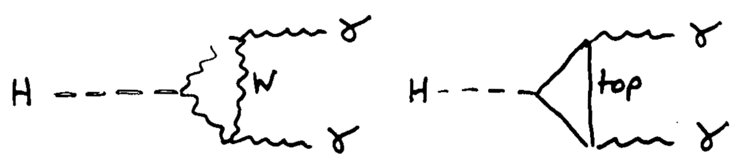
1) $\Gamma(H \to WW, ZZ) \simeq \frac{1}{2}\left(\frac{M_H}{1\,\text{TeV}}\right)^3$

2) DECAY TO HEAVIEST ALLOWED PARTICLE DOMINATES

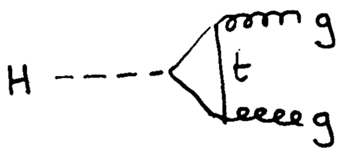
3) FOR $M_H > 2M_Z$

$$\Gamma_W \approx 2\,\Gamma_Z$$

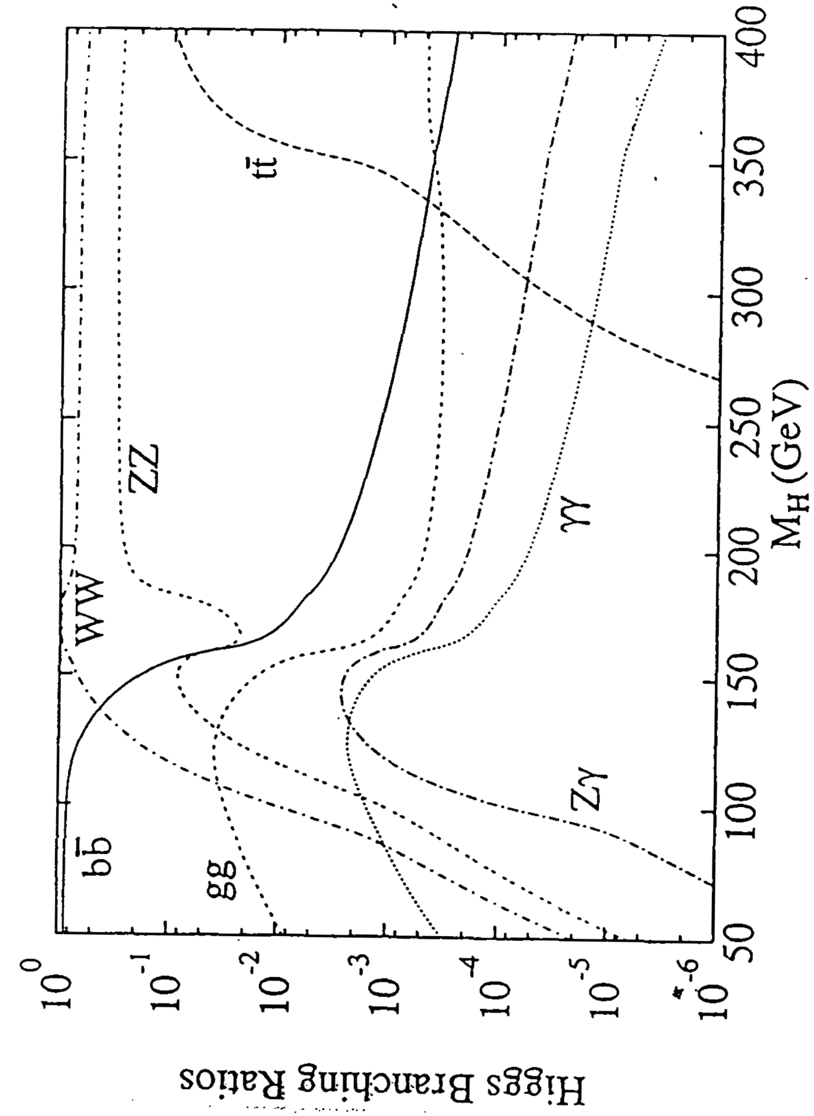
3) RARE DECAYS



— VERY SMALL, BUT USEFUL SIGNAL.

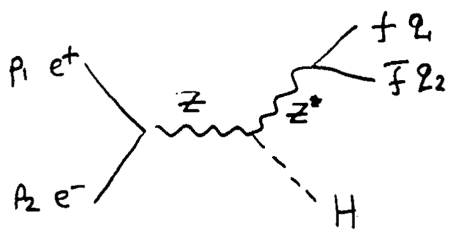


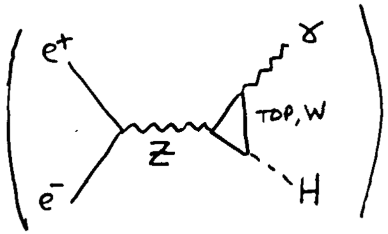
VITAL PRODUCTION MECHANISM AT LHC



HIGGS AT LEP I

— PRODUCED IN Z DECAY





$$\mathcal{M} = \bar{v}(p_1)\gamma_\mu(v-a\gamma_5)u(p_2)\,\frac{g^{\mu\alpha}}{s-M_Z^2+i\Gamma_Z M_Z}$$

$$\cdot\; y_{\alpha\beta}\,\frac{g^{\beta\nu}}{s'-M_Z^2+i\Gamma_Z M_Z}\quad \bar{u}(q_1)\gamma_\nu(v_f-a_f\gamma_5)v(q_2)$$

$$\cdot\; \frac{g^3}{\cos^2\theta_W}\cdot\frac{M_Z^4}{M_W^2}$$

$$\Rightarrow\; \Sigma|M|^2 = \frac{g^6 M_Z^8}{\cos^4\theta_W M_W^4}\;\frac{L^e_{\mu\nu}\,L_f^{\mu\nu}}{[(s-M_Z^2)^2+\Gamma_Z^2M_Z^2][(s'-M_Z^2)^2+\Gamma_Z^2M_Z^2]}$$

$$\frac{\sigma(e^+e^-\to X)}{\sigma(e^+e^-\to\mu^+\mu^-)}$$

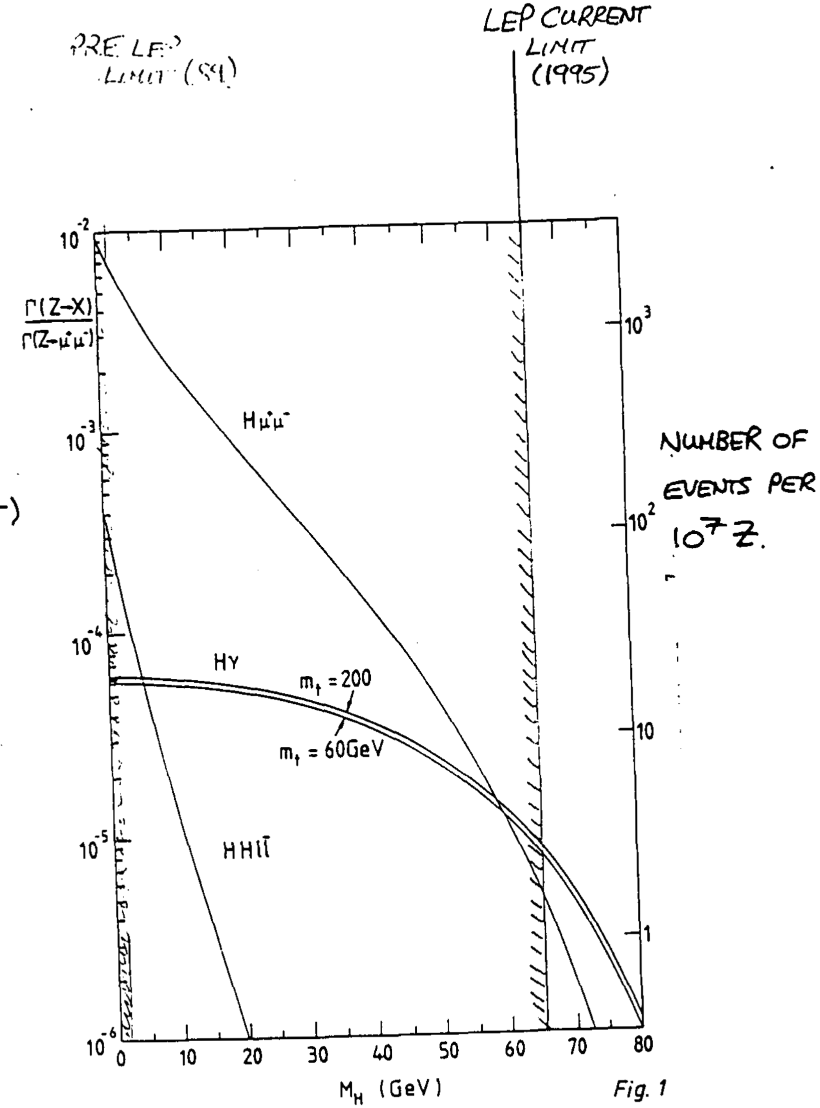
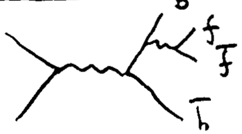


Fig. 1

- SIGNAL $\quad H \to b\bar{b} \quad + f\bar{f} \qquad m_{b\bar{b}} \sim m_H$ (narrow)

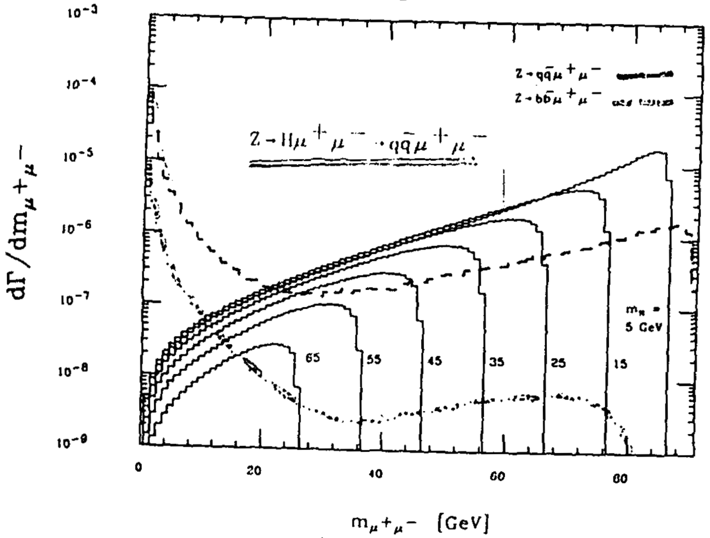
BACKGROUND

e.g.



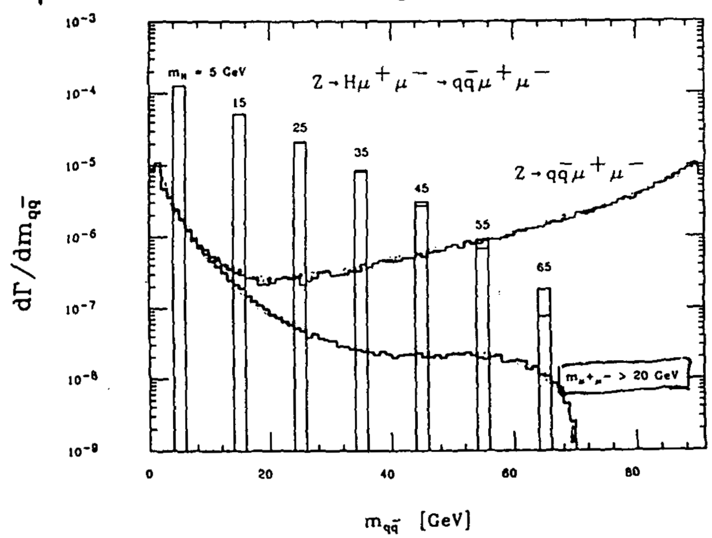
No PEAKING in $m_{b\bar{b}} \sim m_H$

Fig. 6



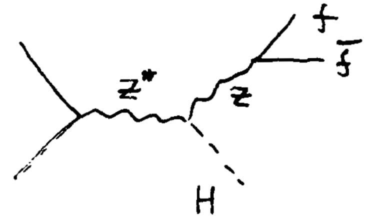
LEP I

Fig. 7



HIGGS AT LEP II

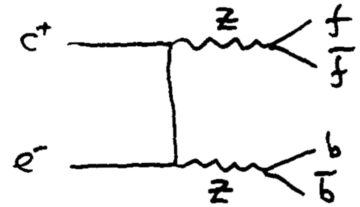
– SAME PRODUCTION PROCESS BUT NOW INITIAL Z OFF SHELL



⇒ SAME MATRIX ELEMENTS

– KINEMATICS $m_{f\bar{f}} \sim M_Z$, $m_{b\bar{b}} \sim m_H$

• BACKGROUND



$m_{f\bar{f}} \sim m_Z$

$m_{b\bar{b}} \sim m_Z$

• HOPE TO SEE

$$m_H \leq \sqrt{s} - 100\ \text{GeV}$$

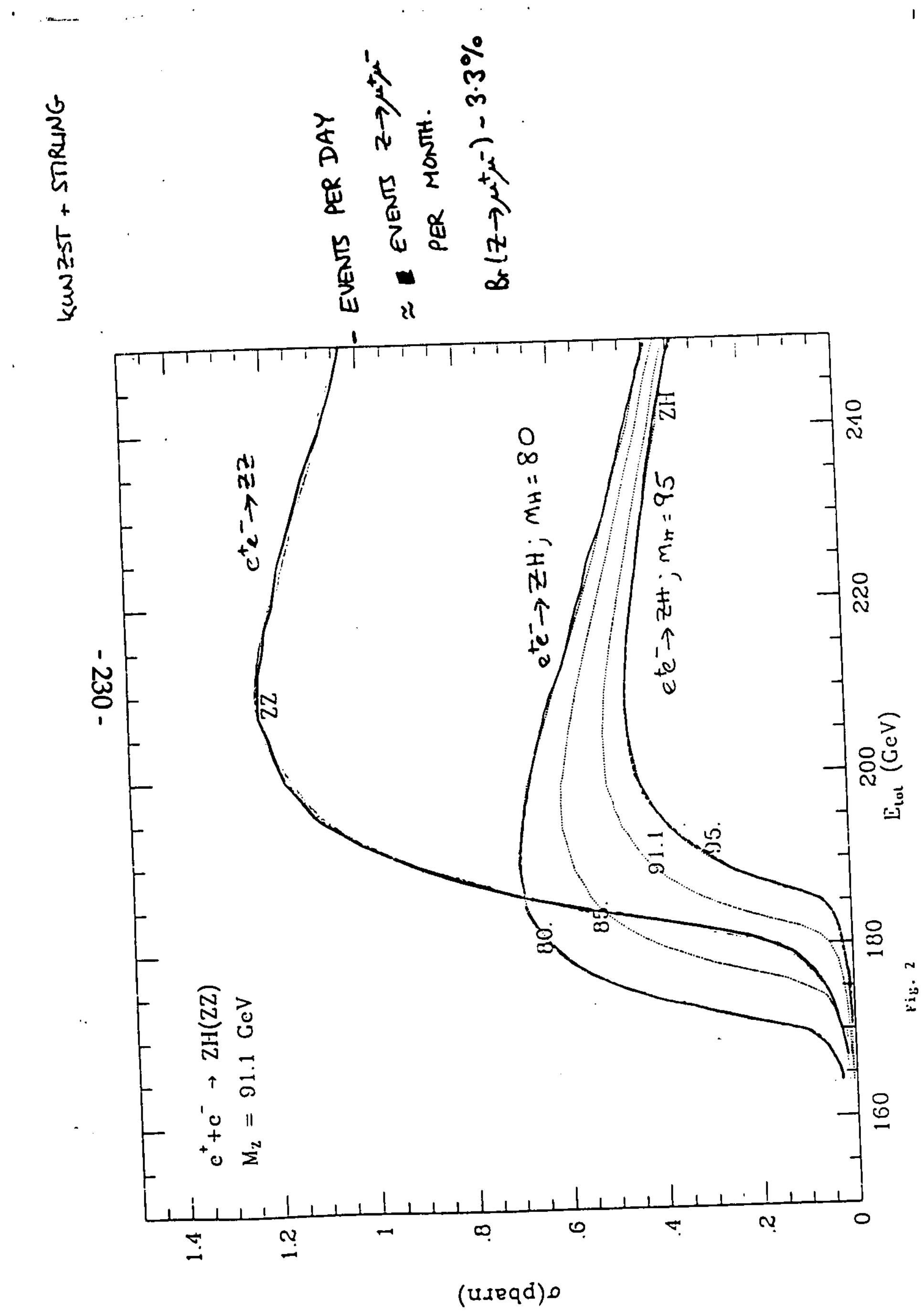


Fig. 2

LEP II

WU ET AL..

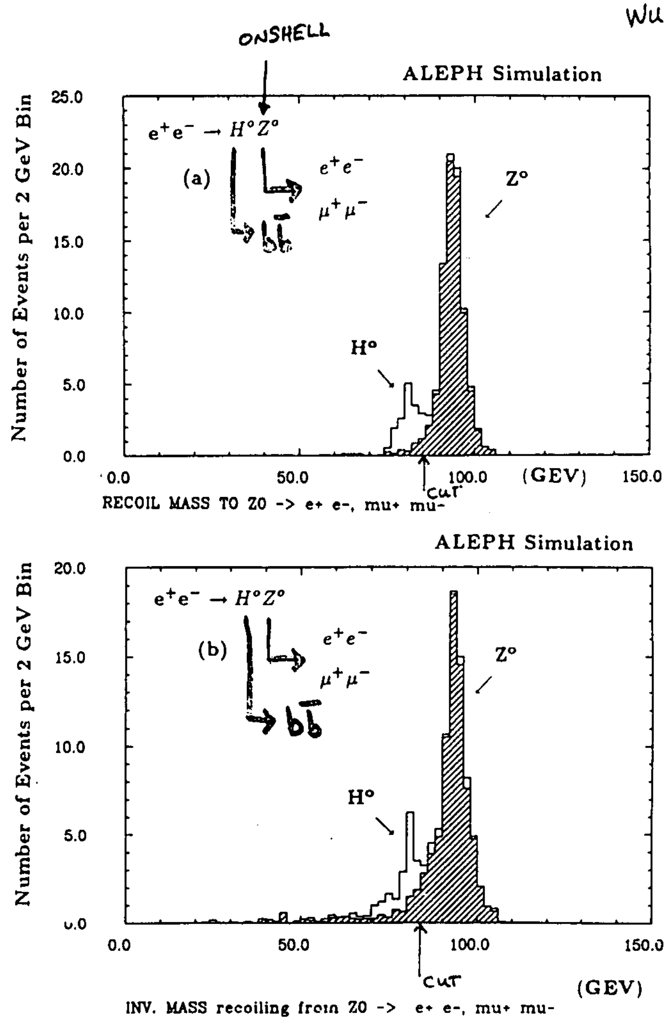
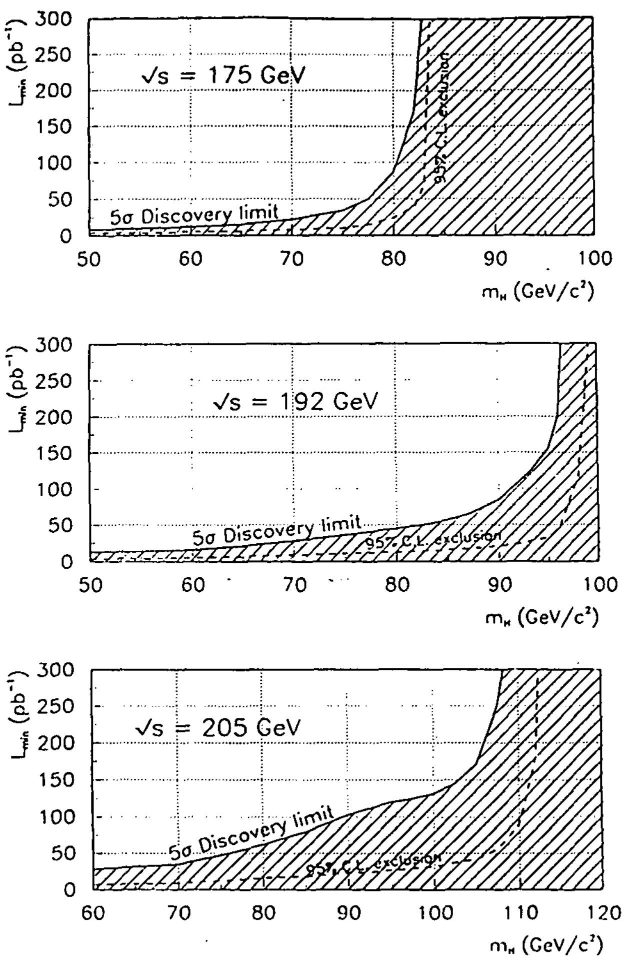


Fig. 17 (a)(b) Same as Fig. 15 for M_{H° = 80 GeV

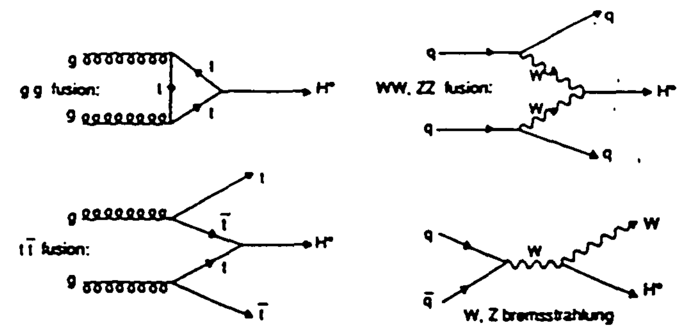
BACKGROUND

$e^+e^- \to ZZ$
$\quad \hookrightarrow e^+e^-, \mu^+\mu^-$
$\quad \hookrightarrow$ JETS

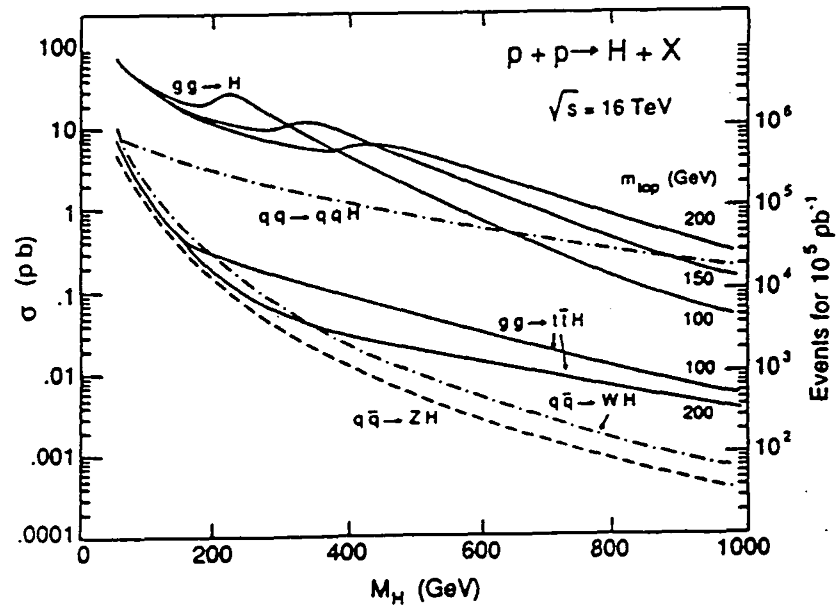


√s = 175 GeV
5σ Discovery limit
95% C.L. exclusion
√s = 192 GeV
5σ Discovery limit
95% C.L. exclusion
√s = 205 GeV
5σ Discovery limit
95% C.L. exclusion
L_{min} (pb^{-1})
m_H (GeV/c^2)

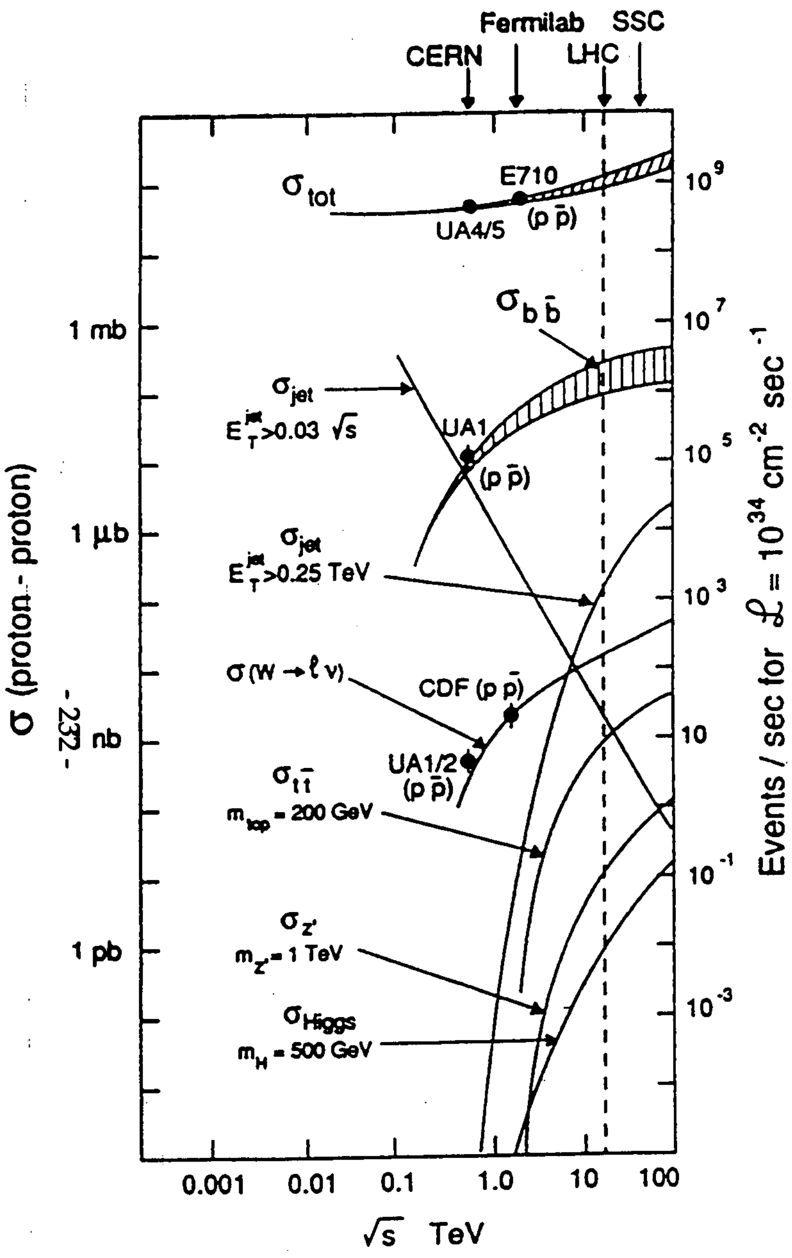
HIGGS AT LHC



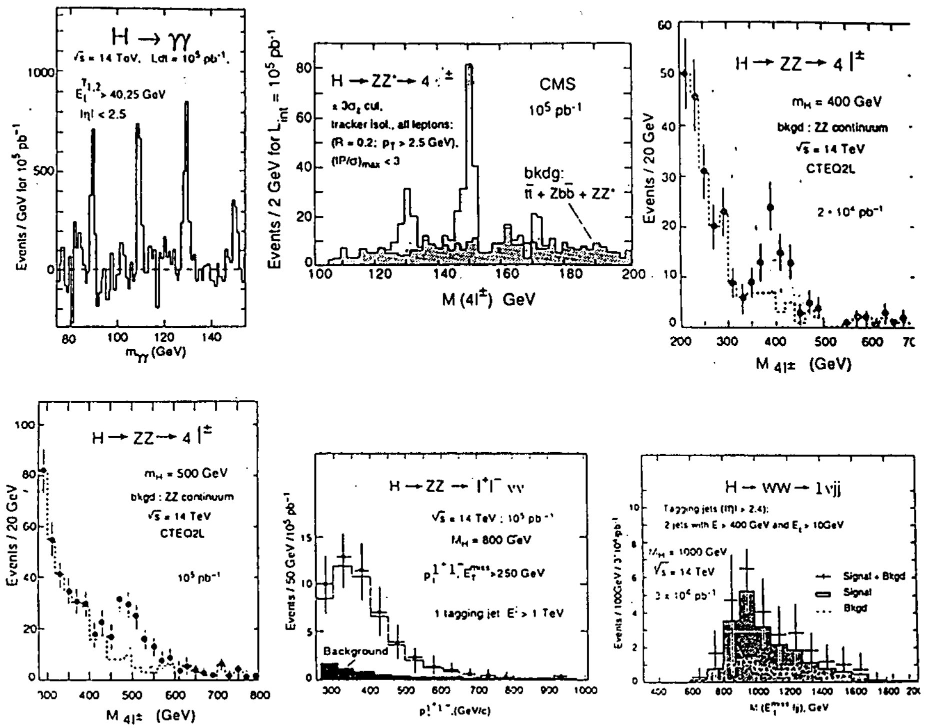
g g fusion:
WW, ZZ fusion:
$t\bar{t}$ fusion:
W, Z bremsstrahlung



p + p → H + X
$\sqrt{s}$ = 16 TeV
g g → H
q q → q q H
g g → $t\bar{t}$H
q $\bar{q}$ → W H
q $\bar{q}$ → Z H
m_{top} (GeV)
σ (p b)
Events for 10^5 pb^{-1}
M_H (GeV)



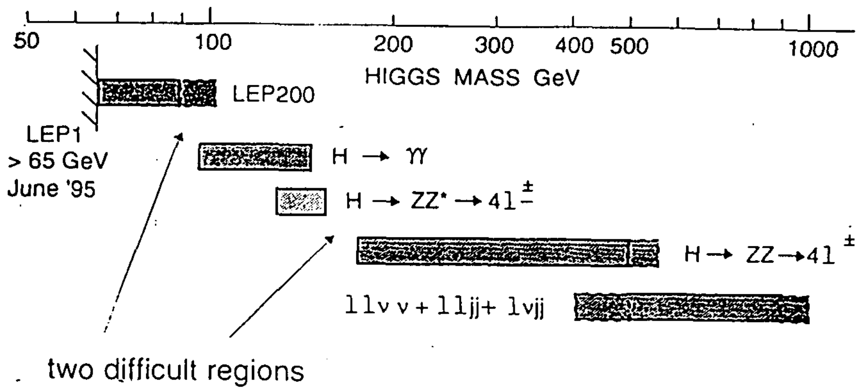
SM Higgs search in CMS



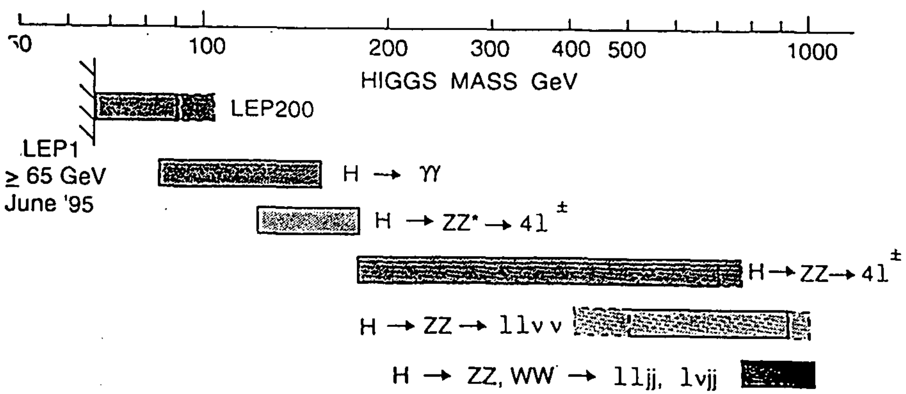
- INTERMEDIATE HIGGS $80 < M_H < 180$ GeV
  - LARGE PRODUCTION CROSS SECTION
  - DETECT SMALL B.R. $H \to \gamma\gamma$, $H \to \ell^+\ell^-\ell^+\ell^-$
- NORMAL HIGGS $180 < M_H < 600$ GeV
  - GOOD RATE, GOLD PLATED $H \to ZZ \to \ell^+\ell^-\ell^+\ell^-$
- WIDE HIGGS $600 < M_H$
  - RATE LOW, WIDE RESONANCE
  - USE BIGGER $ZZ \to \ell\ell\nu\nu$ DECAY

SM Higgs search at LHC

i) mass range explorable at $\sqrt{s}$ =14 TeV with $3 \cdot 10^4$ pb^{-1} taken at 10^{33} $cm^{-2}s^{-1}$ (~ 1 year at $\cdot 10^{33}$



ii) explorable mass range with 10^5 pb^{-1} taken at 10^{34} $cm^{-2}s^{-1}$



SM Higgs search at LHC

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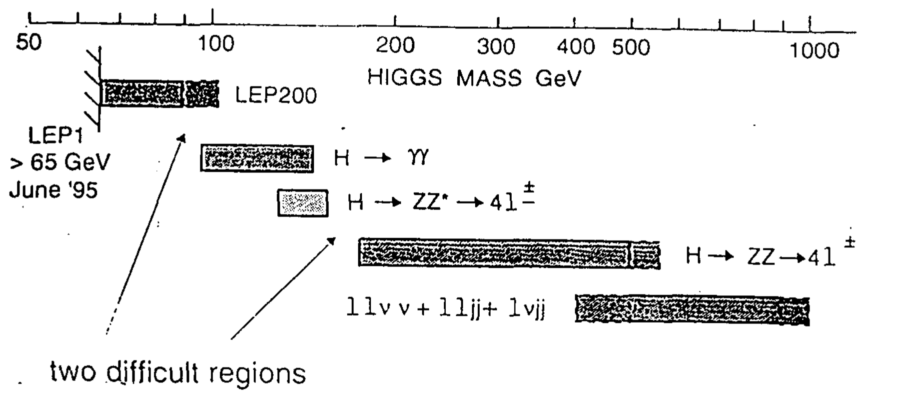

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