CLRC

Technical Report
RAL-TR-95-076

A Generalized Dual Symmetry for Nonabelian Yang-Mills Fields

Chan Hong-Mo et al

December 1995

COUNCIL FOR THE CENTRAL LABORATORY OF THE RESEARCH COUNCILS

The Central Laboratory for the Research Councils
Library and Information Services
Rutherford Appleton Laboratory
Chilton
Didcot
Oxfordshire
OX11 0QX
Tel: 01235 445384 Fax: 01235 446403
E-mail library@rl.ac.uk

ISSN 1358-6254

RAL-TR-95-076

A Generalized Dual Symmetry for Nonabelian Yang-Mills Fields*)

Chan Hong-Mo

Rutherford Appleton Laboratory,
Chilton, Didcot, Oxon, OX11 0QX, UK.

J. Faridani

Department of Physics, University of Toronto,
60 St. George St., Toronto, ON, M5S 1A7, Canada.

Tsou Sheung Tsun

Mathematical Institute, Oxford University,
24-29 St. Giles', Oxford, OX1 3LB, UK.

Abstract

It is shown that classical nonsupersymmetric Yang-Mills theory in 4 dimensions is symmetric under a generalized dual transform which reduces to the usual dual *-operation for electromagnetism. The parallel phase transport $\tilde{A}_\mu(x)$ constructed earlier for monopoles is seen to function also as a potential in giving a full description of the gauge field, playing thus an entirely dual symmetric role to the usual potential $A_\mu(x)$. Sources of A are monopoles of $\tilde{A}$ and vice versa, and the Wu-Yang criterion for monopoles is found to yield as equations of motion the standard Wong and Yang-Mills equations for respectively the classical and Dirac point charge; this applies whether the charge is electric or magnetic, the two cases being related just by a dual transform. The dual transformation itself is explicit, though somewhat complicated, being given in terms of loop space variables of the Polyakov type.

*) Dedicated to the memory of Professor Sir Rudolf Peierls, 1907 - 1995.

1 Introduction

It is well-known that pure electrodynamics is symmetric under the interchange of electricity and magnetism: $E \to -H, H \to E$, or equivalently under the Hodge star operation:[1]

$$ {}^*F_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{1.1}$$

This has led to many interesting consequences which have always intrigued physicists [1]-[9] and have recently again excited much interest due to the work of Seiberg, Witten and many others [10]-[13].

In view of the importance of Yang-Mills theories to particle physics, it is natural to ask whether a similar symmetry exists also for nonabelian gauge fields. This question can be asked at many different levels. Recently, it is most often addressed at the level of quantum fields, where the Yang-Mills theory is embedded in a larger theory, usually supersymmetric and existing in a high-dimensional space-time, in which charges, whether electric or magnetic, appear as 't Hooft-Polyakov solitons [10]. Here, however, we adopt a minimalist approach and ask whether strictly 4-dimensional and nonsupersymmetric Yang-Mills theory may possess a dual symmetry at the classical field and point charge level. Since it is at this level that the Maxwell theory exhibits the well-known dual symmetry, it seems reasonable to ask first whether Yang-Mills theory might possess a generalized version of this symmetry also at the same level.

If duality for nonabelian theories is taken to mean again just the Hodge star operation (1.1), then the answer to the above question is no. The field tensor $F_{\mu\nu}$ in the pure Maxwell theory satisfies the equations:

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x), \tag{1.2}$$

and:

$$\partial^\nu F_{\mu\nu}(x) = 0. \tag{1.3}$$

By virtue of (1.2), $F_{\mu\nu}$ then satisfies the Bianchi identity:

$$\partial^{\nu\,*}F_{\mu\nu}(x) = 0. \tag{1.4}$$

Moreover, because the Hodge star operation is reflexive:

$$ {}^*({}^*F_{\mu\nu}) = -F_{\mu\nu} \tag{1.5}$$

[1] In our convention, $g_{\mu\nu} = (+,-,-,-)$, $\epsilon_{0123} = 1$.

the Maxwell equation of (1.3) can similarly be interpreted in this abelian case as the Bianchi identity for ${}^*F_{\mu\nu}$, which then implies by the Poincaré lemma that there exists a potential $\tilde{A}_\mu$ such that:

$$ {}^*F_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x). \tag{1.6} $$

One sees therefore that $F_{\mu\nu}(x)$ and ${}^*F_{\mu\nu}(x)$ satisfy formally the same equations, or that electromagnetism is dual symmetric. For the pure nonabelian theory on the other hand, the Yang-Mills field tensor satisfies, in parallel to (1.2) and (1.3) for the abelian case, the equations:

$$ F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)], \tag{1.7} $$

and:

$$ D^\nu F_{\mu\nu}(x) = 0, \tag{1.8} $$

where D_μ denotes the usual covariant derivative:

$$ D_\mu = \partial_\mu - ig[A_\mu(x), \quad]. \tag{1.9} $$

Although (1.7) implies again the Bianchi identity:

$$ D^\nu {}^*F_{\mu\nu}(x) = 0, \tag{1.10} $$

this is not the dual of (1.8), since the covariant derivative in (1.10) involves the potential $A_\mu(x)$ and not some "dual potential" appropriate to ${}^*F_{\mu\nu}(x)$. Furthermore, the Yang-Mills equation (1.8) itself can no longer be interpreted as the Bianchi identity for ${}^*F_{\mu\nu}(x)$, nor does it imply the existence of a "dual potential" $\tilde{A}_\mu(x)$ satisfying:

$$ {}^*F_{\mu\nu}(x) \stackrel{?}{=} \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) + i\tilde{g}[\tilde{A}_\mu(x), \tilde{A}_\nu(x)], \tag{1.11} $$

in parallel to (1.7). Indeed, it has been shown by Gu and Yang [14] that for certain cases of $F_{\mu\nu}(x)$ satisfying (1.8) there are no solutions for $\tilde{A}(x)$ in (1.11), which result shows once and for all that dual symmetry of Yang-Mills theory under the Hodge star operation does not hold.

However, it is not excluded that there may be a generalized dual transform which reduces to the Hodge star in the abelian case but for which there is still an electric-magnetic dual symmetry for nonabelian Yang-Mills theory. In fact, in an earlier paper [15], we have already suggested a generalized dual transform which was able to reproduce many of what one may call the dual properties of the abelian theory though not as yet the complete dual symmetry. The missing link in the arguments there for obtaining a nonabelian dual symmetry was again the existence or otherwise of a local dual potential $\tilde{A}_\mu(x)$ for Yang-Mills fields. Although a local quantity $\tilde{A}_\mu(x)$ did appear which functioned as the parallel transport for the phase of colour magnetic charges exactly as a dual potential should, we were unable to show that this $\tilde{A}_\mu(x)$ can reproduce all field quantities - meaning that it gives a complete description of the theory. As a result of this failure our treatment there, though having some desirable features, remained far from being dual symmetric.

What we shall do in this paper is to show that a generalized dual symmetry does exist for nonabelian Yang-Mills theory, and that the dual phase transport $\tilde{A}_\mu(x)$ introduced in [15] does function also as a dual potential in that it gives a full description of the theory and plays an entirely dual symmetric role to the standard gauge potential $A_\mu(x)$. This result is achieved by writing down a dual transform between two new sets of variables which allows us to reformulate the whole theory in an explicitly dual fashion. Indeed, although the new results are derived on the basis of results obtained before, the new dual symmetric formulation is so much neater than the old that we shall find it easier to derive some of the old results again together with the new than to refer back to the older derivations. We shall therefore work throughout with the new dual formulation and only return in the end to sort out the relationship with the older treatment.

A dual symmetry for Yang-Mills fields means in particular that colour electric charges (i.e. ordinary colour charges such as quarks) which are usually taken to be sources of the Yang-Mills field can also be considered as monopoles of the dual field in the same way as colour magnetic charges are monopoles of the Yang-Mills field. It follows therefore that electric and magnetic charges, in nonabelian as in abelian theories, have basically the same dynamics, namely that given by the standard Maxwell and Yang-Mills equations, only formulated in a dual manner. Furthermore, since the relation here between the field and the dual field though somewhat complicated is explicitly known, the result may have brought us one step nearer to realizing the hope of obtaining the strong coupling limit of one formulation from the weak coupling limit of its dual by making use of the generalized Dirac condition:

$$g\tilde{g} = \frac{1}{2N} \tag{1.12}$$

relating the magnitudes of electric and magnetic couplings for a theory with gauge group $SU(N)$.

2 $E_\mu[\xi|s]$ as Variables

In our previous paper [15] on Yang-Mills duality we have relied heavily on a loop space technique developed earlier, using the Polyakov variables $F_\mu[\xi|s]$ to describe the gauge field [16, 17, 18]. These variables $F_\mu[\xi|s]$ take values in the gauge Lie algebra, depend on the parametrized loop ξ only up to the point on ξ labelled by the value s of this parameter, and have only components transverse to the loop at that point. They are known to give a complete description of the Yang-Mills theory but are highly redundant as all loop variables are, and have to be constrained by an infinite set of conditions which is most conveniently stated as the vanishing of the loop space curvature: [17, 18]

$$G_{\mu\nu}[\xi|s] = 0, \tag{2.1}$$

where:

$$G_{\mu\nu}[\xi|s] = \delta_\nu(s)F_\mu[\xi|s] - \delta_\mu(s)F_\nu[\xi|s] + ig[F_\mu[\xi|s], F_\nu[\xi|s]], \tag{2.2}$$

and $\delta_\mu(s)$ denotes the loop derivative $\delta/\delta\xi^\mu(s)$ at s. One great virtue of $F_\mu[\xi|s]$ as variables is that they are gauge independent apart from an innocuous x-independent gauge rotation at the fixed reference point P_0 for the parametrized loops.

In discussing dual properties, however, it was found convenient to introduce another set of quantities $E_\mu[\xi|s]$ which were defined as:

$$E_\mu[\xi|s] = \Phi_\xi(s,0)F_\mu[\xi|s]\Phi_\xi^{-1}(s,0), \tag{2.3}$$

where:

$$\Phi_\xi(s_2, s_1) = P_s \exp ig \int_{s_1}^{s_2} ds A_\mu(\xi(s))\dot{\xi}^\mu(s) \tag{2.4}$$

is the parallel phase transport from the point at s_1 to the point at s_2 along the loop ξ. Hence, in order to exhibit more clearly the dual properties of the theory, it is our intention here to adopt these $E_\mu[\xi|s]$ instead of $F_\mu[\xi|s]$ as field variables. Our first task is to demonstrate that this is possible under conditions which we shall have to specify.

Recall first that the Polyakov variable $F_\mu[\xi|s]$ is defined as:

$$F_\mu[\xi|s] = \frac{i}{g}\Phi^{-1}[\xi]\delta_\mu(s)\Phi[\xi], \tag{2.5}$$

where:

$$\Phi[\xi] = P_s \exp ig \int_0^{2\pi} ds A_\mu(\xi(s))\dot{\xi}^\mu(s), \tag{2.6}$$

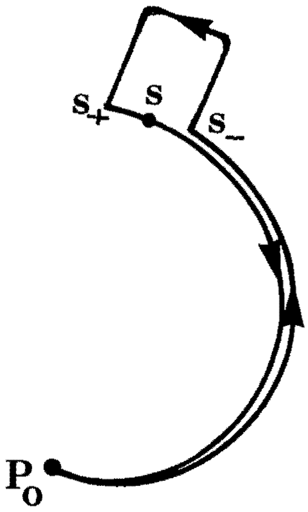

Figure 1: Illustration for $F_\mu[\xi|s]$

or $\Phi_\xi(2\pi, 0)$ as defined in (2.4), so that $F_\mu[\xi|s]$ can be pictured as in Figure 1, where the δ-function $\delta(s-s')$ inherent in our definition of the loop derivative[2] $\delta_\mu(s)$ is represented in the figure as a bump function centred at s with width $\epsilon = s_+ - s_-$. In the same spirit, the quantity $E_\mu[\xi|s]$ defined in (2.3) can be pictured as the bold curve in Figure 2 where the phase factors $\Phi_\xi(s,0)$ in (2.3) have cancelled parts of the circuit in Figure 1. In contrast to $F_\mu[\xi|s]$, therefore, $E_\mu[\xi|s]$ is dependent really only on a "segment" of the loop ξ from s_- to s_+.

The reason for representing the δ-function in Figures 1 and 2 as a bump function is that, as in most functional formulations, our treatment here involves some operations with the δ-function which need to be "regularized" to be given a meaning. Our procedure is to take first the δ-function as a bump function with finite width, and then afterwards take the appropriate zero width limit. For example, we shall need later the loop derivative $\delta_\nu(s)$ of the quantity $E_\mu[\xi|s]$ at the same value of s. Clearly, a loop derivative has a meaning only if there is a segment of the loop on which it can operate. Therefore, to define this derivative, we shall

[2]For any functional $\Psi[\xi]$ of the parametrized loop ξ, we defined [17] the loop derivative $\delta_\mu(s) = \delta/\delta\xi^\mu(s)$ as:

$$\frac{\delta}{\delta\xi^\mu(s)}\Psi[\xi] = \lim_{\Delta\to 0}\frac{1}{\Delta}\{\Psi[\xi'] - \Psi[\xi]\}, \tag{2.7}$$

with:

$$\xi'^\alpha(s') = \xi^\alpha(s') + \Delta\delta^\alpha_\mu\delta(s-s'). \tag{2.8}$$

In case of ambiguity, $\Delta\delta(s-s')$ in the expression above for $\xi'^\alpha(s')$ is replaced by a bump-function with width ϵ and height h, and the limit $\epsilon \to 0$ with $\Delta = \epsilon h$ held fixed is taken first, to be followed by the limit $h \to 0$.

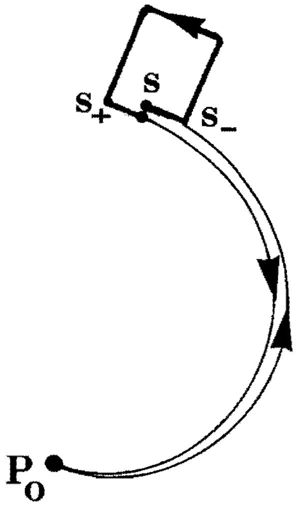

Figure 2: Illustration for $E_\mu[\xi|s]$

first regard $E_\mu[\xi|s]$ as a segmental quantity dependent on the segment of the loop ξ from $s-\epsilon/2$ to $s+\epsilon/2$. We then define the loop derivative $\delta_\nu(s)$ using the normal procedure on this segment, and afterwards take the limit $\epsilon \to 0$. In case a repeated loop derivative of $E_\mu[\xi|s]$ is required at the same s, then the δ-function inherent in the first derivative has again to be represented by a bump function of finite width, say ϵ', so that the second derivative can be defined on this segment of the loop. Afterwards, we take first the limit $\epsilon' \to 0$, and then the limit $\epsilon \to 0$, in that order. In view of these regularization procedures, it is often convenient to picture the quantities $F_\mu[\xi|s]$ and $E_\mu[\xi|s]$ as in Figures 1 and 2.

To show now that $E_\mu[\xi|s]$ do constitute a valid set of variables for a full description of the gauge field, we note first that by (2.3) and (2.5) we have:

$$\delta_\nu(s')E_\mu[\xi|s] = \Phi_\xi(s,0)\{\delta_\nu(s')F_\mu[\xi|s] + ig\theta(s-s')[F_\nu[\xi|s'], F_\mu[\xi|s]]\}\Phi_\xi^{-1}(s,0), \tag{2.9}$$

where $\theta(s)$ is the Heaviside θ-function, so that:

$$G_{\mu\nu}[\xi|s] = \Phi_\xi^{-1}(s,0)\{\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s]\}\Phi_\xi(s,0), \tag{2.10}$$

and the condition (2.1) translated in terms of $E_\mu[\xi|s]$ reads as:

$$\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s] = 0. \tag{2.11}$$

Hence, since we already know that $F_\mu[\xi|s]$ constrained by (2.1) describes the gauge theory, we want now to show that given a set of $F_\mu[\xi|s]$ satisfying (2.1) we recover a set of $E_\mu[\xi|s]$ satisfying (2.11) and vice versa.

The direct statement is easy to see. Given $F_\mu[\xi|s]$ satisfying (2.1), we know from the so-called Extended Poincaré lemma derived in [17] that we can recover a local potential $A_\mu(x)$, from which a parallel transport $\Phi_\xi(s,0)$ by (2.4), and hence also an $E_\mu[\xi|s]$ by (2.3) can be constructed. This $E_\mu[\xi|s]$ will automatically satisfy (2.11) as we wanted.

What is less obvious is the converse statement, namely that given a set of $E_\mu[\xi|s]$ satisfying (2.11), one can also recover a set of $F_\mu[\xi|s]$ satisfying (2.1). To see this, one notes first that given (2.11), it follows that there exists some $W[\xi|s]$ such that:

$$E_\mu[\xi|s] = \delta_\mu(s) W[\xi|s]. \tag{2.12}$$

Indeed, if one writes symbolically:

$$W[\xi|s] = \int_{\xi_0(s)}^{\xi(s)} \delta\xi'^\mu(s) E_\mu[\xi'|s] \tag{2.13}$$

as a line integral with respect to $\delta\xi$ along some path from an arbitrary point $\xi_0(s)$ to the given point $\xi(s)$ then a similar argument as in the usual Stokes' theorem would imply by (2.11) that $W[\xi|s]$ is in fact path-independent and depends only on the end-point $\xi(s)$ as indicated. Furthermore, the derivative of this integral $W[\xi|s]$ would give $E_\mu[\xi|s]$ as desired. If we now take a product of these W's along the loop ξ, thus:

$$\Phi_\xi(s,0) = P_{s'} \prod_{s'=0\to s} \{1 - igW[\xi|s']\}, \tag{2.14}$$

it is seen to satisfy:

$$\Phi_\xi^{-1}(s,0)\delta_\mu(s')\Phi_\xi(s,0) = -ig\theta(s-s')\Phi_\xi^{-1}(s,0)E_\mu[\xi|s]\Phi_\xi(s,0). \tag{2.15}$$

Defining then:

$$F_\mu[\xi|s] = \Phi_\xi^{-1}(s,0)E_\mu[\xi|s]\Phi_\xi(s,0) \tag{2.16}$$

with $\Phi_\xi(s,0)$ given in (2.14), we have:

$$\begin{aligned} &\delta_\nu(s)F_\mu[\xi|s] - \delta_\mu(s)F_\nu[\xi|s] = \\ &\quad \Phi_\xi^{-1}(s,0)\{\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s]\}\Phi_\xi(s,0) - ig[F_\mu[\xi|s], F_\nu[\xi|s]], \end{aligned} \tag{2.17}$$

i.e. (2.10), which by (2.11) means that $G_{\mu\nu}[\xi|s]$ vanishes, as required.

In the above argument, however, we have actually glossed over a rather important point, namely that in writing (2.15) we have used (2.12) in which, by our procedure detailed above, $W[\xi|s]$ ought first to be regarded as a "segmental quantity" depending on a segment of ξ with width $\epsilon = s_+ - s_-$, and only after the loop differentiation has been performed is the segmental width ϵ to be taken to zero. On the other hand, in defining $\Phi_\xi(s,0)$ in terms of $W[\xi|s]$, one wants already in (2.14) to take the limit $\epsilon \to 0$. To assert both statements therefore, we shall need a composition law for W which says that the factor $(1 - igW[\xi|s])$ for a small finite segment is in fact the same as the product of such factors for those infinitesimal segments which make up this small finite segment. That such a composition law holds can be seen by an argument parallel to that given in [17] for deriving the composition law for $\Phi[\xi]$ by writing it in terms of $F_\mu[\xi|s]$ as a surface integral. Here, the line integral in loop space (2.13) representing $W[\xi|s]$ is also in fact a surface integral in ordinary space-time for which a similar argument is seen to apply.

That being the case, we conclude that $E_\mu[\xi|s]$ constrained by (2.11) do constitute a valid set of variables for describing the gauge field, which we shall adopt later for discussing its dual properties. Note that, in contrast to the Polyakov variables $F_\mu[\xi|s]$, the variables $E_\mu[\xi|s]$ are gauge dependent quantities and so, though more convenient than $F_\mu[\xi|s]$ for studying duality, may not be so useful otherwise. We note further that the fact we are able to recover from $E_\mu[\xi|s]$ satisfying (2.11) the Polyakov variables $F_\mu[\xi|s]$ satisfying (2.1) means also by the Extended Poincaré lemma of [17] that there exists a local potential $A_\mu(x)$ such that the parallel transport is indeed given by (2.4). In turn, this implies that:

$$\lim_{\epsilon \to 0} W[\xi|s] = \lim_{s_+ \to s_-} \frac{i}{g}\{\Phi_\xi(s_+, s_-) - 1\} = A_\mu(\xi(s))\dot{\xi}^\mu(s), \tag{2.18}$$

and that:

$$\lim_{\epsilon \to 0} E_\mu[\xi|s] = F_{\mu\nu}(\xi(s))\dot{\xi}^\nu(s), \tag{2.19}$$

with $F_{\mu\nu}(x)$ given as usual in (1.7) in terms of the $A_\mu(x)$ defined in (2.18) above. These two formulae will be of use to us later.

3 Generalized Dual Transform

As noted above in the Introduction, Hodge star duality does not lead to a dual symmetry for nonabelian Yang-Mills theory. We seek therefore a generalized dual transform, if such exists, which may restore dual symmetry to Yang-Mills theory. The experience gained in earlier work leads us to believe that such a transform is best written in terms of the variables $E_\mu[\xi|s]$ introduced in the preceding section.

We seek a dual transform with the following 3 properties. First, we want, of course, that the new dual transform reduces back to the Hodge star (1.1) for the abelian theory, but that it should not do so for the nonabelian case or else the conclusion of Gu and Yang in [14] would be violated. Secondly, in order for the new transform to qualify as a dual transform, we want it to be invertible in the sense that, like the Hodge star, application of the transform twice should give the identity, apart perhaps from a sign. Thirdly, we want the transform to be such that, again like the Hodge star in the abelian case, an electric charge defined as a source of the direct field should appear as a monopole of the dual field, while a magnetic charge defined as a source of the dual field should appear as a monopole of the direct field. This last property seems to us to be the crucial feature which gives dual symmetry to the abelian theory and which, we have reason to believe from past experience, may give dual symmetry also to Yang-Mills fields.

Our suggestion is as follows. Given a set of variables $E_\mu[\xi|s]$ describing the gauge field, we introduce a corresponding dual set of variables $\tilde{E}_\mu[\eta|t]$ labelled by η and t, where η is just another parametrized loop with parameter t which are distinguished here by different symbols from ξ and s for convenience. For given η and t, $\tilde{E}_\mu[\eta|t]$ is defined as:

$$\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t)) = -\frac{2}{\bar{N}}\epsilon_{\mu\nu\rho\sigma}\dot{\eta}^\nu(t)\int \delta\xi ds E^\rho[\xi|s]\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)), \tag{3.1}$$

where $\omega(x)$ is just a local rotational matrix allowing for the freedom of transforming from the "U"-frame in which direct quantities like $E_\mu[\xi|s]$ are represented to a "$\tilde{U}$"-frame in which dual quantities like $\tilde{E}_\mu[\eta|t]$ are represented, and $\bar{N}$ a normalization factor (infinite) defined as: [17, 18, 15]

$$\bar{N} = \int_0^{2\pi} ds \prod_{s' \neq s} d^4\xi(s'). \tag{3.2}$$

As (3.1) involves an implicit regularization procedure, i.e. a fixed order in which various limits are taken, some explanation is in order. The loop integral on the right-hand side of (3.1), as for the loop derivative discussed in the preceding section, needs a segment of the loop ξ on which to operate. Hence, $E_\rho[\xi|s]$ has first again to be regarded as a segmental quantity depending on a little segment of ξ from s_- to s_+ whose width $\epsilon = s_+ - s_-$ is taken to zero only after the integration has been performed. In the same spirit, $\dot{\xi}(s)$ in the integrand is meant to represent the quantity $(\xi(s_+)-\xi(s_-))/\epsilon$ which becomes the tangent to the loop ξ at s when $\epsilon \to 0$. If one is interested only in the value of $\tilde{E}_\mu[\eta|t]$ and not, say, in its derivatives, then $\tilde{E}_\mu[\eta|t]$ can be taken as just a function of the point $\eta(t)$ labelled by t on the loop η and of the tangent $\dot{\eta}(t)$ to the loop at that point. In that case, the δ-function $\delta(\xi(s)-\eta(t))$ on the right says that the segment ξ has to pass through at s the point $\eta(t)$ but is otherwise freely integrated so that $\dot{\xi}(s) = (\xi(s_+) - \xi(s_-))/\epsilon$ can have any direction relative to $\dot{\eta}(t)$, except that the contribution to the integral vanishes when $\dot{\xi}(s)$ is parallel to $\dot{\eta}(t)$ because of the $\epsilon_{\mu\nu\rho\sigma}$ symbol in front. However, if we wish to evaluate the loop derivative $\delta_\alpha(t) = \delta/\delta\eta^\alpha(t)$ of $\tilde{E}_\mu[\eta|t]$ using the formula (3.1), then $\tilde{E}_\mu[\eta|t]$ itself has also to be regarded as a segmental quantity depending on a segment of η from t_- to t_+ with width $\epsilon' = t_+ - t_-$. After the differentiation has been performed, one can then take the limit $\epsilon' \to 0$, and our procedure says that this limit should be taken before the limit $\epsilon \to 0$ for the integral. That being the case, we may take $\epsilon' < \epsilon$, and the δ-function $\delta(\xi(s) - \eta(t))$ should now be interpreted as saying that the segment ξ coincides from $s = t_-$ to $s = t_+$ with the segment η, but outside that interval is still freely integrated so that $\dot{\xi}(s)$ can again have any direction relative to $\dot{\eta}(t)$. Since the integral receives contributions only from ξ segments with $\dot{\xi}$ nonparallel to $\dot{\eta}$, we cannot take $\epsilon' = \epsilon$, otherwise $\dot{\xi}(s) = \dot{\eta}(t)$ and the integral would vanish.

With these clarifications in the interpretation of the dual transform (3.1) let us now examine whether colour electric charges do indeed appear as monopoles of the dual field $\tilde{E}_\mu[\eta|t]$, which property, as stated above, we believe to be crucial for dual symmetry. We recall first that a colour electric charge is usually defined as a source of the Yang-Mills field, namely a nonvanishing covariant divergence $D^\nu F_{\mu\nu}(x)$. Equivalently, according to Polyakov [16], it is a nonvanishing loop divergence $\delta^\mu(s)F_\mu[\xi|s]$ of the loop variable $F_\mu[\xi|s]$. Alternatively again, since (2.9) implies that:

$$\delta^\mu(s)E_\mu[\xi|s] = \Phi_\xi(s,0)\{\delta^\mu(s)F_\mu[\xi|s]\}\Phi_\xi^{-1}(s,0), \tag{3.3}$$

it also means a nonvanishing loop divergence $\delta^\mu(s)E_\mu[\xi|s]$ of the variable $E_\mu[\xi|s]$ adopted here. On the other hand, a colour magnetic charge defined as a monopole of the Yang-Mills field is characterized most easily as a nonvanishing loop space curvature [17, 18] $G_{\mu\nu}[\xi|s]$ as defined in (2.2), or alternatively, by (2.10) in terms of $E_\mu[\xi|s]$, as a nonvanishing "curl" $\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s]$. By a monopole of the dual field $\tilde{E}$ we mean then a nonvanishing curl $\delta_\nu(t)\tilde{E}_\mu[\eta|t] - \delta_\mu(t)\tilde{E}_\nu[\eta|t]$. Hence to show that a colour electric charge is indeed a monopole of the dual field, we need to show that a nonvanishing divergence of E will lead to a nonvanishing curl of the dual variable $\tilde{E}$ as defined by the dual transform (3.1). The parallel for this in the abelian theory is that an electric charge represented by the nonvanishing divergence $\partial^\nu F_{\mu\nu}(x)$ of the Maxwell field can also be interpreted as the violation of the Bianchi identity for the dual field ${}^*F_{\mu\nu}(x)$, which signifies the presence of a monopole in ${}^*F$.

That a nonvanishing divergence of E would generally lead to a nonvanishing curl of $\tilde{E}$ can be seen by direct computation. From (3.1), one can write:

$$\begin{aligned} & \epsilon^{\lambda\mu\alpha\beta}\delta_\lambda(t)\{\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t))\} \\ = & -\frac{2}{\bar{N}}\epsilon^{\lambda\mu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\dot{\eta}^\nu(t)\int\delta\xi ds\{\delta_\lambda(s)E^\rho[\xi|s]\}\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)), \end{aligned} \tag{3.4}$$

where, recalling from the above paragraph that in $\delta(\xi(s)-\eta(t))$ on the right, $\eta(t)$ is first to be interpreted as a little segment which coincides with $\xi(s)$ for $s = t_- \to t_+$, we have put $\delta_\lambda(t) = -\delta_\lambda(s)$ and then performed an integration by parts with respect to $\delta\xi$. Expressing next $\epsilon^{\lambda\mu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}$ as a combination of Kronecker deltas and using the fact that segmental quantities, like loop quantities, have only transverse loop derivatives so that both $\delta_\mu(s)\dot{\xi}^\mu(s)$ and $\delta_\mu(t)\dot{\eta}^\mu(t)$ vanish, we obtain for (3.4):

$$\begin{aligned} & \epsilon^{\lambda\mu\alpha\beta}\delta_\lambda(t)\{\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t))\} \\ = & -\frac{2}{\bar{N}}\int\delta\xi ds\{\dot{\eta}^\beta(t)\dot{\xi}^\alpha(s)-\dot{\eta}^\alpha(t)\dot{\xi}^\beta(s)\}\delta_\rho(s)E^\rho[\xi|s]\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)) \end{aligned} \tag{3.5}$$

On multiplying by $\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}$, we obtain:

$$\begin{aligned} & \omega^{-1}(\eta(t))\{\delta_\nu(t)\tilde{E}_\mu[\eta|t]-\delta_\mu(t)\tilde{E}_\nu[\eta|t]\}\omega(\eta(t)) \\ = & -\frac{1}{\bar{N}}\int\delta\xi ds\epsilon_{\mu\nu\alpha\beta}\{\dot{\eta}^\beta(t)\dot{\xi}^\alpha(s)-\dot{\eta}^\alpha(t)\dot{\xi}^\beta(s)\}\delta_\rho(s)E^\rho[\xi|s]\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)), \end{aligned} \tag{3.6}$$

where the factors $\omega^{-1}(\eta(t))$ and $\omega(\eta(t))$ can be taken outside because loop derivatives vanish for local quantities.[3] One sees thus that the divergence of E is indeed related to the curl of $\tilde{E}$ and that an electric charge characterized by the nonvanishing of the former will in general mean a monopole characterized by a nonvanishing curl of the latter. Conversely, if $\delta^\rho(s)E_\rho[\xi|s] = 0$ then $\delta_\nu(t)\tilde{E}_\mu[\eta|t]-\delta_\mu(t)\tilde{E}_\nu[\eta|t] = 0$, or in other words the absence of sources in E will guarantee the absence of monopoles in $\tilde{E}$, which statement is in fact what is needed for deriving dual symmetry, as we shall see later.

Next, we wish to check that (3.1) reduces to the Hodge star relation when the theory is abelian but not when the theory is nonabelian. To see this, we let the segmental width of $\tilde{E}_\mu[\eta|t]$ in (3.1) go to zero so that we can use the formula (2.19) to write the left-hand side in terms of local quantities:

$$\omega^{-1}(x)\tilde{F}_{\mu\nu}(x)\omega(x) = -\frac{2}{\bar{N}}\epsilon_{\mu\nu\rho\sigma}\int\delta\xi dsE^\rho[\xi|s]\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(x-\xi(s)). \tag{3.7}$$

[3] Although $\omega(\eta(t))$ does vary when η is varied at t, its variation is of measure zero compared with the variation of the loop so long as the δ-function in the definition of the loop derivative is given a finite width, so that the derivative has to be assigned the value zero for consistency with our standard procedure for resolving such ambiguities.

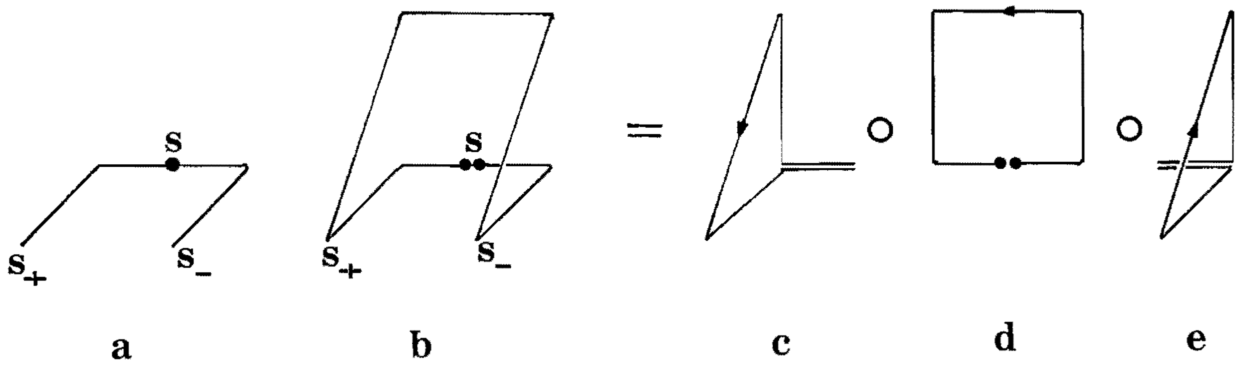

Figure 3: Illustration for the Integrand in Dual Transform

We recall that our procedure is to do the integral before taking the width of the segment in $E_\mu[\xi|s]$ to zero. In other words, within the integral, the loop ξ can still vary by a δ-functional bump as illustrated in Figure 3 (a). For such a ξ, $E_\mu[\xi|s]$, which is obtained by making a δ-functional variation along the direction μ, will take on the shape depicted in Figure 3 (b). This last figure can be expressed as the product of three factors, namely Figures 3(c),(d),(e) in the order indicated. In the abelian theory, the ordering of the factors is unimportant so that the factors of Figures (c) and (e) cancel in the limit when the segmental width $\epsilon \to 0$, leaving only the factor of Figure (d), which can as usual be expressed by (2.19) as $F_{\mu\alpha}(\xi(s))\dot{\xi}^\alpha(s)$, giving:

$$\begin{aligned}\tilde{F}_{\mu\nu}(x) &= -\frac{2}{\bar{N}}\epsilon_{\mu\nu\rho\sigma}\int\delta\xi ds F^{\rho\alpha}(\xi(s))\dot{\xi}_\alpha(s)\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(x-\xi(s)) \\ &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}(x), \qquad (3.8)\end{aligned}$$

which is just the Hodge star relation if we identify $\tilde{F}_{\mu\nu}(x)$ with ${}^*F_{\mu\nu}(x)$. On the other hand, for a nonabelian theory, the factors of Figures 3 (c) and (e) cannot be commuted through the factor of Figure (d) so that the above reduction to the Hodge star relation will not go through.

Lastly, we wish to examine whether the dual transform (3.1) is invertible. From (3.1) we can write:

$$\begin{aligned}&\frac{2}{\bar{N}}\epsilon^{\alpha\beta\mu\lambda}\dot{\zeta}_\beta(u)\int\delta\eta dt\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t))\dot{\eta}_\lambda(t)\dot{\eta}^{-2}(t)\delta(\eta(t)-\zeta(u)) \\ =\;& -\frac{4}{\bar{N}^2}\epsilon^{\alpha\beta\mu\lambda}\epsilon_{\mu\nu\rho\sigma}\dot{\zeta}_\beta(u)\int\delta\eta dt\dot{\eta}_\lambda(t)\dot{\eta}^\nu(t)\dot{\eta}^{-2}(t)\delta(\eta(t)-\zeta(u)) \\ &\int\delta\xi ds E^\rho[\xi|s]\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)). \qquad (3.9)\end{aligned}$$

By integrating first over all directions of $\dot{\eta}(t)$ which we recall from the explanation given after (3.1) is admissible, we obtain a factor $\bar{N}\delta^{\nu}_{\lambda}/4$, so that the right-hand side reduces to:

$$\frac{2}{\bar{N}}\{\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}\}\dot{\zeta}_{\beta}(u)\int \delta\xi ds E^{\rho}[\xi|s]\dot{\xi}^{\sigma}(s)\dot{\xi}^{-2}\delta(\xi(s)-\eta(t)). \tag{3.10}$$

Using the argument in the paragraph above, one can show that the integral in (3.10) is antisymmetric in the indices ρ and σ giving then just twice the first term where, since $\dot{\zeta}$ and $\dot{\xi}$ are no longer forbidden to be parallel, we may put them equal using $\delta(\xi(s)-\zeta(u))$ so that the whole expression reduces to just $E^{\alpha}[\zeta|u]$, giving:

$$\omega(\zeta(u))E_{\alpha}[\zeta|u]\omega^{-1}(\zeta(u)) = \frac{2}{\bar{N}}\epsilon_{\alpha\beta\mu\lambda}\dot{\zeta}^{\beta}(u)\int \delta\eta dt \tilde{E}^{\mu}[\eta|t]\dot{\eta}^{\lambda}(t)\dot{\eta}^{-2}(t)\delta(\eta(t)-\zeta(u)), \tag{3.11}$$

as required.

We have now shown that the generalized dual transform suggested in (3.1) does indeed have all the 3 properties that we desired.

4 Pure Yang-Mills Theory

With the variables E and $\tilde{E}$ introduced in the two preceding sections, let us now examine the dual properties of the pure Yang-Mills theory. Since the theory in the standard (direct) formulation has a local potential $A_{\mu}(x)$, it follows that if the theory is symmetric under the dual transform (3.1) introduced above, then there must also be a local potential $\tilde{A}_{\mu}(x)$ in the dual formulation. Now, in the abelian theory, it was the equation of motion (1.3) which guaranteed via the Poincaré lemma the existence of the dual potential $\tilde{A}_{\mu}(x)$; so we can hope that here too in the nonabelian theory, it is the Yang-Mills equation of motion, namely (1.8), which guarantees the existence of the local potential $\tilde{A}_{\mu}(x)$. We shall now show that this is indeed the case.

According to Polyakov [16], the Yang-Mills equation (1.8) can be written in terms of the loop variables $F_{\mu}[\xi|s]$ as:

$$\delta^{\mu}(s)F_{\mu}[\xi|s] = 0. \tag{4.1}$$

By (3.3) it follows that

$$\delta^{\mu}(s)E_{\mu}[\xi|s] = 0. \tag{4.2}$$

Hence by (3.6) the dual variables $\tilde{E}_{\mu}[\eta|t]$ have to satisfy the condition:

$$\delta_{\nu}(t)\tilde{E}_{\mu}[\eta|t] - \delta_{\mu}(t)\tilde{E}_{\nu}[\eta|t] = 0. \tag{4.3}$$

However, we know from Section 2 that this is exactly the condition for these variables to possess a local potential. Indeed, according to the arguments there, (4.3) implies the existence of a $\tilde{W}[\eta|t]$ such that:

$$\tilde{E}_\mu[\eta|t] = \delta_\mu(t)\tilde{W}[\eta|t], \tag{4.4}$$

and the local potential $\tilde{A}_\mu(x)$ is given by the dual analogue of (2.18):

$$\tilde{A}_\mu(\eta(t))\dot{\eta}^\mu(t) = \lim_{\epsilon\to 0} \tilde{W}[\eta|t]. \tag{4.5}$$

One sees thus that the existence of a local dual potential $\tilde{A}_\mu(x)$ is indeed guaranteed.

From previous work [17, 18, 20, 15], we have learned that it is possible, and in fact even convenient for deriving the dynamics of colour charges, to reformulate the Yang-Mills theory in terms of loop variables. This was done for the Polyakov variables $F_\mu[\xi|s]$. Let us do it now in terms of the variables $E_\mu[\xi|s]$. We have shown already in Section 2 that they give a complete description of the theory although they have to be constrained by the curl-free condition (2.11). Suppose then we start with the standard Yang-Mills action:[4]

$$\mathcal{A}_F^0 = -\frac{1}{16\pi}\int d^4x \,\mathrm{Tr}\{F_{\mu\nu}(x)F^{\mu\nu}(x)\}, \tag{4.6}$$

which in terms of the Polyakov variables $F_\mu[\xi|s]$ takes the familiar form:

$$\mathcal{A}_F^0 = -\frac{1}{4\pi\bar{N}}\int \delta\xi ds \,\mathrm{Tr}\{F_\mu[\xi|s]F^\mu[\xi|s]\}\dot{\xi}^{-2}(s), \tag{4.7}$$

we have from (2.3) in terms of $E_\mu[\xi|s]$:

$$\mathcal{A}_F^0 = -\frac{1}{4\pi\bar{N}}\int \delta\xi ds \,\mathrm{Tr}\{E_\mu[\xi|s]E^\mu[\xi|s]\}\dot{\xi}^{-2}(s). \tag{4.8}$$

Incorporating the constraint (2.11) into the action by means of Lagrange multipliers $W_{\mu\nu}[\xi|s]$, we obtain:

$$\mathcal{A}_F = \mathcal{A}_F^0 + \int \delta\xi ds \,\mathrm{Tr}\{W^{\mu\nu}[\xi|s](\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s])\}, \tag{4.9}$$

the extremization of which with respect to the variables $E_\mu[\xi|s]$ yields then the equation of motion in parametric form:

$$E_\mu[\xi|s] = -(4\pi\bar{N}\dot{\xi}^2(s))\delta^\nu(s)W_{\mu\nu}[\xi|s]. \tag{4.10}$$

[4]For $su(2)$, our convention is: $B = B^i t_i, t_i = \tau_i/2$, $\mathrm{Tr}B = 2\times$ sum of diagonal elements, so that $\mathrm{Tr}(t_i t_j) = \delta_{ij}$. Our results are given explicitly for $su(2)$ although they can be trivially extended to any $su(N)$.

The parameter $W_{\mu\nu}[\xi|s]$ being antisymmetric in its indices μ, ν, (4.10) is easily seen to imply (4.1), or in other words the Yang-Mills equation (1.8) as expected.

Now earlier work has shown that the Lagrange multipliers in such a formulation often play the role of a dual potential [15]. If so, we expect that the dual potential $\tilde{A}_\mu(x)$ should be expressible in terms of the parameters $W_{\mu\nu}[\xi|s]$. For reasons which will be made clear later when we deal with colour charges, we anticipate that $\tilde{A}_\mu(x)$ is expressible in terms of $W_{\mu\nu}[\xi|s]$ as:

$$\tilde{A}_\mu(x) = -8\pi \int \delta\xi ds \epsilon_{\mu\nu\rho\sigma}\omega(\xi(s))W^{\rho\sigma}[\xi|s]\omega^{-1}(\xi(s))\dot{\xi}^\nu(s)\dot{\xi}^{-2}\delta(\xi(s)-\eta(t)). \quad (4.11)$$

However, we have already given a formula for $\tilde{A}_\mu(x)$ in terms of $\tilde{W}[\eta|t]$ in (4.5). To see that these two expressions agree, substitute the expression (4.10) above into the dual transform (3.1) obtaining:

$$\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t)) = 8\pi\epsilon_{\mu\nu\rho\sigma}\dot{\eta}^\nu(t)\int \delta\xi ds \delta_\alpha(s)W^{\rho\alpha}[\xi|s]\dot{\xi}^\sigma(s)\delta(\xi(s)-\eta(t)), \quad (4.12)$$

where for:

$$^*W_{\mu\nu}[\xi|s] = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}[\xi|s], \quad (4.13)$$

one can rewrite:

$$\epsilon_{\mu\nu\rho\sigma}\delta_\alpha(s)W^{\rho\alpha}[\xi|s] = -\{\delta_\mu(s)^*W_{\nu\sigma}[\xi|s] + \delta_\nu(s)^*W_{\sigma\mu}[\xi|s] + \delta_\sigma(s)^*W_{\mu\nu}[\xi|s]\}. \quad (4.14)$$

However, since loop quantities by definition have only loop derivatives transverse to the loop, the last two terms inside the bracket on the right-hand side of (4.14) give zero contributions when substituted into (4.12) giving:

$$\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t)) = -8\pi\delta_\mu(t)\int \delta\xi ds \dot{\eta}^\nu(t)^*W_{\nu\sigma}[\xi|s]\dot{\xi}^\sigma(s)\delta(\xi(s)-\eta(t)), \quad (4.15)$$

where we have performed an integration by parts with respect to $\delta\xi$. It follows then from (4.4) that, apart from a constant term:

$$\omega^{-1}(\eta(t))\tilde{W}[\eta|t]\omega(\eta(t)) = -8\pi\dot{\eta}^\mu(t)\int \delta\xi ds \epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}[\xi|s]\dot{\xi}^\nu(s)\delta(\xi(s)-\eta(t)), \quad (4.16)$$

from which we obtain easily through (4.5) the relation (4.11) as desired.

The structure of the preceding arguments is set out on the left-hand side of Chart I, where the $\tilde{U}$-invariance will be demonstrated later. The similarity with Chart I of [15] for the abelian case is obvious.

Next, we explore whether a similar structure is also obtained if we go over into the dual formulation in terms of $\tilde{E}$. Substituting the expression (3.11) for E

in terms of $\tilde{E}$ into the action $\mathcal{A}_F^0$ in (4.8), we obtain on integrating over ξ and summing over indices:

$$\mathcal{A}_F^0 = \frac{1}{4\pi\bar{N}} \int \delta\eta dt \, \mathrm{Tr}\{\tilde{E}_\mu[\eta|t]\tilde{E}^\mu[\eta|t]\}\dot{\eta}^{-2}(t), \tag{4.17}$$

where we have used the fact that $\tilde{E}_\rho[\eta|t]$ has only components transverse to the loop η. Apart from a sign, this is formally the same as the action (4.8) in terms of E. Hence, if we extremize this action under the constraint (4.3) ensuring that $\tilde{E}$ is curl-free to remove the redundancy of these variables, we see that the problem will formally be exactly the same as for the direct formulation in terms of E, producing the structure shown on the right-hand side of Chart I. In other words, one has an exact dual symmetry as hoped.

5 Yang-Mills Theory with Charges

Monopoles in gauge theories have by virtue of their topological nature an intrinsic interaction with the gauge field, and Wu and Yang [1] have suggested a criterion whereby equations of motion for monopoles can be derived as consequences of the topology without introducing an explicit interaction term into the action. The criterion has already been repeatedly applied with success in earlier work [17, 20, 15]. In case a theory is dual symmetric, then both electric and magnetic charges are monopoles in the appropriate fields so that the Wu-Yang criterion can be applied to both giving dual symmetric equations as the result. This was the case in the abelian theory, and since we now claim that the Yang-Mills theory is symmetric under the new generalized duality, it should be true here also, which is what we wish now to demonstrate.

Let us start with a colour magnetic charge which is a monopole in the Yang-Mills field, appearing as a topological obstruction with nontrivial loop space holonomy, or equivalently non-zero loop space curvature $G_{\mu\nu}[\xi|s]$, constructed from the Polyakov variable $F_\mu[\xi|s]$ as connection [17, 18]. This in turn means non-zero curl for $E_\mu[\xi|s]$. The statement that there is a classical (colour) magnetic point charge $\tilde{g}$ moving along a world-line $Y^\mu(\tau)$ can thus be explicitly expressed as:

$$\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s] = -4\pi J_{\mu\nu}[\xi|s], \tag{5.1}$$

with:

$$J_{\mu\nu}[\xi|s] = \tilde{g}\epsilon_{\mu\nu\rho\sigma} \int d\tau \mathcal{K}(\tau) \frac{dY^\rho(\tau)}{d\tau} \dot{\xi}^\sigma(s)\delta(\xi(s) - Y(\tau)), \tag{5.2}$$

where $\mathcal{K}(\tau)$ is an algebra-valued quantity satisfying the conditiom $\exp i\pi\mathcal{K} = -1$. [17]

The Wu-Yang criterion stipulates that equations of motion are to be derived by imposing this definition (5.1) of the monopole as a constraint on the free action, which is for the classical point particle:

$$\mathcal{A}^0 = \mathcal{A}^0_F - m\int d\tau. \tag{5.3}$$

Incorporating then the constraint (5.1) by means of Lagrange multipliers $W_{\mu\nu}[\xi|s]$ into the action, we have:

$$\mathcal{A} = \mathcal{A}^0 + \int \delta\xi ds\, \mathrm{Tr}[W^{\mu\nu}[\xi|s]\{\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s] + 4\pi J_{\mu\nu}[\xi|s]\}]. \tag{5.4}$$

We notice that at every space-time point not on the world-line $Y^\mu(\tau)$ of the monopole, the condition (5.1) says that the curl of E vanishes, which is exactly the constraint we need to impose on the E variables to remove their intrinsic redundancy. Hence, in the action (5.4), where this constraint has already been incorporated, $E_\mu[\xi|s]$ can now be taken as independent variables.

Extremizing then $\mathcal{A}$ in (5.4) with respect to the variables $E_\mu[\xi|s]$ and $Y^\mu(\tau)$, we obtain again (4.10) together with:

$$\begin{aligned} m\frac{d^2Y^\mu(\tau)}{d\tau^2} &= -8\pi\tilde{g}\int \delta\xi ds \epsilon^{\mu\nu\rho\sigma}\delta^\lambda(s)\,\mathrm{Tr}\{W_{\lambda\rho}[\xi|s]\mathcal{K}(\tau)\} \\ &\quad\times\frac{dY_\nu(\tau)}{d\tau}\dot{\xi}_\sigma(s)\delta(\xi(s)-Y(\tau)). \end{aligned} \tag{5.5}$$

From these the Lagrange multipliers $W_{\mu\nu}[\xi|s]$ can be eliminated giving the Polyakov equation (4.1) or (4.2) together with:

$$\begin{aligned} m\frac{d^2Y^\mu(\tau)}{d\tau^2} &= \frac{2\tilde{g}}{\bar{N}}\int \delta\xi ds \epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\{E_\rho[\xi|s]\mathcal{K}(\tau)\} \\ &\quad\times\dot{\xi}_\sigma(s)\dot{\xi}^{-2}(s)\frac{dY_\nu(\tau)}{d\tau}\delta(\xi(s)-Y(\tau)), \end{aligned} \tag{5.6}$$

where one sees that $E_\rho[\xi|s]$ appears in the combination:

$$\frac{2}{\bar{N}}\int \delta\xi ds \epsilon^{\mu\nu\rho\sigma}E_\rho[\xi|s]\dot{\xi}_\sigma(s)\dot{\xi}^{-2}(s)\delta(\xi(s)-Y(\tau)), \tag{5.7}$$

which is exactly what appeared also in the dual transform (3.1) if one takes there the zero segmental width limit ($\epsilon \to 0$) and put $\eta(t) = Y(\tau)$. However, the other field equation of motion (4.2) has already been shown via the dual transform to imply the existence of a local gauge potential $\tilde{A}_\mu(x)$ for $\tilde{E}_\mu[\eta|t]$, so that by (2.19) in the limit of zero segmental width:

$$\tilde{E}_\mu[\eta|t] \longrightarrow \tilde{F}_{\mu\nu}(\eta(t))\dot{\eta}^\nu(t), \tag{5.8}$$

with:

$$\tilde{F}_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) + i\tilde{g}[\tilde{A}_\mu(x), \tilde{A}_\nu(x)]. \tag{5.9}$$

Whence, it follows that (5.6) reduces to:

$$m\frac{d^2 Y^\mu(\tau)}{d\tau^2} = -\tilde{g}\,\mathrm{Tr}\{K(\tau)\tilde{F}_{\mu\nu}(Y(\tau))\}\frac{dY_\nu(\tau)}{d\tau}, \tag{5.10}$$

with:

$$K(\tau) = \omega(Y(\tau))\mathcal{K}(\tau)\omega^{-1}(Y(\tau)), \tag{5.11}$$

and $\tilde{F}_{\mu\nu}(Y(\tau))$ as given by (5.9), which is the dual of the Wong equation[5] [19].

Conversely, if we start with a colour electric charge considered as a monopole of $\tilde{E}_\mu[\eta|t]$, we will obtain via exactly the same arguments the dual of the above equations, namely:

$$\delta^\mu(t)\tilde{E}_\mu[\eta|t] = 0, \tag{5.12}$$

which guarantees the existence of the potential $A_\mu(x)$ and is equivalent to the "dual Yang-Mills equation":

$$\tilde{D}^\nu \tilde{F}_{\mu\nu}(x) = 0, \tag{5.13}$$

with:

$$\tilde{D}_\nu = \partial_\nu - i\tilde{g}[\tilde{A}_\nu(x),\quad], \tag{5.14}$$

[5]This equation (5.10) should be clearly distinguished from the equation with ${}^*F_{\mu\nu}(x)$ in place of the $\tilde{F}_{\mu\nu}(x)$ here which we used to write in previous work [17, 20, 15] prefaced by a warning that it was meant only as illustration and should not be taken literally because ${}^*F_{\mu\nu}(x)$ is patched and cannot be given a meaning at the position $Y(\tau)$ of the monopole. The present equation (5.10) does not suffer from these faults since $\tilde{F}_{\mu\nu}(x)$ is covariant with respect to $\tilde{U}$- but invariant with respect to U-transformations so that in the presence of the magnetic charge (which is a monopole of E but only a source of $\tilde{E}$) it need not be patched at all and can exist even at the position $Y(\tau)$ of the magnetic charge, just as in the dual situation the Yang-Mills field $F_{\mu\nu}(x)$ requires no patching when only electric charges are present. Whatever patching that was needed has been absorbed into the transformation matrix $\omega(x)$ which has itself to be patched in the presence of the magnetic charge, as was shown in Section 6 of [15]. One notes further that the appearance of ${}^*F_{\mu\nu}(x)$ in (5.10) instead of $\tilde{F}_{\mu\nu}(x)$ would make the equation non-dual-symmetric since according to Gu and Yang [14] a "dual potential" to ${}^*F_{\mu\nu}(x)$ sometimes cannot exist. On the other hand, by virtue of the Yang-Mills equation or (4.2), a potential for $\tilde{F}_{\mu\nu}(x)$ is known to exist through the arguments in Section 2, thus restoring the symmetry with $F_{\mu\nu}(x)$ which is endowed with a potential right from the beginning of the standard (direct) formulation. Technically, what had gone wrong in "deriving" the old equation with ${}^*F_{\mu\nu}(Y(\tau))$ was that one had to take first the limit of the segmental width $\epsilon \to 0$ and apply the formula (2.19) in the expression (5.7) before performing the integral, whereas the rule of the game as we understand it now requires that the integral has to be first performed before the $\epsilon \to 0$ limit is taken, a rule to which we have now adhered.

together with the Wong equation:

$$m\frac{d^2Y^\mu(\tau)}{d\tau^2} = -g\,\mathrm{Tr}\{I(\tau)F^{\mu\nu}(Y(\tau))\}\frac{dY_\nu(\tau)}{d\tau}. \tag{5.15}$$

The dynamics of a classical point charge is thus seen to be entirely dual symmetric.

Consider next a Dirac particle carrying a colour magnetic charge. The logical steps for deriving its equations of motion in the gauge field using the Wu-Yang criterion are the same as for the classical point particle, except that the free action $\mathcal{A}^0$ is now: [20, 15]

$$\mathcal{A}^0 = \mathcal{A}_F^0 + \int d^4x\bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x), \tag{5.16}$$

and the "current" $J_{\mu\nu}[\xi|s]$ in (5.4) is now the quantum current:

$$J_{\mu\nu}[\xi|s] = \tilde{g}\epsilon_{\mu\nu\rho\sigma}\{\bar{\psi}(\xi(s))\omega(\xi(s))\gamma^\rho t^i\dot{\xi}^\sigma(s)\omega^{-1}(\xi(s))\psi(\xi(s))\}t_i, \tag{5.17}$$

both depending on the wave function $\psi(x)$ of the particle. Extremizing the action (5.4) with respect to $E_\mu[\xi|s]$ yields again the equation (4.10) which is equivalent to the Polyakov equation (4.1) or the Yang-Mills equation (1.8). Extremizing $\mathcal{A}$ with respect to $\bar{\psi}(x)$ on the other hand yields:

$$(i\partial_\mu\gamma^\mu - m)\psi(x) = -\tilde{g}\tilde{A}_\mu(x)\gamma^\mu\psi(x), \tag{5.18}$$

where $\tilde{A}_\mu(x)$ is as given in (4.11) and has already been shown there to be the same as the dual potential. This equation is thus exactly the dual of the Yang-Mills-Dirac equation for $\psi(x)$.

Starting with a colour electric charge considered as a monopole of $\tilde{E}[\eta|t]$ and following exactly the same arguments will lead easily to the dual equations to the above, namely the condition (4.3) which guarantees the existence of the local gauge potential $A_\mu(x)$ together with the Yang-Mills-Dirac equation for $\psi(x)$:

$$(i\partial_\mu\gamma^\mu - m)\psi(x) = -gA_\mu(x)\gamma^\mu\psi(x). \tag{5.19}$$

We have thus also for the quantum particle exact dual symmetry as we had hoped.

The result in this section is summarized in Chart II, which is seen to be quite symmetric on left and right and entirely analogous to the Chart II of [15] for electrodynamics.

6 $U \times \tilde{U}$ Invariance

That there is a dual doubling of the gauge symmetry in Yang-Mills theory has already been shown previously [20, 15]. Our task here is merely to outline how this gauge symmetry operates in terms of the new formulation, which turns out in fact to be considerably simpler than it has appeared before.

Under simultaneous infinitesimal U and $\tilde{U}$ local transformations parametrized respectively by the gauge parameters $\Lambda(x)$ and $\tilde{\Lambda}(x)$, the variables $E_\mu[\xi|s]$ and $\tilde{E}_\mu[\eta|t]$ transform as:

$$E_\mu[\xi|s] \longrightarrow [1 + ig\Lambda(\xi(s))]E_\mu[\xi|s][1 - ig\Lambda(\xi(s))], \tag{6.1}$$

$$\tilde{E}_\mu[\eta|t] \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(\eta(t))]\tilde{E}_\mu[\eta|t][1 - i\tilde{g}\tilde{\Lambda}(\eta(t))], \tag{6.2}$$

while the rotation matrix $\omega(x)$ transforms as:

$$\omega(x) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]\omega(x)[1 - ig\Lambda(x)]. \tag{6.3}$$

It is clear then that the dual transform (3.1) and its inverse (3.11) are both gauge covariant. Further, recalling that the gauge parameters $\Lambda(\xi(s))$ and $\tilde{\Lambda}(\xi(s))$, being local quantities, have zero loop derivatives (see the footnote in Section 3), one sees that the relation (3.6) giving the curl of $\tilde{E}$ in terms of the divergence of E which is so crucial for our duality arguments is also gauge covariant. That being the case, we need henceforth consider the invariant properties for only one half of the dual symmetric Charts I and II, since those for the other half will follow automatically.

Consider first Chart I for pure Yang-Mills fields. It is obvious that the free field term in the action (4.9) is gauge invariant. The only question then is how the Lagrange multipliers $W_{\mu\nu}[\xi|s]$ in the constraint term will transform. We put:

$$W_{\mu\nu}[\xi|s] \longrightarrow [1+ig\Lambda(\xi(s))]\{W_{\mu\nu}[\xi|s]+i\tilde{g}\epsilon_{\mu\nu\rho\sigma}\delta^\rho(s)\tilde{\Lambda}^\sigma[\xi|s]\}[1-ig\Lambda(\xi(s))], \tag{6.4}$$

where we notice that in addition to a U-gauge rotation there is an inhomogenious $\tilde{U}$-term parametrized by a vector quantity $\tilde{\Lambda}^\sigma[\xi|s]$. Under a pure $\tilde{U}$-transformation (i.e. for $\Lambda = 0$ in (6.4)) the transformation of $W_{\mu\nu}[\xi|s]$ is that of the tensor potential [6] discovered some years ago first in supersymmetry theory [23]. On substituting (6.4) into the action (4.9), the U-gauge rotation factors cancel, while the extra increment due to $\tilde{\Lambda}^\sigma[\xi|s]$, after an integration by parts with respect to ξ, is seen to vanish by virtue of the identity satisfied by the curl of E, namely:

$$\epsilon^{\mu\nu\rho\sigma}\delta_\rho(s)(\delta_\nu(s)E_\mu[\xi|s] - \delta_\mu(s)E_\nu[\xi|s]) = 0, \tag{6.5}$$

leaving thus the whole action invariant.

[6]Indeed, the Yang-Mills action when formulated in loop space (4.7) is entirely analogous to the Freedman-Townsend action with $W_{\mu\nu}[\xi|s]$ here playing the role of the Freedman-Townsend tensor potential [21, 22].

The Lagrange multiplier $W_{\mu\nu}[\xi|s]$, however, is related to the dual potential $\tilde{A}_\mu(x)$ by the relation (4.11) so that its transformation in (6.4) will induce a transformation in the dual potential. The result is:

$$\tilde{A}_\mu(x) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]\tilde{A}_\mu(x)[1 - i\tilde{g}\tilde{\Lambda}(x)] - 2i\tilde{g}\partial_\mu \int \delta\xi ds \tilde{\Lambda}_\nu[\xi|s]\dot{\xi}^\nu(s)\delta(\xi(s) - x), \tag{6.6}$$

where we have used the fact that $\tilde{\Lambda}_\nu[\xi|s]$ has only transverse derivatives and performed an integration by parts with respect to ξ. Hence, we see that $\tilde{A}_\mu(x)$ transforms as a gauge potential should, if we put:

$$\tilde{\Lambda}(x) = -8\pi \int \delta\xi ds \tilde{\Lambda}_\nu[\xi|s]\dot{\xi}^\nu(s)\delta(\xi(s) - x). \tag{6.7}$$

Given that it is this dual potential $\tilde{A}_\mu(x)$ which is coupled to the wave function $\psi(x)$ of the magnetic charge, it is clear then that the action (5.4) on Chart II is also invariant when the above transformations are coupled with the usual transformations for the Wong "charge":

$$K(\tau) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]K(\tau)[1 - i\tilde{g}\tilde{\Lambda}(x)], \tag{6.8}$$

and for the wave function:

$$\psi(x) \longrightarrow [1 + i\tilde{g}\tilde{\Lambda}(x)]\psi(x). \tag{6.9}$$

This last observation then completes our task.

7 Concluding Remarks

Compared with our earlier work [15] the present paper has gone further in yielding an actual dual symmetry which had previously eluded us and in giving simpler derivations of the old results. The basis for this improvement is the dual transform of (3.1) which allows one to switch at will from one formulation of the theory to its dual. In terms of this language, our previous treatment is only a half-way house where only part of the dual transform has been carried out. Thus, for example, the so-called dual potential $T_{\mu\nu}[\xi|s]$ of [15], which is essentially our $W_{\mu\nu}[\xi|s]$ here, has in the present treatment to undergo a further transform, namely (4.11) which is analogous to (3.1), in order to give the genuine dual potential $\tilde{A}_\mu(x)$. It is the realization of this step which eventually reveals the full dual symmetry.

Since the relationship between the two treatments can be worked out, given the relation (2.3) between the variables $E_\mu[\xi|s]$ used here and the Polyakov variables

$F_\mu[\xi|s]$ adopted in the earlier paper, no detailed comparison need be given[7]. There is one point, however, concerning the phase factor $\Phi_\xi(s_+,0)$ occuring only in [15] which puzzled us at first and deserves perhaps a mention. The factor $\Phi_\xi(s_+,0)$ appeared first in [15] in the defining constraint for the "magnetic" current:

$$G_{\mu\nu}[\xi|s] = -4\pi J_{\mu\nu}[\xi|s], \tag{7.1}$$

where for a classical point charge we had:

$$J_{\mu\nu}[\xi|s] = \tilde{g}\kappa[\xi|s]\epsilon_{\mu\nu\rho\sigma}\int d\tau \frac{dY^\rho(\tau)}{d\tau}\dot{\xi}^\sigma(s)\delta(\xi(s)-Y(\tau)), \tag{7.2}$$

with:

$$\kappa[\xi|s] = \Phi_\xi^{-1}(s_+,0)\mathcal{K}(\tau)\Phi_\xi(s_+,0), \tag{7.3}$$

and $\mathcal{K}(\tau)$ a local quantity, while for a Dirac point charge we had:

$$J_{\mu\nu}[\xi|s] = \tilde{g}\epsilon_{\mu\nu\rho\sigma}[\bar{\psi}(\xi(s))\omega(\xi(s))\gamma^\rho t^i\omega^{-1}(\xi(s))\psi(\xi(s))]\Phi_\xi^{-1}(s_+,0)t_i\Phi_\xi(s_+,0). \tag{7.4}$$

These expressions differ from (5.2) and (5.17) of this paper by the factor $\Phi_\xi(s_+,0)$ and its inverse, where we note that the argument is s_+ and not s as elsewhere in this paper.[8] That these factors should be there in (7.2) and (7.4) for consistency but not in (5.2) and (5.17) can be seen as follows. The loop space curvature $G_{\mu\nu}[\xi|s]$ as exhibited in (2.2) satisfies the Bianchi identity:

$$\epsilon^{\mu\nu\rho\sigma}\mathcal{D}_\rho(s)G_{\mu\nu}[\xi|s] = 0, \tag{7.5}$$

where $\mathcal{D}_\mu(s)$ denotes the "covariant loop derivative":

$$\mathcal{D}_\mu(s) = \delta_\mu(s) - ig[F_\mu[\xi|s], \quad]. \tag{7.6}$$

Hence, the current $J_{\mu\nu}[\xi|s]$ on the right-hand side of (7.1) must also satisfy this identity, which it does if it contains the factors $\Phi_\xi(s_+,0)$ and $\Phi_\xi^{-1}(s_+,0)$ as shown in (7.2) and (7.4), but will not do so without these factors. On the other hand, although in the equation (5.1) which is the equivalent to (7.1) in terms of $E_\mu[\xi|s]$, the current must also satisfy a similar identity (6.5), this involves only the ordinary loop derivative $\delta_\mu(s)$, and not the covariant loop derivative $\mathcal{D}_\mu(s)$. The expressions (5.2) and (5.17) have thus no need for the phase factors $\Phi_\xi(s_+,0)$ and $\Phi_\xi^{-1}(s_+,0)$.

[7] We note that, for convenience, we have used the same symbols in some cases to denote related but not identical quantities in the two papers, but this we think should not lead to any confusion.

[8] In [15, 20], we had actually written $\omega(s_+)$ instead of $\omega(s)$ as we do here to indicate that it was not affected by loop differentiation, but this is in fact unnecessary in view of the footnote of Section 3.

This difference between the "currents" in the two treatments means that the corresponding Lagrange multipliers, namely $L_{\mu\nu}[\xi|s]$ in the old and $W_{\mu\nu}[\xi|s]$ in the new, are also related by a conjugation with respect to $\Phi_\xi(s_+, 0)$, from which it follows that the dual potential $\tilde{A}_\mu(x)$ defined in [15], in spite of appearances, is in fact identical to that defined here in (4.11).

The above observation serves as a further example for the delicate handling often required in loop space operations, which we consider as a weakness of the whole loop space approach. Although we believe we have considerably improved our understanding in the present work, sufficiently in fact to clarify one or two subtle points such as that in the Wong equation noted in the footnote of Section 5 which we have not been able to make clear before, we still feel strongly the lack of a general calculus for handling complex loop space operations, the construction of which however is unfortunately beyond our present capability.

Apart from this reservation, we find the result of the present paper rather gratifying in that it seems to have answered the long-standing question whether there is a dual symmetry for Yang-Mills theory and gives even an explicit, though rather complicated, transformation between dual variables, which is being sought for in other duality contexts. For us in particular, it seems to have answered also a question that we have been asking on and off for some years concerning the dynamical properties of nonabelian monopoles. The answer to this turns out to be staggeringly simple, namely that monopole dynamics is the same as that described by the standard theory for Yang-Mills sources, only formulated in the dual fashion. In consequence, one need not enquire, at least at the classical field level so far studied, whether the charges one sees in nature are sources or monopoles unless both types exist, for otherwise there will be no way to distinguish them. This is a rather unexpected result in view of the fact that sources and monopoles are initially conceived as very different objects, the former being essentially algebraic and the latter topological, and that the dynamics is determined here via the Wu-Yang criterion by the topology in an entirely different fashion from the manner that interactions for sources are usually introduced.

Acknowledment

One of us (TST) thanks the Wingate Foundation for partial support, while another (JF) thanks the Particle Theory Group of the Rutherford Appleton Laboratory for hospitaility during her summer visit there when part of this work was done.

REFERENCES

[1] T.T. Wu and C.N. Yang, Phys. Rev. D **14**, 437 (1976).

[2] C. Montonen and D. Olive, Phys. Lett. **72B**, 117 (1977).

[3] M. Halpern, Nucl. Phys. **B139**, 477 (1978).

[4] S. Deser, J. Phys. A: Math. Gen. **15**, 1053 (1982).

[5] S. Strominger, Nucl. Phys. **B343**, 167 (1990); C.G. Callan, J.A. Harvey, and A. Strominger, *ibid* **B359**, 611 (1991); **B367**, 60 (1991).

[6] M.J. Duff and J.X. Lu, Nucl. Phys. **B354**, 129 (1991); **B357**, 534 (1991).

[7] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, Nucl. Phys. **B424**, 155 (1994).

[8] J.P. Gauntlett and J.A. Harvey, hepth-9407111.

[9] A. Sen, Nucl. Phys. **B404**, 109 (1993); Phys. Lett. B **303**, 22 (1993); Mod. Phys. Lett. A **8**, 2023 (1993); J.H. Schwarz and A. Sen, Nucl. Phys. **B411**, 35 (1994).

[10] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B431**, 484 (1994).

[11] K. Intriligator and N. Seiberg, Nucl. Phys. **B431**, 551 (1994); N. Seiberg, *ibid.* **B435**, 129 (1995); N. Seiberg, in *Particles, Strings, and Cosmology*, Proceedings of the Workshop, Syracuse, New York, 1994, edited by K.C. Wali (World Scientific, Singapore, 1995).

[12] C. Vafa and E. Witten, Nucl. Phys. **B431**, 3 (1994); E. Witten, Math. Res. Lett. **1**, 769 (1994).

[13] O. Aharony, Phys. Lett. **351B**, 220 (1995); L. Girardello, A. Giveon, M. Porrati, and A. Zaffaroni, Nucl. Phys. **B448**, 127 (1995), hep-th/9507064; A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen hep-th/9412200; J.A. Harvey, G. Moore and A. Strominger, hep-th/9501022; R.G. Leigh and M.J. Strassler, Nucl. Phys. **B447**, 95 (1995); E. Comay hep-th/9503234; D. Kutasov and A. Schwimmer, Phys. Lett. **354B**, 315 (1995); K. Intriligator and P. Pouliot, Phys. Lett. **353B**, 471 (1995); K. Intriligator, Nucl. Phys. **B448**, 187 (1995); C. Gomez and E. Lopez, Phys. Lett. **357B**, 558 (1995); E. Witten, hep-th/9505186; G.W. Gibbons and D.A. Rasheed, hep-th/9506035; K. Intriligator, R.G. Leigh and M.J. Strassler, hep-th/9506148; N. Mohammedi, hep-th/9507040; E. Alvarez, L. Alvarez-Gaumé and I. Bakas,

hep-th/9507112; Y. Lozano, hep-th/9508021; D. Olive, hep-th/9508091; C. Ford and I. Sachs, hep-th/9508140. The above is still only a partial list of the many articles which have recently appeared on the subject.

[14] C.H. Gu and C.N. Yang, Sci. Sin. **28** (1975) 483; C.N. Yang, Phys. Rev. **D1** (1970) 2360.

[15] Chan Hong-Mo, J. Faridani and Tsou Sheung Tsun, Phys. Rev. **D52** (1995).

[16] A.M. Polyakov, Nucl. Phys. **164**, 171 (1980).

[17] Chan Hong-Mo, P. Scharbach and Tsou Sheung Tsun, Ann. Phys. (N.Y.) **167** (1986) 454; *ibid.* **166** (1986) 396; Chan Hong-Mo and Tsou Sheung Tsun, Act. Phys. Pol. **B17**, 259 (1986).

[18] Chan Hong-Mo and Tsou Sheung Tsun, *Some Elementary Gauge Theory Concepts* (World Scientific, 1993).

[19] S.K. Wong, Nuovo Cimento **A65**, 689 (1970).

[20] Chan Hong-Mo, J. Faridani and Tsou Sheung Tsun, Phys. Rev. **D51** (1995) 7040.

[21] D. Freedman and P.K. Townsend, Nucl. Phys. **B177**, 282 (1981).

[22] Chan Hong-Mo, J. Faridani and Tsou Sheung Tsun, Rutherford Appleton Laboratory preprint RAL-93-080 (1993) (unpublished).

[23] K. Hayashi, Phys. Lett. **44B**, 497 (1973); M. Kalb and P. Ramond, Phys. Rev. D **9**, 2273 (1974); E. Cremmer and J. Scherk, Nucl. Phys. **B72**, 117 (1974); Y. Nambu, Phys. Rep. **23**, 250 (1976); M. Henneaux and C. Teitelboim, Phys. Lett. **206B**, 650 (1988).

Chart I

Pure Yang-Mills Theory

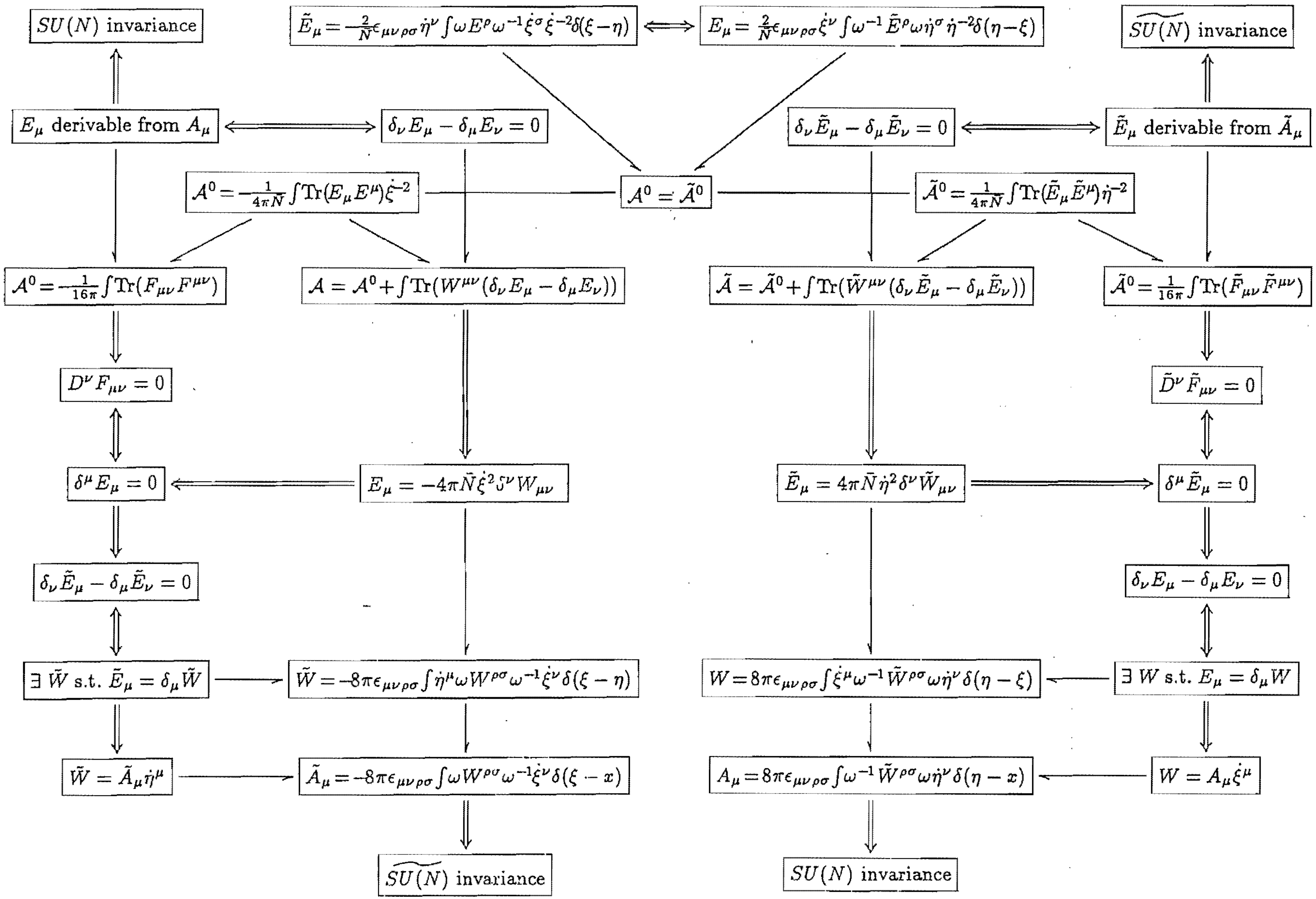

Chart II

Yang-Mills Theory with Charges

COLOUR MAGNETIC CHARGE

COLOUR ELECTRIC CHARGE

Potential $\mathcal{A}$ patched

$\delta_{[\nu} E_{\mu]} = -4\pi J_{\mu\nu}$

$\tilde{E}_\mu = -\frac{2}{\bar{N}} \epsilon_{\mu\nu\rho\sigma} \dot{\eta}^\nu \int \omega E^\rho \omega^{-1} \dot{\xi}^\sigma \dot{\xi}^{-2} \delta(\xi - \eta)$

$\delta_{[\nu} \tilde{E}_{\mu]} = -4\pi \tilde{J}_{\mu\nu}$

Potential $\tilde{\mathcal{A}}$ patched

$J_{\mu\nu} = \tilde{g} \epsilon_{\mu\nu\rho\sigma} \int K \frac{dY^\rho}{d\tau} \dot{\xi}^\sigma \delta(\xi - Y)$

$J_{\mu\nu} = \tilde{g} \epsilon_{\mu\nu\rho\sigma} (\bar{\psi} \omega \gamma^\rho t_i \omega^{-1} \psi) t^i \dot{\xi}^\sigma$

$\tilde{J}_{\mu\nu} = g \epsilon_{\mu\nu\rho\sigma} (\bar{\psi} \omega^{-1} \gamma^\rho t_i \omega \psi) t^i \dot{\eta}^\sigma$

$\tilde{J}_{\mu\nu} = g \epsilon_{\mu\nu\rho\sigma} \int I \frac{dY^\rho}{d\tau} \dot{\eta}^\sigma \delta(\eta - Y)$

$\mathcal{A}_F^0 = -\frac{1}{4\pi \bar{N}} \int \mathrm{Tr}(E_\mu E^\mu) \dot{\xi}^{-2}$

$\mathcal{A}_F^0 = \tilde{\mathcal{A}}_F^0$

$\tilde{\mathcal{A}}_F^0 = \frac{1}{4\pi \bar{N}} \int \mathrm{Tr}(\tilde{E}_\mu \tilde{E}^\mu) \dot{\eta}^{-2}$

$\mathcal{A}^0 = \mathcal{A}_F^0 - m \int d\tau$

$\mathcal{A}^0 = \mathcal{A}_F^0 + \int \bar{\psi}(i \not\partial - m)\psi$

$\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}_F^0 + \int \bar{\psi}(i \not\partial - m)\psi$

$\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}_F^0 - m \int d\tau$

$\mathcal{A} = \mathcal{A}^0 + \int \mathrm{Tr} W^{\mu\nu} (\delta_{[\nu} E_{\mu]} + 4\pi J_{\mu\nu})$

$\mathcal{A} = \mathcal{A}^0 + \int \mathrm{Tr} W^{\mu\nu} (\delta_{[\nu} E_{\mu]} + 4\pi J_{\mu\nu})$

$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^0 + \int \mathrm{Tr} \tilde{W}^{\mu\nu} (\delta_{[\nu} \tilde{E}_{\mu]} + 4\pi \tilde{J}_{\mu\nu})$

$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^0 + \int \mathrm{Tr} \tilde{W}^{\mu\nu} (\delta_{[\nu} \tilde{E}_{\mu]} + 4\pi \tilde{J}_{\mu\nu})$

$E_\mu = -4\pi \bar{N} \dot{\xi}^2 \delta^\nu W_{\mu\nu}$

$\tilde{E}_\mu = 4\pi \bar{N} \dot{\eta}^2 \delta^\nu \tilde{W}_{\mu\nu}$

$\delta^\mu E_\mu = 0$

$\delta^\mu \tilde{E}_\mu = 0$

$m \frac{d^2 Y^\mu}{d\tau^2} = -\tilde{g} \mathrm{Tr}(K \underline{\tilde{F}}^{\mu\nu}) \frac{dY_\nu}{d\tau}$

$(i \not\partial - m)\psi = -\tilde{g} \tilde{\not{A}} \psi$

$\tilde{A}_\mu = -8\pi \epsilon_{\mu\nu\rho\sigma} \int \omega W^{\rho\sigma} \omega^{-1} \dot{\xi}^\nu \delta(\xi - x)$

$(i \not\partial - m)\psi = -g \not{A} \psi$

$A_\mu = 8\pi \epsilon_{\mu\nu\rho\sigma} \int \omega^{-1} \tilde{W}^{\rho\sigma} \omega \dot{\eta}^\nu \delta(\eta - x)$

$m \frac{d^2 Y^\mu}{d\tau^2} = -g \mathrm{Tr}(I \underline{F}^{\mu\nu}) \frac{dY_\nu}{d\tau}$

$\tilde{E}_\mu$ derivable from potential

$\widetilde{SU(N)}$ invariance

$SU(N)$ invariance

E_μ derivable from potential