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Rutherford Appleton Laboratory

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Proceedings of the School for Young High Energy Physicists

F E Close I G Halliday T Jones and C Maxwell

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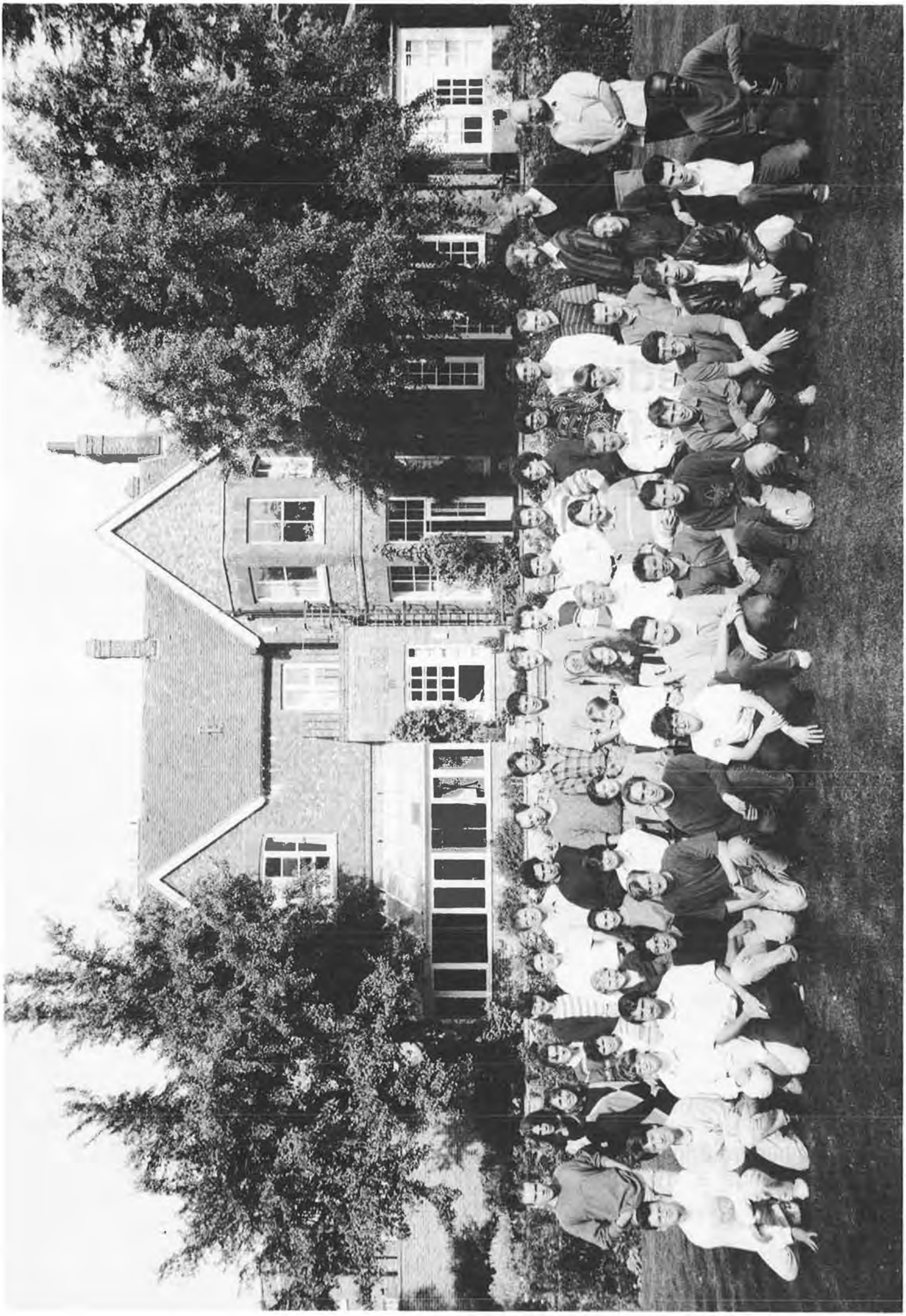

HEP SUMMER SCHOOL FOR YOUNG HIGH ENERGY PHYSICISTS

RUTHERFORD APPLETON LABORATORY/THE COSENER'S HOUSE, ABINGDON:

6 - 19 SEPTEMBER 1992

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PREFACE

The 1992 School for young High Energy Physicists took place from September 6 - 19, at the Cosener's House, Abingdon, and was attended by virtually all UK 1st year graduate students in the field of Experimental Particle Physics. It was organised and funded by the Rutherford Appleton Laboratory, whose assistance is very gratefully acknowledged.

Published here are the lectures that were given in the mornings. These were supplemented and reinforced by the work in the afternoons, which were devoted to problems and tutorials. At the end of the intensive two week course the students emerged exhausted, but with a thorough grounding in the Standard Model of Elementary Particle Physics, on which most of them are performing their experimental work.

In the evenings the students all gave short talks on the research they were doing, and we also had seminars from Dr John Hassard, on Nuclear proliferation and verification, from Dr John Mulvey, on some aspects of the history of HEP in the UK and the scientific policy making that went with it, and from George Kalmus of RAL. This last was a warm-up for a half-day visit to the laboratory.

The continued success of this school is due to the dedicated enthusiasm of the lecturers and tutors, to the staff of Cosener's House, and to the organisers at RAL responsible for this school, especially Mrs Ann Roberts.

Dr R J Barlow

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QUANTUM FIELD THEORY: CANONICAL AND PATH INTEGRAL APPROACHES

By Prof I G Halliday
University of Wales, Swansea

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1992

Quantum Field Theory:
Canonical and Path Integral Approaches.

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Introduction.

This short lecture course is aimed at connecting the later courses to your undergraduate knowledge of Quantum Mechanics, Maxwell's equations, etc. I also have to provide a language and a series of results for the later lectures,- a tall order. So, I would like to show you how to quantise an arbitrary theory, in particular why gauge theories are hard. On the other hand I would like to demonstrate Salam's remark that he has been surprised at how little he has had to change his ideas! So the highlight of the course will be a "proof" of Feynman rules for the perturbative evaluation of field theories such as scalar fields, Yang-Mills and the Higgs effect. I will make a point of starting far enough back to be comprehensible to everybody - I expect riots otherwise. A constant theme will be the way that different views make different aspects transparent ! The prime example being the way Hamiltonian methods make Quantum Mechanics easy but hide symmetries. Lagrangian methods make symmetries easy but lose direct physical contact with your old view of Quantum Mechanics as a theory of operators, states, eigenvectores, etc..

Synopsis.

0) The examples done before coming to Coseners
   - (1) Harmonic Oscillator
   - (2) Gaussian Integrals
   - (3) Pictures in Quantum Mechanics
   - (4) Newton implies Lagrange
   - (5) Dirac δ function

1) Classical Mechanics - Lagrange vs Hamilton

2) Quantum Mechanics
   - (1) Schrödinger vs Heisenberg vs Dirac pictures
   - (2) Hamilton vs Lagrange, Dirac and Feynman
   - (3) Heisenberg Harmonic Oscillator

3) Free Boson
   - (1) Classical
   - (2) Quantum

4) Interacting Boson
   - (1) Feynman diagrams by operators
   - (2) Feynman diagrams by Functional integrals

5) Groups and Algebras
   - (1) Definitions and Examples

6) Gauge Theories
   - (1) ClassicalMaxwell Theory -Lorentz invariance
   - (2) Lagrangian formalism
   - (3) U(1) Covariant derivative
   - (4) Non-Abelian Gauge Theories

(5) Feynman Rules
(6) Gauge Fixing

7) Higgs

(1) U(1)
(2) $SU(2) \times U(1)$ The Salam Weinberg model

Acknowledgements

It is a great pleasure to thank Roger Barlow for organising and running the school so well. His gentle humour contributes greatly to the atmosphere. The students, of course, made the school. I was impressed by their thirst for knowledge, etc., but not by their tennis. Finally I would thank that unsung heroine Ann Roberts who is well versed, having been given much opportunity to practise, in the art of applying pressure to recalcitrant authors.

Chapter 0

The prerequisites for the course

The purpose of this section of the notes is to provide you with a summary of what I expect you to know already. Everything here will be used in one form or another in the course. You must do all the examples before you arrive at Coseners. I stress that these examples and exercises are at the heart of quantum field theory either in operator form or the more trendy Functional or Path integral method. You must do all of these problems and two finger exercises. If not, you will end up having to understand both the mathematics and the field theory ideas simultaneously. With the exercises behind you the maths should be second nature. The mathematics at the heart of quantum field theory is constructed out of many oscillators. So the first aim is to be absolutely sure that you understand one oscillator by itself. Hence the way your undergraduate courses pounded away at this problem, probably without explaining why. To jump the gun slightly, the basic reason is that we have a photon, pion, Higgs of energy $E(\underline{k}) = \hbar\omega(\underline{k})$ for momentum $\underline{k}$ then n such particles have energy $n\hbar\omega(\underline{k})$. The energy levels of a Harmonic Oscillator of frequency ω are $\hbar\omega(n+\frac{1}{2})$. Apart from the constant $\frac{1}{2}\hbar\omega$ these agree !

In the lectures I will use the Path integral formalism extensively to construct Feynman diagrams, fix gauges,... This trickery leans totally on a knowledge of Gaussian integrals. So again I include a revision section on these integrals and the two standard tricks of completing the square and introducing new parameters.

Many of you will have already have studied the concept of pictures in Quantum Mechanics; in particular the Schrödinger and Heisenberg pictures. We will use a new picture due to Dirac to construct perturbation theory. So here I revise the standard material.

Finally I will use extensively the ideas of Lagrangian and Hamiltonian mechanics. For completeness I give the standard derivation of Lagrange's equations from Newton's. In Quantum Field Theory Lagrangians play a major role because of the way they make the symmetries of the problem manifest. In Quantum Mechanics we usually start from the Hamiltonian which often hides the symmetries. We will thus develop a new formalism, the path integral formalism, which is the Lagrangian variant of Quantum Mechanics. This will make many calculations much easier and slicker. Here I remind you of the classical connection between Lagrangians and Hamiltonians.

We will use many properties of Dirac δ-functions. I remind you of the definitions, proofs and the results we will use.

0.1) Harmonic Oscillator

So let us redo the Harmonic Oscillator in Quantum Mechanics using an operator formalism. We use the most common picture in undergraduate texts - the Schrodinger picture - where operators are independent of time.

$$\hat{p} = -i\hbar\frac{\partial}{\partial q}$$

$$\hat{q} = q$$

The notation may look confusing, but here and in the lecture, I will try to avoid a standard confusion. The above equations are usually written

$$
\begin{aligned}
\hat{p} &= -i\hbar\frac{\partial}{\partial x} \\
\hat{x} &= x
\end{aligned}
\tag{0.1.1}
$$

where $\hat{x}$ is the dynamical variable corresponding to the position of a point particle. Later in Field theory we will have dynamical variables $\hat{\Phi}(\underline{x},t)$ where $\underline{x}$ is not a dynamical variable but merely a label. i.e.$\hat{\Phi}(\underline{x},t)$ is an operator at some fixed point $\underline{x}$ of space and $\underline{x}$ undergoes no dynamics, although $\hat{\Phi}(\underline{x},t)$ certainly does. To avoid this confusion the dynamical variables for point particles are usually rewritten as momentum p, position q. The only property of (0.1.1) I will use is the commutation relation

$$
[\hat{q},\hat{p}] = i\hbar \tag{0.1.2}
$$

and from now on we choose units such that $\hbar = 1$. In Quantum Mechanics the starting point is usually the Hamiltonian or energy operator

$$
\hat{H}(\hat{p},\hat{q}) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 \tag{0.1.3}
$$

written as a function of the position and momentum. We will solve for the energy eigenstates and normalised wave functions of (0.1.3) **only using (0.1.2) and not (0.1.1).** This will be important later. Define

$$
\begin{aligned}
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}(\hat{q}\sqrt{m\omega} - i\frac{\hat{p}}{\sqrt{m\omega}}) \\
\hat{a} &= \frac{1}{\sqrt{2}}(\hat{q}\sqrt{m\omega} + i\frac{\hat{p}}{\sqrt{m\omega}})
\end{aligned}
\tag{0.1.4}
$$

Since $\hat{q}^\dagger = \hat{q}$ and $\hat{p}^\dagger = \hat{p}$ it is clear that $\hat{a},\hat{a}^\dagger$ are, in fact, Hermitian conjugates of one another so the notation makes sense. Now compute

$$
[\hat{a},\hat{a}^\dagger] = \frac{1}{2}[\hat{q},-i\hat{p}] + \frac{1}{2}[i\hat{p},\hat{q}] = 1 \tag{0.1.5}
$$

by (0.1.2) Moreover

$$
\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{2}(\frac{\hat{p}^2}{m\omega} + \hat{q}^2 m\omega + i[\hat{q},\hat{p}]) \\
&= \frac{1}{\omega}(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 - \frac{1}{2}\omega)
\end{aligned}
\tag{0.1.6}
$$

where the $\hbar$ can be resurrected by dimensional analysis if required. Now let us compute the eigen values of $\hat{H}$ suppose we have an eigenstate $|\alpha\rangle$ such that

$$
\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha|\alpha\rangle
$$

Then clearly

$$\hat{H}|\alpha\rangle = \hbar\omega(\alpha + \frac{1}{2})|\alpha\rangle$$

so an eigenstate of $\hat{a}^\dagger\hat{a}$ is an eigenstate of $\hat{H}$ and vice versa. Now I claim that $\hat{a}^\dagger|\alpha\rangle$ is also an eigenstate of $\hat{H}$:

$$\begin{aligned}\hat{a}^\dagger\hat{a}\{\hat{a}^\dagger|\alpha\rangle\} &= \hat{a}^\dagger\{\hat{a}\hat{a}^\dagger\}|\alpha\rangle \\ &= \hat{a}^\dagger\{\hat{a}^\dagger\hat{a}+1\}|\alpha\rangle \\ &= \hat{a}^\dagger\{\alpha+1\}|\alpha\rangle; \qquad by(0.6) \\ &= (\alpha+1)\{\hat{a}^\dagger|\alpha\rangle\}\end{aligned} \tag{0.1.7}$$

because $(\alpha+1)$ is a number. Similarly

$$\hat{a}^\dagger\hat{a}\{\hat{a}|\alpha\rangle\} = (\alpha-1)\{\hat{a}|\alpha\rangle\} \tag{0.1.8}$$

Thus given any eigenstate with eigenvalue α we can easily construct eigenstates with eigenvalues $\alpha, \alpha+1, \alpha+2, \cdots, \alpha-1, \alpha-2, \cdots$, by multiple applications of $\hat{a}^\dagger$ and $\hat{a}$. Do these sequences ever stop? To fix the limits, consider

$$\begin{aligned}\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle &= \int dq\Psi^*_\alpha(q)(\hat{a}^\dagger\hat{a})\Psi_\alpha(q) \\ &= \int dq\big(\hat{a}\Psi_\alpha(q)\big)^*\big(\hat{a}\Psi_\alpha(q)\big) \\ &= \int dq\Phi^*\Phi \geq 0\end{aligned} \tag{0.1.9}$$

where we have used the fact that the Hamiltonian conjugate of $\hat{a}^\dagger$ is $\hat{a}$ and have written $\Phi(q) = \hat{a}\Psi_\alpha(q)$. Note that in (0.1.9) we get zero if and only if $\Phi(q) = 0$. On the other hand, if $|\alpha\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$, the left hand side is

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle \tag{0.1.10}$$

Now above we constructed eigenstates $|\alpha\rangle, |\alpha-1\rangle, |\alpha-2\rangle\cdots$ and the above can be applied to any of them. So in (0.1.9) we have, using (0.1.10) that

$$(\alpha-n)\langle\alpha-n|\alpha-n\rangle \geq 0 \qquad ; n > 0 \tag{0.1.11}$$

Since $\langle\alpha-n|\alpha-n\rangle \geq 0$ we have $(\alpha-n) \geq 0$ for all n. Clearly an absurdity for sufficiently larger n. So somewhere above there is a mistake. Can you see it? Don't turn the page and cheat. The mistake is a standard mistake that lies at the heart of angular momentum theory, the theory of Lie algebras, Kac-Moody!

The way out is that we proved

$$\hat{a}^\dagger \hat{a}(\hat{a}^\dagger|\alpha\rangle) = (\alpha+1)(\hat{a}^\dagger|\alpha\rangle)$$

which seems to prove $\hat{a}^\dagger|\alpha\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$. **But** remember for any operator $\hat{O}$ the eigenvalue equation

$$\hat{O}\Phi(q) = \alpha\Phi(q)$$

always has the trivial solution $\Phi(q) = 0$. So we must check that $\hat{a}^\dagger|\alpha\rangle \neq 0$. Since we are getting a contradiction, in fact for some n we must always get, for the first time,

$$(\hat{a})^n|\alpha\rangle = 0$$

for some positive integer n. Then consider the state $|\beta\rangle = (\hat{a})^{n-1}|\alpha\rangle \neq 0$ such that $\hat{a}|\beta\rangle = 0$ and hence $\hat{a}^\dagger\hat{a}|\beta\rangle = 0|\beta\rangle$. Notice that the $(\hat{a}^\dagger)^n|\alpha\rangle$ eigenstates give us no problem; the analogue of (0.1.11) is

$$(\alpha+n)\langle\alpha+n|\alpha+n\rangle \geq 0 \qquad n \geq 0$$

which gives no sign problems. So we now construct everything on the basis of the lowest state $|\beta\rangle$ which must satisfy $\hat{a}|\beta\rangle = 0$. Then

$$\begin{aligned}\hat{H}(\hat{a}^\dagger)^n|\beta\rangle &= \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})(\hat{a}^\dagger)^n)|\beta\rangle \\ &= \omega(n+\frac{1}{2})(\hat{a}^\dagger)^n|\beta\rangle)\end{aligned}$$

So, as promised, the eigenstates of $\hat{H}$ are $\omega(n+\frac{1}{2})$. If you want to construct the wave functions, the $|\beta\rangle$ equation

$$\hat{a}|\beta\rangle = 0 \Rightarrow (\sqrt{m\omega}q + \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial q})\Psi(q) = 0$$

or

$$\Psi(q) = e^{\frac{-q^2 m\omega}{2}}$$

Higher eigenstates can be computed by applying

$$\hat{a}^\dagger = (\sqrt{m\omega}q - \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial q})$$

to $\Psi(q)$.

An interesting exercise is to check the normalisation of the states. If we call the lowest states $|0\rangle, |1\rangle, |2\rangle$, of energy e-value $\frac{1}{2}\hbar\omega, \frac{3}{2}\hbar\omega, \frac{5}{2}\hbar\omega$ then the normalised state is

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|O\rangle$$

The proof of this is straightforward using the commutation relatives and induction. Take the normalised $|0\rangle$ state $\langle 0|0\rangle = 1$. Then

$$\begin{aligned}\langle 1|1\rangle &= \langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\ &= \langle 0|(\hat{a}^\dagger\hat{a} + 1)|0\rangle \\ &= \langle 0|0\rangle = 1\end{aligned}$$

because $\hat{a}|0\rangle = 0$. Then assume $\langle n|n\rangle = 1$ and compute

$$\begin{aligned}\langle n+1|n+1\rangle &= \frac{1}{(n+1)}\langle n|\hat{a}\hat{a}^\dagger|n\rangle \\ &= \frac{1}{(n+1)}\langle n|(\hat{a}^\dagger\hat{a} + 1)|n\rangle \\ &= \frac{1}{(n+1)}\langle n|(n+1)|n\rangle \\ &= \langle n|n\rangle = 1\end{aligned}$$

This innocent looking $\sqrt{n!}$ is at the heart of all of laser physics. In that context $\hat{a}^\dagger, \hat{a}$ correspond to operators creating and annihilating a single photon of energy ω. Thus any atomic physics process where atoms decay and give off a photon must have an interaction proportional to $\hat{a}^\dagger$; a process where photons are absorbed must have an interaction proportional to $\hat{a}$. So consider two matrix elements

$$\begin{aligned}\langle n+1|\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\langle n+1|n+1\rangle \\ &= \sqrt{n+1} \\ \langle n-1|\hat{a}|n\rangle &= \sqrt{n}\langle n|n\rangle = \sqrt{n}\end{aligned}$$

The first, when squared to give the probability of the transition, gives a factor of (n+1) i.e. the atom decays even if n = 0, increasing the number of photons from 0 to 1. **But**, if there already exist 10^{15} photons in that mode, the probability increases by 10^{15} i.e. stimulated emission and possibly lasing. So later we will see that a free scalar field corresponds to

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\hat{a}^\dagger(\underline{k})\hat{a}(\underline{k})$$

i.e. the Hamiltonian is a sum of Harmonic Oscillators, one for each momentum $\underline{k}$. The odd normalisation is due to us having to enforce relativistic invariance.

0.2) Gaussian Integrals

This is the mathematical trickery necessary to establish Feynman diagram expansions in field theory without driving yourself crazy commuting operators past each other. The method was one of Feynman's favourite ways of doing integrals. We build up slowly from simple 1 dimensional integrals to integrals over infinite numbers of variables.

a) Compute:

$$I(\alpha) = \int_{-\infty}^{+\infty} dx e^{-\alpha x^2}$$

Trick

$$I^2 = \int_{-\infty}^{+\infty} dx e^{-\alpha x^2} . \int_{-\infty}^{+\infty} dy e^{-\alpha y^2} = \int_{-\infty}^{+\infty} dx dy e^{-\alpha(x^2+y^2)}$$

Now change to polar coordinates r, θ.

$$I^2 = \int_0^{+\infty} r dr \int_0^{2\pi} d\theta e^{-\alpha r^2} = 2\pi \frac{1}{2} \int_0^{+\infty} dr^2 e^{-\alpha r^2} = \frac{\pi}{\alpha} e^{-\alpha r^2} \Big|_0^{\infty} = \frac{\pi}{\alpha}$$

so that

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

b) Compute:

$$I(\alpha, \beta) = \int_{-\infty}^{+\infty} dx e^{-\alpha x^2 + \beta x} = \int_{-\infty}^{+\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2 + \alpha \frac{\beta^2}{4\alpha^2}}$$

Change variables to $y = x - \frac{\beta}{2\alpha}$, then

$$I(\alpha, \beta) = I(\alpha) e^{\frac{\beta^2}{4\alpha}} = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

c) Now we can compute any integral

$$I_m(\alpha) = \int_{-\infty}^{+\infty} x^m e^{-\alpha x^2} dx = \left\{ \left(\frac{\partial}{\partial \beta} \right)^m I(\alpha, \beta) \right\} \Big|_{\beta=0}$$

since differentiating with respect to β inside the integral just brings down extra factors of x.

$$I_1 = \frac{\partial}{\partial \beta} \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}} \Big|_{\beta=0} = 0$$

This is trivially correct as the integrand is odd.

$$\begin{aligned} I_2 &= \frac{\partial^2}{\partial \beta^2} \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}} \Big|_{\beta=0} \\ &= \sqrt{\frac{\pi}{\alpha}} \frac{\partial}{\partial \beta} \left\{ \frac{\beta}{2\alpha} e^{\frac{\beta^2}{4\alpha}} \right\} \Big|_{\beta=0} \\ &= \sqrt{\frac{\pi}{\alpha}} \left\{ \frac{1}{2\alpha} e^{\frac{\beta^2}{4\alpha}} + \frac{\beta^2}{4\alpha^2} e^{\frac{\beta^2}{4\alpha}} \right\} \Big|_{\beta=0} \\ &= \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \end{aligned} \tag{0.2.1}$$

This example deserves close scrutiny. What we are doing is swapping powers of x in the integrand for derivitives with respect to a paramater in the exponent. This is absolutely characteristic of Functional Integrals. Moreover in equation (0.2.1) above, we very characteristically get two terms. The first comes from differentiating the term brought down from the derivative of the exponent leaving no β term in the product. The second comes from differentiating the exponent twice. This leaves β factors which vanish. This trivial remark is at the heart of the construction of the Feynman diagram expansion.

d) *The Most Important Integral in the world.*

$$L = \int \prod_1^n d\phi_i e^{-\sum_{i,j} \phi_i K_{ij} \phi_j + \sum_k J_k \phi_k} \tag{0.2.2}$$

We have switched to integration variables called ϕ_i. Later these will be the values of scalar fields and i will be a label picking space-time points. If you were a lattice person the i's would label sites in the lattice. The K_{ij} form an $n \times n$ symmetric matrix and the J_k a column vector with n entries. Amazingly the above integral can be done for all K_{ij}. The answer is

$$L = \frac{\pi^{\frac{n}{2}}}{\sqrt{det(K)}} e^{-\sum_{i,j} J_i (K^{-1})_{ij} J_j}$$

This result is the simplest and most elegant way of devoloping Feynman diagrams. The $(K^{-1})_{ij}$ will be the Feynman propagator from space point i to space point j. To prove this formula we reduce it to n copies of $I(\alpha, \beta)$. Since K_{ij} is a real symmetric matrix we can diagonalise it by an orthogonal matrix U such that

$$U^{-1} K U = K'$$

where

$$K' = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix}$$

Here $U^T = U^{-1}$, $detU = 1$ Now define

$$\phi_i = \sum_j U_{ij} \phi'_j \qquad or \qquad \phi = U\phi'$$

Then

$$L = \int \prod d\phi'_i e^{-\sum_j \phi'_j \lambda_j \phi'_j + (\sum_{k,j} J_k U_{kj}) \phi'_j}$$

since the Jacobian from ϕ to ϕ' is det U = 1.

Thus we have n copies of $I(\alpha, \beta)$. These give factors of $\sqrt{\pi}$ n-times and $\sqrt{\lambda_i}$. Now $detK = detK' = \prod \lambda_i$ so we get the correct $detK$ factor. Finally in the exponent we get

$$J^T.U.K'^{-1}.U^T.J = J^T.K^{-1}.J$$

as is easily checked. The announced answer.

0.3) Pictures in Quantum Mechanics

Normally in elementary Quantum Mechanics we are given the operators $\hat{p}$ and $\hat{q}$ as $-i\hbar\frac{\partial}{\partial x}$ and x operating on wave functions $\Phi(x)$. The first hint that this may not be the most general way of thinking about things is the realisation that we could equally well talk about wave functions $\Psi(p)$. This led Dirac to introduce the idea of pictures.

Schrödinger Picture

Normal Quantum Mechanics has operators that are independent of time, such as $\hat{p}$ and $\hat{q}$ as above. Wave functions however depend on time through the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}(\hat{p},\hat{q})\Psi(x,t) \tag{0.3.1}$$

The formal solution of this is easily written out setting $\hbar = 1$

$$\Psi(x,t) = e^{-i\hat{H}t}\Psi(x,0) \tag{0.3.2}$$

If this wave function happens to be an eigenstate of $\hat{H}$ then the exponent becomes a simple numerical phase e^{-iEt}. So usually the easiest way to compute the time dependence of any state is to expand in energy eigenstates and then give each term in the expansion its appropriate phase.

Heisenberg Picture

Here we switch all the time dependence to the operators leaving time independent wave functions. Remember Heisenberg's version of Quantum Mechanics was called matrix mechanics i.e. all the dynamics was in the time dependence of matrices (= operators) not wave functions.

Define, for any Schrödinger operator $\hat{O}_S$ the equivalent Heisenberg operator $\hat{O}_H$ by

$$\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} \tag{0.3.3}$$

and the Heisenberg wave function

$$\Psi_H = e^{i\hat{H}t}\Psi_S(t) \tag{0.3.4}$$

where I have suppressed the coordinate dependence of the wave functions and only explicitly shown the all important time dependence.

The crucial result is that all physical quantities such as probabilities, matrix elements etc. are unchanged by the switch of pictures. For example let us calculate the average value of an operator $\hat{O}$ in a state Ψ in each picture

$$\begin{aligned} Average &= \int \Psi_H^* \hat{O}_H(t)\Psi_H dq \\ &= \int \left(e^{i\hat{H}t}\Psi_S\right)^* \left(e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}\right)\left(e^{i\hat{H}t}\Psi_S\right) dq \\ &= \int \Psi_S^*(t)\hat{O}_S\Psi_S(t) dq \end{aligned}$$

The Heisenberg wave functions are now time independent. The factor in eqn (0.3.4) was clearly chosen to cancel the explicit time dependence coming from the solution (0.3.2) of the Schrödinger equation.

The dynamics is now all hiding in the time dependence of the operators. So we need the equation of motion of the Heisenberg operators. Since the Schrödinger operators are time independent this is just a matter of differentiating the definition of the Heisenberg operators.

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{O}_H(t) &= i\frac{\partial}{\partial t}\{e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}\} \\ &= -\hat{H}\hat{O}_H + \hat{O}_H\hat{H} \\ &= [\hat{O}_H, \hat{H}] \end{aligned}$$

It is also easy to check that the commutation relations of any two operators are usually unchanged in the two pictures. Thus

$$[\hat{q}, \hat{p}] = i$$

is true whether in Heisenberg or Schrödinger pictures. In particular in later lectures we will use the fact that the $\hat{a}^\dagger$ and $\hat{a}$ commutation relations are unchanged in the two pictures.

0.4) Classical Mechanics: Newton implies Lagrange

In order to gain an understanding of what the theoretical physicists are up to, we need to go back and quickly understand the developments of Newton's equations due to Lagrange. More particularly to see what the point was! Newton insists on inertial coordinates. Lagrange says any will do!

Suppose we have a system of masses with coordinates $\underline{x}_i \quad i = 1, \cdots N$. We parametrise these by coordinates $q_1, q_2, \cdots, q_n$, so that

$$\underline{x}_i = \underline{f}_i(q_i, \cdots q_n, t)$$

Now Newton's equations say

$$m_i\ddot{\underline{x}}_i = \underline{F}_i$$

where the $\underline{F}_i$ is the force on the i'th mass. Dot each side with $\frac{\partial \underline{f}_i}{\partial q_r}$ and sum over i.

$$\sum_i m_i \frac{\partial \underline{f}_i}{\partial q_r}.\ddot{\underline{x}}_i = \sum_i \frac{\partial \underline{f}_i}{\partial q_r}.\underline{F}_i$$

. For ease we use the Einstein convention that repeated indices are summed, so that $\sum_r q_r q_r \equiv q_r q_r$. But

$$\begin{aligned} \frac{\partial \dot{\underline{x}}_i}{\partial \dot{q}_r} &= \frac{\partial}{\partial \dot{q}_r}\left[\frac{\partial f_i}{\partial q_l}\dot{q}_l + \frac{\partial f_i}{\partial t}\right] \\ &= \frac{\partial f_i}{\partial q_r} \\ &= \frac{\partial x_i}{\partial q_r} \end{aligned}$$

So

$$\begin{aligned}
\ddot{\underline{x}}_i.\frac{\partial \underline{f}_i}{\partial q_r} &= \ddot{\underline{x}}_i.\frac{\partial \dot{\underline{x}}_i}{\partial \dot{q}_r} \\
&= \frac{d}{dt}\left[\dot{\underline{x}}_i.\frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right] - \dot{\underline{x}}_i.\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right) \\
&= \frac{d}{dt}\left[\dot{\underline{x}}_i.\frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right] - \dot{\underline{x}}_i.\left\{\frac{\partial^2 \underline{x}_i}{\partial q_l \partial q_r}\dot{q}_l + \frac{\partial^2 \underline{x}_i}{\partial q_r \partial t}\right\} \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\frac{1}{2}\dot{\underline{x}}_i{}^2\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{\underline{x}}_i{}^2\right)
\end{aligned}$$

Write $T = \frac{1}{2}\sum_i m_i \dot{\underline{x}}_i^2$. Therefore

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_r}) - \frac{\partial}{\partial q_r}T = \sum_i \frac{\partial \underline{f}_i}{\partial q_r}.\underline{F}_i$$

Now

$$\sum_i \frac{\partial \underline{f}_i}{\partial q_r}.\underline{F}_i \delta q_r$$

is the work done in a δq_r shift for the other q's fixed, since

$$\delta \underline{x}_i = \frac{\partial \underline{f}_i}{\partial q_r}.\delta q_r$$

Several remarks are now in order

.

a) Many forces never do any work e.g. the tension in the string of a pendulum, reactions at fixed points. They never show in Lagrange's equations, thus usually simplifying the problem greatly.

b) If the forces are conservative then the work done can be written directly for $V(q_i \cdots q_n)$

$$-\frac{\partial V}{\partial q_r}.\delta q_r$$

So define L = T - V and Newton reduces to

Lagrange:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_r}\right] - \frac{\partial L}{\partial q_r} = 0.$$

These equations are true for any coordinates you like, such that fixing $q_1 \cdots q_n$ fixes the $\underline{x}_i$. These coordinates can be accelerating, rotating, whatever. Lagrange takes care of it all.

Consider a simple pendulum for small oscillations. For θ the angle away from the vertical, $l =$ length,

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}ml^2\dot{\theta}^2 - mg(1-\cos\theta)l \\ &\approx \frac{1}{2}ml^2\dot{\theta}^2 - mg\theta^2\frac{l}{2} \qquad for \qquad \theta \ll 1 \end{aligned}$$

Lagrange:

$$\begin{aligned} \frac{\partial}{\partial t}\left[ml^2\dot{\theta}\right] + mgl\theta = 0 \\ \ddot{\theta} + \frac{g}{l}\theta = 0 \end{aligned}$$

Simple harmonic motion - solutions

$$\theta = A\cos(\omega t + \delta); \qquad \omega = \sqrt{\frac{g}{l}}$$

Notice, unlike the Newtonian analysis, we don't need to introduce the tension in the string, but we also learn nothing about it.

Summary

Lagrange is superior to Newton in that any coordinates can be used. The Lagrangian L = T - V is a function $L(q_i, \dot{q}_i)$ of the coordinates and their derivatives.

0.5) Dirac δ-function

We need later many simple properties of the famous Dirac δ-function. Here I rehearse and make explicit the results we will use.

I remind you that the δ-function is defined by

$$\int_{-\infty}^{+\infty} f(x).\delta(x)dx = f(0)$$

for all functions f. More complicated integrals are always performed by using standard variable changes to rewrite them in this form.

For example, in

$$I = \int f(x).\delta(ax)dx$$

we set $y = ax$ and

$$I = \int_{-\infty}^{+\infty} f(\frac{y}{a})\delta(y)\frac{dy}{|a|} = \frac{1}{|a|}f(0)$$

The more complicated case of

$$\int f(x)\delta(g(x))dx$$

where $g(x)$ is zero, once only, at $x = x_0$ is solved by setting $y = g(x)$, $y_0 = g(x_0) = 0$.

$$\int f(g^{-1}(y))\delta(y)\frac{dy}{|\frac{dg}{dx}|_{x=x_0}} = f(x_0)\frac{1}{|\frac{dg}{dx}|_{x=x_0}}$$

Multi-dimensional integrals are done the same way.

$$H = \int f(x_1, x_2, \cdots, x_n)\prod_{i=1}^{n}\delta(A_{ij}x_j - B_i)\prod_i dx_i$$

Define $y_i = A_{ij}x_j$ so that $x_j = \left(A^{-1}\right)_{jk}y_k$

$$H = \int f(x_1, x_2, \cdots, x_n)\prod\delta(y_i - B_I)\frac{\prod dy}{|detA_{ij}|}$$

$$= \frac{1}{|detA_{ij}|}.f((A^{-1})_{jk}B_k)$$

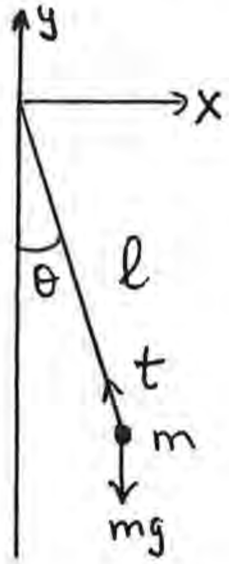
If f=1 notice this integral is independent of B !

Chapter 1

Classical Mechanics

In this brief chapter I would like to run through, without detailed proof, the status of Newton's equations versus Lagrange's methods and finally Hamilton's approach. These different methods already show effects that are of importance in Quantum Mechanics but there are important differences between the Classical and Quantum cases.

Since the Harmonic oscillator will play so large a rôle in our life for the next few days let me take it as the pedagogical example.



In Newton's method we need to take inertial coordinates i.e. coordinates where Newton's equations are correct. This excludes rotating coordinates which change the form of the equations from

$$m\ddot{x} = Force$$

Thus we are forced into coordinates such as (x, y) in the diagram. So the equations are, where t is the string tension,

$$\begin{aligned} m\ddot{x} &= t\sin\theta \\ m\ddot{y} &= -mg + t\cos\theta \end{aligned}$$

For small angle θ these reduce to

$$\begin{aligned} ml\ddot{\theta} &= t\theta \\ ml\frac{d^2}{dt^2}(1 - \frac{\theta^2}{2} + \cdots) = 0 &= -mg + t \qquad ; ignoring \qquad \theta^2, etc. \end{aligned}$$

Thus we get $\ddot{\theta} = -\frac{g}{l}\theta$, the standard equation for simple harmonic motion.

In Lagrangian form we can use any old coordinates and Lagranges equations will sort out the mess. So for small oscillations in terms of the coordinate θ we have

$$T = \frac{1}{2}m(l\dot{\theta})^2$$
$$V = +\frac{1}{2}mgl(1-\cos\theta) \approx \frac{1}{2}mgl\theta^2$$

So Lagrange's equation, and notice we only have one at this stage, unlike Newton where we had two, becomes

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\theta}}L\right\} - \frac{\partial}{\partial\theta}L = 0$$
$$\frac{d}{dt}\{ml^2\dot{\theta}\} - \{-mgl\theta\} = 0$$
$$ml^2\ddot{\theta} + mgl\theta = 0$$

The same as Newton !

This is a second order equation in time t. Hamilton was interested in obtaining two first order equations instead. Let me show you how this works in the simple case . I will then give a general proof.

The general idea is to define a new variable $p = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$. Then recognising this as the argument in the first term of Lagrange's equation we get, for free, that $\frac{d}{dt}p = \frac{d}{d\theta}L = -mgl\theta$. This then gives as before that $ml^2\ddot{\theta} = mgl\theta$. If we define $H = T + V = \frac{p^2}{2ml^2} + \frac{1}{2}mgl\theta^2$, then the equations of motion are

$$\dot{q} = -\frac{\partial H}{\partial\theta} \qquad \dot{\theta} = \frac{\partial H}{\partial p}$$

This trick holds in general.

There is now a classic calculation which the thermodynamic whizzes among you should recognise. It is the equivalent of changing variables from V,S to V,T in going from the energy $dE = TdS - PdV$ to the free energy equation $dF = -SdT - PdV$. Here we go from $q, \dot{q}$ to q, p. So calculate

$$\delta L = \frac{\partial L}{\partial q_r}\delta q_r + \frac{\partial L}{\partial\dot{q}_r}\delta\dot{q}_r$$
$$= \dot{p}_r\delta q_r + p_r\delta\dot{q}_r$$
$$= \delta\{\sum_r p_r\dot{q}_r\} + \sum_r\{\dot{p}_r\delta q_r - \dot{q}_r\delta p_r\}$$

So that shuffling terms

$$\delta\{-L + \sum p_r\dot{q}_r\} = \sum \dot{q}_r\delta p_r - \sum \dot{p}_r\delta q_r$$

This gives, as in thermodynamics,

$$\dot{q}_r = \frac{\partial H}{\partial p_r}$$
$$\dot{p}_r = -\frac{\partial H}{\partial q_r}$$

where $H = \sum_r p_r \dot{q}_r - L$. In many cases $h = T + V$.

These are the two first order equations which we promised. Notice H contains no $\dot{p}$ or $\dot{q}$ dependence. In particular the Hamiltonian $H(p_r, q_r)$ is now to be thought of as a function of p_r and q_r. The Lagrangian L is a function of $q_r, \dot{q}_r$. This is the classical reason for H appearing when we introduce p's and q's.

Now the big advantage of the Lagrangian method was it's ability to make use of any old coordinates. What is the equivalent statement for Hamiltonian systems. Define for any two functions of the p's and q's, say $u(p, q)$ and $v(p, q)$, the Poisson brackets

$$\{u, v\} = \sum_r \left(\frac{\partial u}{\partial q_r}\frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r}\frac{\partial v}{\partial q_r} \right)$$

With this definition it is easy to see that

$$\{q_i, q_j\} = 0$$
$$\{q_i, p_j\} = \delta_{i,j}$$
$$\{p_i, p_j\} = 0$$

These should set bells off in your head as they look awfully like the q, p commutation relations. The statement of invariance of Hamilton's equations can now be simply stated. Given a change of variables from (q, p) to $Q(q, p), P(p, q)$ then Hamilton's equations remain invariant if and only if the Q, P have the same Poisson brackets as q, p. In his book Dirac claims that to Quantise any theory you just replace the Poisson brackets by $\frac{1}{i}[,]$. This is an enormously influential statement; which is unfortunately wrong. Using the above, any classical theory can be written using any canonically equivalent coordinates. But in general, if you use the Dirac prescription in one coordinate system, you do **not** get the result the Dirac prescription would predict for the other set of coordinates. In other words, although Classical Mechanics is invariant under Canonical Transformations, the equivalent quantised theories are not invariant under canonical changes due to operator ordering. This is a problem which has driven theorists mad for 70 years.

Chapter 2

Quantum Mechanics

The idea behind this section is to set up the two classic ways of computing perturbative expansions in Field Theory. Now Field Theory is just an example of Quantum Mechanics. So I will give the two formalism in general Quantum Mechanics.

The physics behind these mathematical manipulations is actually rather complex and not well understood. As you gradually understand Quantum Field Theory you will slowly realise that it is an astonishingly complex mathematical structure. So either your understanding will go round and round in a convergent spiral or your head will spin.

The naïvest idea is that to first approximation, say in Quantum Electrodynamics, the electrons and photons can move about with little or no interaction. Thus it makes sense to split the Hamiltonian into two pieces. The first, soluble piece corresponds to free electrons and photons. We will see how to solve such a Hamiltonian in the next chapter. The interactions between them may then be treated as a small perturbation.

In the prerequisites I asked you to revise (or learn for the first time) the concept of a picture. In this chapter, in general Quantum Mechanics, I will introduce you to a third picture, the Dirac picture, which is explicitly defined to make perturbative calculations simple; well almost ! In the second section I will introduce you to the ideas of Feynman Path Integrals or Functional integrals; these, after you have tunnelled through a conceptual barrier, **are** the easy way of doing perturbative calculations. They are also the route which enables the Lattice people to simulate Quantum Field Theories on computers.

2.1) The Dirac Picture

We are in a Quantum Mechanical system where the Hamiltonian $\hat{H}$ is a sum of two parts. One soluble , one small.

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

Here $\hat{H}_I$ is usually called the interaction term. The Dirac picture is often called the interaction picture. The idea, starting from the Schrödinger picture is to switch to the Heisenberg picture but only using the $\hat{H}_0$ term. Thus define

$$\begin{aligned}\hat{O}_I(t) &= e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}\\ &= e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{O}_H(t)e^{i\hat{H}t}e^{-i\hat{H}_0 t}\\ &= U(t,0)\hat{O}_H(t)U^{-1}(t,0)\end{aligned}$$

I stress here that $\hat{O}_H$ is defined from the Schrödinger operator using the full Hamiltonian. The operator

$$\hat{U}(t,0) = e^{i\hat{H}_0 t}e^{-i\hat{H}t}$$

is crucial in what follows. Similarly we define for states

$$|a,t\rangle_I = e^{i\hat{H}_0 t}|a,t\rangle_S = U(t,0)|a\rangle_H$$

So we get the interaction picture from the Schrödinger operator by the free $\hat{H}_0$. Thus it satisfies

$$i\frac{\partial}{\partial t}\hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0]$$

Since $\hat{H}_0$ is soluble we can calculate this easily.

To calculate the Dirac picture operators we clearly need $U(t,0)$ so let us calculate its equation.

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,0) &= -\hat{H}_0 e^{i\hat{H}_0 t}e^{-i\hat{H}t} + e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{H} \\ &= e^{i\hat{H}_0 t}\hat{H}_I e^{-i\hat{H}t} \\ &= \hat{H}_I^I U(t,0) \end{aligned}$$

In the last eqution we easily see that $\hat{H}_I^I = \hat{H}_I(\hat{O}_I)$ i.e. the interaction Hamiltonian in the interaction picture is obtained by writing the interaction Hamiltonian in terms of interaction picture operators.

The crucial point is that this equation can easily be solved perturbatively. So we write

$$U(t,0) = 1 + U_1 + U_2 + U_3 + \cdots$$

where these terms are of order 0,1,2,3...in powers of the small $\hat{H}_I$. Substitute in the equation for U and compare equal powers of $\hat{H}_I$ on the two sides. The first term is clearly 1 since if $\hat{H}_I$ is 0 then U = 1.

$$i\frac{\partial}{\partial t}U_1 = \hat{H}_I(t)$$

Hence

$$U_1 = -i\int_0^t \hat{H}_I(t_1)dt_1$$

$$i\frac{\partial}{\partial t}U_2 = \hat{H}_I(t)U_1(t)$$

Hence

$$U_2(t) = (-i)^2\int_0^t dt_2 \int_0^{t_2} dt_1 \hat{H}_I(t_2)\hat{H}_I(t_1)$$

You can guess the rest ?

Now let us massage this result into the standard form. Define the time ordered product of any two operators by

$$\begin{aligned} T(\hat{A}(t_1), \hat{B}(t_2)) &= \hat{A}(t_1)\hat{B}(t_2); t_1 > t_2 \\ &= \hat{B}(t_2)\hat{A}(t_1); t_2 > t_1 \end{aligned}$$

In general for many operators you move the earliest to the right, then the next earliest and so on. This has a beautiful effect, inside a time ordered expression we can permute operators in an arbitary way. The result is unchanged under such permutations. Notice in our expression for U that $t_1 < t_2$. So the integrand looks asymmetric. A much more symmetric way to write it is as follows

$$\begin{aligned} U_n &= (-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} dt_1 T(\hat{H}_I(t_n) \cdots \hat{H}_I(t_1)) \\ &= \frac{1}{n!} \int_0^t \prod_i dt_i T(\hat{H}_I(t_1)\hat{H}_I(t_2) \cdots \hat{H}_I(t_n)) \end{aligned} \tag{2.1.1}$$

These terms sum into an exponential

$$U = T\{exp(-i \int_0^t \hat{H}_I(t)dt)\} \tag{2.1.2}$$

To define this exponential we expand into the U_n terms. These are polynomials in $\hat{H}_I$ and we can apply the definition of the time-ordering T.

In the next chapter we will use this formula extensively in the context of relativistic quantum field theory. I reiterate that in this case the physical model is of free physical particles which interact weakly. These interaction can then be treated as small perturbations. The theory itself will, of course, tell us whether this is internally consistent. Indeed many of the later lectures in QCD and the Salam Weinberg model will revolve around this problem.

2.2) Lagrangian Quantum Mechanics

In the preliminary reading for the course and the beginning of my lectures I stressed the importance of different views of dynamics. Above we have been very much concerned with the view of Quantum mechanics you were taught as undergraduates. Thus the equations are full of Hamiltonians and time dependence comes via Schrödinger equations. Unfortunately this is intrinsically non-relativistic in appearance. In special relativity space and time are supposed to be treated on an equal footing. This is impossible in a Hamiltonian approach. We need to switch back to Lagrangians. This means we need to address the problem of Lagrangian quantum mechanics. The pay-off will be a manifestly Lorentz symmetric formalism. I fact this is the chosen method for all problems with symmetries of any kind. Since Gauge symmetries dominate modern particle physics this is another reason for learning this method. The whole Faddeev Popov method comes from manipulation of Feynman integrals. I hasten to add that the Hamiltonian method, Schrödinger equation and all, is perfectly Lorentz symmetric but it is not manifestly symmetric. The Lagrangian methods solve the same equations and get the same answers. But the manifest imposition of symmetries often makes things easier to see.

I will develop the method first for a single dynamical degree of freedom q and its associated momentum p. Thus you should first understand these notes in this case. However if you now visualise q, p as column vectors for a finite number of degrees of freedom the proofs will be seen to be valid in this case also. Finally to reach field theory we need to make an intellectual leap and use the results for the infinite degrees of freedom implicit in Field theory. I will lead you up the garden path quite gently !

So, for the moment, consider a Quantum sytem with Hamiltonian

$$\hat{H}(\hat{p},\hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

We would like to compute the amplitude for the particle to start at q_i at t=0 and move to q_f at time $t = t_f$. In the Schrödinger picture this is given by the amplitude $A = \langle q_f|e^{-i\hat{H}t}|q_i\rangle$ where $|q\rangle$ is the time independent eigenstate of position . Thus, in words, we start in the position eigenstate at $t = 0$, propagate in time for a time $t = t_f - t_i$ through the exponent and finally compute the overlap with the time independent final eigenstate q_f. The tricky bit is calculating the exponent. The Feynman trick is to split it into a lot of little steps. To each such step we can then apply perturbation theory. Thus write

$$e^{-i\hat{H}t} = e^{-i\hat{H}\Delta}.e^{-i\hat{H}\Delta}.\cdots.e^{-i\hat{H}\Delta}$$

with n terms in the product and $\Delta = \frac{(t_f - t_i)}{n}$. Then $\hat{H}\Delta$ is small if we take n large enough; and we will eventually let $n \to \infty$ We write

$$A = \langle q_f|e^{-i\hat{H}\Delta}.e^{-i\hat{H}\Delta}.\cdots e^{-i\hat{H}\Delta}|q_i\rangle$$

Now insert, many times, the Quantum Mechanical representations of 1

$$\sum_{q_i} |q_i\rangle\langle q_i| = 1$$
$$\sum_{p_i} |p_i\rangle\langle p_i| = 1$$

$$\sum_{q_i,p_i} \langle q_f|p_n\rangle\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle\langle q_{n-1}|p_{n-1}\rangle\langle p_{n-1}|e^{-i\hat{H}\Delta}|q_{n-2}\rangle\cdots|q_1\rangle\langle q_1|p_1\rangle\langle p_1|e^{-i\hat{H}\Delta}|q_i\rangle$$

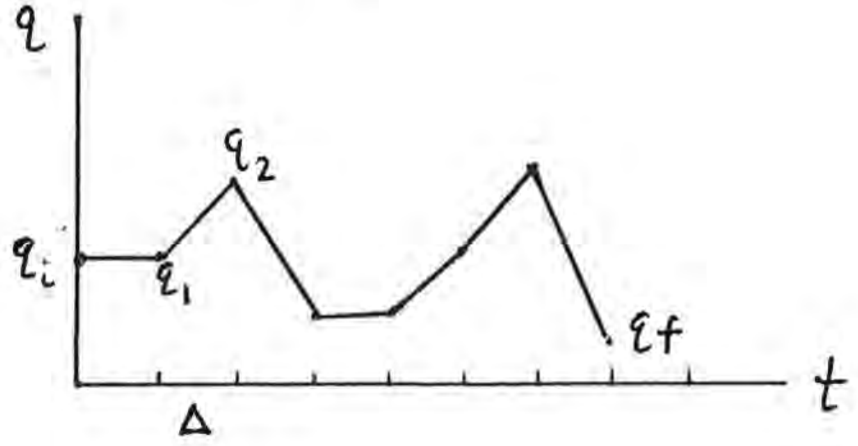
Now $\langle q_n|p_n\rangle = e^{iq_np_n}$ (c.f. $\langle x|\psi\rangle = \Psi(x)$). Thus we can rewrite

$$\begin{aligned}\langle p_n|e^{-i\hat{H}\Delta}|q_{n-1}\rangle &= \langle p_n|(1 - i\hat{H}(\hat{p},\hat{q})\Delta)|q_{n-1}\rangle \\ &= \langle p_n|(1 - i\hat{H}(p_n, q_{n-1}))|q_{n-1}\rangle \\ &= e^{-iH(p_n,q_{n-1})}.e^{-ip_n\cdot q_{n-1}}\end{aligned}$$

Thus, substituting this in the expression for A, we get

$$\begin{aligned}A &= \int_{q_0=q_i,q_n=q_f} \prod_1^n dp_i \prod_1^{n-1} dq_i\{e^{-i\sum H(p_j,q_{j-1})\Delta}.e^{-i(q_{j-1}\cdot p_j)}.e^{iq_j\cdot p_j}\} \\ &= \int \prod_1^n dp_i \prod_1^{n-1} dq_i\left[e^{i\sum \Delta\frac{(q_n - q_{n-1})p_n}{\Delta} - H(p_j,q_{j-1})}\right] \qquad (2.2.1) \\ &= \int [dpdq]e^{i\int dt[p\dot{q} - H]}\end{aligned}$$

The last line comes from the approximation that $\frac{q_n - q_{n-1}}{\Delta}$ becomes $\dot{q}$ in the limit $\Delta \to 0$. The exponent is now the time integral of the Lagrangian. This is the Action. Thus the Lagrangian appears in Quantum Mechanics. The integrals correspond to integrating over all the paths connecting q_i with q_f.



Those of you wide awake should be saying "Hey, you said the Lagrangian was a function of $q, \dot{q}$ not q, p !". So we need one last trick. Rather than give you a general proof, let me consider a simple case. Assume $H = \frac{p^2}{2m} + V(q)$ i.e. a simple problem of a particle moving in a potential V. Then the p-integrals above are Gaussian.

$$\int \prod dp_i e^{-i\frac{\Delta p_i{}^2}{2m}} . e^{ip_i(q_i - q_{i-1})} \approx e^{\frac{-i(q_i - q_{i-1})^2 m}{2\Delta}}$$

The p_i integral is trivially performed by replacing p_i by $\frac{(q_i - q_{i-1}).m}{\Delta} = \dot{q}_i.m$ in the Lagrangian. This is the Classical prescription in this simple case. Thus finally we get the Lagrangian expression for our amplitude

$$A = \int [dq] e^{i \int dt L(q,\dot{q})} \tag{2.2.2}$$

We will use this formula extensively in Quantum Field Theory. It is the basic Quantum Mechanical input into the Functional method. In terms of Quantum Mechanics QED or QCD are just special choices of L. The lattice people spend all their lives trying to do these functional integrals numerically. Next we will show how this is a brilliant formalism for discussing Lorentz and Gauge symmetries. So this is the crucial modern starting point for all the discussions of Gauge fixing; the Faddeev Popov trickery. We will return to this.

2.3) Heisenberg Harmonic Oscillator's

In the pre-school exercises I asked you to make sure that you could solve the Harmonic Oscillator in what we now know as the Schrödinger representation. Here for pedagogical purposes and to ease your way into Field theory I solve it in the Heisenberg picture. So we have a Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$. In the Heisenberg picture we have two time-dependent operators $\hat{q}(t)$ and $\hat{p}(t)$. These must satisfy the Heisenberg equations of motion

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{q}(t) &= [\hat{q}(t), \hat{H}] \\ i\frac{\partial}{\partial t}\hat{p}(t) &= [\hat{p}(t), \hat{H}] \end{aligned} \tag{2.3.1}$$

Given the commutation relation $[\hat{q}, \hat{p}] = i$, which is unaffected by the switch to the Heisenberg picture we easily see

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{q}(t) &= \frac{1}{m}i\hat{p} \\ i\frac{\partial}{\partial t}\hat{p}(t) &= m\omega^2(-i\hat{q}) \end{aligned}$$

Differentiate one of these and use the other gives

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\hat{q}(t) &= -\omega^2\hat{q}(t) \\ \frac{\partial^2}{\partial t^2}\hat{p}(t) &= -\omega^2\hat{p}(t) \end{aligned}$$

Although these are operator equations our true and tried methods work.

$$\begin{aligned} \hat{q}(t) &= \frac{1}{2m\omega}(e^{i\omega t}\hat{A}^\dagger + e^{-i\omega t}\hat{A}) \\ \hat{p}(t) &= \frac{1}{2m\omega}(im\omega e^{i\omega t}\hat{A}^\dagger - im\omega e^{-i\omega t}\hat{A}) \end{aligned}$$

Solving for $\hat{A}^\dagger, \hat{A}$ we get

$$\begin{aligned} \hat{A} &= \sqrt{\frac{1}{2}}(\sqrt{m\omega}\hat{q} + i\frac{\hat{p}}{\sqrt{m\omega}}) \\ \hat{A}^\dagger &= \sqrt{\frac{1}{2}}(\sqrt{m\omega}\hat{q} - i\frac{\hat{p}}{\sqrt{m\omega}}) \end{aligned}$$

It is easy to check that these have the same commutation relations as the $\hat{a}, \hat{a}^\dagger$ of the prerequisites. Hence they have the same eigenvalues and eigenstates.

Chapter 3

Free Boson or Scalar Field Theory

The time has come to raise the stakes. We have been studying general Quantum Mechanics so far. Now we pick the special Lagrangians and Hamiltonians designed to give relativistically invariant field Theories. We will see that these provide exactly the language to describe particle production as you see at LEP. They will also have manifest symmetry properties. Altogether they provide the standard theoretical language to understand our world. I stress these results all come from standard Quantum Mechanics.

As always the first thing to get clear, even before writing down the Hamiltonian, is the set of independent degrees of freedom. In other words for which variables will we have to solve Heisenberg's equations ? In field theory the variables are the values of the fields at the different space points $\underline{x}$. You are used to this idea from Maxwell's equations where the values of the electric and magnetic field form the dynamical variables. In Quantum Mechanics we have to decide which variables turn into operators which then satisfy Heisenberg's equations of motion. Thus we have variables, for a single scalar field, $\phi(\underline{x}, t)$.

3.1) Classical treatment

First let us treat the problem classically. Then ϕ is a single real variable at each space-time point. Thus ϕ at any given point is the direct analogue of q for a single point particle. We expect a Kinetic energy of

$$T = \int d^3x \left(\frac{\partial \phi}{\partial t}\right)^2 \qquad c.f. \frac{\dot{q}^2}{2m}$$

The potential energy will have two terms

$$V = \int d^3x \phi(\underline{x}, t)^2 \qquad c.f. \frac{1}{2} m\omega^2 q^2$$
$$+ \int d^3x \sum_i \frac{c^2}{2} \left(\frac{\partial \phi}{\partial x^i}\right)^2$$

The gradient terms are necessary to enforce Lorentz invariance. Thus we have the Lagrangian

$$L = \int d^3x \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \sum_i \frac{c^2}{2}\left(\frac{\partial \phi}{\partial x_i}\right)^2 - \frac{\mu^2}{2}\phi^2(\underline{x}, t)\right]$$

Then the action S can be written

$$S = \int dt L = \int d^3x dt \mathcal{L}(\underline{x}, t)$$

where the Lagrangian density $\mathcal{L}$ can easily be written down from above. Now classically we have a Lagrangian density that is a function of the dynamical variables ϕ, their time derivatives $\frac{\partial\phi}{\partial t}$ but also, unexpectedly, their spatial derivatives $\frac{\partial\phi}{\partial x_i}$ for each $i = 1,2,3$. Thus we must redo, say the Stationary action derivation of Lagrange, to obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right] + \frac{\partial}{\partial x^i}\left[\frac{\partial\mathcal{L}}{\partial(\frac{\partial\phi}{dx^i})}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Which in the case of our Lagrangian density gives

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial {x_i}^2} + \mu^2\phi = 0$$

Several comments are in order at this point. We use the Einstein summation convention so that the repeated i indices are summed. The above equation is Lorentz invariant if ϕ is a Lorentz scalar. Indeed if $\mu = 0$ this is the wave equation for light with c the velocity of light.

The general solution is not hard to write down. The expected result is that any solution will be a sum of plane waves. Notice the equation is linear in ϕ so that, given any two solutions, any linear combination is also a solution. Think of light.

So try a plane wave solution for ϕ.

$$\phi(\underline{x},t) = Ae^{i(\underline{k}.\underline{x}-\omega(\underline{k})t)}$$

Substituting in the equation of motion gives

$$A\big[-\omega^2(\underline{k}) + c^2k^2 + \mu^2\big] = 0$$

In order for the solution to be non trivial we must have $\omega(\underline{k}) = \pm\sqrt{\mu^2 + c^2\underline{k}^2}$. From now on ω will stand for the positive root of this equation. Then the general solution will be given, by superposition, as

$$\phi(\underline{x},t) = \int\frac{d^3k}{(2\pi)^3.2\omega(\underline{k})}\big[a(\underline{k})e^{i(\underline{k}.\underline{x}-\omega t)} + a^*(\underline{k})e^{-i(\underline{k}.\underline{x}-\omega t)}\big]$$

The factors of 2π and ω are conventional, but make Lorentz invariance manifest later. For the moment just think of them as factors extracted from a, a^*. The fact that ϕ is a real valued variable is guaranteed by a^* being the complex conjugate of a.

To switch to Hamiltonian methods we need to compute the momentum conjugate to $\phi(\underline{x},t)$.

$$\Pi(\underline{x},t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\underline{x},t)} = \dot{\phi}(\underline{x},t)$$

Writing this in terms of the a's we get

$$\Pi(\underline{x},t) = \int\frac{d^3k}{2\pi^3.2\omega(\underline{k})}\left[-i\omega a(\underline{k})e^{i(\underline{k}.\underline{x}-\omega t)} + i\omega a^*e^{-i(\underline{k}.\underline{x}-\omega t)}\right]$$

Then we can easily write

$$\hat{H} = \int d^3y \left[\frac{\hat{\Pi}^2(\underline{y},t)}{2} + \frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial y}^2\right) + \frac{\mu^2}{2}\hat{\phi}^2 \right] \tag{3.1.1}$$

Now you can quantise in your sleep. We have the Hamiltonian in standard form written in terms of coordinates $\phi(\underline{x},t)$ and their conjugate momenta $\Pi(\underline{x},t)$.

3.2) Quantum Mechanics

Now we switch to Quantum Mechanics. As always the classical variables turn into operators. So corresponding to the classical single particle $\hat{q}$ we have an operator at each point of space time $\hat{\phi}(\underline{x})$. In the Schrödinger representation these will be time independent; in the Heisenberg representation they will be time dependent. Corresponding to the single particle $\hat{p}$ we will have a momentum corresponding to each $\phi(\underline{x})$. I stress again that the dynamical variables here are the values at the points $\underline{x}$ not the variables $\underline{x}$. Thus we have the momenta $\hat{\Pi}(\underline{x},t)$. Then directly copying your Undergraduate Quantum course we have the commutation relations.

$$\begin{aligned}
\left[\hat{\phi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right] &= 0 \\
\left[\hat{\Pi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right] &= -i\delta^3(\underline{x}-\underline{y}) \\
\left[\hat{\Pi}(\underline{x},t),\hat{\Pi}(\underline{y},t)\right] &= 0
\end{aligned} \tag{3.2.1}$$

In other words the variables at different space points all commute, the only non-zero commutator is between a variable and its momentum at the same point. The normalisation is not obvious at this point, but we will see how natural it is later. Clearly the numerical factors can be changed by scaling ϕ. Here, because of the manifest time dependence, we have used Heisenberg operators. We showed before that commutation relations at equal times are unaffected by the switch of pictures.

Now we solve the Heisenberg equations of motion for the time dependence of these variables.

$$\begin{aligned}
i\dot{\hat{\phi}}(\underline{x},t) &= \left[\hat{\phi}(\underline{x},t),\hat{H}\right] \\
&= \int d^3y \left[\hat{\phi}(\underline{x},t), \frac{\hat{\Pi}^2(\underline{y},t)}{2}\right] \\
&= \int d^3y \left[\hat{\phi},\hat{\Pi}\right]\hat{\Pi}(\underline{y},t) \\
&= \int d^3y\, i\delta^3(\underline{x}-\underline{y})\hat{\Pi}(\underline{y},t) \\
&= i\hat{\Pi}(\underline{x},t)
\end{aligned}$$

$$
\begin{aligned}
i\dot{\hat{\Pi}} &= \left[\hat{\Pi}(\underline{x},t),\hat{H}\right] \\
&= \int d^3y\left[\hat{\Pi}(\underline{x},t),\frac{\partial\hat{\phi}(\underline{y},t)}{\partial y}\right]\frac{\partial\hat{\phi}}{\partial y} + \mu^2\int d^3y\left[\hat{\Pi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right]\hat{\phi}(\underline{y},t) \\
&= \int d^3y\left\{-i\frac{\partial}{\partial y}\delta^3(\underline{x}-\underline{y}).\frac{\partial\hat{\phi}}{\partial y} - i\mu^2\hat{\phi}(\underline{x},t)\right\} \\
&= i\frac{\partial^2\hat{\phi}}{\partial y^2} - i\mu^2\hat{\phi}(\underline{x},t)
\end{aligned}
$$

So if we put these together we get the equations for $\hat{\phi}$ and $\hat{\Pi}$ separately.

$$\frac{\partial^2\hat{\phi}}{\partial t^2} = \nabla^2\phi - \mu^2\hat{\phi}^2$$

and

$$\hat{\Pi}(\underline{x},t) = \dot{\hat{\phi}}(\underline{x},t)$$

You often hear nonsense, particularly in elementary field theory books, about the simple Schrödinger equation being changed into a relativistic wave equation. I stress here that this relativistic equation has been derived directly from the Heisenberg equation, in the Heisenberg picture. It is completely equivalent to the usual Schrödinger equation. What has changed is the Hamiltonian. Quantum mechanics is unchanged.

We can solve these operator equations exactly as in the classical case. Firstly, they are linear equations, so superpositions of solutions are solutions. We get, again with a funny choice of normalisation of the coefficients,

$$
\begin{aligned}
\hat{\phi}(\underline{x},t) &= \int\frac{d^3k}{(2\pi)^3.2\omega(\underline{k})}\left[\hat{a}(\underline{k})e^{i(\underline{k}.\underline{x}-\omega t)} + \hat{a}^\dagger(\underline{k})e^{-i(\underline{k}.\underline{x}-\omega t)}\right] \\
\hat{\Pi}(\underline{x},t) &= \int\frac{d^3k}{(2\pi)^3.2}\left[-i\hat{a}(\underline{k})e^{i(\underline{k}.\underline{x}-\omega t)} + i\hat{a}^\dagger(\underline{k})e^{-i(\underline{k}.\underline{x}-\omega t)}\right]
\end{aligned}
\tag{3.2.2}
$$

Given the commutation relations for the ϕ and Π we can compute those for the a and $a^\dagger$. Then we obtain

$$
\begin{aligned}
\left[\hat{a}(\underline{k}),\hat{a}(\underline{k}')\right] &= 0 \\
\left[\hat{a}^\dagger(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= 0 \\
\left[\hat{a}(\underline{k}),\hat{a}^\dagger(\underline{k}')\right] &= (2\pi)^3.2\omega.\delta^3(\underline{k}-\underline{k}')
\end{aligned}
\tag{3.2.3}
$$

This is the result we are after. We have an infinite set of Harmonic Oscillators. For different $\underline{k}$ they commute. The next move is to compute the Hamiltonian. This is easily (well actually lengthily and tediously) proved, by substituting for $\hat{\phi}$ and $\hat{\Pi}$, to be given by

$$\hat{H} = \int\frac{d^3k}{(2\pi)^3}\hat{a}^\dagger(\underline{k})\hat{a}(\underline{k}) + Constant \tag{3.2.4}$$

The Hamiltonian is a sum of independent commuting Harmonic Oscillators. The vacuum state is the state where all the oscillators are in their ground states. The excited states are obtained by applying the raising operators one or more at a time. For example $\hat{a}^\dagger(\underline{k})$ creates a particle of momentum $\underline{k}$ and energy $\hbar\omega(\underline{k})$. We can easily check that

$$\hat{H}\hat{a}^\dagger(\underline{p})|0\rangle = \omega(\underline{p})\hat{a}^\dagger(\underline{p})|0\rangle$$

The Hilbert space of the theory is given by the states $|0\rangle$, the vacuum; $\hat{a}^\dagger(\underline{k})|0\rangle$, the one particle states; $\hat{a}^\dagger(\underline{k})\hat{a}^\dagger(\underline{p})|0\rangle$, the two particle states; and so on. The commutation relations incidentally guarantee the two particle states are even under exchange i.e. the particles are guaranteed to be Bosons.

In the above Lorentz invariance was a bit hidden. We ended up with a nice covariant equation for ϕ but this felt rather accidental. The Heisenberg equations clearly treat $\underline{x}$ and t differently. The resolution for this lies in the Lagrangian method. So let us compute the Lagrangian or, in fact, the Action

$$Action = \int d^4x \left[\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{\mu^2}{2}\phi^2\right]$$

This is manifestly Lorentz invariant. The measure d^4x is and so is the Lagrangian density. Thus the Functional method will start with a big advantage. Already, before doing anything, the formalism looks invariant. Contrast this with normal Quantum Mechanics where both the Schrödinger and Heisenberg equations treat time very differently to space coordinates. This would lead you to think they could not be invariant. They are invariant but not manifestly. This can lead to tedious, apparently non Lorentz invariant calculations, which mysteriously come right at the end. An example is the Heisenberg quation for ϕ which came out as the wave equation, eventually.

Before leaving scalars let me generalise slightly and introduce a pair of scalars. These will be necessary in the Salam-Weinberg theory. So we write down

$$\mathcal{L} = \sum_{r=1}^{2}\left[\frac{1}{2}(\partial^\mu\phi_r)(\partial_\mu\phi_r) - \frac{\mu^2}{2}\phi_r^2\right]$$

which involves two independent scalr fields ϕ_1 and ϕ_2. For the purposes of gauge invariance it is convenient to also have a formalism where we instead have two complex valued fields defined by

$$\chi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$
$$\chi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

An easy exercise is to check that

$$\mathcal{L} = \partial_\mu\chi^\dagger\partial_\mu\chi - \mu^2\chi^\dagger\chi$$

Then we can write down Heisenberg's equations as before and solve them. We find

$$\hat{\chi} = \int \frac{d^3x}{(2\pi)^3.2\omega}\left(\hat{b}(\underline{k})e^{-ik.x} + \hat{d}^\dagger(\underline{k})e^{ik.x}\right)$$

Since χ is not real, we have two independent operators $\hat{b}$ and $\hat{d}$. The Hermitian conjugate field is obtained by Hermitian conjugation.

The commutation relations can also be computed. The only non-zero terms come from

$$\begin{aligned}
[\hat{b}(\underline{k}), \hat{b}^\dagger(\underline{q})] &= 2\omega(2\pi)^3\delta^3(\underline{k}-\underline{q}) \\
[\hat{d}(\underline{k}), \hat{d}^\dagger(\underline{q})] &= 2\omega(2\pi)^3\delta^3(\underline{k}-\underline{q})
\end{aligned}$$

and, most importantly, the Hamiltonian is given by

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3} \left(b^\dagger(\underline{k})b(\underline{k}) + d^\dagger(\underline{k})d(\underline{k})\right)$$

So again we have an independent set of oscillators. An interesting problem, is to determine the electric charge carried by each particle.

Chapter 4

Interacting Bosons

In the previous chapter we discussed the Quantum Mechanics of the Free Scalar Boson. This was a theory that was Lorentz invariant and corresponded to a set of free non-interacting particles. This means that it is a very boring theory. Particles never interact. No particles are created. In this chapter we introduce the interactions. We will do this first in the mathematically simplest theory of interacting scalar particles. You can think of this as a theory of Higgs particles, if you like. Later we will consider more realistic theories.

We will consider, for pedagogical reasons, a theory with a single scalar ϕ and a complex pair $\chi, \chi^\dagger$. So the Lagrangian density will be given by

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\chi + \mathcal{L}_{int}$$

where the interaction term is given by $\mathcal{L}_{int} = -g\hat{\chi}^\dagger\hat{\phi}\hat{\chi}$. This is Lorentz invariant since each field is a scalar. It is not hard to check that the Heisenberg equations of motion are

$$((\partial)^2 + \mu^2)\hat{\phi} + g\hat{\chi}^\dagger\chi = 0$$
$$((\partial)^2 + \mu^2)\hat{\chi} + g\hat{\phi}\hat{\chi} = 0$$

These are horrible non-linear operator equations. A Nobel prize for any solution. Only two ways to get information from this are known to man or woman. One is to assume the interaction term is small. The other is to put this on a lattice and do the functional integrals by brute force computing. My mission here is to explain perturbation theory. By being ingenious theorists have shown this is a correct move in a surprising number of cases. In fact it works much better than we have any right to expect.

So I will prove first, that the interaction term will lead to scattering, production and absorption of new particles. All the time the formalism will keep energy and momentum conservation correct.

I will first do a simple case using operators and an obvious Quantum Mechanical formalism. Then I will redo the calculation in the much slicker Functional formalism.

4.1) Feynman Diagrams from Operators

So, in Quantum Mechanics, in the Heisenberg picture, the crucial object to calculate is the $\hat{U}$ operator of Chapter 2. In lowest order of perturbation theory it is given by

$$\begin{aligned}\hat{U}(t_i, t_f) &= -i\int_{t_i}^{t_f} \hat{H}_I(t)dt \\ &= -ig\int d^4x\hat{\chi}^\dagger\hat{\phi}\hat{\chi}\end{aligned}$$

This, remember, is the operator which will propagate a t_i state to a t_f state. We apply it to the decay of a ϕ into a $\chi\chi^\dagger$ pair. For this to be kinematically possible, we must have the ϕ mass at least twice the χ mass. Then we take $t_i = -\infty$ and $t_f = +\infty$.

The states are therefore given by

$$
\begin{aligned}
|t = -\infty\rangle &= \hat{a}^\dagger(\underline{k})|0\rangle \\
|t = +\infty\rangle &= \hat{b}^\dagger(\underline{p})\hat{d}^\dagger(\underline{q})|0\rangle \\
\langle t = \infty| &= \langle 0|\hat{d}(\underline{q})\hat{b}(\underline{p})
\end{aligned}
$$

Being Heisenberg physical states they have no time dependence. So we need to calculate the matrix element

$$\langle t = \infty|\hat{U}(\infty, -\infty)|t = -\infty\rangle = -ig\langle|\hat{d}(\underline{q})\hat{b}(\underline{p})\left(\int d^4x \hat{\chi}^\dagger\hat{\phi}\hat{\chi}\right)\hat{a}(\underline{k})|0\rangle$$

substituting from above.

$$
\begin{aligned}
&= -ig\langle 0|\hat{d}(\underline{q})\hat{b}(\underline{p})\int d^4x \int \not{d}p' \int \not{d}k' \int \not{d}q' \{\hat{d}(\underline{p}')e^{-ip'x} + \hat{b}^\dagger(\underline{p}')e^{-ip'.x}\} \\
&\{\hat{a}(\underline{k}')e^{-ik'.x} + \hat{a}^\dagger(\underline{k}')e^{ik'.x}\}\{\hat{b}(\underline{q}')e^{-iq'.x} + \hat{d}^\dagger(\underline{q}')e^{iq'.x}\}\hat{a}^\dagger(\underline{k})|0\rangle
\end{aligned}
$$

Now remember that an annihilation operator acting on $|0\rangle$ gives zero as does a creation operator acting on $\langle 0|$. Thus we can throw away the $\hat{a}^\dagger$, $\hat{b}$ and $\hat{d}$ terms. We have also introduced the Imperial notation $\not{d}^3k$ for $\frac{d^3k}{(2\pi)^3.2\omega}$ to save writing.

$$= -ig\langle 0|\hat{d}(q)\hat{b}(p)\int d^4x \int \not{d}p' \ \not{d}k' \ \not{d}q' \hat{b}^\dagger(\underline{p}')\hat{a}(\underline{k}')\hat{d}^\dagger(\underline{q}')\hat{a}^\dagger(\underline{k})|0\rangle e^{ix.(p'-k'-q')}$$

Now we have a creation and an annihilation operator for a,b,d. The only non-zero contribution comes from the right hand side of each commutator. Finally commuting all the terms past until they annihilate the vacuum we get

$$= -ig\delta^4(p - k - q)\langle 0|0\rangle$$

The δ^4 contains 4-momentum conservation. This is our first real perturbative calculation. The method is general. Write down the initial and final states in terms of creation and annihilation operators. Write down the relevant term in $\hat{U}$. Start commuting all annihilation operators to the right. When they reach $|0\rangle$ they give zero. When a creation operator reaches the left and hits $\langle 0|$ it too gives zero. This is a finite procedure. It clearly needs systematic organisation. In operator formalism this goes under the name of Wick's theorem. We will duck this and use a much slicker Functional proof.

However, before we leave operators, I would like to present the operator formulation of the propagator. In the Interaction picture we saw how we obtained expressions of the form

$$\int dt_1 dt_2 \cdots dt_n T(H_I(t_1)H_I(t_2)H_I(t_3)\cdots)$$

Using the above trick we see that a crucial object will be the difference between two time ordered operators as appear in U and the so-called normal ordered form where we carry out the above procedure and commute all annihilation operators to the right of all creation operators.

Let us investigate this for two scalar fields. The time ordered form

$$T(\phi(x)\phi(y))$$

is quadratic in creation and annihilation operators. Moving all the annihilation operators to the right gives a normal ordered form plus non-operator terms which depend on x and y. Let us compute the non-operator term. Since it does not depend on any operator it is most easily extracted by taking the vacuum expectation value.

$$\langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle$$

The normal ordered terms all give zero leaving the constant term.

$$\langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle = i\Delta_F(x,y)$$

by definition of the Feynman propagator. Let us calculate it.

$$i\Delta_F = \langle 0| \int \not{d}k \int \not{d}q \big(\hat{a}(\underline{k})e^{i(\underline{k}.\underline{x}-\omega t)}\big)\big(\hat{a}^\dagger(\underline{q})e^{-i(\underline{q}.\underline{x}-\omega\tau)}\big)|0\rangle$$

The missing terms in the expansions of ϕ all give zero either on the initial or final vacuum. The two times are called t and τ. We assume $t \geq \tau$. Now commute the two terms past each other.

$$i\Delta_F(x,y) = \int \not{d}k \ \not{d}q (2\pi)^3 2\omega.\delta^3(\underline{k}-\underline{q}).e^{i\underline{k}.(\underline{x}-\underline{y})-i(t-\tau)\omega}$$

The delta function lets us do one of the momentum integrals. Notice that the above calculation assumed $t \geq \tau$ to put the operators in the above order. Thus we can write in general for any times

$$i\Delta_F(x,y) = \int \frac{d^3k}{(2\pi)^3.2\omega}\left[\theta(t-\tau)e^{i\underline{k}.(\underline{x}-\underline{y})-i(t-\tau)\omega} + \theta(\tau-t)e^{-i\underline{k}.(\underline{x}-\underline{y})-i(\tau-t)\omega}\right]$$

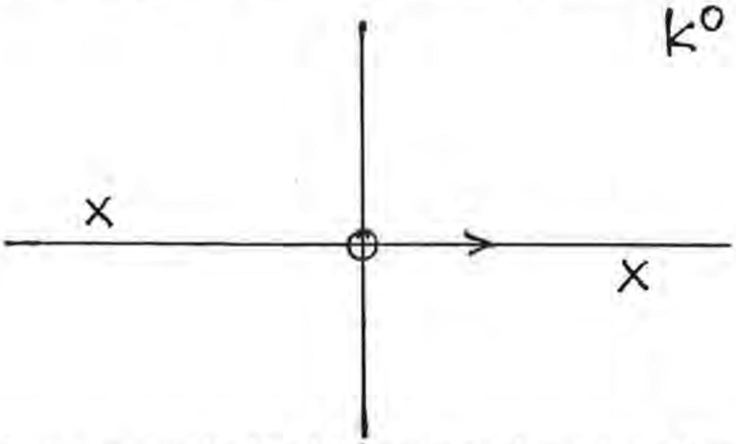
I now claim that

$$i\Delta_F(x,y) = \int \frac{d^4k e^{-ik.(x-y)}}{k^2-\mu^2+i\epsilon}$$

where we now use relativistic four vector notation and $a.b = a^0b^0 - \underline{a}.\underline{b}$. The proof of this statement is most easily given from Cauchy's theorem on complex integrals. In the above formula the $i\epsilon$ term fixes which side of the real k^0 contour integral contains the poles. The poles occur at solutions for k^0 of the equation

$$k^{0\,2} - \underline{k}^2 - \mu^2 + i\epsilon$$

These occur at $k^0 = \pm\sqrt{\underline{k}^2 + \mu^2 - i\epsilon}$ or $k^0 = \pm\omega(\underline{k}) \mp i\epsilon$. So the k^0 plane looks like



The contour can now be lifted to ∞ up or down depending on the sign of $t - \tau$. If $t - \tau > 0$ then we close in the lower plane, if $t - \tau < 0$ then we close in the upper half plane. We pick up one pole in either case. To give the result

$$\theta(t-\tau).2\pi i \int \frac{-d^3k e^{i\underline{k}.(\underline{x}-\underline{y})-i\omega(t-\tau)}}{2\omega} + \theta(\tau - t).2\pi i \int \frac{d^3k e^{i\underline{k}.(\underline{x}-\underline{y})+i\omega(t-\tau)}}{-2\omega}$$

Again we see that an apparently non Lorentz invariant formalism gives invariant answers. The Feynman propagator is Lorentz invariant. Notice also the curious effect that suddenly we integrate not over d^3k but d^4k. In other words the Feynman propagator corresponds to particles off their mass-shell. Their energy, momentum does **not** satisfy $E^2 - \underline{p}^2 = m^2$.

Before leaving the operator methods let us very roughly outline how a more complicated process might go. This calculation also carries a health warning. We are going to calculate the fourth order contribution to the vacuum to vacuum transition matrix element. The operator expression for this is proportional to the integrals over the times x^0, y^0, z^0 and u^0.

$$\langle 0|T(H_I(x^0)H_I(y^0)H_I(z^0)H_I(u^0))|0\rangle$$

And each such Hamiltonian is an integral over the space components of the energy density $-g\chi^\dagger\phi\chi$. Thus we get the term

$$\int d^4x d^4y d^4z d^4u T\Big\{\chi^\dagger\phi\chi..\chi^\dagger\phi\chi..\chi^\dagger\phi\chi..\chi^\dagger\phi\chi\Big\}$$

This is to be sandwiched between vacuum states. So when we turn the time ordering into the normal ordering no operators must be left. So as we commute terms past we must always pick up the Feynman terms, not the normal ordered terms. Crudely then we get the Feynman pairings of the points x, y, z, u in all possible ways. For example one possible term in the answer is

$$\int d^4x d^4y d^4z d^4u \Delta_\phi(x-y)\Delta_\phi(z-u)\Delta_\chi(x-y)\Delta_\chi(z-u)\Delta_\chi(x-u)\Delta_\chi(y-z)$$

There is a systematic procedure for writing down all possible Feynman diagrams. First draw all possible diagrams with vertices where $\phi, \chi, \chi^\dagger$ meet at each vertex. For each vertex a factor -ig, for each propagator a factor $\frac{i}{p^2-\mu^2+i\epsilon}$. At each vertex four momentum is conserved. This gives the overall conservation of four momentum.

4.2) The Functional Method of Deriving Feynman Diagrams

Now we turn to the standard modern approach to this problem of constructing Feynman diagrams. I assume you revised the Gaussian integrals section of the prerequisites. I claim that any free field theory reduces to doing Gaussian integrals. We can do any Gaussian integral using *the most important integral in the world.* I then would like to use the Gaussian trickery to calculate Feynman diagrams.

First let us write down the path integral for N particles of positions q_i. The canonical object to study is the function

$$W(J_i) = \int \prod_i [dq_i] e^{i \int_{t_i}^{t_f} L(q_i, \dot{q}_i) dt + \sum J_i q_i} \tag{4.2.1}$$

where $[dq_i]$ is the path integral over the i'th coordinate. The J_i are fake parameters put in to let us play Gaussian tricks. Thus

$$\frac{\partial W}{\partial J_l} = \int [dq_i] q_l e^{iS}$$

where S is the action. Thus J_i derivatives let us pull down factors of q_i in a systematic way. Note that we need to put $J_i = 0$ to get back to the original action S.

The generalisation to field theory is instantaneous if we remember what the dynamical degrees of freedom actually are. The analogues of the q_i are the field values $\phi(\underline{x})$. Just as i counts the individual degrees of freedom for Quantum Mechanics so $\underline{x}$ counts the degrees of freedom in field theory.

Thus the Functional integral in field theory is

$$W(J(x)) = \int \prod_x [d\phi] e^{-\int d^4x . \mathcal{L} + \int d^4x J(x) \phi(x)}$$

In other words, given a function J(x), we compute a number W. Thus we map from a function to a number...the old fashioned definition of a functional. As before we can take derivatives with respect to J(x) to pull down factors of $\phi(x)$.

Some care has to be taken with these derivatives. We need the concept of a Functional derivative rather than a normal derivative. Let us study a simple case. A functional is a map from a function to a number. The simplest example, with which you are familiar, is a normal integral. Given a function $J(x)$ the integral returns a number

$$W[J(x)] = \int \phi(x) . J(x) dx$$

for an arbitrary function $J(x)$ and any fixed function $\phi(x)$. The Functional derivative is defined to be the limit as $\epsilon \to 0$ of

$$\frac{\delta W}{\delta J(y)} = lim \ \frac{W[J(x) + \epsilon\delta(x-y)] - W[J(x)]}{\epsilon} \tag{4.2.2}$$

In our integral case this gives

$$\begin{aligned}\frac{\delta W}{\delta J(y)} &= \int \delta(x-y)\phi(x)dx \\ &= \phi(y)\end{aligned}$$

The Dirac delta is necessary to give a non zero answer under the integral sign. The result of taking a Funtional derivative of a constant Functional is a function of y, the point where the Functional derivative was evaluated.

So now let us turn to our free scalar first. Then we will show how to derive perturbation theory. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

The first three terms correspond to our free Boson, which we quantised previously by operators. The ϕ^4 term corresponds to the interaction term. In terms of Feynman diagrams we expect it to correspond to vertices where four particles meet.

First we solve the free part, then we add the perturbations. From the Gaussian prerequisites we know how to compute Gaussian integrals so we first rewrite the exponent in the form

$$\phi.\mathbf{Operator}.\phi$$

copying the Gaussian form

$$\sum \phi_i K_{ij} \phi_j$$

Since we now have an infinite number of degrees of freedom, labelled by $\underline{x}$ rather than i, we expect the $\sum$ to be replaced by $\int dx$. Thus

$$\int d^4x \left(\frac{\partial\phi}{\partial t}\right)^2 = -\int d^4x\phi\frac{\partial^2\phi}{\partial t^2} + Surface\ terms$$

Playing this game throughout and dropping surface terms we get, for the free theory,

$$W_0(J) = \int [d\phi]exp[-\frac{1}{2}\int d^4xd^4y\phi(x).K(x,y).\phi(y) + \int d^4zJ(z)\phi(z)]$$

with

$$K(x,y) = \delta^4(x-y)(-\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2) \tag{4.2.3}$$

The answer to the standard Gaussian is given in terms of the inverse matrix. So here we need the inverse operator to $K(x,y)$. In other words the solution to

$$\int d^4 y K(x,y)\Delta(y,z) = \delta(x-z) \tag{4.2.4}$$

the analogue of

$$\sum_j K_{ij} K^{-1}_{jk} = \delta_{i,k}$$

in the discrete case.

It should come as no great surprise that the solution to this is our old friend the Feynman propagator.

$$\Delta_F = \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{e^{ik.(x-y)}}{k^2+\mu^2} \right\} \tag{4.2.5}$$

The only subtlety is that we take $k = (ik_0, \underline{k})$ i.e. Euclidean four vectors to guarantee exponential convergence of the integrals. To get physical answers we must continue back. So we see that the inverse operator is just the Fourier transform of the Feynman propagator. This is the basic result of the functional method. Later when we do gauge theories and when you add Dirac particles there will be additional indices for the gauge degrees of freedom, the spin indices, charge, etc., ... The propagator is the inverse, summing over all these degrees of freedom, as we will see.

So the exponent in the answer will be

$$\int dx dy J(x)\Delta_F(x,y)J(y)$$

Now let us turn to interacting theory. Then we must calculate our perturbation series for U. To compute objects like

$$\int dt_1 dt_2 \langle 0|T(\phi(x)\phi(z)H_I(t_1)H_I(t_2))|0\rangle$$

we take Functional derivatives

$$\frac{\delta}{\delta J(x)} \cdot \frac{\delta}{\delta J(z)} \cdot \frac{\delta^4}{\delta J(u)^4} \cdot \frac{\delta^4}{\delta J(w)^4} \tag{4.2.6}$$

to bring down all the operators, and then integrate $d^4u.d^4w$ to recover the two Hamiltonians integrated over time.

At the end of the calculation we must set all the $J=0$. Thus, after all the derivatives have been performed, no factors of J must remain in the numerator, such terms go away as $J \to 0$. Since the exponent is quadratic, this means that each term in the exponent must be differentiated twice. This must be done in all possible ways. Thus we get a sum of terms. In each term the derivatives are paired in that they both operate on the same term from the exponent. Such a paired derivative gives a factor

$$\frac{\delta}{\delta J(u)} \cdot \frac{\delta}{\delta J(v)} . e^{\frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)} = \Delta(u,v) \tag{4.2.7}$$

So each such pairing brings down a Feynman propagator, and the different pairings give all Feynman diagrams.

Chapter 5

Groups and Algebras

The big revolution in my lifetime has been the dominance of gauge theories. When I was a graduate student the philosophy was that the lightest, and therefore longest range hadron, was the pion. So the dynamics of this scalar particle was seen to be the most important. So everybody rushed around writing papers on the dynamics of scalar fields. Gauge theories were seen as **bad**. Seminars at CERN predicting that large p_T, which occurs automatically in gauge theories, might be interesting were treated with barely disguised derision.

Similarly a band of nutters went around writing down non-renormalisable, apparently non-predictive, gauge theories of the weak interactions. They were called Salam and Weinberg. This is all changed. The norm is gauge theories in all directions, as far as the eye can see. The nutters now do string theory. However being a nutter is not a sufficient reason to expect success.

The basic property of gauge theories is a vast symmetry called the Gauge Group. Except in a very few cases little is known about this vast symmetry group. The usual trickery of group theory is at a loss. Only the fearless theoretical physicists plunge into the unknown. The basic building blocks are the usual Lie groups. Since some of you are unfamiliar with these let me survey a couple of simple cases. These give a generic feel for the general case.

The language of symmetry in Quantum Mechanics is Group theory. So let us start with the definition of a group.

5.1) Definitions and Examples

A **Group** is a set of objects with a multiplication defined such that if a,b,c are arbitrary objects in the set then a.b is also in the set (closure), a^{-1} is also in the set (inverse), a unit e is in the set and we have the properties

$$a.(b.c) = (a.b).c$$

always,

$$a.a^{-1} = e = a^{-1}.a$$

$$e.a = a.e = a$$

e is often written 1.

Examples

a) The numbers $\{1, -1\}$ under multiplication.

b) The integers under addition.

c) More interesting. The set of 2×2 unitary matrices of determinant 1. Unitary means $a^\dagger = a^{-1}$. This is SU(2).

d) The set of 3×3 orthogonal matrices of determinant 1. Orthogonal means $a^T = a^{-1}$. This is SO(3). Since any rotation of three vectors can be written as a 3×3 orthogonal matrix this group is isomorphic to the rotation group.

The last two examples are known as Lie groups since the matrices depend smoothly on a finite number of parameters. Thus any rotation can be written as a product of rotations around the x,y, or z axes.

The rotation Group and Algebra

Such rotations can be parametrised

$$\begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

Now an interesting object appears if we consider the limit as the angles get small. This object is called the Lie algebra. This is important since it is the world inhabited by Gauge fields. Expanding the above we get $1+\theta_3\Sigma_3, 1+\theta_2\Sigma_2, 1+\theta_1\Sigma_1$ where

$$\Sigma_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Sigma_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The interesting structure is not the product, but the commutator,

$$\begin{aligned} [\Sigma_1, \Sigma_2] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\Sigma_3 \end{aligned}$$

Similarly $[\Sigma_2, \Sigma_3] = -\Sigma_1$ and $[\Sigma_3, \Sigma_1] = -\Sigma_2$. Thus these elements are closed under commutation. If we redefine $\Sigma \to i\Sigma$ then we retain the commutation relations of the angular momentum operators.

Thus the Lie algebra of the rotation group is the angular momentum algebra you have studied in great detail in your Quantum Mechanics courses. There is another way to see that angular momentum and rotations are closely linked. Consider a rotation of axes in the x,y plane such that

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \approx x + \theta.y \\ y' &= -x\sin\theta + y\cos\theta \approx -x.\theta + y \end{aligned}$$

Then a wave function transforms as

$$\Psi(x', y') = \Psi(x, y) + \theta(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})\Psi(x, y) + O(\theta^2)$$

for small θ. The operator carrying out this transform is

$$(1 - i\theta \hat{L}_z \cdots)$$

In other words the infinitesimal rotations are generated by our old friend the angular momentum operator. This is the real reason that it appears everywhere in Quantum Mechanics. If a Quantum system is rotationally invariant then the Hamiltonian commutes with the Rotation operators i.e. the angular momentum operators. They can therefore be simultaneously measured. Hence eigenstates of energy can also be labelled by the total angular momentum.

SU(2) and su(2)

The most general SU(2) matrix can be written

$$g = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \tag{5.1.1}$$

where a,b,c,d are real numbers satisfying $a^2+b^2+c^2+d^2=1$. The unit matrix corresponds to $a \approx 1$ so, to get the algebra, we expand around $b, c, d \approx 0$. Thus we get

$$g \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ib \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + id \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + ic \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Thus the elements of the algebra are the Pauli matrices. These also satisfy the angular momentum algebra. Thus the Lie Algebra of SU(2) is the same as the Lie Algebra of SO(3).

SU(3)

The group behind QCD is of course SU(3). So we need a quick survey of the group and its algebra. The group is the set of 3×3 unitary, determinant 1 matrices. The first obvious question is to compute the number of independent parameters or angles. Assume there are n $\theta_i; i = 1, \cdots, n$. Then expand an arbitrary SU(3) matrix in terms of the θ_i, assuming $\theta_i = 0$ corresponds to the unit matrix. Then $g = 1 + \sum \theta_i L_i \cdots$. Now it is easily seen that the condition that det g = 1 is equivalent to trace $L_i = 0$ (the trace of a matrix is the sum of the diagonal terms) while the condition that g is unitary implies that L_i are Hermitian i.e $L_i^\dagger = L_i$. There are only 8 independent traceless, Hermitian matrices. Gell-Mann wrote down a convenient set, which physicists have used ever since.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2 = \begin{pmatrix} o & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{5.1.2}$$

These are closed under commutation ! The elements of the Lie algebra su(3) are linear combinations of these terms. This is the linear space in which the Gauge fields of SU(3) live. So be warned.

Lie's theorems

The heart of the theory is the realisation that, although the algebra can be computed in terms of infinitesimal group elements, from the algebra we can recover the Group. Let us see how this works for rotations. The matrix corresponding to a rotation by ψ about the unit vector $\underline{\hat{n}}$ is given by

$$\begin{aligned} R(\psi,\underline{\hat{n}}) &= \left[R(\frac{\psi}{N},\underline{\hat{n}})\right]^N \\ &= \left(1+i\sum\frac{\theta_i}{N}\Sigma_i\right)^N \\ &= e^{i\sum\theta_i\Sigma_i} \end{aligned} \tag{5.1.3}$$

where we have used the formula

$$\begin{aligned} \left(1+\frac{a}{N}\right)^N &= i+N.\frac{a}{N}+\frac{N(N-1)}{2}.\frac{a^2}{N^2}+\frac{N(N-1)(N-2)}{3!}.\frac{a^3}{n^3} \\ &\approx 1+a+\frac{a^2}{2}++\frac{a^3}{3!}+\cdots \\ &= e^a \end{aligned}$$

The basic idea is that a big rotation can be constructed from a lot of small ones. The small ones are fixed by the algebra. In the case of su(2) this gives, because $(\underline{\sigma}.\underline{\hat{n}})^2 = 1$, that

$$e^{i\frac{\psi}{2}\underline{\sigma}.\hat{n}} = \cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\underline{\sigma}.\underline{\hat{n}} \tag{5.1.4}$$

This is Hamilton's representation of finite rotations by quaternions.

Representations

In the above we defined a group as an abstract set with abstract properties. The Lie algebras were defined by expansions of explicit matrices about the identity. We could give a definition of the Lie Algebra as a linear vector space closed under commutation. Such things have been classified. The actual examples we had, in terms of matrices, are what mathematicians call representations. So in terms of Σ and the Pauli matrices we had two representations of su(2) in terms of 3×3 and 2×2 matrices. Lie lets us turn these into representations of SU(2). They correspond to the spin 1 and spin $\frac{1}{2}$ representations of SU(2).

Below we will see that, to define a gauge theory, we put the Gauge fields into the algebra and must prescribe which representations contain the other particles.

Chapter 6

Gauge theories

The particle physics interest is by now manifest. QCD is an SU(3) gauge theory. The Salam-Weinberg model is a gauge theory with gauge group $SU(2) \times U(1)$ Such theories are renormalisable, just like QED. They have remarkable properties which will be explored in the other lectures. Quark confinement, running coupling constants, chiral symmetry breaking, the Higgs effect are all properties of gauge theories. So we start at the beginning and follow our scalar route. First classical equations, then Lagrangians, then functional integrals, then compute perturbation theory.

6.1) Classical Maxwell Theory

First let us revise the Grand-daddy of all the gauge theories, due to Mr Maxwell. We will first check that it is Lorentz invariant. Secondly we show how to derive it from a Lagrangian. Thirdly we discuss the U(1) gauge symmetry of this theory. Then we are all set to stick this classical Lagrangian into our Functional integral and derive the Feynman rules for the Maxwell U(1) Gauge theory.

The first thing to do is rewrite Maxwell in manifestly Lorentz invariant form. I choose units in which Maxwell's equations are

$$\begin{aligned}
\nabla.\underline{B} &= 0 \\
\nabla \wedge \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\
\nabla.\underline{E} &= \rho \\
\nabla \wedge \underline{B} &= \underline{j} + \frac{\partial \underline{E}}{\partial t}
\end{aligned}$$

Two of these equations can be solved by writing $\underline{B} = \nabla \wedge \underline{A}$ and $\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t}$. Now I remind you of the four vectors $x^\mu = (t, \underline{x})$, $x_\mu = (t, -\underline{x})$, $j^\mu = (\rho, \underline{j})$ and $j_\mu = (\rho, -\underline{j})$. Now write $A^\mu = (\phi, \underline{A})$ and $A_\mu = (\phi, -\underline{A})$. Then we define the objects $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. These naturally give the combinations $F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\dot{A}_x - \frac{\partial}{\partial x} A_0 = -E_x$ and $F_{12} = \partial_1 A_2 - \partial_2 A_1 = -(\nabla \wedge \underline{A})_z = -\underline{B}_z$ so filling in all the terms we get

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Then the covariant looking equation you might guess, would be $\partial_\mu F^{\mu\nu} = j^\nu$. Then we get for $\nu = 1$

$$
\begin{aligned}
\partial_\mu F^{\mu 1} &= \frac{\partial}{\partial t} F^{01} + \frac{\partial}{\partial x} F^{11} + \frac{\partial}{\partial y} F^{21} + \frac{\partial}{\partial z} F^{31} \\
&= -\dot{\underline{E}}_x + \frac{\partial}{\partial y} B_z + \frac{\partial}{\partial z}(-B_y) \\
&= (-\dot{\underline{E}} + \nabla \wedge \underline{B})_x \\
&= j_x \\
&= j^1
\end{aligned}
$$

So this is one of Maxwell's equations. Now check the $\nu = 0$ term.

$$
\begin{aligned}
\frac{\partial}{\partial t} F^{00} + \frac{\partial}{\partial x} F^{10} + \frac{\partial}{\partial y} F^{20} + \frac{\partial}{\partial z} F^{30} &= 0 + \frac{\partial}{\partial x}(E_x) + \frac{\partial}{\partial y}(E_y) + \frac{\partial}{\partial z}(E_z) \\
&= \nabla . \underline{E} \\
&= j^0 \\
&= \rho
\end{aligned}
$$

Thus two of Maxwell's equations are subsumed in

$$\partial_\mu F^{\mu\nu} = j^\nu$$

The other two are subsumed in

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

We thus have our goal of a manifestly Lorentz invariant formalism . Note however that the equations have a funny little symmetry: $A_\mu \to A_\mu + \partial_\mu \phi$ leaves $F_{\mu\nu}$ unchanged. This was the first sign of a Gauge invariance. In classical physics it is a curiosity, since all of classical physics can be written safely in terms of the unaffected $\underline{E}, \underline{B}$ fields. In Quantum Mechanics it is a different matter as the $\underline{A}$ field is directly measurable.

The classic experiment to demonstrate this is due to Bohm and Aharanov.

An electron two-slit experiment is carried out with the addition of a long solenoid between the slits. The solenoid carries current and so there is a magnetic field $\underline{B}$ inside the coil. Outside the coil there is no magnetic field but $\underline{A}$ is not zero. The integral of $\underline{A}$ around a loop encircling the coil is given by Stokes

$$\int \underline{A}.d\underline{l} = \int d\underline{S}.(\nabla \wedge \underline{A}) = \int d\underline{S}.\underline{B} = B flux$$

So the A field is non zero outside the coil.

The interference pattern changes when the field is switched on. Hence, in Quantum physics, the dynamical variables are not merely the $\underline{E}, \underline{B}$ fields.

6.2) The Lagrangian Formalism

Now we return to rewrite the classical Maxwell theory in Lagrangian formalism. From the adverts above, we expect a Lorentz invariant Lagrangian density.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

for the simple case of no fields except the Maxwell fields. The independent degrees of freedom are, at first sight, the four A_μ fields. So we get only four equations from Lagrange but remember that the A_μ already solve two of Maxwell's equations.

Thus the A_0 equation gives

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{A}_0}\right] + \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{L}}{\partial(\partial_x A_0)}\right] + \cdots - \frac{\partial \mathcal{L}}{\partial A_0} = 0 \\ 0 + \frac{\partial}{\partial x}\Big[-F_{01}\Big] + \frac{\partial}{\partial y}\Big[-F_{02}\Big] \cdots - 0 = 0 \\ \nabla.\underline{E} = 0 \end{aligned}$$

As promised, one of Maxwell's equations. There is however a problem, $\Pi_{A^0} = 0$ since the Lagrangian has no dependence on $\frac{\partial A^0}{\partial t}$. So what on earth can the commutation relations be ?

Now turn to the $\underline{A}_x$ equation.

$$\frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\underline{A}}_x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{L}}{\partial(\partial_x A_x)}\right] + \frac{\partial}{\partial y}\left[\frac{\partial \mathcal{L}}{\partial(\partial_y A_x)}\right] + \frac{\partial}{\partial z}\left[\frac{\partial \mathcal{L}}{\partial(\partial_z A_x)}\right] - \frac{\partial \mathcal{L}}{\partial A_x} = 0$$

$$\frac{\partial}{\partial t}\Big[-F_{01}\Big] + 0 + \frac{\partial}{\partial y}\Big[-F_{12}\Big] + \frac{\partial}{\partial z}\Big[F_{31}\Big] = 0$$

$$\frac{\partial}{\partial t}\Big[-\underline{E}_x\Big] + \frac{\partial}{\partial y}\underline{B}_z - \frac{\partial}{\partial z}\underline{B}_y = 0$$

$$\frac{\partial}{\partial t}\underline{E} = \nabla \wedge \underline{B}$$

The other Maxwell equation. Thus this Lagrangian gives the expected answers. So we have connected Maxwell's equations to our world of Lagrangians and, if necessary, Hamiltonians. The only fly in the ointment is the bizarre non-existence of the momentum conjugate to the scalar electric potential A^0. If we treated Gauge theories in Hamiltonian formulation this would cause us serious aggravation. The jargon, again due to Dirac, is the language of constrained systems and Dirac Brackets as opposed to Poisson brackets. We will avoid this problem by sticking to our Lagrangian, functional treatment. These problems will not go away but will reappear in a different disguise. The no free lunch theorem.

6.3) The U(1) Covariant Derivative

In a short while we will launch into the complexities of Non-Abelian gauge theories. Before we do this I introduce a crucial construction, the covariant derivative, in the simple U(1) case. We saw above that Maxwell's equations were invariant under the transformation

$$A^\mu(x) \to A'^\mu = A^\mu(x) + \partial^\mu \chi(x).$$

Now consider some field $\psi(x)$ which transforms as

$$\psi(x) \to \psi'(x) = e^{-ig\chi(x)}\psi(x)$$

for an arbitrary real function $\chi(x)$. Notice that unlike in the case of Lorentz transformations or rotations no changes happen to $\underline{x}$.

We would like to construct Lagrangians, with interaction terms involving both A_μ and ψ. So first turn to the transformation of the $\partial_\mu\psi(x)$ term.

$$\begin{aligned}
\partial_\mu\psi(x) \to \partial_\mu\psi'(x) &= \partial_\mu\left\{e^{-ig\chi(x)}\psi(x)\right\} \\
&= \frac{\partial\psi}{\partial x^\mu}.e^{-ig\chi(x)} - ig\frac{\partial\chi}{\partial x}.e^{-ig\chi}.\psi \\
&= \left\{\frac{\partial\psi}{\partial x^\mu} - ig\frac{\partial\chi}{\partial x^\mu}.\psi\right\}.e^{-ig\chi(x)} \\
&\neq e^{-ig\chi(x)}.\frac{\partial\psi}{\partial x^\mu}
\end{aligned}$$

So the derivative does not transform in the same way as the original function. But try

$$\begin{aligned}
\left(\partial^\mu + igA^\mu\right)\psi \to \left(\partial^\mu + ig\big(A^\mu + \partial^\mu\chi(x)\big)\right).e^{-ig\chi(x)}.\psi& \\
= e^{-ig\chi(x)}\left[\partial^\mu\psi - ig\partial^\mu\chi.\psi + iga^\mu.\psi + ig\partial^\mu\chi.\psi\right]& \\
= e^{-ig\chi}\left[\partial^\mu + igA^\mu\right]\psi&
\end{aligned}$$

So we see that the combination $D^\mu\psi(x) = (\partial^\mu + igA^\mu)\psi(x)$ transforms in exactly the same way as ψ. This makes it easy to construct gauge invariant Lagrangian densities. This is the covariant derivative. Those of you who have studied general relativity will recognise the name, the style, but not the details.

For example, the standard kinetic term for a Free spin $\frac{1}{2}$ Dirac particle is $\bar{\psi}\gamma_\mu\partial^\mu\psi$. Now ψ transforms as $e^{-ig\chi}\psi(x)$ and the kinetic term is not gauge invariant. However the combination $\bar{\psi}\gamma_\mu\left[\partial^\mu + igA^\mu\right]\psi$ is gauge invariant. The imposition of gauge invariance then ties the free quadratic term to the interacting term containing three fields. In this sense gauge invariance fixes the interactions, given the free terms.

6.4) Non-abelian Gauge Theories

We take the case of SU(2). More general theories are easy once you understand this case. The trick, as above, is to construct the covariant derivative. Suppose we have a field ψ transforming as

$$\psi(x) \to \psi'(x) = U(x)\psi(x) \tag{6.4.1}$$

where $U(x)$ is an element of SU(2). This element can be different at every point of space time. In other words this gauge symmetry will turn out to be a vast group. The elements of the group are fields with group values. In other words at every point in space time we attach an SU(2) matrix, possibly different at every point. Group multiplication is defined by multiplying the elements at each space time point

$$U(x).V(x) = U.V(x)$$

giving another set of SU(2) matrices at each space time point. We can choose independent SU(2) rotations at every point in space and have them depend arbitrarily on time. Compare this with the angular momentum /rotation operator which rotates all points of space by the same amount. This is why the gauge symmetry is often called a local symmetry. We can choose to have a gauge transformation which is 1 (i.e. no transformation) everywhere except a finite local region. We can carry out symmetry transformations independently on the moon and on earth.

So we introduce the non-abelian gauge field for SU(2). As promised it lives in the su(2) algebra. Thus the Gauge field can be thought of as

$$W^\mu = \frac{\tau_1}{2}W_1^\mu + \frac{\tau_2}{2}W_2^\mu + \frac{\tau_3}{2}W_3^\mu = \frac{1}{2}\underline{\tau}.\underline{W}^\mu \tag{6.4.2}$$

Here the three τ's are the three Pauli matrices of the su(2) algebra. They are given a different name just to avoid confusion with any angular momentum Pauli matrices that might be around. There are 12 W fields. One for each Pauli matrix and each such term is a four vector, hence the Lorentz indices μ. You will therefore often see the gauge field given as 2×2 matrix W^μ which can be rewritten as a linear combination of Pauli matrices with coefficients the actual fields W_i^μ.

Now put these together to construct the covariant derivative.

$$D^\mu.\psi = \left(\partial^\mu + iW^\mu\right)\psi \tag{6.4.3}$$

Unlike Maxwell we do not know how the gauge field must transform under gauge transformations. We let the covariant derivative tell us. Thus we assume that after the gauge transformation

$$D^\mu\psi \to D'^\mu\psi' = U(x).D^\mu\psi$$

From this we see that

$$D' = UDU^{-1}$$

Rewriting this in terms of the fields W^μ and W'^μ, we arrive at

$$\partial^\mu + i\frac{\underline{\tau}}{2}.\underline{W}'^\mu = U.\left(\partial^\mu + i\frac{\underline{\tau}}{2}.\underline{W}^\mu\right)U^{-1}$$

or in terms of the 2×2 matrices W^μ

$$W'^\mu = W^{\mu U} = U.W^\mu U^{-1} - iU(x).\partial^\mu U^{-1}(x) \tag{6.4.4}$$

This somewhat strange formula is defined entirely to make the covariant derivative of a gauge field transform in the same way as the field.

Now we construct the analogue of the $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ field. This is defined to be

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - i[W^\mu, W^\nu] \tag{6.4.5}$$

This is done with only one aim in mind; to construct gauge invariant Lagrangians. So how does $F^{\mu\nu}$ transform ? The result is

$$F'^{\mu\nu} = U.F^{\mu\nu}.U^{-1} \tag{6.4.6}$$

Notice that all these terms are 2×2 matrices which do not necessarily commute. In the Abelian, commuting case the U's would cancel. This happens in the Maxwell case where the U's are just phases.

To check this by brutal calculation we just substitute the transformation properties of the W's into the definition of the covariant derivative and grind.

$$\begin{aligned}
F'^{\mu\nu} &= \partial^\mu W'^\nu - \partial^\nu W'^\mu + i[W'^\mu, W'^\nu] \\
&= \partial^\mu\left(U.W^\nu.U^{-1} - iU.\partial^\nu U^{-1}\right) \\
&- \partial^\nu\left(U.W^\mu.U^{-1} - iU.\partial^\mu U^{-1}\right) \\
&+ i\left[U.W^\mu.U^{-1} - iU\partial^\mu U^{-1}, U.W^\nu.U^{-1} - iU.\partial^\nu U^{-1}\right] \\
&= U\left[\partial^\mu W^\nu - \partial^\nu W^\mu + i[W^\mu, W^\nu]\right]U^{-1} \\
&+ \partial^\mu U.W^\nu.U^{-1} + U.W^\nu.\partial^\mu U^{-1} - i\partial^\mu U.\partial^\nu U^1 - iU.\partial^\mu\partial^\nu U^{-1} \\
&- \partial^\nu U.W^\mu.U^{-1} - U.W^\mu.\partial^\nu U^{-1} + i\partial^\nu U.\partial^\mu U^1 + iU.\partial^\mu\partial^\nu U^{-1} \\
&+ i\left[U.W^\mu.U^{-1}, -iU.\partial^\nu U^{-1}\right] + \left[U.\partial^\mu U^{-1}, U.W^\nu.U^{-1}\right] - i\left[U.\partial^\mu U^{-1}, U.\partial^\nu U^{-1}\right]
\end{aligned}$$

The first line is the expected answer. We have to show the rest cancels. The trick here is to take the derivative of the identity relation $\partial^\mu(U(x).U^{-1}(x)) = \partial^\mu 1 = 0$. Which gives $\partial^\mu U.U^{-1} + U.\partial^\mu U^{-1} = 0$ so that $\partial^\mu U^{-1} = -U.\partial^\mu U.U^{-1}$. This lets us get rid of all derivatives of U^{-1} in the above, to give for the right hand side

$$= U.F^{\mu\nu}.U^{-1} + \partial^\mu U.W^\nu.U^{-1} - U.W^\nu.U^{-1}.\partial^\mu U.U^{-1} + i\partial^\mu U.U^{-1}.\partial^\nu U.U^{-1}$$

$$-iU.\partial^\mu\partial^\nu U^{-1} - \partial^\nu.W^\mu.U^{-1} + U.W^\mu.U^{-1}.\partial^\nu U.U^{-1} - i\partial^\nu U.U^{-1}.\partial^\mu U.U^{-1}$$

$$+iU.\partial^\mu\partial^\nu U^{-1} - U.W^\mu.U^{-1}.\partial^\nu U.U^{-1} + \partial^\nu U.U^{-1}.U.W^\mu.U^{-1} - U.U^1.\partial^\mu U.U^{-1}.U.W^\nu U^{-1}$$

$$+U.W^\nu.U^{-1}\partial^\mu U.U^{-1} - i\partial^\mu U.U^{-1}.\partial^\nu U.U^{-1} + i\partial^\nu U.U^{-1}\partial^\mu U.U^{-1}$$

$$= U.F^{\mu\nu}.U^{-1}$$

It is now an easy exercise to check that the following is a gauge invariant, Lorentz invariant Lagrangian density.

$$\mathcal{L} = -\frac{1}{4}trace\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{6.4.7}$$

This is the **Yang-Mills** Lagrangian density.

Here trace is the sum of the diagonal terms in the matrices. It is easy to prove the crucial property that tr(A.B.C)=tr(C.A.B). This term is also Lorentz invariant due to the way we contracted the Lorentz $\mu\nu$ indices. So we are in good shape with both symmetries manifest.

If we wanted to do QCD with its SU(3) symmetry we would have taken W^μ to be a linear superposition of the 8 Gell-Mann matrices. There would have been 4(Lorentz)×8(su(3) generators) gauge fields. Similar arguments apply to any Gauge group.

6.5) Feynman Rules

We are now ready to compute the Feynman rules for the gauge field. Before plunging into details let us look crudely at the Lagrangian. We see there are terms quadratic in the fields W^μ of the form $(\partial^\mu W^\nu - \partial^\nu W^\mu)^2$ plus terms cubic and quartic in the W's. The quadratic terms will describe free fields. The others, the unavoidable self-interaction terms of a non-abelian Yang-Mills theory. They are the reason why QCD, even without fermions, is a highly non-trivial field theory.

We follow our Gaussian tricks to the end. So we need again to write the quadratic part of the action as **Field.Operator.Field**. Consider the Abelian U(1) case. The problem here lies in the Lorentz indices not the gauge group indices.

$$\int (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) d^4x$$

$$= \int \Big\{ A_\nu(-\partial_\mu\partial_\mu A_\nu + A_\nu\partial_\mu\partial_\nu A_{\mu\nu} + A_\mu\partial_\nu\partial_\mu A_\nu - A_\mu\partial_\nu\partial_\nu A_\mu \Big\} d^4x$$

So in terms of operators we need the inverse of the operator

$$\int d^4xd^4yA_\mu(x).\Big[-2(\partial_\mu\partial_\nu)g_{\mu\nu}+2\partial_\mu\partial_\nu\Big]\delta^4(x-y).A_\nu(y)$$

Remember we solved the free spin zero particle by Fourier transforms. Here we have the added complication of the Lorentz indices. The operator is a two index tensor in Lorentz indices. So we need the inverse of this matrix/tensor in momentum space. After Fourier we get, in the numerator,

$$P_{\mu\nu}=(k_\mu k_\nu-k^2.g_{\mu\nu})$$

This has an unfortunate property. Compute the product of two of them

$$\begin{aligned}P_{\mu\nu}P_{\nu\rho}&=\big(k_\mu k_\nu-k^2g_{\mu\nu}\big)\big(k_\nu k_\rho-k^3g_{\nu\rho}\big)\\&=k_\mu k_\rho.k^2-k^2.k_\mu k_\rho-k^2k_\mu k_\rho+k^4.g_{\mu\rho}\\&=-k^2\big(k_\mu k_\rho-k^2g_{\mu\rho}\big)\end{aligned}$$

Now any matrix whose square is proportional to itself does not have an inverse. Try multiplying the equation by the inverse. So the propagator does not exist !

What's up ? There are several views on this.

1). Canonical quantisation is in trouble. Remember we got $\Pi_{A_0}=0$. Which is inconsistent with the standard commutation relations. We need to start again using Dirac's theory of constraints.

2). Another way of saying the same thing is the fact that there are only actually two photon states c.f. right and left polarised; whereas we have a vector A^μ describing the photon i.e. four degrees of freedom.

3) In path integral formalism we should not integrate over the gauge equivalent field configurations. Thus if W^μ is gauge equivalent to W'^μ then to count both in the path integral is a bit strange. The Faddeev Popov trick is a systematic way of removing this double counting.

6.6) Gauge Fixing or Faddeev Popov

Let me start with a trivial, rotationally invariant, integral and use a large sledgehammer to crack it. The sledgehammer will however also crack the gauge problem.

So consider the integral

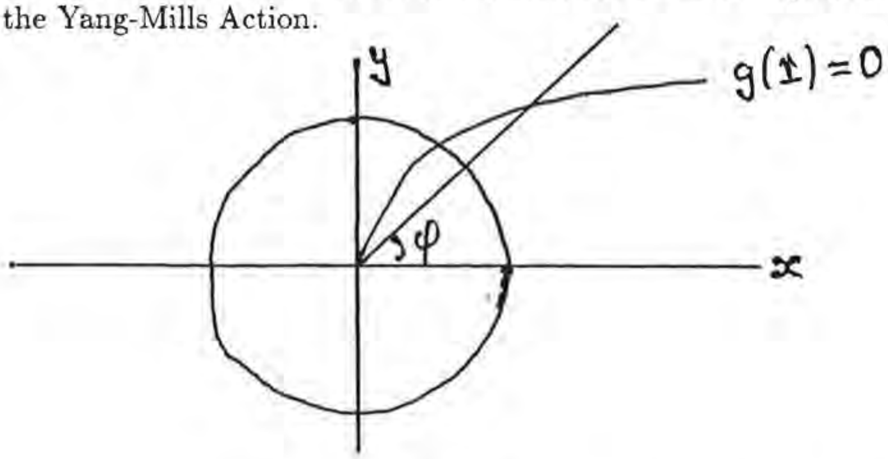
$$I=\int f(x,y)dxdy$$

where f is invariant under rotations. So in terms of polar coordinates $f(x,y)=F(r,\theta)=F(r)$. So trivially

$$I=\int F(r)rdrd\theta=2\pi\int F(r)rdr$$

Here we think of 2π as the volume of the rotation group.

I would like to rewrite this familiar calculation in the language of invariance and group transformations. In this case trivial rotations, in the general case these will be full gauge transformations. The rotational invariance of the function f will be replaced by the gauge invariance of the Yang-Mills Action.



First I redo the trivial calculation. Then I will use a general coordinate change. These will correspond to different choices of **gauges** in the Yang-Mills case. Given a point $\underline{r} = (r, \theta)$ we define a rotated point $\underline{r}_\phi = (r, \theta + \phi) = R(\phi)\underline{r}$. Thus R corresponds to an operator which rotates the coordinates. The function f is clearly invariant under such an operation; in fact this is what rotational invariance means. We only want to count one point from each circle. All other points on a given circle have the same value of f. The jargon is that the circles form the orbits of the rotation group.

So define

$$W_\phi = \int d^2 r f(x, y)\delta(\theta - \phi) = \int r dr d\theta F(r, \theta)\delta(\theta - \phi) = \int r dr F(r, \phi)$$

It is clear immediately that

$$W = \int d^2 r f(x, y) = \int d\phi W_\phi$$

W_ϕ corresponds, obviously, to integrating straight out along a radial path at angle ϕ. Because of the invariance of F it is immediate that $W_\phi = W_{\phi'}$ for all ϕ, ϕ'. Thus the integral

$$W = \int d\phi W_\phi = 2\pi . W_\phi$$

since W_ϕ is independent of ϕ. This trick, of using the invariance to extract the group volume, is what we need to use in general.

Consider the same example but integrated along a arbitrary curve which cuts each orbit once and once only. Suppose the equation of this curve is $g(\underline{r}) = 0$. For example the x-axis is $g(x, y) = y = 0$.

First consider the integral

$$\Delta_g^{-1}(\underline{r}) = \int d\phi \delta(g(\underline{r}_\phi))$$

The reason for the name will become clear. At each radius there exists a ϕ which rotates an arbitrary $\underline{r}$ onto our curve. Thus this δ function has a zero in ϕ. The answer is

$$\frac{1}{|\frac{\partial g}{\partial \theta}|_{g=0}}$$

Hence the name, it is the inverse Jacobian.

But now we prove that Δ is invariant i.e. unchanged by rotations of $\underline{r}$.

$$\Delta_g^{-1}(\underline{r}_{\phi'}) = \int d\phi \delta(g(\underline{r}_{\phi+\phi'})) = \int d\psi \delta(g(\underline{r}_\psi)) = \Delta_g^{-1}(\underline{r})$$

where we changed integration variable to $\psi = \phi + \phi'$. Thus, inserting a factor of 1,

$$W = \int d\phi W_\phi$$

where

$$W_\phi = \int d^2 r f(x,y).\Delta_g(\underline{r})\delta(g(\underline{r}_\phi)) = W_{\phi'}$$

The proof of this latter statement goes as follows. First calculate the rotation R that takes $\underline{r}_\phi \to \underline{r}_{\phi'}$. Then change variables from $\underline{r}$ to $R.\underline{r} = \underline{r}'$. The three terms d^2r,$f(\underline{r})$ and Δ are all invariant while the δ term changes as required. In the general case we will need to prove the invariance of the measure, the function and the Δ term. Thus finally we have

$$W = 2\pi.W_\phi$$

The 2π is, once again, the group volume and W_ϕ integrates once over each orbit as required.

Now turn to the gauge case. We want to integrate over the W^μ. But we only want to count the gauge equivalent fields once. Thus we need the equivalent of g above. Equations such that they only have one solution under gauge transformations.

Before plunging into the details let me give two examples.

a). **Axial Gauge**: For each W_a^μ field we set $g^a = W_a^3 = 0$. In other words if we call the field W_a^μ, transformed by the gauge transformation U, $W_a^{\mu U}$ we substitute this in g and solve for U. At each space-time point we can rotate the W field so that it's third space component is zero for each su(2) index. At this point we will lose manifest Lorentz invariance. This is possible because the gauge transformations of (6.4.4) are x dependent and so show up differently in the different Lorentz components even although they originally operate on the SU(2) labels.

b). **Covariant Gauge**: This is defined by taking $g^a = \partial_\mu A_a^\mu$.

Now with a wild generalisation we write

$$\Delta_g^{-1}(W^\mu) = \int [\prod_x dU(x)] \prod_x \delta(g^a(W_a^{\mu U})) = \Delta_g^{-1}(W^{\mu V})$$

for all gauge transformations V. Moreover we can show, as above, that

$$\Delta_g^{-1}(W^\mu) = \int [\prod dU] \prod \delta(g_a(W_a^{\mu U}) - B_a) = \Delta_g^{-1}(W^{\mu V})$$

independent of V and B. This follows from standard properties of the δ-function as summarised in the prerequisites. The $\prod$ over all space time points makes the notation cumbersome so we drop it, but keep in mind that we now have a functional integral over separate degrees of freedom at each space time point. An essential part of our simple rotation model was that $d\theta = d(\theta + \phi)$ or in other words that the measure is invariant under rotations. Similarly above we integrate dU over the gauge group. It is a known result that all Lie groups like SU(2), SU(3) have invariant integration measures. These are called Haar measures. So into our gauge field functional integral we insert the factors of

$$1 = \Delta_f(W_a^\mu). \int [dU] \delta(g_a(W_a^{\mu VU}) - B_a)$$

$$Const = \int [dB] e^{-\frac{1}{2\xi}\int d^4x B^2(x)}$$

giving

$$\int [dW_a^\mu]\,[dU]\,e^{-Action}.\Delta_g(W_a^\mu) \prod \delta(g_a(W_b^{\mu VU}) - B_a). \int [dB]\,e^{-\frac{1}{2\xi}\int d^4x B^2(x)}$$

$$= \int [dW_a^\mu]\,[dU]\,e^{-Action}\Delta_g(W_a^\mu)\delta(g_a(W_a^\mu) - B_a). \int [dB]\,e^{-\frac{1}{2\xi}d^4x B^2(x)}$$

$$= \int [dW_a^\mu]\,[dU]\,e^{-Action}\Delta_g(W_a^\mu)e^{-\frac{1}{2\xi}\int d^4x (g_a)^2}$$

$$= \left\{Volume\ of\ gauge\ Group\right\} \times \int [dW_a^\mu]\,e^{-Action}\Delta_g(W_a^\mu)e^{-\frac{1}{2\xi}\int d^4x g_a^2}$$

The \$64,000 question at this point is whether we have solved the propagator problem? In other words, throwing away the gauge group measure above, does the quadratic term operator now have an inverse? The real change is the additional term g_a^2. In covariant gauge this alters the Fourier transformed operator numerator into

$$k_\mu k_\nu \{1 - \frac{1}{\xi}\} - k^2.g_{\mu\nu}$$

This now has an inverse and the propagator in momentum space is

$$\frac{\left[g^{\mu\nu}k^2 - (1-\xi)k^\mu k^\nu\right]}{k^2 + i\epsilon}$$

The other term in our new integral Δ gives rise to ghosts although in some gauges there is no such contribution. I do not have time for this. There is clearly a large scale industry in trying funny gauges. Some people develop this to an art. Occasionally you can eliminate almost all Feynman graphs by clever choices.

Chapter 7

Higgs

In this section we reach the Standard model. We are interested in symmetry breaking. In other words having constructed a great edifice of symmetry we apparently now pull it all down. This is a totally false impression. At a simple level the various constraints that symmetry imposes are alive and well but heavily disguised. At a more subtle level there is a theorem, due to Elitzur, that says that a gauge symmetry is never spontaneously broken.

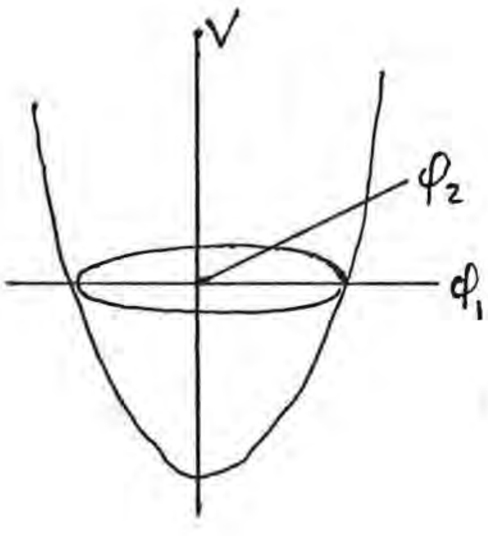
Our tool is still perturbation theory. But we are more careful about what is meant by perturbation theory.

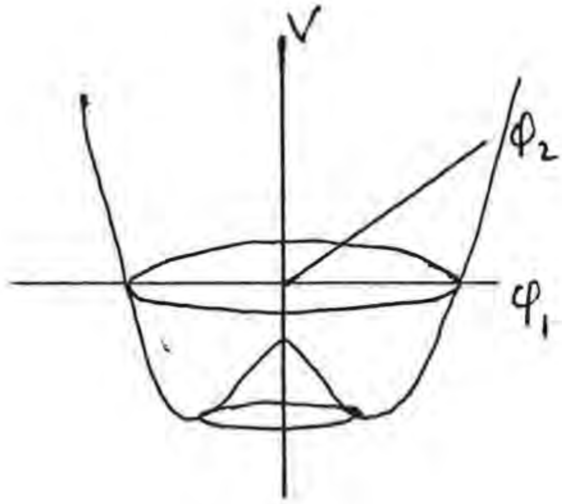
7.1) The U(1) Higgs model

As you all know the Higgs is a scalar particle. Our old friends of Chapters 3,4. You probably have heard words about negative massses, vacuum expectation values, magnets,...

The standard approach says consider the potential part of the Lagrangian density $\mathcal{L}$ with a complex scalar field χ

$$V = -\frac{m^2}{2}|\chi^2| + \frac{\lambda^2}{4}|\chi|^4 = \frac{\lambda^2}{4}\left(|\chi|^2 - \frac{m^2}{\lambda^2}\right)^2 + \frac{m^2}{\lambda^2}$$





There are clearly two options depending on the sign of m^2. If it is negative we have no real minimum of the potential away from $|\chi| = 0$. If it is positive then there is a line of minima at $|\chi| = \frac{m}{\lambda}$.

It now looks natural to expand around this minimum. The fact that there is a circle of minima comes from the U(1) invariance if V. If we are dealing with a global, x-independent phase invariance

$$\chi \to e^{i\alpha}\chi$$

with α a constant then we can choose a specific direction in χ space to expand around. This is equivalent to a magnet magnetising. The Hamiltonian for a magnet is rotationally invariant but when the temperature is lowered it "spontaneously" chooses a direction for its north pole. In real life this direction is determined by residuals from previous magnetisations.

In our perturbative terms the philosophy is that it is worth a try. As I stressed at the beginning, perturbation theory is really a set of assumptions that certain terms are small. We have no current analytical control over this statement so the wise physicist just steams ahead.

This is the standard description of the breaking of a global symmetry. We turn now to the Higgs mechanism looking first at the simple Abelian U(1) case before turning to the Salam Weinberg model.

Now we replace the derivatives by covariant derivatives. Let us start with the U(1) part. Then

$$\mathcal{L} = \left[(\partial_\mu + ieA_\mu)\chi\right]^\dagger \left[(\partial^\mu + ieA^\mu)\chi\right] + \mu^2\chi^\dagger\chi - \lambda^2(\chi^\dagger\chi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Now write

$$\chi = \frac{1}{\sqrt{2}}(f + \rho(x))e^{-\frac{i}{f}\xi(x)}$$

$$f = \frac{\mu}{\lambda}$$

In the spirit of Lagrange this is just a change of variables from χ to ρ, ξ. We can now stuff this into the Lagrangian and get a Lagrangian density in terms of the new variables. To simplify the algebra keep only free quadratic terms.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) - \mu^2\rho^2 + \frac{1}{2}(\partial_\mu\xi - feA_\mu)(\partial^\mu - feA^\mu)\cdots$$

This has curious $A_\mu\partial^\mu\xi$ terms. They correspond to transitions between photons and ξ's. Clearly the Hamiltonian is not diagonal.

The trick is to go back to eqn.(7.2.2). By performing a gauge transformation we can clearly set $\xi = 0$. Or in other words choose

$$A'_\mu = A_\mu - \frac{1}{ef}\partial_\mu\xi$$

Then the quadratic part becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) - \mu^2\rho^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{f^2e^2}{2}A_\mu A^\mu$$

So we now have a massive vector A and a massive scalar the ρ instead of a massless A and two scalars.

7.2) The Standard Model

Now we play the same game in the SU(2)×U(1) model. This will be cursory. Later lecturers will fill in the details. So we write down a model invariant under both SU(2) and U(1) gauge transformations. The model contains a set of scalars and the gauge particles.

$$\mathcal{L} = \left(D_\mu \phi\right)^\dagger \left(D^\mu \phi\right) + \frac{1}{2}\mu^2 \phi^\dagger \phi - \lambda^2 (\phi^\dagger \phi)^2 - \frac{1}{4} tr\left(F^{\mu\nu} F_{\mu\nu}\right) - \frac{1}{4}\left(f^{\mu\nu} f_{\mu\nu}\right)$$

The covariant derivative now contains three gauge vector fields

$$D_\mu \phi = \left(\partial_\mu + i\frac{g}{2}\underline{\tau}.\underline{A}_\mu + ig' B_\mu\right)\phi$$

with the two terms corresponding to the SU(2) and U(1) parts of the gauge symmetry. We now write the complex field ϕ which is an SU(2) doublet as

$$\phi = e^{-\frac{i}{2f}\underline{\tau}.\underline{\xi}(x)} \begin{pmatrix} 0 \\ f + \rho(x) \end{pmatrix}$$

where we have used the SU(2) rotation to write ϕ as a rotation applied to the τ_3 negative eigenstate. We now use the freedom of gauge invariance to cancel this term. So finally

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(f + \rho(x)) \end{pmatrix}$$

Substitute into $\mathcal{L}$ and keep only quadratic terms

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu \rho)(\partial^\mu \rho) - \frac{\mu^2}{2}\rho^2 - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} \\ &+ \frac{f^2 g^2}{2}(A^1_\mu A^\mu_1 + A^2_\mu A^\mu_2) + \frac{f^2}{8}(gA^3_\mu - gB_\mu)(gA^\mu_3 - gB_\mu)\end{aligned}$$

So we read off a massive vector pair with equal masses $M_{W^\pm} = \frac{fg}{2}$ and, if we define

$$Z^\mu_0 = A^\mu_3 \cos\theta_W - B^\mu \sin\theta_W$$

$$A^\mu = A^\mu_3 \sin\theta_W + B^\mu \cos\theta_W$$

of masses $M_z = \frac{f}{2}\sqrt{g^2 + g'^2}$ and zero respectively.

The famous Weinberg angle is defined by $\tan\theta_W = \frac{g'}{g}$.

Questions and Exercises

0.1.1) Prove (0.1.2).

0.1.2) Compute the normalised wave functions $\Psi_1(q), \Psi_2(q)$ for the first and second excited states of the Harmonic Oscillator.

0.1.3) Check explicitly that $(\hat{a})^2\Psi_1(q) = 0$.

0.1.4) Prove that the Hermitian conjugate of $\hat{a}$ is $\hat{a}^\dagger$.

0.2.1) Compute

$$\int_{-\infty}^{+\infty} x^4 e^{-\alpha x^2}.$$

0.3.1) Consider a free quantum mechanical particle, of mass m, moving along a 1 dimensional line. Given a wave function $\Psi = e^{-x^2}$ at t = 0, calculate, in both Heisenberg and Schrödinger pictures, the probability density of finding it at x = 0 at time t.

0.3.2) Check $[\hat{q}, \hat{p}] = i$ in both Schrödinger and Heisenberg pictures.

0.3.3) Check that the $\hat{a}, \hat{a}^\dagger$ commutation relations are the same in the two pictures.

0.4.1) Derive the Lagrangian for a particle falling, under gravity, near the surface of the earth. Derive the equations of motion, and solutions, in both Lagrangian and Hamiltonian form.

0.5.1) Compute

$$\int_{-\infty}^{+\infty} dx e^{-x^2}\delta(2x-1)$$

$$\int_{-\infty}^{+\infty} dx e^{-x^2}\delta(2x^2-1)$$

0.5.2) Prove that

$$lim_{\lambda\to\infty}\cdot\sqrt{\frac{\lambda}{\pi}}e^{\frac{-x^2}{2\lambda}} = \delta(x)$$

0.5.3) Compute

$$\int_{-\infty}^{+\infty} dx dy e^{-(x^2+y^2)}\delta(2x-y-1)\delta(x+y-2)$$

1.1.1) Using Lagrange's equation, solve for the motion of a double pendulum undergoing small oscillations.

1.2.1) What are the momenta conjugate to θ, ϕ?

1.3) Compute the Poisson brackets

$$\{q^2, p\}, \{q, p^2\}$$

How do they compare with

$$[\hat{q}^2, \hat{p}], [\hat{q}, \hat{p}^2]$$

1.4) Prove that Hamilton's Classical equations can be rewritten

$$\dot{q}_i = \{q_i, H\}$$

$$\dot{p}_i = \{p_i, H\}$$

Does this satisfy Dirac's Classical to Quantum prescription?

2.1.1) Derive an expression for $U_3(t)$. Prove it is given by the time ordered prescription.

12) Prove that

$$\sum_{q_i} |q_i\rangle\langle q_i| = 1$$

2.2.2) For a free particle with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

write down the Δ - time slice approximation explicitly for (2.2.1) and (2.2.2)

2.2.2) Prove that if $A(t_i, t_f, q_i, q_f)$ is the amplitude to get from position $q_i at t_i$ to position q_f at t_f then this function satisfies

$$\int dq' A(t_i, t', q_i, q') A(t', t_f, q', q_f) = A(t_i, t_f, q_i, q_f)$$

for $t_i < t' < t_f$.

2.3.1) Check that equations (2.3.1) are obtained by applying Dirac's quantisation prescription to Hamilton's Classical equations.

3.1.1) Derive equation 3.1.1.

3.2.2) Derive equation 3.2.1 from 3.2.3.

3.2.3) Derive (3.2.4)

3.2.4) Check the ϕ field describes a boson.

4.1.1) Using the free Lagrangian for ϕ but a new interaction $\lambda\phi^4$ compute, by commutation, the amplitude for 2 $\phi's$ of momenta q_1, q_2 to scatter into two ϕ's of momenta p_1, p_2 in lowest order.

4.2.1) Compute $\frac{\delta}{\delta\phi(y)}$ and $\frac{\delta^2}{\delta\phi(y)\delta\phi(z)}$ of

$$a) \int \phi(x) J(x) dx$$

$$b) \left[\int \phi(x) J(x) dx\right]^2$$

$$c) \int \phi^2(x) J(x) dx$$

4.2.2) Prove (4.2.7)

5.1.1) Check the product of 2 matrices of the form 5.1.1 gives another matrix of the same form.

5.1.2) Check that the commutators of the Gell-Mann λ_i matrices give linear combinations of the λ_i.

5.1.3) Prove 5.1.4.

5.1.4) Prove that the operators ($F_i = \frac{\lambda_i}{2}$)

$$T_\pm = F_1 \pm iF_2, U_\pm = F_6 \pm iF_7, V_\pm = F_4 \pm iF_5$$

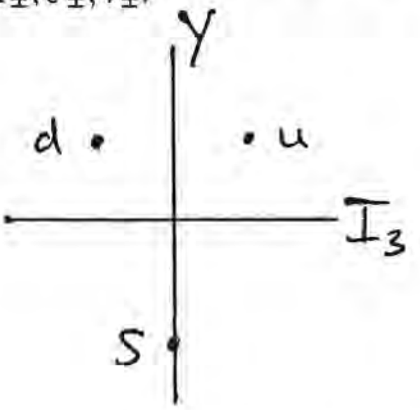
satisfy ($T_3 = F_3, Y = \frac{2}{\sqrt{3}}F_8$)

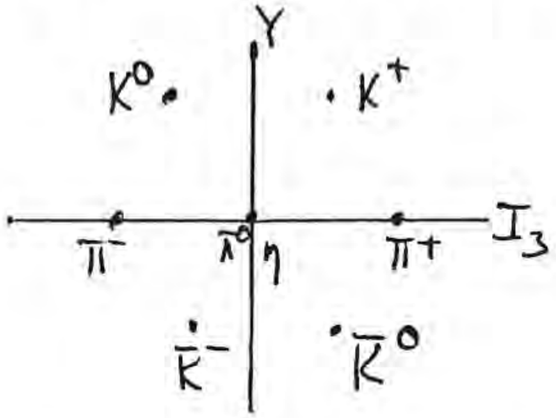
$$[T_3, T_\pm] = \pm T_\pm \qquad [Y, T_\pm] = 0$$

$$[T_3, U_\pm] = \mp\frac{1}{2}U_\pm \qquad [Y, U_\pm] = \pm U_\pm$$

$$[T_3, V_\pm] = \pm\frac{1}{2}V_\pm \qquad [Y, V_\pm] = \pm V_\pm$$

Hence show that they operate as raising and lowering operators on eigenstates of T_3, Y. Hence show how the states in the 3 dim., 8 dim. representations of $SU(3)$ are related by $T_\pm, U_\pm, V_\pm$.





6.1.1) You are used to the electric field due to a static charge q being given by a scalar potential

$$\phi(\underline{r}, t) = \frac{1}{4\pi\epsilon_0}.\frac{q}{r}$$

$$\underline{A}(\underline{r}, t) = 0$$

Prove it is equally well described by

$$\phi(\underline{x}, t) = 0$$

$$\underline{A}(\underline{r}, t) = \frac{tq\underline{r}}{4\pi\epsilon_0.r^3}$$

So a voltmeter had better not measure ϕ ! What does it measure?

6.4.1) Another way of defining $F_{\mu\nu} = [D_\mu, D_\nu]$. Prove that this is equivalent to 6.4.5. Use this to prove 6.4.6.

RELATIVISTIC QUANTUM MECHANICS, QED AND QCD

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Relativistic Quantum Mechanics, QED and QCD.

1. Introduction.
2. Relativistic Wave Equations.
3. The Dirac Equation.
4. Free Particle Solutions.
5. Negative Energy Solutions.
6. Single Particle Solutions and Spin.
7. Lorentz Covariance.
8. Bilinear Covariants and the Clifford Algebra.
9. Fermi's Golden Rule.
10. Cross-sections and Decay Rates.
11. QED: $e^-\mu^-\to e^-\mu^-$.
12. Higher Orders?
13. Other QED Processes.
14. Electron-PROTON Elastic Scattering.
15. Renormalisation.
16. QCD.
17. Asymptotic Freedom.
18. $e^+e^- \to$ hadrons.

1. INTRODUCTION.

This course is mostly about the techniques of relativistic quantum mechanics and quantum field theory. The aim is to get to the point that the Feynman rules can be "read off" a Lagrangian and used to calculate amplitudes and cross-sections. It is therefore complementary to (and dependent on) other lectures at this school, particularly those of Ken Barnes, wherein the connection between Lagrangians and Feynman rules is established via the full apparatus of canonical quantisation.

Through time and space constraints, the material is confined to the parity-conserving sector of the standard model: QED and QCD. Weak interactions and electroweak "unification" will be covered in David Bailin's lectures.

After introductory material on the Dirac equation (sections 2-8) and Fermi's Golden Rule and phase space calculations (sections 9-10), the Feynman rules of QED are used (section 11) to calculate the leading contribution to the cross-section for an elementary process ($e^-\mu^-\to e^-\mu^-$). Then other QED processes and aspects of renormalisation are discussed (sections 12-15).

Finally the QCD Lagrangian is introduced, and an attempt made to demonstrate how it comes about, through asymptotic freedom, that reliable perturbative calculations of strong interaction effects can be made for certain processes (sections 16-18).

Turning to the question of conventions, I will made heavy use of 4-vector notation:

$$x^\mu = (ct,\ \boldsymbol{x})$$

except that throughout $c = \hbar = 1$.

I use metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

$$g_{\mu\nu}x^{\mu}x^{\nu} = t^2 - \mathbf{x}^2.$$

Generally, my conventions are those of Bjorken and Drell <u>except</u> for the spinor normalisation where I follow the usual modern practice: see eq. (6.14). The main practical consequences of this are that

$$\sum_{spins} \frac{u\bar{u}}{v\bar{v}} = \not{p} \pm m, \text{ rather than } \frac{\not{p} \pm m}{2m} \quad (B\text{-}D),$$

and that (10.1) holds for both bosons and fermions.

Relevant books are:

Bjorken & Drell Vol. 1

Aitchison and Hey

and of course Prof. Sachrajda's lectures at previous schools.

2. RELATIVISTIC WAVE EQUATIONS.

One way to develop non-relativistic quantum mechanics is to start with the fundamental commutation relations for position and momentum:-

$$[r_i, p_j] = i\delta_{ij} \tag{2.1}$$

and energy and time:-

$$[t, E] = -i \tag{2.2}$$

and observe that they are satisfied if we identify

$$\boldsymbol{p} \to -i\nabla \tag{2.3}$$

and

$$E \to i\frac{\partial}{\partial t} \tag{2.4}$$

then if we substitute into the nonrelativistic energy-momentum relation:

$$E = \frac{\boldsymbol{p}^2}{2m} \tag{2.5}$$

we generate Schrödinger's equation:

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi. \tag{2.6}$$

Since (2.5) is a nonrelativistic approximation, it's clear that (2.6) will not be valid for relativistic phenomena.

However (2.3), (2.4) are perfectly compatible with relativity; they amount to a relationship between two 4-vectors:

$$p^{\mu} \Rightarrow i\frac{\partial}{\partial x_{\mu}} \equiv i\partial^{\mu}$$

where (2.7)

$$p^{\mu} = (p^{0}, \boldsymbol{p}) = (E, \boldsymbol{p})$$

and

$$x_{\mu} = g_{\mu\nu}x^{\nu} = (x^{0}, -\boldsymbol{x}) = (t, -\boldsymbol{x}) .$$

The obvious guess for a relativistic generalisation of (2.6) is to take the relativistic energy momentum relation:

$$E^2 = m^2 + \boldsymbol{p}^2$$

(2.8)

and use (2.7) (or (2.3) and (2.4)), giving

$$-\frac{\partial^2}{\partial t^2}\phi = (m^2 - \nabla^2)\phi$$

or

$$(\Box + m^2)\phi = 0$$

(2.9)

where

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial x_{\mu}\partial x^{\mu}} = \partial^{\mu}\partial_{\mu} .$$

(2.9) is called the Klein-Gordon equation. It has solutions of the form

Although one can derive an equation with the right form:-

$$\frac{\partial \rho}{\partial t}+\nabla . \boldsymbol{J} = 0 \, [\equiv \partial_\mu J^\mu = 0]$$

(2.12)

where

$$\rho = i [\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}]$$

(2.13)

and

$$\boldsymbol{J} = -i (\phi^* \nabla \phi - \phi \nabla \phi^*)$$

(2.14)

ρ cannot be a probability density because, for example with (2.10) we have

$$\rho = 2 |N|^2 E$$

(2.15)

which is not positive definite, since we have already decided that E can be negative.

It turned out that these problems were not really problems at all, and that the negative energy solutions could be reinterpreted as antiparticles. It was the attempt to circumvent these (non-) problems which led to the Dirac equation - which (it turned out) also permitted negative energy solutions!

3. THE DIRAC EQUATION.

Dirac looked for an equation that was linear in the time derivative (like Schrödinger's equation) but at the same time LORENTZ COVARIANT. Hence it had to be linear in space derivatives too, and his starting point was

$$i\frac{\partial\psi}{\partial t} = -i\boldsymbol{\alpha}.\nabla\psi + \beta m\psi \tag{3.1}$$

or (with the identification (2.3), (2.4)):

$$E\psi = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)\psi = H\psi \tag{3.2}$$

To demonstrate Lorentz covariance of (3.1) we will need to know how ψ transforms: it will turn out that the simple scalar rule (2.11) doesn't work. What this means is that a necessary consequence of the postulate (3.1) is that the corresponding particle has spin. We will return to this point later.

Now if we assume that ψ describes a free particle, it is natural to ask that ψ satisfy the KG equation in order that the particle obey the usual energy-momentum relation, (2.8). This leads to relationships among α, β as follows:-

$$\begin{aligned} &(i\frac{\partial}{\partial t} + i\boldsymbol{\alpha}.\nabla - \beta m)\psi = 0 \\ \rightarrow &(i\frac{\partial}{\partial t} - i\boldsymbol{\alpha}.\nabla + \beta m)(i\frac{\partial}{\partial t} + i\boldsymbol{\alpha}.\nabla - \beta m)\psi = 0 \\ \rightarrow &(-\frac{\partial^2}{\partial t^2} + \alpha_i\alpha_j\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \beta^2 m^2)\psi + im(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} \\ &= 0 \end{aligned} \tag{3.3}$$

So ψ will obey the KG equation if

$$\frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) = \delta_{ij}$$

(3.4a)

$$\alpha_i\beta + \beta\alpha_i = 0$$

(3.4b)

$$\beta^2 = 1$$

(3.4c)

It is clear that α, β cannot be numbers. Although we can proceed treating them as abstract entities satisfying (3.4) it is often useful to use an explicit realisation, in the form of matrices.

Note that hermiticity of the Hamiltonian ((3.2)) means that α, β must be hermitian. There is an obvious set of matrices satisfying (3.4a): the Pauli matrices:-

$$\alpha_i = \sigma_i, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(3.5)

Note that they are traceless and have eigenvalues ± 1. This is quite general for matrices satisfying (3.4). (see problem (2)).

The Pauli matrices are adequate for the case $m = 0$, since then we wouldn't have the terms involving β in (3.3). This means that massless fermions can be described by two-component spinor objects ψ satisfying

$$(i\frac{\partial}{\partial t} + i\sigma.\nabla)\psi = 0.$$

(3.6)

Unfortunately there is no β that satisfied (3.4b) if we choose $\alpha_1 = \sigma_1$; we have to go to 4-dimensional matrices for the minimal realisation. A popular representation satisfying (3.4) is:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.7}$$

called the PAULI REPRESENTATION.

In (3.7) each entry represents a 2 x 2 block, so that for example

$$\boldsymbol{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

NB the 1 in the (3.7) form of β represents

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \textit{not} \ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The block form is convenient for doing algebra, since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$$

In other words, you can multiply as if the blocks were numbers, as long as you maintain matrix ordering correctly.

Another popular representation is the <u>chiral</u> one:

$$\boldsymbol{\alpha} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

The chiral representation is more convenient in theories with parity violation (like the standard model).

Given α, β as 4 x 4 matrices, the structure of (3.1) dictates that ψ be a 4-component object: which (with hindsight) we term a spinor.

The first indication of success is that we can derive a continuity equation in the form (2.12), with $\rho = \psi^\dagger\psi$ and $\mathbf{J} = \psi^\dagger\alpha\psi$ where ρ is positive definite and hence a candidate probability density.

4. FREE PARTICLE SOLUTIONS.

We seek plane wave solutions in the form

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} e^{-ip.x} = \begin{pmatrix}\phi\\ \chi\end{pmatrix} \quad e^{-i(Et-\boldsymbol{p}.\boldsymbol{x})} \tag{4.1}$$

where

$$\phi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \ \chi = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}.$$

are independent of x^μ. If we plug this in (3.2), using the Pauli rep, we get:

$$E\begin{pmatrix}\phi\\ \chi\end{pmatrix} = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}m & \boldsymbol{\sigma}.\boldsymbol{p}\\ \boldsymbol{\sigma}.\boldsymbol{p} & -m\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} \tag{4.2}$$

So

$$(E + m)\chi = \boldsymbol{\sigma}.\boldsymbol{p}\,\phi \tag{4.3}$$

and

$$(E - m)\,\phi = \boldsymbol{\sigma}.\boldsymbol{p}\,\chi\ . \tag{4.4}$$

Let's look first at p = 0 solutions.

Then we have either

(i)

$$E = m,\ \chi = 0$$

or (ii)

$$E = -m,\ \phi = 0$$

So we have for particles at rest, positive energy solutions

$$\psi = \begin{pmatrix}\phi\\ 0\end{pmatrix} e^{-imt}$$

(4.5)

and negative energy solutions

$$\psi = \begin{pmatrix}0\\ \chi\end{pmatrix} e^{imt}$$

(4.6)

Contrary to Dirac's original motivation, we still have the negative energy solutions. ϕ, χ are two-components constant spinors; we will see that these two components correspond to the possible spin states of the particle and antiparticle respectively.

For $\mathbf{p} \neq 0$, $| E | \neq m$ and it is easy to show from (4.3), (4.4) that

$$E^2 = m^2 + \boldsymbol{p}^2 \tag{4.7}$$

(which we knew already, since we imposed that ψ satisfied the KG equation) and $(E \neq -m)$

$$\psi(x) = \begin{pmatrix}\phi\\ \dfrac{\boldsymbol{\sigma}.\boldsymbol{p}}{E+m}\phi\end{pmatrix} e^{-ip.x} \tag{4.8}$$

or $(E \neq m)$

$$\psi(x) = \begin{pmatrix} \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{E-m}\chi \\ \chi \end{pmatrix} e^{-ip.x}$$

(4.9)

As in the KG case we have $E = \pm (p^2 + m^2)^{½}$.

The forms (4.8), (4.9) are convenient for representing the positive and negative energy solutions respectively, as is evident from (4.5), (4.6). At this point it is convenient to introduce the u,v spinors and in doing so we need to make a slight change in notation. From now on, we will always define E so as to have E > 0. Then while in (4.8) we have simply

$$\psi(x) = u(E,p)\, e^{-ip.x} = \begin{pmatrix} \phi \\ \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{E+m}\phi \end{pmatrix} e^{-ip.x},$$

(4.10)

we represent <u>negative energy</u> solutions by (instead of (4.9))

$$\psi(x) = v(E,p)\, e^{ip.x} = \begin{pmatrix} \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{E+m}\chi \\ \chi \end{pmatrix} e^{ip.x}$$

(4.11)

where in both (4.10) and (4.11), $p.x = Et-\mathbf{p}.\mathbf{x}$. Thus (4.11) represents a solution with energy -E and momentum -**p**. Both u and v depend on the spin state of the particle, as we shall see in section 6.

5. NEGATIVE ENERGY SOLUTIONS: the Feynman interpretation.

Dirac had a way of thinking about the negative energy solutions based on the idea that all the negative energy states were full so that if one electron was promoted to a positive energy state the resulting 'hole' in the Dirac 'sea' of negative energy states would be a (positive energy) positron.

The problem with this is that it doesn't work for bosons (and hence the Klein-Gordon equation) since they have no exclusion principle.

Feynman gave a simpler interpretation. To understand it, recall that f(ct - x) is a R-moving wave and f(ct + x) a L-moving one. Then our single-particle solution

$$\psi(\boldsymbol{x}, t) = u\, e^{-i(Et-\boldsymbol{p}.\boldsymbol{x})} \tag{5.1}$$

is a plane wave if $E = (p^2 + m^2)^{½}$ moving in the direction of p

Then

$$\psi'(\boldsymbol{x}, t) = u\, e^{-i(E't-\boldsymbol{p}.\boldsymbol{x})} \tag{5.2}$$

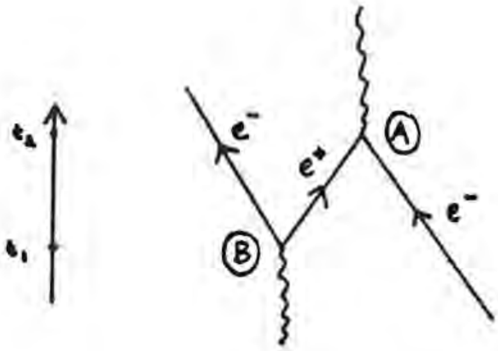
where $E' = -(p^2 + m^2)^{½}$,

can just be written

$$\psi'\ (\boldsymbol{x}, t) = u\, e^{-i(Et' - \boldsymbol{p}.\boldsymbol{x})} \tag{5.3}$$

which is identical with ψ, except that because $t' = -t$, if we think of ψ', as a wave propogating in the p direction then the wave is propogating "BACKWARDS IN TIME"!

This dangerous sounding conclusion is ameliorated by the realisation that a proper interpretation follows when we introduce antiparticles. Consider the following sequence of events:



An e^+e^- pair is created at time t_1; The e^+ is annihilated by a different electron at time t_2.

The mathematics of these events is as follows: an electron propogates to point A, emits a photon and thus becomes a negative energy state; propogates backwards in time to B where it absorbs a photon and becomes positive energy again.

Note that antiparticles necessarily have opposite charge to particles, since if charge (+Q) is emitted this is equivalent to (-Q) being absorbed. (We could pursue this by looking at the coupling of a Dirac fermion to an electromagnetic field).

It is this $\pm$ Q sign flip that in the KG case enables us to save the interpretation of ρ (eq. (2.15)) as a probability density: we just modify ρ by multiplying by Q:

$$\rho \rightarrow Q.2|N|^2.E.$$

Now both +ve and -ve energy solutions correspond to +ve ρ.

In the Dirac case ρ was already +ve; but here there's another minus sign associated with the anticommuting nature of the fermion fields.

The upshot of all this is that when we write the plane wave expansion of a Dirac fermion operator of spin s in quantum field theory:

$$\Psi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3 . 2E} \left[\hat{b}(p, s) u(p, s) e^{-ip.x} + \hat{d}^{\dagger}(p, s) v(p, s) e^{ip.x} \right]$$

(5.4)

where $p.x = Et - \mathbf{p}.\boldsymbol{x}$, then $\hat{b}$ is an operator that annihilates a fermion of spin s and 4-momentum $p^{\mu} = (E, \mathbf{p})$. Then rather than interpreting $d^{\dagger}$ as an operator that annihilates a fermion of 4-momentum $(-E, -\mathbf{p})$, we interpret it as <u>creating</u> an <u>antifermion</u> of 4-momentum $(E, \mathbf{p})$. Where the <u>physics</u> enters is when we define the vacuum to be the state $|0\rangle$ such that

$$\hat{b}|0\rangle = \hat{d}|0\rangle = 0$$

(5.5)

instead of

$$\hat{b}|0\rangle = \hat{d}^{\dagger}|0\rangle = 0 .$$

The relationship between the field operator and the wave function ψ we have dealt with is that the latter is the matrix element of ψ between the vacuum and a single particle state. The fact that $d^{\dagger}$ creates an <u>antifermion</u> (i.e. a particle with opposite charge to the fermion) becomes clear if we calculate the charge operator:

$$\hat{Q} \sim \int d^3x \;\; \Psi^{\dagger}\Psi \sim \int d^3p [b^{\dagger}b - d^{\dagger}d] \; .$$

(5.6)

The minus sign in (5.6) arises because d, $d^{\dagger}$ obey anticommutation relations rather than commutation relations. Thus while the result is analagous to the bosonic one (see Professor Barnes's eq. (2.61)), the origin of the minus sign is different.

6. SINGLE PARTICLE SOLUTIONS AND "SPIN".

The angular momentum operator L for a particle is given by

$$L = r \wedge p \tag{6.1}$$

for a particle with momentum p.

[Note: when we take classical objects over into quantum mechanics we normally have to worry about operator-ordering ambiguities if we have a product of operators. Doesn't arise here, even though r and p don't commute. Why not?]

If we take the Dirac Hamiltonian (3.2), and calculate [L, H], we get: (using 2.1))

$$\begin{aligned}
[L_i, H] &= \epsilon_{ijk}\left[r_j p_k, \alpha_l p_l\right] \\
&= \epsilon_{ijk} p_k \alpha_l \left[r_j, p_l\right] \\
&= i\epsilon_{ijk} p_k \alpha_j \\
\therefore [L, H] &= i\alpha \wedge p \quad .
\end{aligned} \tag{6.2}$$

Thus eigenstates of L^2 and L_3 (say) are not eigenstates of the Hamiltonian :- angular momentum is not conserved. This is another clue to the existence of intrinsic spin.

If we define

$$J = L + S \tag{6.3}$$

where

$$S = -\frac{i}{2}\alpha_1\alpha_2\alpha_3\boldsymbol{\alpha}$$

(6.4)

then it's easy to check that

$$[\boldsymbol{S}, \beta] = 0, \quad [S_i, \alpha_j] = i\epsilon_{ijk}\alpha_k$$

(6.5)

Using (6.5) we find

$$[\boldsymbol{S}, H] = -i\boldsymbol{\alpha}\wedge\boldsymbol{p}$$

(6.6)

and hence

$$[\boldsymbol{J}, H] = 0 \ .$$

(6.7)

It is natural to interpret S as a contribution to the angular momentum <u>intrinsic</u> to the particle.

[To really check this, we should look at the Dirac equation in an electromagnetic field. We should also do an experiment, of course.]

In the Pauli representation,

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$$

(6.8)

so

$$\boldsymbol{S}^2 = \frac{3}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad .$$

(6.9)

Recalling that the eigenvalue of J^2 in a state of spin j is j(j+1) we see that S represents spin ½.

Let's look at our free particle solutions again. Taking (4.10) and putting in

$$\phi = N\begin{pmatrix}1\\0\end{pmatrix}, \qquad N\begin{pmatrix}0\\1\end{pmatrix}$$

and also choosing $p = (0, 0, p_z)$, we have

$$\psi = u_\uparrow e^{-ip.x}, \qquad\qquad u_\downarrow e^{-ip.x} \tag{6.10}$$

where

$$u_\uparrow = N\begin{pmatrix}1\\0\\ \dfrac{p_z}{E+m}\\0\end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix}0\\1\\0\\ \dfrac{-p_z}{E+m}\end{pmatrix}$$

The reason for the $\uparrow,\downarrow$ notation becomes clear when we observe that

$$S_z u_\uparrow = \frac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} u_\uparrow = \frac{1}{2}\, u_\uparrow \tag{6.11}$$

and

$$S_z u_\downarrow = \frac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} u_\downarrow = -\frac{1}{2} u_\downarrow \ . \tag{6.12}$$

So $u\uparrow$, $u\downarrow$ represent spinors with spin up and down respectively along the z axis.

For "negative energy" solutions once again choosing

$$p_x = p_y = 0 ; \chi = N\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$

we have

$$\psi = v_\uparrow e^{ip.x}, \qquad v_\downarrow e^{ip.x}$$

where

$$v_\uparrow = N\begin{pmatrix}0\\-p_z/E+m\\0\\1\end{pmatrix}, \; v_\downarrow = N\begin{pmatrix}p_z/E+m\\0\\1\\0\end{pmatrix} \quad . \tag{6.13}$$

Once again $v_\uparrow$, $v_\downarrow$ are eigenstates of S with eigenvalues ∓ 1. This apparently perverse choice is because $v_\uparrow$ represents the <u>absence</u> of a spin down negative energy electron, i.e. a spin <u>up</u>, positive energy position.

Finally we turn to the question of normalisation: i.e. the choice of N. We have (from (4.10)).

$$u^\dagger u = \phi^\dagger\phi + \phi^\dagger\frac{(\boldsymbol{\sigma}.\boldsymbol{p})^2}{(E+m)^2}\phi \quad .$$

So

$$u^\dagger u = N^2\left(1 + \frac{p^2}{(E+m)^2}\right)$$

and we choose N so that

$$u^\dagger u = 2E,$$

i.e. (6.14)

$$N = (E+m)^{\frac{1}{2}} .$$

This normalisation corresponds to having 2E particles per unit volume.

The reason it's a convenient one is that, as we shall see in the next section, $u^\dagger u$ transforms like E under Lorentz transformations (i.e. as the "time" component of a four-vector).

The v spinor is normalised in the same way, so that $v^\dagger v = 2E$.

7. LORENTZ COVARIANCE AND THE DIRAC EQUATION.

It is convenient to replace β, α by new objects γ^μ where

$$\gamma^0 = \beta, \ \gamma^i = \beta\alpha_i \quad . \tag{7.1}$$

In the Pauli rep,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{7.2}$$

In terms of γ^μ, the fundamental relations (3.4) become

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{7.3}$$

This is called a CLIFFORD algebra.

For subsequent use, we note the form of the Dirac equation for spinors u, v:-

From
$$E\psi = (\boldsymbol{\alpha}.\boldsymbol{p} + \beta m)\psi$$
we have
$$E\gamma^0\psi = (\gamma^i p^i + m)\psi$$
or
$$\gamma^\mu p_\mu \psi = m\psi .$$
or
$$(\not{p} - m)\psi = 0 \quad (\not{p} = \gamma^\mu p_\mu)$$
thus
$$(\not{p} - m)\, u(p) = 0 \tag{7.4a}$$
and
$$(\not{p} + m)\ v\ (p) = 0 \ . \tag{7.4b}$$

To get (7.4b), recall that v is a spinor with (-E, -p). We now want to see if it is consistent to require that the Dirac equation preserves its form under Lorentz transformations. That is, we want that the wave function used by an observer in a different reference frame to describe a given electron obeys the Dirac equation in his reference frame. This is a non-trivial exercise, as those of you who have done the same exercise for

Maxwell's equations will realise. The key is to determine how the wave-function itself transforms. (Just as in the Maxwell case it is to determine how the E, B fields transforms).

Under a L.T.,

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{7.5}$$

[This is shorthand for the familiar

$$t' = \gamma\left(t - \frac{vx}{c^2}\right),\ x' = \gamma\,(x - vt)$$
$$y' = y,\, z' = z$$

for the special case of a LT along a common x axis.]

Note that from the requirement that

$$g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu}$$

we have that
$$g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta} \tag{7.6}$$

(this is analagous to the orthogonality property that defines a rotation matrix)

Now the Dirac equation is

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0. \tag{7.7}$$

we want to have $\psi'(x')$ such that

$$(i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0.$$

(7.8)

The question is, how is $\psi'(x')$ related to $\psi(x)$?

If $\psi(x)$ was a LORENTZ SCALAR, we would expect that $\psi'(x') = \psi(x)$.

What happens with a vector field $A^\mu(x)$ is that we get

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x);$$

(7.9)

i.e. the components of the vector field mix up under the transformation.

So we might expect that the components of ψ also mix, i.e. that

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$$

(7.10)

where $S(\Lambda)$ is a 4 x 4 matrix acting on the spinor index of ψ.

Now a covariant four-vector, A_μ transforms as

$$A_\mu'(x') = (\Lambda^{-1})^\nu{}_\mu A_\nu(x)$$

(7.11)

or

$$A_\nu(x) = \Lambda^\mu{}_\nu A'_\mu(x').$$

so in particular

$$\partial_\mu = \Lambda^\sigma{}_\mu \partial_\sigma',$$

(7.12)

and (7.7) can be written

$$(i\gamma^\mu \Lambda^\sigma{}_\mu \partial'_\sigma - m)\psi(\Lambda^{-1}x') = 0.$$

(7.13)

Now we multiply by the matrix S:

$$(iS\gamma^\mu S^{-1}\Lambda^\sigma{}_\mu \partial'_\sigma - m)\ S\psi(\Lambda^{-1}x') = 0$$

(7.14)

which is

$$(iS\gamma^\mu S^{-1}\Lambda^\sigma{}_\mu \partial'_\sigma - m)\ \psi'(x') = 0.$$

(7.15)

So all we need is that, for any Λ, we can find an $S(\Lambda)$ such that

$$S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu \ .$$

(7.16)

What we now need to show is that for any given Lorentz transformation Λ, there is an S that satisfies (7.16). If there is, then the Dirac equation is Lorentz covariant. Notice that replacing S by the unit matrix will <u>not</u> satisfy (7.16); so ψ is <u>not</u> a scalar field.

A quickie way of seeing that it must be true is as follows.

Define $$\hat{\gamma}^\nu = \Lambda^\nu{}_\mu \gamma^\mu.$$

Then

$$
\begin{aligned}
\{\hat\gamma^\mu, \hat\gamma^\nu\} &= \Lambda^\mu{}_\alpha \gamma^\alpha \Lambda^\nu{}_\beta \gamma^\beta + \mu \leftrightarrow \nu \\
&= \gamma^\alpha\gamma^\beta (\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta + \Lambda^\nu{}_\alpha \Lambda^\mu{}_\beta) \\
&= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha) \\
&= 2g^{\alpha\beta}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = 2g^{\mu\nu}.
\end{aligned}
$$

where we have used (7.3) and (7.5). $\hat\gamma^\mu$ obeys the same algebra as γ^μ; so it's just another set of 4 matrices that would do perfectly well for the Dirac matrices.

Now we have to invoke the theorem that all such sets are equivalent, and can be connected by a similarity transformation:

$$
S\hat\gamma^\mu S^{-1} = \gamma^\mu \tag{7.17}
$$

But this is just (7.16). So assuming all possible sets of γ^μ can be related via (7.17), then there exists an S satisfying (7.16). To construct S explicitly (in the infinitesimal case) let

$$
\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + O(\lambda^2) \tag{7.18}
$$

then
$$
g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}
$$

gives
$$
\lambda_{\mu\nu} + \lambda_{\nu\mu} = 0. \tag{7.19}
$$

Then let

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\lambda^{\mu\nu} + 0(\lambda^2)$$
$$S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\lambda^{\mu\nu} + 0(\lambda^2)$$

(7.20)

where $\sigma_{\mu\nu}$ is some (as yet unknown) set of 4 x 4 matrices. Sticking this in (7.16) we get

$$-\frac{i}{4}\sigma_{\alpha\beta}\lambda^{\alpha\beta}\gamma^{\nu} + \frac{i}{4}\gamma^{\nu}\sigma_{\alpha\beta}\lambda^{\alpha\beta} + \gamma^{\nu} + \gamma^{\mu}\lambda^{\nu}{}_{\mu} = \gamma^{\nu}$$

whence

$$\gamma^{\mu}\lambda^{\nu}{}_{\mu} = +\frac{i}{4}\lambda^{\alpha\beta}\ [\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}],$$

(7.21)

$$LHS = \lambda^{\nu\mu}\gamma_{\mu} = g^{\nu}{}_{\alpha}g^{\mu}{}_{\beta}\lambda^{\alpha\beta}\gamma_{\mu}$$
$$= \frac{1}{2}(g^{\nu}{}_{\alpha}g^{\mu}{}_{\beta} - g^{\mu}{}_{\alpha}g^{\nu}{}_{\beta})\lambda^{\alpha\beta}\gamma_{\mu}$$

So finally

$$g^{\nu}{}_{\alpha}\gamma_{\beta} - g^{\nu}{}_{\beta}\gamma_{\alpha} = \frac{i}{2}[\sigma_{\alpha\beta}, \gamma^{\nu}]$$

(7.22)

You can check that

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$$

satisfies (7.22), and so:

$$S = 1 + \frac{1}{8}\lambda^{\alpha\beta}[\gamma_{\alpha}, \gamma_{\beta}]$$

(7.23)

is the appropriate infinitesimal transformation.

Note that S is <u>NOT</u> unitary in general, but

$$\gamma^0 S^\dagger \gamma^0 = S^{-1} . \tag{7.24}$$

This can be seen from (7.23), but in fact follows quite generally from (7.16).

So S <u>is</u> unitary if we restrict to rotations, i.e. we choose both α, β to be spatial, (ijk) indices. This is to do with the "deep" fact that rotations correspond to the group O(3): a unitary group. The full Lorentz group, including boosts, is not unitary.

Given that

$$\psi' = S\psi$$

we have

$$\psi'^\dagger = \psi^\dagger S^\dagger$$

or, using (7.24),

$$\bar{\psi}' = \bar{\psi} S^{-1} \tag{7.25}$$

So the bilinear $\bar{\psi}\,\psi$

is LORENTZ INVARIANT:

$$\bar{\psi}'\psi' = \bar{\psi}\psi . \tag{7.26}$$

In the next section we'll consider the Lorentz transformation properties of general bilinears.

8. BILINEAR COVARIANTS AND THE CLIFFORD ALGEBRA

If we introduce $\sigma^{\mu\nu}$ and γ_5, given by

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]$$

and (8.1)

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

then the set of 16 matrices (8.2)

$$\Gamma : \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

form a linearly independent basis for products of the γ-matrices.

The elements of the set Γ can be used to construct fermion bilinears which have characteristic transformation properties with respect to the Lorentz group, as follows:-

$$\begin{array}{lll}
\bar\psi\psi \to \bar\psi'\psi' & = \bar\psi\psi & \textit{SCALAR} \\
\bar\psi\gamma_5\psi \to \bar\psi'\gamma_5\psi' & = (\det\Lambda)\ \bar\psi\gamma_5\psi & \text{PSEUDOSCALAR} \\
\bar\psi\gamma^\mu\psi \to \bar\psi'\gamma^\mu\psi' & = \Lambda^\mu{}_\nu\bar\psi\gamma^\nu\psi & \textit{VECTOR} \\
\bar\psi\gamma^\mu\gamma_5\psi \to \bar\psi'\gamma^\mu\gamma_5\psi' & = (\det\Lambda)\,\Lambda^\mu{}_\nu\psi\gamma^\nu\gamma_5\psi & \textit{AXIAL VECTOR} \\
\bar\psi\sigma^{\mu\nu}\psi \to \bar\psi'\sigma^{\mu\nu}\psi' & = \Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma\bar\psi\sigma^{\lambda\sigma}\psi & \textit{TENSOR}
\end{array}$$

(8.3)

The fact that $\bar\psi\gamma^\mu\psi$ is a four-vector has the following consequence. Notice that

$$\bar\psi\gamma^0\psi = \bar\psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$$

so $\psi^\dagger\psi$ transforms like the zeroeth component of a four-vector under Lorentz transformations. Hence the normalisation convention (6.14) is a Lorentz covariant one.

From (8.3), notice that the only difference between scalars and pseudoscalars is det Λ; and it is easy to prove (from eq. (7.6)) that

$$\det\Lambda = \pm 1 .$$

(8.4)

Transformations with det $\Lambda = -1$ are clearly not of the form (7.18); they involve a space reflection. If we define

$$\Lambda_p = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

(8.5)

which corresponds to reflecting all 3 space coordinates, can we find a new wave function ψ_p which obeys the Dirac equation in the reflected system?

From (7.16), we see that what we need is an S such that

$$\begin{aligned} S\gamma^0 S^{-1} &= \gamma^0 \\ S\gamma^i S^{-1} &= -\gamma^i \end{aligned}$$

(8.6)

By inspection, a solution is

$$S = \gamma^0$$

(8.7)

and hence

$$\psi_p(x') = \gamma^0 \psi(x)$$

(8.8)

or explicitly

$$\psi_p(x^0, -\boldsymbol{x}) = \gamma^0 \psi(x^0, \boldsymbol{x}) .$$

(8.9)

By the reasoning developed in eqs. (7.8), (7.10), (7.13)-(7.15) it follows that $\psi_p(x')$, obeys the Dirac equation in the reflected coordinate system: hence the Dirac equation preserves parity. (When Dirac set up his equation no-one dreamt that parity could be anything but a good symmetry).

For massless fermions the spinors

$$\psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\ \psi$$

(8.10)

are helicity eigenstates. The helicity operator is

$$\mathrm{H} = \frac{\boldsymbol{S}.\boldsymbol{p}}{p}$$

(8.11)

where S is given in eq. (6.4); it is straightforward to show that

$$\boldsymbol{S}.\boldsymbol{p} = -\frac{1}{2}\gamma^0\gamma_5\boldsymbol{\gamma}.\boldsymbol{p}$$

So that

$$\boldsymbol{S}.\boldsymbol{p}\ \psi = -\frac{1}{2}\gamma^0\gamma_5\boldsymbol{\gamma}.\boldsymbol{p}\ \psi = \frac{1}{2}\gamma_5|\boldsymbol{p}|\psi$$

(using the Dirac eqn. for m = 0).

Similarly

$$\boldsymbol{S}.\boldsymbol{p}\gamma_5\psi = +\frac{1}{2}|\boldsymbol{p}|\psi$$

So

$$H\psi_{R,L} = \pm\frac{1}{2}\psi_{R,L}$$

(8.12)

From (8.9) it is easy to show that for a right (left) handed spinor, the corresponding parity transformed spinor is left (right) handed. Now we will see in Dr. Maxwell's lectures that a crucial ingredient of the standard model is that ψ_L and ψ_R have <u>different</u> gauge interactions for all quarks and leptons. This is possible, because, of course parity is not a good symmetry.

Finally we should note another important symmetry of the Dirac equation: charge conjugation invariance. For c-numbers this means simply complex conjugation; but ψ transforms as follows:

$$\psi(x) \to \psi_c(x) = C\gamma^0\psi^*$$

(8.13)

where C is a Dirac matrix satisfying the relation

$$(C\gamma^0)\gamma^{\mu*}(C\gamma^0)^{-1} = -\gamma^\mu.$$

(8.14)

Once again, this invariance is not maintained in the standard electroweak model.

9. FERMI'S GOLDEN RULE.

The process by which amplitudes are turned into cross-sections begins with Fermi's Golden rule, which follows in turn from time-dependent perturbation theory in quantum mechanics.

Suppose we have

$$i\frac{\partial\psi}{\partial t} = H\psi = (H_o + H')\psi \tag{9.1}$$

where H' is a perturbation, and the problem with H_o alone is solved:

$$H_0\Phi = i\frac{\partial\Phi}{\partial t}, \tag{9.2}$$

i.e.

$$H_o\phi_n = E_n\phi_n, \qquad \Phi = \sum\nolimits_n \phi_n(\boldsymbol{x})\, e^{-iE_nt}. \tag{9.3}$$

Then we can always write

$$\psi\ (\boldsymbol{x}, t) = \sum\nolimits_n C_n(t)\,\phi_n(\boldsymbol{x})\, e^{-iE_nt} \tag{9.4}$$

by some theorem that says that the eigenstates of a Hamiltonian form a complete set.

Then substituting (9.4) in (9.1), you can show that

$$i\dot{C}_m = \sum\nolimits_n C_n\, H'_{mn}\, e^{i(E_m-E_n)t} \tag{9.5}$$

where

$$H'_{mn} = \int d^3x \; \phi_m^* H' \; \phi_n$$

(9.6)

and you need to use

$$\int d^3x \; \phi_m^* \; \phi_n = \delta_{mn}.$$

Suppose at $t = -T$, say $C_p = 1$, $C_n = 0 \quad n \neq p$

i.e. system starts in an eigenstate of H_o.

Then

$$i\dot{C}_m = H'_{mp} e^{i(E_m - E_p)t}$$

(9.7)

(this assumes that C_p remains close to 1 for all t of interest).

Then

$$iC_m(t) = H'_{mp} \int_{-T}^{t} e^{i(E_m - E_p)t'} dt' \; .$$

(9.8)

Here we have assumed H' has no explicit t-dependence.

The transition rate R_{mp} is the probability/unit time that the system begins in one state (p) and ends in another (m):

$$R_{mp} = \lim_{T \to \infty} \frac{|C_m(T)|^2}{T} = \lim_{T \to \infty} \frac{1}{2T} |H'_{mp}|^2 \left| \int_{-T}^{T} e^{i\omega_{mp}t} dt \right|^2$$

(9.9)

where $\omega_{mp} = E_m - E_p$.

Using the result

$$\lim_{a \to \infty} \frac{\sin^2 ax}{ax^2} = \pi \delta (x)$$

(9.10)

it is easy to show that

$$R_{mp} = 2\pi |H'_{mp}|^2 \delta (\omega_{mp}) \; .$$

(9.11)

Then if the number of final states with a specific energy E is given by a density function ρ_f (E) then we have for the transition i → f a total rate

$$R = 2\pi |H'_{fi}|^2 \rho_f (E_f) \, \delta (E_i - E_f)$$

(9.12)

This is Fermi's Golden rule. As we will see in the next section, the various terms in R have analogs in the relativistic quantum field theory case.

The effect of including higher orders of perturbation theory in the solution of (9.5) by the replacement in (9.12):

$$H'_{fi} \to H'_{fi} + \sum_{n \neq i} \frac{H'_{fn} H'_{ni}}{\omega_{ni}} + \; .\,.\,.\,.\,.$$

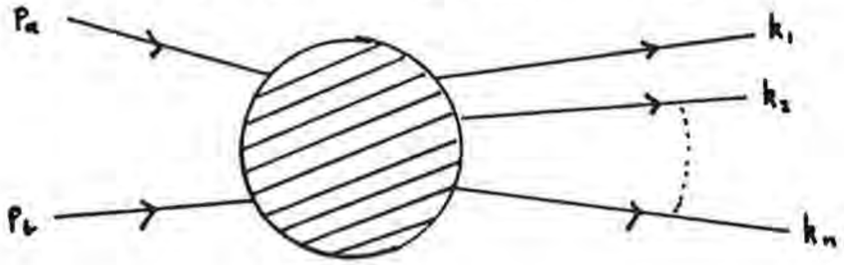
10. CROSS-SECTIONS AND DECAY RATES.

We now want to generalise Fermi's Golden Rule to the case of a relativistic quantum field theory.

For the process

$$a + b \rightarrow c + d + \ldots$$

$$(p_a) \quad (p_b) \qquad (k_1) \quad (k_2) \quad \ldots \; (k_n)$$



The result for the differential cross-section is

$$d\sigma = \left[\frac{1}{|\mathbf{v}_a - \mathbf{v}_b|} \quad \frac{1}{2E_a 2E_b}\right]|M|^2 \quad (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_i k_i) \;\; \rho S \tag{10.1}$$

where

$$\rho = \prod_{i=1}^{n} \quad \frac{d^3 k_i}{(2\pi)^3 2E_i}$$

and

$$S = \prod \frac{1}{m!}$$

for m identical particles in the final state.

The flux factor converts a transition rate to a cross-section. (The 1/2E factors are there because of our choice of normalisation of 2E particles/unit volume).

The flux factor (F) can be written in manifestly covariant form:-

$$F = \frac{1}{4E_aE_b|\mathbf{v}_a-\mathbf{v}_b|} = \frac{1}{4\left((p_a\cdot p_b)^2 - m_a^2m_b^2\right)^{\frac{1}{2}}} = \frac{1}{2\lambda(s,m_a^2,m_b^2)}$$

(10.2)

where

$$\lambda^2(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

(10.3)

and

$$s = (p_a + p_b)^2 \ .$$

(10.4)

(Note that the first expression in (10.2) is valid only in a frame in which a, b are collinear).

The phase space factor $\mathcal{P}$ is also (in spite of appearances) Lorentz invariant. It originates as follows:-

The number of states available for a particle in the momentum range p to p + dp is

$$d^3n = \frac{d^3\mathbf{p}}{(2\pi)^3}$$

in non-relativistic QM.

This is not a Lorentz invariant expression; but when we recall that our normalisation of single particle states corresponded to having 2E particles per unit volume, we have that the phase space available <u>per particle</u> is

$$\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2E}$$

To see that this is Lorentz invariant, we rewrite it:

$$\int \frac{d^3p}{2E} = \int d^4p\ \delta(p^2 - m^2)\theta(p^o)$$

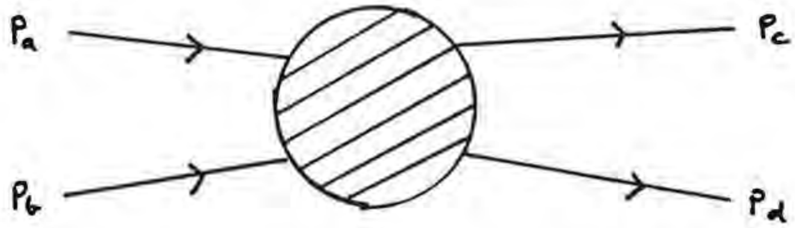
(10.5)

$$(where\ \theta(p^o) = 1\ if\ p^o>0, 0\ if\ p^o<o)$$

which is manifestly invariant under LTs preserving sign (p^o).

For the decay of a particle of mass M, the differential partial width in its rest frame, $d\Gamma$ is given by the same expression as for $d\sigma$, (10.1), except that the flux factor F is replaced by $(2M)^{-1}$.

<u>2 → 2 scattering</u>



In this important special case we can (by picking a Lorentz frame) explicitly do four of the six integrations using the Dirac δ-function. The result is, in the centre of mass frame (defined by $\mathbf{p}_a + \mathbf{p}_b = 0$):

$$\frac{d\sigma}{d\Omega^*} = \frac{\lambda(s, m_c^2, m_d^2)}{64\pi^2 s\ \lambda(s, m_a^2, m_b^2)} |M|^2$$

(10.6)

where Ω^* is the scattering solid angle in the centre of mass frame, and $s = (p_a + p_b)^2$. (9.6) is valid for any $|M|^2$; but if $|M|$ is a constant, then the total cross-section is just

$$\sigma = \frac{1}{16\pi s} \frac{\lambda(s, m_c^2, m_d^2)}{\lambda(s, m_a^2, m_b^2)} |M|^2$$

(10.7)

In $2 \to 2$ scattering it is traditional to use the following Lorentz invariant variables:-

$$
\begin{aligned}
s &= (p_a + p_b)^2 \\
t &= (p_a - p_c)^2 \\
u &= (p_a - p_d)^2
\end{aligned}
\tag{10.8}
$$

In fact only 2 of these are independent;

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \tag{10.9}$$

[For N particles there are 3N-10 independent variables.]

In the centre of mass frame, when we neglect the particle masses, we have simply

$$
\begin{array}{ll}
p_a = (E, \mathbf{p}) & p_b = (E, -\mathbf{p}) \\
p_c = (E, \mathbf{p}') & p_d = (E, -\mathbf{p}')
\end{array}
\tag{10.10}
$$

where

$$|\mathbf{p}| = |\mathbf{p}'| = E$$

Then

$$
\begin{aligned}
s &= 4E^2 \\
t &= -2E^2(1 - \cos\theta) \\
u &= -2E^2(1 + \cos\theta)
\end{aligned}
\tag{10.11}
$$

where θ is the scattering angle.

The region of the (s, t, u) variables of current interest is the region large s, t, u; fixed θ which we might call the QCD region. In olden days, there was more interest in the region large s, fixed t (hence $\theta \to 0$). This was called the Regge region. It turns out that the peculiar properties of QCD mean that predictions for the differential cross-section can typically be made in the QCD but not the Regge region.

A final remark on 2 → 2 scattering. <u>If</u> we assume that $|M|^2$ is independent of the azimuthal angle ϕ^*, then (10.6) can be written in the following form:-

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\left[\lambda\,(s, m_a^2, m_b^2)\right]^2}\ |M|^2 \tag{10.12}$$

It is instructive to check that this is consistent with (10.7) for the case of constant $|M|^2$: the key is in the limits of the t-integration.

11. QUANTUM ELECTRODYNAMICS.

The Lagrangian is (for a particle of charge q):

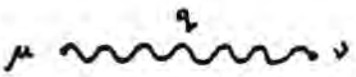
$$L = -\frac{1}{4}F_{\mu\nu}^2 + i\overline{\Psi}\not{D}\psi - m\overline{\Psi}\psi - \frac{1}{2}(\partial^\mu A_\mu)^2 \tag{11.1}$$

where

$$\not{D}\psi = \gamma^\mu(\partial_\mu + iqA_\mu)\psi$$

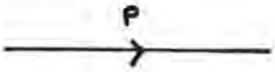
For the electron, $q = -e$ where $e \approx 0.3$.

FEYNMAN RULES:

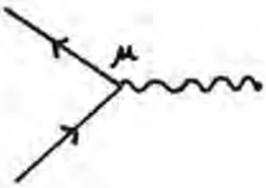


$$\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$$

(11.2)



$$\frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}$$



$$-iq\gamma^\mu$$

INCOMING FERMION (ANTIFERMION) WITH MOMENTUM k : $u(k)$ $(\bar{v}(k))$

OUTGOING " " " " " : $\bar{u}(k)$ $(v(k))$

INCOMING PHOTON WITH MOMENTUM k : $\epsilon^\mu(k)$

OUTGOING " " " " : $\epsilon^{\mu *}(k)$.

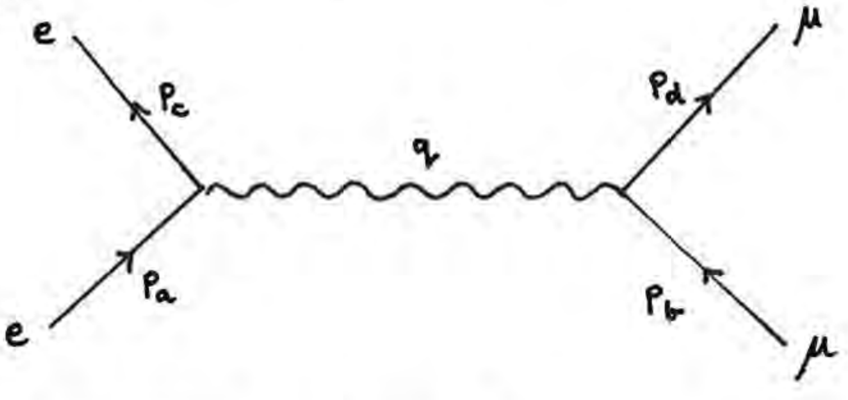
FOR A LOOP:

$$\int d^4k/(2\pi)^4$$

where k is the loop momentum.

Now we combine these rules with section 10 to calculate a specific process: electron-muon scattering.

To leading order this process is governed by a single Feynman diagram:



The corresponding matrix element is:

$$M = (ie)^2\ \overline{u}_c\gamma^\mu u_a\overline{u}_d\gamma^\nu u_b\frac{(-ig_{\mu\nu})}{q^2} \tag{11.3}$$

In (11.3) a single index is used to represent both momentum and spin of a given spinor u. One must also remember that u_a, u_c are electron spinors while u_b, u_d are muon ones.

Then

$$|M|^2 = \frac{e^4}{q^4}\ L^{\mu\nu}(e)\,L_{\mu\nu}(\mu) \tag{11.4}$$

where

$$L^{\mu\nu}(e) = \overline{u}_c\gamma^\mu u_a(\overline{u}_c\gamma^\nu u_a)^*$$

(similarly $L^{\mu\nu}(\mu)$)

Then using the relationship

$$\gamma^{o}\gamma^{\mu\dagger}\gamma^{o} = \gamma^{\mu} \tag{11.5}$$

it is straightforward to show that

$$L^{\mu\nu}(e) = \overline{u}_c\gamma^{\mu}u_a\overline{u}_a\gamma^{\nu}u_c$$

and hence

$$|M|^2 = \frac{e^4}{q^4}(\overline{u}_c\gamma^{\mu}u_a\overline{u}_a\gamma^{\nu}u_c)(\overline{u}_d\gamma_{\mu}u_b\overline{u}_b\gamma_{\nu}u_d) \tag{11.6}$$

There is a nice simplification if (as is often the case) we are working with unpolarised beams of particles and do not measure the final state particle polarisations. If that is so, we replace

$$|M|^2 \; by \; \frac{1}{2^2}\sum |M|^2$$

(The $\frac{1}{2^2}$ is because we <u>average</u> over the initial spins.)

and we can use the relation (for a spinor u (p, s))

$$\sum_{spins} u_{\alpha}\overline{u}_{\beta} = (\not{p} + m)_{\alpha\beta} \tag{11.7}$$

(There is an analagous formula for v:
$$\sum v_{\alpha}\overline{v}_{\beta} = (\not{p} - m)_{\alpha\beta}$$

whereupon

$$\frac{1}{2^2}\sum |M|^2 = \frac{e^4}{4q^4} Tr[\gamma^\nu (\not{p}_c+m_e)\gamma^\mu(\not{p}_a+m_e)] . \tag{11.8}$$
$$Tr[\gamma_\mu (\not{p}_b+m_\mu)\gamma_\nu(\not{p}_d+m_\mu)]$$

It is straightforward to show that

$$Tr[\gamma^\mu (\not{p}_1+m)\gamma^\nu(\not{p}_2+m)]$$
$$= 4(p_1{}^\mu p_2{}^\nu + p_1{}^\nu p_2{}^\mu - g^{\mu\nu}(p_1 . p_2 - m^2)) \tag{11.9}$$

(using

$$tr(\not{a}\not{b}\not{c}\not{d}) = 4(a.b\ c.d - a.c\ b.d + a.d\ b.c)$$

whereupon

$$\frac{1}{2^2}\sum |M|^2 = \frac{4e^4}{q^4}(p_c{}^\mu p_a{}^\nu + p_c{}^\nu p_a{}^\mu - g^{\mu\nu}(p_a . p_c - m_e{}^2)) .$$
$$(p_{b\mu}p_{d\nu} + p_{b\nu}p_{d\mu} - g_{\mu\nu}(p_b . p_d - m_\mu^2)) . \tag{11.10}$$

After some algebra we find, using (10.8):

$$\frac{1}{2^2}\sum M^2 = \frac{2e^4}{t^2}\left[s^2+u^2-4\,(u+s)\,(m_e^2+m_\mu^2)+6\,(m_e^2+m_\mu^2)^2\right]$$

(11.11)

so that using (10.7) we have

$$\frac{d\sigma}{d\Omega^*} = \frac{\alpha^2}{2st^2}\left[s^2+u^2-4\,(u+s)\,(m_e^2+m_\mu^2)+6\,(m_e^2+m_\mu^2)^2\right]$$

(11.12)

If s, u >> m_e^2, m_μ^2 then

$$\frac{d\sigma}{d\Omega}* = \frac{e^4}{32\pi^2 s}\quad\frac{s^2+u^2}{t^2}$$

(11.13)

The presence of the $\frac{1}{t^2}$ term in (11.12) raises the question of whether t can be zero. In the case of equal masses $m_a = m_b = m_c = m_d = m$, it's easy to show that

$$\begin{aligned} s &= 4\,(m^2+\boldsymbol{p}^2) \\ t &= -2\boldsymbol{p}^2(1-\cos\theta^*) \\ u &= -2\boldsymbol{p}^2(1+\cos\theta^*) \end{aligned}$$

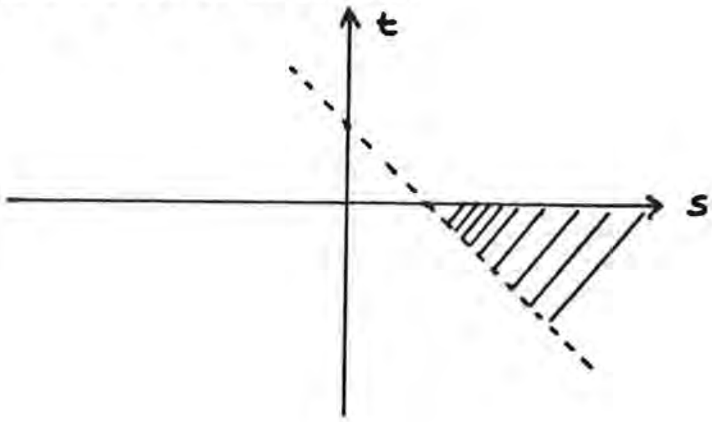
(11.14)

where θ^* is the centre of mass scattering angle and p is the CMS momentum of any one of the particles. Thus we see that

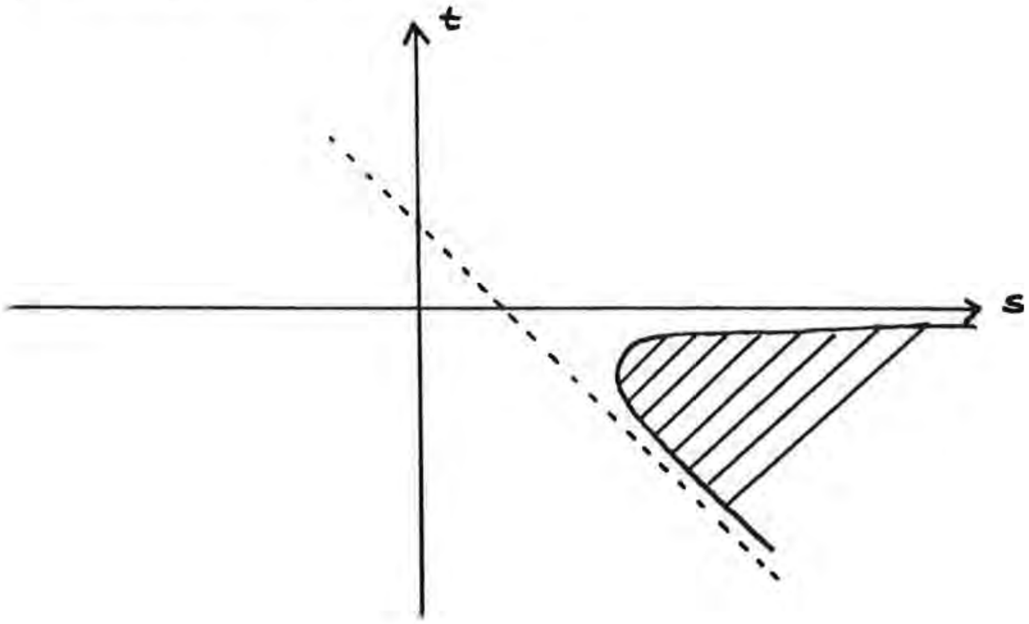
$$t^2 = 16\,(\boldsymbol{p}^2)^2\sin^4\theta^*/2$$

(11.15)

and evidently $t = 0$ when $\theta^* = 0$. This divergence of the differential cross-section in the forward direction is familiar from the Rutherford cross-section, and corresponds to elastic scattering. You can draw the allowed region for (s, t) as follows:

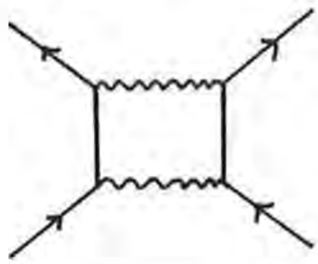


In the general case $m_a \neq m_b \neq m_c \neq m_d$ you get something like this;



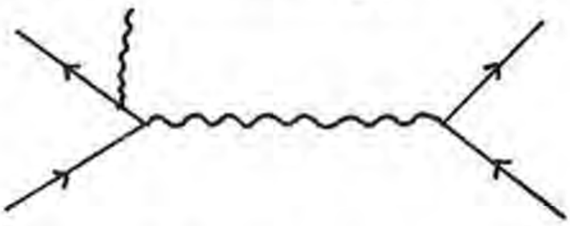
12. HIGHER ORDERS?

The accuracy of leading order QED calculations is a (fortunate) consequence of the fact that $\alpha \approx \frac{1}{137}$ is a small number. Thus graphs of $O(\alpha^2)$ etc. like



give contributions which are (in general) smaller by a factor of α. Certain graphs, for example

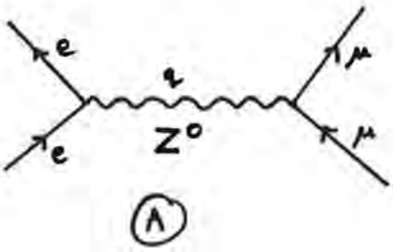
require special treatment: they are DIVERGENT in the limit that the second photon's four-momentum $k^\mu \to \infty$. This is the problem of ULTRAVIOLET DIVERGENCES in QED, which we will return to presently. This graph is also divergent when $k^\mu \to 0$. The treatment of this problem is subtle, involving cancellations between graphs for quite distinct (in principle) processes: one must include the bremsstrahlung processes such as:

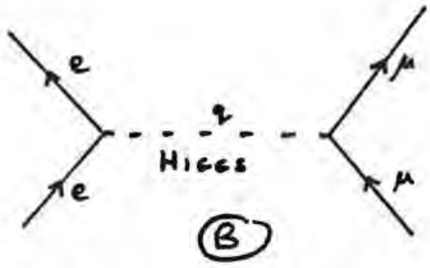


where the final state e (or μ) emits a photon.

This is a large subject which unfortunately we won't have time for. It's very important though: QED corrections must be applied to LEP experimental results, for example. Certainly the agreement between LEP results and the standard model depends on inclusion of radiative corrections, and thus this agreement provides compelling (indirect) evidence for the field-theoretic aspects of QED.

Going beyond QED, there are in the standard model other graphs contributing to the same process:-





In fact

$$\frac{\text{amplitude(A)}}{\text{amplitude(QED)}} \sim \frac{q^2}{q^2 - M_Z^2}$$

and

$$\frac{\text{amplitude(B)}}{\text{amplitude(QED)}} \sim \frac{q^2}{q^2 - m_H^2}\left(\frac{m_e m_\mu}{M_W^2}\right)$$

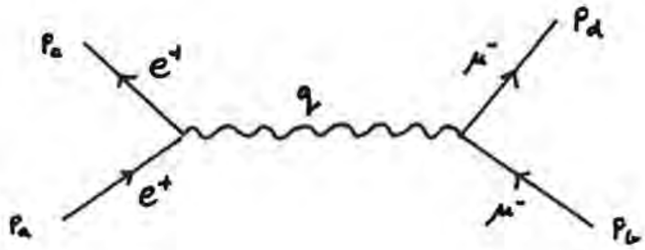
So at low q^2 both are negligible; but if $|q^2| \sim M_z^2$ then graph 1 is competitive with the pure QED one.

Of course $q^2 < 0$ in the above process; in the "crossed" process, $e^+e^- \to \mu^+\mu^-$, $q^2 > 0$ and so the graph with the Z dominates as $q^2 \to M_z^2$.

13. OTHER QED PROCESSES.

Let's look at some other simple QED calculations, noting the different points that arise.

(a) $e^+\mu^- \to e^+\mu^-$



This is a similar calculation. The matrix element is:

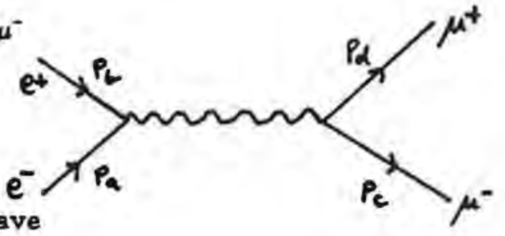
$$M = -(ie)^2\, \overline{v}_a\, \gamma^\mu v_c \frac{(-ig_{\mu\nu})}{q^2} \overline{u}_d \gamma^\nu u_b \tag{13.1}$$

<u>Note</u>

1. The reversed matrix ordering on the positron line.
2. The extra minus sign. This is because (compared to (11.3)), we have "interchanged" the two e-lines.

   In this case the minus sign makes no difference. In the limit that $s,\ u \gg m_e^2,\ m_\mu^2$ it's easy to see that we just get (11.13) again.

(b) $e^+e^- \to \mu^+\mu^-$



$Q = p_a + p_b$

This time we have

$$M = (ie)^2 \frac{(-ig_{\mu\nu})}{Q^2}\ \overline{v}_b \gamma^\mu u_a \overline{u}_c \gamma^\nu v_d \tag{13.2}$$

In the approximation that $s = Q^2 \gg m_e^2,\ m_\mu^2$, one finds for the <u>total</u> cross-section:

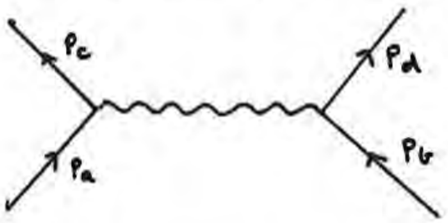
$$\sigma = \frac{4\pi\alpha^2}{3s}$$

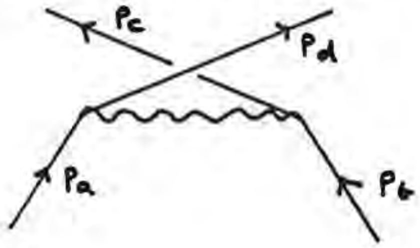
(13.3)

This is an important result in the quark model context, as we will see later.

(c) e^-e^- <u>scattering</u>.

You might guess that this amounts to just replacing m_μ by m_e in section (11) but this would be WRONG. There are now two graphs at leading order:-





The matrix elements for (i) and (ii) are

$$M_{(i)} = (ie)^2 \frac{(-ig_{\mu\nu})}{(p_a-p_c)^2} \, \overline{u}_c\gamma^\mu u_a \overline{u}_d \gamma^\nu u_b$$

(13.4)

$$M_{(ii)} = -(ie)^2 \frac{(-ig_{\mu\nu})}{(p_a-p_d)^2} \, \overline{u}_d\gamma^\mu u_a \overline{u}_c \gamma^\nu u_b$$

(13.5)

Notice the minus sign, which has the same origin as in (13.1). BUT now it <u>makes</u> a big difference, because

$$|M|^2 = |M_{(i)} + M_{(ii)}|^2$$

(13.6)

If you get the sign wrong in (13.5), the interference term in (13.6) will have the wrong sign.

14. Electron-PROTON elastic scattering.

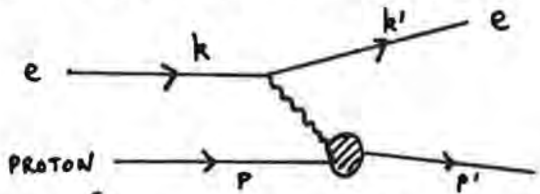
What can we say about ep scattering without a detailed understanding of the strong interactions? Surprisingly, quite a lot: using only the following pieces of information:

(i) the electron has no strong interactions.

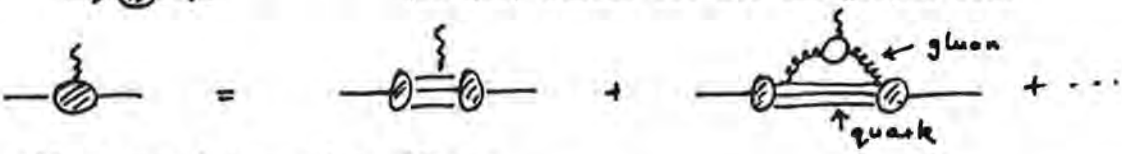
(ii) the proton has spin ½.

(iii) the process is (to leading order) electromagnetic, so parity and charge conjugation invariance are respected.

The matrix element must take the form:

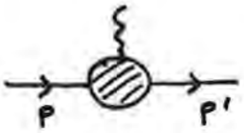


where  (in QCD) stands for all the graphs like



i.e., all strong interaction effects.

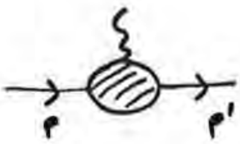
We can try assuming that the proton behaves like an elementary Dirac fermion with mass M_p : ie



$$= \overline{u}(p')\,[-ie\gamma^{\mu}]\,u(p)\;\ldots\;? \tag{14.1}$$

It turns out that this is not a very good approximation.

What if we just assume (i) ... (iii) and Lorentz invariance? We know that



$$= +\overline{u}(p')\,O^{\mu}u(p) \tag{14.2}$$

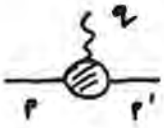
where O^μ is a matrix in Dirac space, and may also depend on p^ν and $p^{\nu'}$, and also Lorentz invariants constructed from them. Of these there is only one independent one, which we can choose to be $q^2 = (p-p')^2$. [Of course $p^2 = p'^2 = m_p^2$].

It turns out there are four independent possible combinations of gamma matrices and p, p′ consistent with Lorentz invariance and CURRENT CONSERVATION $(q_\mu \bar{u}(p') O^\mu u(p) = 0)$.

These are:

$$\begin{array}{llll} (a) & \gamma^\mu & (c) & \sigma^{\mu\nu}\gamma_5 q_\nu \\ (b) & \sigma^{\mu\nu} q_\nu & (d) & \gamma_5 (q^2\gamma^\mu - \not{q} q^\mu) \end{array} \tag{14.3}$$

(c) and (d) are ruled out by the fact that electromagnetic interactions conserve parity. We thus have that



$$= (-ie)\,\overline{u}(p')\left[\gamma^\mu F_1(q^2) + i\kappa_p \frac{F_2(q^2)}{2m_p}\sigma^{\mu\nu}q_\nu\right] u(p) \tag{14.4}$$

F_1, F_2 are dimensionless functions.

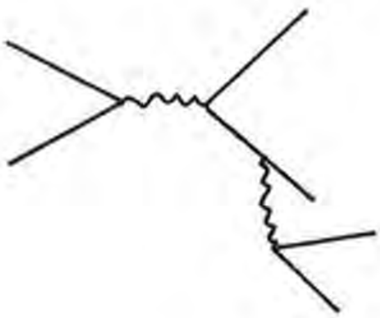
κ_p is chosen so that $F_2(0) = 1$. In fact $\kappa_p = 1.79$. Also, $F_1(0) = 1$, basically because the proton charge is e.

All our ignorance of the strong interactions is incorporated into F_1 and F_2. We could proceed and calculate $d\sigma/d\Omega$ as a function of $F_1(q^2)$, $F_2(q^2)$, q^2, and scattering angle: F_1, F_2 can then be extracted from experiment.

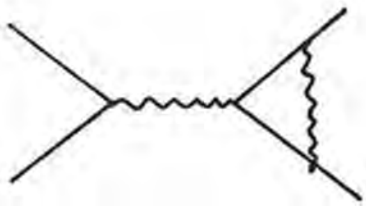
Any electromagnetic (or electroweak) process involving hadrons can be described so that the strong interaction corrections are parameterised by a set of invariant functions: the task of a strong interaction theory is to predict these functions.

15. RENORMALISATION (of QED).

The rules we have been using are in principle sufficient for the extraction of unambiguous results from any tree graph (which means a graph without a closed loop of propagators).



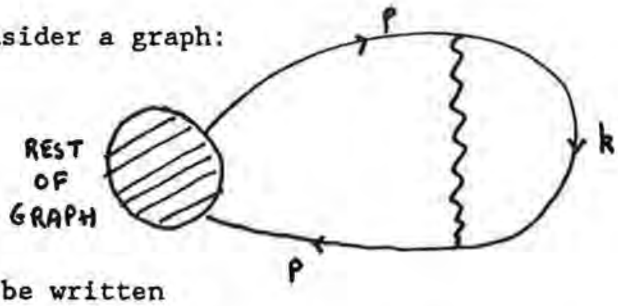
TREE GRAPH



ONE-LOOP GRAPH

A loop involves an integration over the possible four momentum flowing through it.

Consider a graph:



This can be written

$$(\ldots)\ \frac{i}{\not{p}-m} \sum (p)\ \frac{i}{\not{p}-m}\ (\ldots)$$

where the (...) represents the rest of the graph, and

$$\sum (p) = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{k}+m)\gamma_\mu}{[k^2-m^2+i\epsilon][(p-k)^2+i\epsilon]} \tag{15.1}$$

In the special case that $p = 0$, we have

$$\sum = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{[4m]}{[k^2+i\epsilon]\,[k^2-m^2+i\epsilon]} \tag{15.2}$$

Now the evaluation of this integral is still non trivial, because of the poles at $k^2 = 0$ and $k^2 = m^2$. These are controlled by the $i\epsilon$'s, but unfortunately the integral is still DIVERGENT, because of what happens at large k, since then

$$\sum \approx \int \frac{d^4k}{k^4} \approx \int \frac{dk}{k} \approx \ln\Lambda/m$$

Λ is called a CUTOFF.

Thus

$$\sum (p = o) \approx -e^2 m \ln\Lambda/m.$$

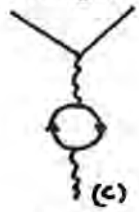
This represents a <u>divergent</u> radiative correction to the electron mass. How do we deal with this?

We say the following: we know that the electron mass we <u>measure</u> is a) finite b) includes all radiative corrections. Therefore the parameter in the Lagrangian that we <u>thought</u> was the electron mass must in fact not be. In fact <u>it</u> must be divergent, in just such a way as to cancel the divergence in the loop graph (and all higher order ones).

The same thing happens for the electron charge, via graphs like

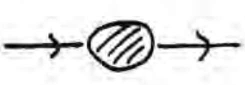
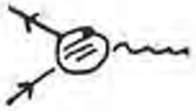

(a)


(b)


(c)

In fact the infinite parts of (a), (b) cancel so that charge renormalisation comes from (c) (at one loop).

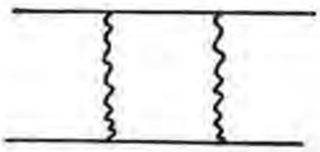
How is it then possible to make finite predictions? The reason is that the only graphs which are divergent are ones of the form

 ,  ,  .

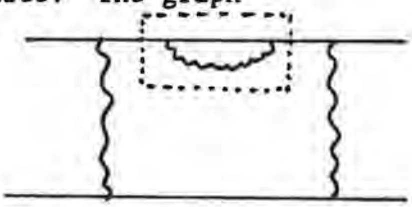
which renormalise the objects

(1) $\bar{\psi}\not{\partial}\psi$, $m\bar{\psi}\psi$, (2) $F_{\mu\nu}{}^2$, (3) $\bar{\psi}\gamma^\mu\psi A_\mu$ respectively.

Anything else is convergent and unambiguously calculable. Thus the graph



is perfectly finite. The graph



isn't finite, but the divergence is in a subgraph of the type $\bar{\psi}\psi$. Since we have to make this finite anyway, we don't need to worry about it when it's inserted in some other graph.

Mathematically, what we are doing amounts to the following. We start with

$$L = -\frac{1}{4}F_{\mu\nu}^2 \ (A_\mu^B) + \bar{\psi}^B[i\not{\partial}+e\not{A}^B]\psi^B + m_B\bar{\psi}^B\psi^B$$

(for simplicity we ignore the gauge fixing term) and we let

$$\psi^B = Z_2^{\frac{1}{2}}\psi$$

$$A_\mu{}^B = Z_3^{\frac{1}{2}}A_\mu$$

$$e_B = Z_e e = \frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}}e$$

$$m_B = Z_m m$$

So that

$$L = -\frac{1}{4}Z_3 F_{\mu\nu}^2 + iZ_2\bar{\psi}\partial\!\!\!/\psi + eZ_1\bar{\psi}A\!\!\!/\psi + Z_m Z_2 m\bar{\psi}\psi\,.$$

Then we adjust $Z_1 \dots Z_3$, Z_m so that all amplitudes give <u>finite results</u>. As an example, the graph

$$= \bar{u}(p)\ \ [A\gamma^\mu + B\sigma^{\mu\nu}q_\nu + Cq^2\gamma^\mu + \dots]\,u(p')$$

Now if we do the calculation, we'll find that A is divergent; but that's OK since we just absorb it into Z_1. We can't do this with B or C, since, for example, there's no term in the Lagrangian of the form:-

$$\frac{1}{m}\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$$

which is what we'd need to deal with B. So it's just as well that B comes out to be finite, and quite unambiguous: in fact this is the calculation of the correction to the magnetic dipole moment of the electron, first done by Schwinger. Kinoshita has been doing the 4 loop calculation since 1977: latest results are:

(Theory)

$(g-2)_e = 1,159,652,140(28) \times 10^{-12}$

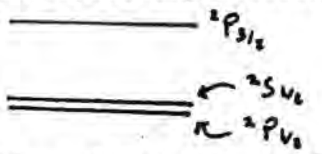
(Experiment)

$(g-2)_e = 1,159,652,188.4(4.3) \times 10^{-12}$

Most of the theoretical error arises from uncertainty in the fine structure constant.

Faced with this it is hard to argue that QFT is not basic to the operation of the universe).

The C term is also finite and calculable - it corresponds to the LAMB SHIFT in the hydrogen atom; the splitting between the $^2S_{\frac{1}{2}}$ and $^2P_{\frac{1}{2}}$ levels:



It was Bethe's calculation of the Lamb shift that convinced physicists that quantum field theory really worked.

16. QUANTUM CHROMODYNAMICS (QCD).

In the 1960's the majority of practising theorists lost interest in QFT. Even though the vector/axial nature of the weak interactions was understood, they were discouraged by the apparent non-renormalisability of massive vector boson theories. In the case of strong interactions, their very strength and the large number of hadrons seemed to preclude a QFT description. In my view, while the development of the GSW electroweak model can be seen (with hindsight!) as a natural chain of events, the development of QCD is a most unforseen and extraordinary conceptual achievement. The fact that (as most theorists believe) so simple a Lagrangian as

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \sum \overline{\psi}_F (i\not{D} - m_F) \psi_F \tag{16.1}$$

can in principle give a complete description of all strong interaction phenomena is a strange and profound thing.

In (16.1),

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{16.2}$$

and

$$(D_\mu \psi_F)^i = \partial_\mu \psi^i_F + ig(T^a)^{ij} \psi^j_F A^a_\mu$$

A_μ^a are the gluons (a: 1 ...8) and ψ_F^i are the quarks. The i index (i:1..3) is called colour. The F index stands for (u, d, c, s, t, b). T^a is a set of 8 matrices (3 x 3) that obey the SU(3) algebra:

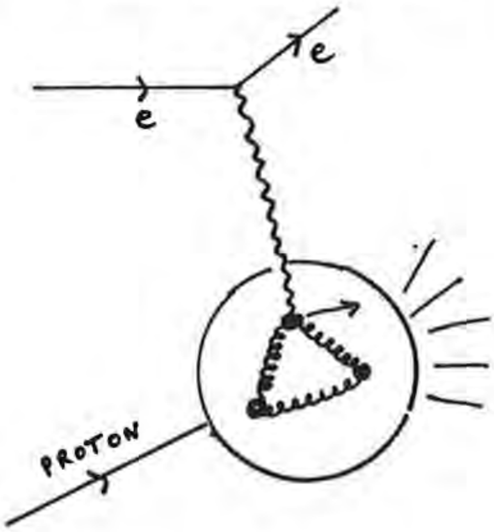
$$[T^a, T^b] = i f^{abc} T^c \tag{16.3}$$

where f^{abc} are the structure constants of SU(3). The T^a are hermitian, and traceless. They are usually written $T^a = \lambda^a/2$.

There are no Higgs bosons; the only relic of the standard model Higgs relevant if we are considering strong interactions is the presence of the mass terms,

$$\sum_F m_F \overline{\Psi}_F \Psi_F.$$

With QCD, hadrons are believed to consist of sets of 3 quarks (baryons) or quark-antiquark pairs (mesons), bound by their interaction with gluons, and having no net colour. This is quite analagous to atoms, which have no net charge. Where the analogy fails is that atoms are readily persuaded to dissociate into an ion and one (or more) free electron. It is generally believed, however, that the "binding energy" of a quark in a hadron is infinite - there is no such thing as a free quark. This is called CONFINEMENT. One of the most beautiful features of the theory is the resolution of the apparent paradox posed by the conjunction of this picture of "tightly bound" constituents with the fact that in high energy scattering experiments the quarks behave in many respects as (almost) free particles. This phenomenon is known as ASYMPTOTIC FREEDOM.

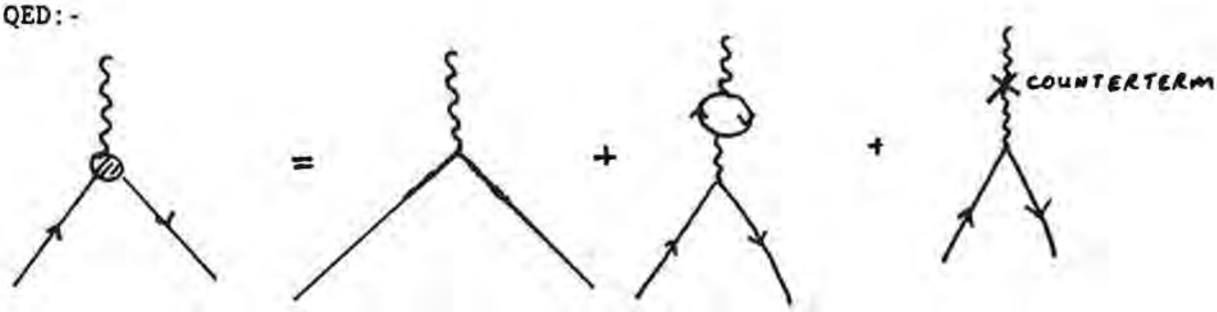


Deep inelastic scattering. As $|Q^2| \to \infty$, the interaction of the photon with the quark is as if the quark were free. But as the "free" quark sets off out of the proton, the strong interactions become strong again - "Hadronisation" sets in.

17. ASYMPTOTIC FREEDOM.

Nonabelian gauge theories (like QCD) differ crucially from other renormalisable theories in four dimensions with respect to their high energy behaviour. This behaviour is a consequence of asymptotic freedom and leads to the important concept of a running coupling constant. Although this concept came into prominence only with the advent of QCD, it in fact is equally valid in other field theories such as QED.

Consider the set of graphs that determine the electron's charge in QED:-



$$= \gamma^\mu \left[e - \frac{2}{3}\frac{e^3}{16\pi^2} \ln\frac{\Lambda^2}{q^2} + \frac{2}{3}\frac{e^3}{16\pi^2} \ln\frac{\Lambda^2}{\mu^2} + \dots \right] \tag{17.1}$$

Here e is the renormalised (i.e. finite) coupling constant from the Lagrangian, and $q^2 = -Q^2$. Λ is a cutoff. The last term is the counterterm contribution introduced to make the total finite; you can think of it as arising from a redefinition of the bare charge,

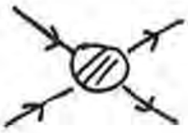
$$e_B = e + \frac{2}{3}\frac{e^3}{16\pi^2} \ln\frac{\Lambda^2}{\mu^2}$$

The most important thing about the counterterm is that it necessarily involves a choice of mass-scale, μ. This is a "new" parameter, whose presence in the theory is not evident from the original Lagrangian. It is there in all field theories (except finite ones), fundamentally because radiative corrections break a symmetry called conformal invariance.

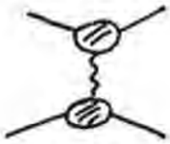
We have

$$= \gamma^\mu \left[e + \frac{2}{3} \frac{e^3}{16\pi^2} \ln\frac{q^2}{\mu^2} + \ldots \right] \tag{17.2}$$

So when we stick graphs together to give scattering amplitudes:



$$= \quad$$



$$+ \quad \cdots$$

it's clear that the result will depend on μ! How can this be, since μ is arbitrary?

The answer is that e depends on μ in just such a way that the scattering amplitudes don't. Thus

$$\mu \frac{\partial}{\partial\mu} \left[e + \frac{2}{3} \frac{e^3}{16\pi^2} \ln\frac{q^2}{\mu^2} + \ldots \right] = 0 \tag{17.3}$$

or

$$\mu \frac{\partial e}{\partial \mu} - \frac{4}{3} \frac{e^3}{16\pi^2} + \frac{2}{3} \frac{3e^2}{16\pi^2} \mu \frac{\partial e}{\partial \mu} \ln\frac{q^2}{\mu^2} \ldots = 0$$

To leading order

$$\mu \frac{\partial e}{\partial \mu} = \frac{4}{3} \frac{e^3}{16\pi^2} \tag{17.4}$$

or with $\alpha = e^2/4\pi$,

$$\mu \frac{\partial \alpha}{\partial \mu} = \frac{2}{3} \frac{\alpha^2}{\pi}$$

We call

$$\mu \frac{\partial \alpha}{\partial \mu}$$

the beta-function: $\beta(\alpha)$.

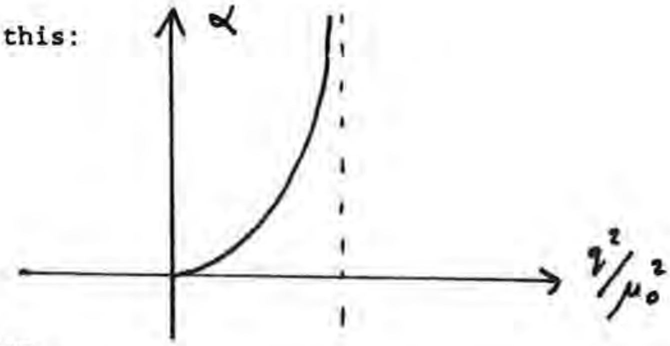
The solution of (17.4) is

$$\alpha(\mu^2) = \frac{\alpha(\mu_0^2)}{1 - \frac{1}{3}\frac{\alpha(\mu_0^2)}{\pi}\ln\frac{\mu^2}{\mu_0^2}} \tag{17.5}$$

Now in a given experiment, what is the appropriate value of μ to choose? Looking back at the original expression for  , we see that <u>if</u> we choose $\mu^2 = q^2$, then the logarithm disappears; and (as long as $\alpha(\mu)$ is small) the tree approximation will be a good one. Thus we write

$$\alpha(q^2) = \frac{\alpha(\mu_0^2)}{1-\frac{1}{3}\frac{\alpha(\mu_0^2)}{\pi}\ln\frac{q^2}{\mu_0^2}} \tag{17.6}$$

Let's plot this:



The pole when

$$\ln\frac{q^2}{\mu_o^2} = \frac{3\pi}{\alpha(\mu_o^2)}$$

is called the LANDAU pole. We must remember, however, that it may not really be there since <u>we can't trust our expression</u> if $\alpha(\mu^2) > 1$, because higher orders in perturbation theory will become important.

In fact because α is so small, the effect on α is small even at LEP energies. The change is from 1/137 (at low energies) to 1/128 (at LEP). If you want to check this, you need to take into account the contribution to $\beta(\alpha)$ of all the quarks and leptons, and the fact that β changes as the b-threshold is crossed.

In QCD things are more or less the same except for one little sign:

NB

$$= \gamma^\mu\left[g + \left(\frac{1}{3}F - \frac{11}{2}\right)\frac{g^3}{16\pi^2}\ln\frac{q^2}{\mu^2} + \ldots\right]$$

Here F stands for the number of quark flavours.

Repeating the steps that led to (17.4) we find

$$\mu\frac{\partial g}{\partial\mu} = \left(\frac{2}{3}F - 11\right)\frac{g^3}{16\pi^2},$$

or

$$\mu\frac{\partial\alpha_s}{\partial\mu} = \left(\frac{1}{3}F - \frac{11}{2}\right)\frac{\alpha_s^2}{\pi}, \quad \alpha_s = \frac{g^2}{4\pi}$$

so that

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \left(\frac{11}{4} - \frac{1}{6}F\right)\frac{\alpha_s(\mu^2)}{\pi}\ln\frac{q^2}{\mu^2}} \tag{17.7}$$

Now as long as $F < \frac{33}{2}$, we have that $\alpha_s(q^2)$ looks like this:-

This means that as long as $\alpha_s(\mu^2)$ is "small" for some μ^2, then $\alpha_s(q^2)$ is certainly smaller: strong interactions get weak at higher energies.

Eq. (17.7) is often rewritten as follows:

$$\alpha_s(q^2) = \frac{12\pi}{(33-2F)\ln\left(\frac{q^2}{\Lambda^2}\right)} \tag{17.8}$$

where

$$\ln\Lambda^2 = \ln\mu^2 - \left(\frac{12\pi}{33-2F}\right)\frac{1}{\alpha_s(\mu^2)}$$

this Λ is not to be confused with the cutoff Λ in (17.1)!

(17.8) can be used to extract Λ from the data. Since $\alpha_s \to \infty$ as $q^2 \to \Lambda^2$, we can think of Λ as setting the scale for confinement. (Although of course (17.8) is not valid once $\alpha_s(q^2)>1$).

When higher order effects are included, it turns out that Λ depends on the subtraction scheme. A popular one is MS ("modified minimal subtraction").

Recent measurements of jet multiplicities in Z^0 decays at LEP give

$$\alpha_s(M_z) = 0.115 \pm 0.005(\exp)\,{}^{+0.012}_{-0.010}(theor.)$$

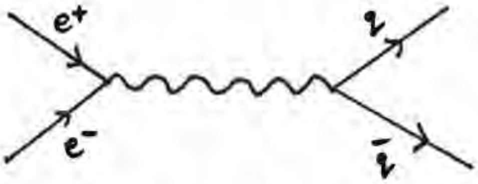
or

$$\Lambda_{\overline{MS}} = 190^{+60}_{-50}(\exp)\,{}^{+170}_{-\ 90}(theor)\,MeV.$$

The fact that QCD provides a correct description of strong interactions, and the crucial role of asymptotic freedom, was first realised in the context of a phenomenon known as"Bjorken scaling", identified in deep inelastic lepton-hadron scattering experiments in the late 60's. (The 1990 Nobel prize was recognition of the significance of this work.) In the next section, however, we consider a somewhat simpler application.

18. e^+e^- hadrons.

Ignoring the strong interactions, we'd have (to leading order in α) that $e^+e^- \to$ hadrons proceeds via the graph



which clearly gives total cross-section

$$\sigma = 3.\frac{4\pi}{3}\ \frac{\alpha^2}{Q^2}\sum_q Q_q^2$$

(18.1)

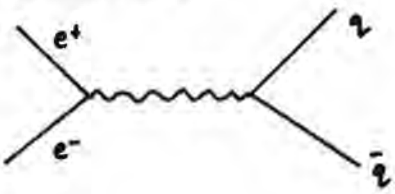
where Q_q is the charge of the quark q, and the factor of 3 arises from a sum over colours. We thus have

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2$$

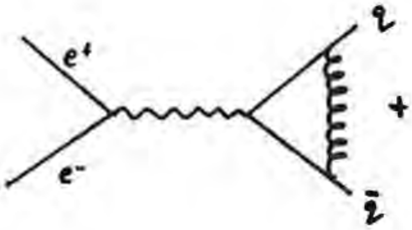
(18.2)

For $\sqrt{s} > 10$ Gev, the data is indeed in reasonable agreement with the predicted value of R. (Of course resonances cause large perturbations.) This result is a success for QCD because QCD unambiguously predicts that, at large Q^2, the strong interaction corrections to the naive prediction will indeed be small; but can we do better, and say something about the corrections?

If α_s is small and we are not near a hadronic resonance, then we hope that calculating



$+$



$+ \cdots$

will give us the leading correction. The result is

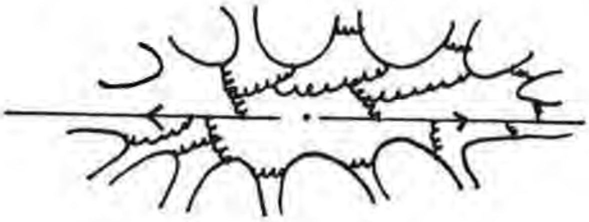
$$R = 3\sum Q_q^2 \{1 + \frac{\alpha_s}{\pi}(Q^2) + \ldots\} \tag{18.3}$$

This suggests that R approaches the naive prediction from above. The data isn't good enough to extract α_s this way, but the general trend is compatible.

More dramatic is the fact that QCD makes a specific prediction for the nature of the final state: that there are <u>jets</u>. In the leading order graph, the final state consists of a 'back to back' $q\bar{q}$ pair:

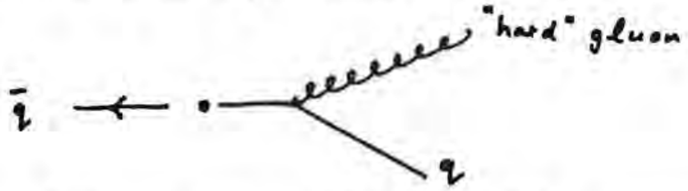
$$\bar{q} \leftarrow \cdot \rightarrow q$$

Of course this has to change into hadrons:



But because <u>$\alpha_s(Q^2)$ is small when Q^2</u> is <u>large</u>, this hadronisation occurs without substantial transfer of momentum between the q and $\bar{q}$. Consequently the final state is mostly two back-to back jets. If you have done problem 13, you will not be surprised to hear that the jets show the characteristic $(1 + \cos^2\theta^*)$ distribution in the centre-of-mass frame.

However there will be a few 3 jet events:



The observation of these events at DESY in 1980 is sometimes called the "discovery of the gluon". Much more on this and other aspects of standard model phenomenology will be found in Dr. Robert's lectures.

PRESCHOOL PROBLEMS.

1. PROBABILITY DENSITY AND CURRENT DENSITY.

Starting from the Schrödinger equation for a wave function $\psi(\mathbf{x}, t)$, show that the probability density $\rho = \psi^*\psi$ satisfies the continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla.\boldsymbol{J} = 0$$

where

$$\boldsymbol{J} = \frac{\overline{h}}{2im}\left[\psi^*\nabla\psi - (\nabla\psi^*)\,\psi\right]$$

how do we interpret J?

2. ROTATIONS AND THE PAULI MATRICES.

Show that a 3-dimensional rotation can be represented by an orthogonal matrix (with determinant +1).

[Start with $x'_i = R_{ij}x_j$, and impose that $x'_i x'_i = x_i x_i$]; [here using Einstein summation convention].

Show that if we write $R = 1 + iA$, where 1 is the unit matrix, then if A is infinitesimal ($A^2 \sim AA^T \sim 0$) then A is <u>antisymmetric</u>. [The i is there to make A hermitian.]

If we write

$$A = \begin{pmatrix} 0 & -a_3 i & +a_2 i \\ a_3 i & 0 & -a_1 i \\ -a_2 i & a_1 i & 0 \end{pmatrix} = \sum_i a_i L_i$$

then show that the matrices L_i obey the <u>algebra</u> (O(3) or SU(2))

$$[L_i, L_j] = i\epsilon_{ijk}L_k.$$

Verify that the <u>Pauli matrices</u> $\frac{1}{2}\sigma_i$ obey the same algebra.

3. REPRESENTATIONS OF SU(2).

From the angular momentum algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

show that the operators

$$L\pm = L_1 \pm iL_2$$

satisfy

$$[L_+, L_-] = 2L_3$$

(1)

$$[L_+, L_3] = -L_+$$

(2)

$$[L_-, L_3] = L_-.$$

(3)

and show that

$$[L^2, L_3] = 0$$

(4)

where

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

From (4) it follows that there can be states ψ_{lm} that are simultaneous eigenstates of L^2 and L_3. Given that:

$$L^2\psi_{lm} = l(l+1)\psi_{lm}$$

and

$$L_3\psi_{lm} = m\psi_{lm}$$

Show that

$$L\pm\psi_{lm}$$

are also eigenstates of L^2 (with eigenvalue $l(l+1)$) and L_3 (with eigenvalue $m \pm 1$).

4. FOUR VECTORS.

A Lorentz transformation on the coordinates x^μ : x^0 ($=$ ct), x^1 ($=$ x), x^2 ($=$ y), x^3 ($=$ z) is defined as follows:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1}$$

where $\Lambda^\mu{}_\nu$ is a 4 x 4 matrix.

For the usual case of a boost along the x axis is:

$$x' = \gamma(x-vt) \qquad t' = \gamma\left(t-\frac{vx}{c^2}\right),\ y' = y,\ z' = z$$

Show that

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta = \frac{v}{c}$ for this special case. (2)

By imposing that

$$g_{\mu\nu}x'^\mu x'^\nu = x^\mu x^\nu g_{\mu\nu} \tag{3}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

show that

$$g_{\mu\nu}\Lambda^{\mu}{}_{\pi}\Lambda^{\nu}{}_{\delta} = g_{\pi\delta} \quad ,$$

or

$$\Lambda^{T}g\Lambda = g \quad .$$

Check that (2) satisfies this.

[This is the analog of the orthogonality condition for rotations.]

Now introducing

$$x_{\mu} = g_{\mu\nu}x^{\nu},$$

Show that

$$x'_{\mu} = x_{\nu}\,(\Lambda^{-1})^{\nu}{}_{\mu}$$

[Easiest way is by reconsidering (3), using $x^{\mu}x_{\mu}$.]

Vectors A^{μ}, B_{μ} that transform like x^{μ}, x_{μ} are called CONTRAVARIANT AND COVARIANT respectively.

A particularly important COVARIANT vector is formed by taking $\frac{\partial}{\partial x^{\mu}}$ of a scalar. Thus

$$\frac{\partial}{\partial x^{\mu}}\,\Phi = \partial_{\mu}\Phi$$

is <u>COVARIANT</u>.

<u>Show</u> this: i.e. show that $\frac{\partial}{\partial x^{\mu}}$ transforms like x_{μ}, not x^{μ}.

5. <u>ELECTROMAGNETISM</u>.

The fundamental laws of electromagnetism are

(1) div $\mathbf{E} = \dfrac{\rho}{\epsilon_0}$ (2) div $\mathbf{B} = 0$.

(2) curl $\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (4) curl $\mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$.

Show that div $\mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$.

What is the significance of this equation?

Verify that it can be written in manifestly covariant form

$$\partial_\mu \mathcal{J}^\mu = 0$$

where

$$\mathcal{J}^\mu = (c\rho, \mathcal{J}).$$

If we introduce scalar and vector potentials ϕ, $\mathbf{A}$ by defining $\mathbf{B} =$ curl $\mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \dfrac{\partial \mathbf{A}}{\partial t}$, show that if we assume ϕ,$\mathbf{A}$ form a 4-vector: $A^\mu = (\dfrac{\phi}{c}, \mathbf{A})$, then Maxwell's equations are all satisfied if we choose the <u>gauge</u> $\partial_\mu A^\mu = 0$ and

$$\Box A^\mu = \frac{1}{\epsilon_0 c^2} \mathcal{J}^\mu,$$

where

$$\Box \equiv \partial^\mu \partial_\mu$$

If we define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, then rewrite $F_{\mu\nu}$ in terms of $\mathbf{E}$, $\mathbf{B}$. Show that

$$F_{\mu\nu}^2 \sim \frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2,$$

and

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \sim \boldsymbol{E}.\boldsymbol{B}.$$

6. GROUP THEORY: IN PARTICULAR SU(N)

Unitary matrices are matrices U such that $UU^\dagger = I$ (unit matrix) and they form a group; verify this by showing that $W = UV$ is unitary if U, V are. To show they form a group you should also show that every U has an inverse in the group: but this is obvious from the definition (isn't it?).

U(N) is the group of N x N unitary matrices; SU(N) is the group of N x N matrices with determinant equal to + 1.

If we write a unitary matrix in the form $U = e^{iG}$, then check that unitarity of U means G is hermitian. Also check that if $U \in SU(N)$, (i.e. det U = 1) then G is traceless [use the fact that det = exp tr ln].

An N x N, traceless, hermitian matrix can be expressed as a linear combination of a chosen basis set. Thus for any G, we can choose numbers α_i such that

$$G = \sum_i \alpha_i T_i$$

i runs from 1 to $N^2 - 1$, (why?) and T_i are a set of N x N, traceless, hermitian matrices.

Show that $[T_i, T_j]$ is antihermitian and traceless.

Hence $[T_i, T_j] = if_{ijk}T_k$.

This is called the algebra of SU(N), and f_{ijk} are the structure constants.

They depend on the choice we make of λ_i.

Verify that

$$[T_i, [T_j, T_k]] + [T_j, [T_k, T_i]] + [T_k, [T_i, T_j]] = 0$$

and show that this means that the f_{ijk}'s obey an identity:

$$f_{jkl}f_{ilm} + f_{kil}f_{jlm} + f_{ijl}f_{klm} = 0.$$

called the JACOBI identity.

Show that the set of $N^2 - 1 \times N^2 - 1$ matrices

$$(\hat{T}^i)_{jk} = -if_{ijk}$$

(THIS IS CALLED THE ADJOINT REPRESENTATION)

obeys the same algebra as the T's.

Write down a set of 3 T's for SU(2) and a set of 8 T's for SU(3).

Notice that the algebra of SU(2) is the same as the algebra of rotations described in (2). This is because the groups SU(2) and O(3) are closely related: there is a HOMOMORPHISM

$$SU(2) \rightarrow O^+(3)$$

$O^+(3)$ means rotations with det + 1: i.e. without reflections.

PROBLEMS.

1. Derive the continuity equation (2.12) for $\phi(r,t)$ satisfying the KG equation. (See (2.13), (2.14).) [Take the KG equation $\otimes\phi^*$; and then subtract this equation from its complex conjugate.]

2. Prove that any set of matrices (α,β) satisfying (3.4) have to be traceless, with eigenvalues ± 1. Hence prove that they are necessarily even dimensional.

   They also have to be Hermitian, in order that the Hamiltonian be so. Does hermiticity for α, β follow from (3.4) alone?

3. Verify that the representation (3.7) for α, β satisfies (3.4).

4. Derive the continuity equation for ψ satisfying Dirac's equation and $\rho = \psi^\dagger\psi$.

5. Show that the non-relativistic Hamiltonian for a free particle, $H = \frac{p^2}{2m}$, satisfies

   $[L,H] = 0$, using (6.1) and (2.1).

   If the particle is in a potential, so that $H = \frac{p^2}{2m} + V(r)$, show that $[L, H] \neq 0$ in general. For which special class of $V(r)$ is $[L,H] = 0$?

6. Verify (6.5) $\rightarrow$ (6.7).

7. Verify (7.23): i.e., work through the algebra from (7.18) - (7.23).

8. Derive (9.5).

9. Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi$$

   Hence verify (9.10).

11. Derive (10.6), the cross-section for $2 \to 2$ in the center of mass frame, from the general $2 \to n$ formula (10.1).

[This is a non-trivial exercise. In the CM frame, $p_a = (E_a, p)$ and $p_b = E_b, -p)$. Then the δ-function becomes $\delta(E_a + E_b - E_c - E_d)\, \delta^{(3)}(p_c + p_d)$. You can do the $\delta^{(3)}(p_c + p_d)$, which sets $p_c = -p_d = p'$, say. The last δ-function then takes some thought. You will need the result that

$$\int dx\, \delta(f(x))\, g(x) = \frac{g(x_0)}{|\frac{df}{dx}|_{x-x_o}}$$

where $f(x_0) = 0$.

12. Work through the algebra from (11.6) $\to$ (11.11). You will need the following trace theorems:

$$(i)\ Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$
$$(ii)\ Tr[\gamma^\mu\gamma^\nu\gamma^\lambda] = 0$$
$$(iii)\ Tr[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = 4[g^{\mu\nu}g^{\lambda\rho}+g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho}].$$

13. Derive eq. (13.3), the total cross-section for $e^+e^- \to \mu^+\mu^-$ (neglecting m_e, m_μ).

[This proceeds much as section 11, but with much less algebra if you neglect m_e, m_μ from the beginning.

Start by showing that

$$\frac{1}{2^2}\sum_{spins} |M|^2 = \frac{2e^4}{s^2}(u^2+t^2)$$

and then use (10.6)]

14. Verify that (a) - (d) in (14.3) all represent <u>conserved</u> currents, i.e. that

$$q_\mu \overline{u} O^\mu u = 0.$$

15. Find the solution of the differential equation

$$\frac{d\alpha}{dt} = b\alpha^2, \qquad \alpha(t=0) = \alpha_o.$$

Hence check eqs. (17.5), (17.7).

Acknowledgements.

The enthusiasm and commitment of the students, tutors, other lecturers and the organiser Roger Barlow made participation in this school a unique experience. My thanks go to them, and also to Chris Sachrajda, whose lectures at previous schools I plagiarised shamelessly.

Finally I am particularly grateful to Rosemary Morris, who displayed patience and heroism beyond ordinary mortals in the preparation of this manuscript.

THE STANDARD MODEL AND BEYOND

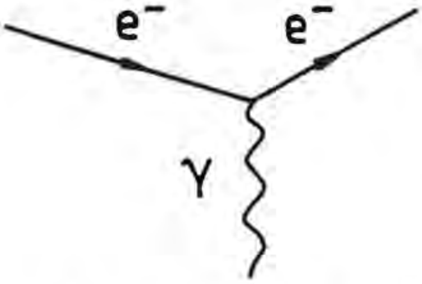
By Dr C Maxwell
University of Durham

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1992

The Standard Model and Beyond

1. Preliminaries

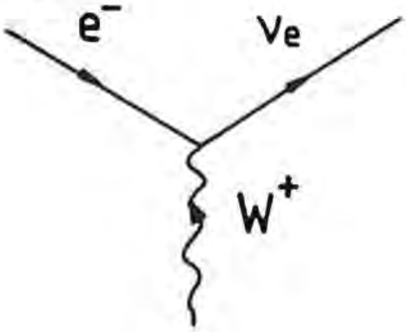
We shall begin with a discussion of weak interactions. The prototypical weak interaction process is beta-decay, $n \rightarrow p + e^- + \bar{\nu}_e$. Pauli in 1932 speculated that an undetected massless neutral spin-$\frac{1}{2}$ fermion had to be emitted in this process in order to conserve energy and momentum, and this he named the neutrino. (In fact in modern parlance it is an anti-neutrino that is emitted along with the electron.) In 1934 Fermi postulated a charged vector current interaction for weak processes. From Tim's course you know that the electromagnetic current which couples to the photon field in QED is



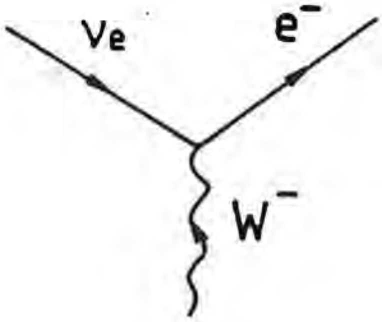
$$J_\mu^{em} = \bar{\psi}\,\gamma_\mu\,\psi = \bar{e}\,\gamma_\mu\,e$$

We shall often use particle letters to denote Dirac spinors throughout these lectures.

Fermi guessed that in weak interactions one would have similar vector currents.



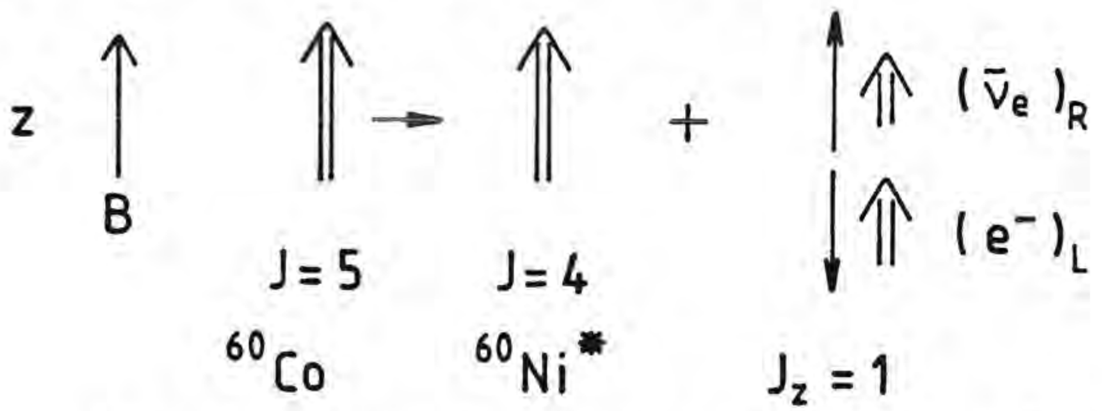
$$J_\mu^+ = \bar{\nu}\,\gamma_\mu\,e$$



$$J_\mu^- = \bar{e}\,\gamma_\mu\,\nu$$

A charge raising ($e^- \rightarrow \nu$) current with a (+) superscript and a charge lowering ($\nu \rightarrow e^-$) with a (-) superscript. Fermi had of course no reason to hypothesise a massive exchanged V boson and so constructed a four-point interaction for beta-decay with a charge raising and charge lowering current contracted together with no propagator. He introduced an overall effective coupling with dimensions of M^{-2}, $G_F \simeq 10^{-5}$ GeV^{-2}.

This four-point interaction together with the effective dimensional coupling constitute an effective weak interaction theory valid for momentum transfers $q^2 \ll M_V^2$ where the V propagator $\frac{1}{q^2-M_V^2}$ can be replaced by $\frac{1}{M_V^2}$ which is absorbed into $G_F = \frac{\sqrt{2}g^2}{8M_V^2}$, where $g^2 \simeq 4\ e^2$ in the Standard Model is the dimensionless charged current coupling constant, with $e \simeq .303$ the analogous QED coupling. The q^2 involved in beta decay, muon decay, and many other weak processes, is sufficiently small that the effective Fermi theory can be applied (see Exercise 3). However one crucial modification to the original suggestion is first required. It was convincingly demonstrated by Mme. Wu and her collaborators at Brookhaven in 1956 that weak interactions are maximally parity violating, and hence a vector interaction is ruled out. A grossly simplified summary of this Cobalt 60 experiment is shown below.



Cobalt 60 nuclei are polarized with their spins in the +z direction using a magnetic field. They decay into an excited state of Nickel 60, $\bar{\nu}_e$ and e^-. The underlying process is beta-decay of a neutron in Cobalt 60. It is found that the electron is preferentially emitted in the direction opposite to the +z polarization direction. In fact one finds an angular distribution $\sim 1 - \frac{v}{c}\cos\theta$, where v is the electron velocity and θ the electron angle with respect to +z. If we pretend that the electron is massless then we can apply helicity (J_z) conservation to deduce that the $\bar{\nu}_e, e^-$ have net $J_z = +1$, and hence must be right-handed and left-handed respectively. The implication is that only right-handed anti-neutrinos occur in nature. Notice that in the Standard Model the neutrinos are taken to be massless, although this is not required by gauge invariance as for the photon. It is only for massless particles that one can have states of definite helicity, for a massive particle a Lorentz boost along the momentum direction can of course change $L \leftrightarrow R$.

It required a separate experiment by Goldhaber, Grodzius and Sonyar (1958) using the K-capture reaction

$$
\begin{array}{lll}
e^- + {}^{150}\mathrm{Eu}\ (J=0) & \longrightarrow & {}^{152}\mathrm{Sm}^*(J=1) + \nu_e \\
 & & \quad\searrow \\
 & & \qquad {}^{152}\mathrm{Sm}\ (J=0) + \gamma
\end{array}
$$

to demonstrate that only left-handed neutrinos can occur.

A vector charged current interaction is therefore ruled out and we need to project out left handed particles and right-handed anti-particles only. To see how to do this let us begin by defining ψ_L and ψ_R, chiral components of an arbitrary (possibly massive) spin-$\frac{1}{2}$ fermion field ψ.

We recall the 4 × 4 Dirac gamma matrices γ^μ $\mu = 0,1,2,3$ from Tim's course. They satisfy the anti-commutation algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

with $g^{\mu\nu}$ the metric which we take to be diag(1,-1,-1,-1). We introduce the object $\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$. It is easy (see Exercise 1) to show that $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5^2 = 1$ (the 4 × 4 unit matrix), γ_5 is a hermitean matrix, $\gamma_5^\dagger = \gamma_5$. This follows directly because γ^0 is hermitean and γ^i anti-hermitean (i = 1,2,3), $\gamma_i = -\gamma_i^\dagger$.

We define the chiral projection operators P_L, P_R, where $P_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)$. These satisfy $P_L + P_R = 1$, $P_R^2 = P_R$, $P_L^2 = P_L$, $P_R P_L = P_L P_R = 0$. We then <u>define</u> $\psi_L = P_L\psi$, $\psi_R = P_R\psi$. For massless particles, e.g. ν's, ψ_L and ψ_R are the helicity eigenstates of $\vec{\sigma}\cdot\vec{p}$. For the adjoint spinors $\bar{\psi} = \psi^\dagger\gamma_0$ one has $\bar{\psi}_L = \bar{\psi}P_R$, $\bar{\psi}_R = \bar{\psi}P_L$. To see this consider $\bar{\psi}_L = \psi_L^\dagger\gamma_0 = (P_L\psi)^\dagger\gamma_0 = \psi^\dagger P_L^\dagger\gamma_0 = \psi^\dagger P_L\gamma_0 = \psi^\dagger\gamma_0 P_R = \bar{\psi}P_R$. $P_L^\dagger = P_L$ since γ_5 is hermitean. $P_L\gamma_0 = \gamma_0 P_R$ follows from the anti-commutation of γ_0 and γ_5.

To produce a chiral theory where only left handed particles and right-handed anti-particles are involved one uses the projection operator P_L and changes the charge raising and lowering currents Fermi proposed to

$$J_\mu^+ = \bar{\nu}\gamma_\mu P_L e = \bar{\nu}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)e = \bar{\nu}_L\gamma_\mu e_L\ ,$$

$$J_\mu^- = \bar{e}\gamma_\mu P_L \nu = \bar{e}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)\nu = \bar{e}_L\gamma_\mu \nu_L\ .$$

In deriving the final form of the current involving only left-handed fields one uses $P_L = P_L^2$ and takes one P_L through the γ_μ using $\gamma_\mu P_L = P_R\gamma_\mu$, then $\bar{\nu}P_R = \bar{\nu}_L$. The resulting weak interaction theory is referred to as 'V-A' since the set of four numbers $\bar{\psi}\gamma_\mu\psi$ transform as the co-ordinates of a vector and $\bar{\psi}\gamma_\mu\gamma_5\psi$ as co-ordinates of an axial vector under a co-ordinate transformation.

Notice that QED is a non-chiral U(1) gauge theory. If we write out the QED Lagrangian in terms of the chiral components of the fields we have a manifest $L \leftrightarrow R$ symmetry. For instance $J_\mu^{em} = \bar{\psi}\gamma_\mu\psi = \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R$ (check you understand why by plugging in P_L's and P_R's!), similarly for the Dirac mass term $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$. The QED Lagrangian is invariant under the local gauge transformation $\psi \longrightarrow \psi' = e^{i\Lambda(x)}\psi$ ($\bar{\psi} \longrightarrow \bar{\psi}' = e^{-i\Lambda(x)}\bar{\psi}$). For the chiral components separately we will have

$$\psi_L \longrightarrow \psi_L' = e^{i\Lambda(x)}\psi_L \quad \text{and}$$

$$\psi_R \longrightarrow \psi_R' = e^{i\Lambda(x)}\psi_R .$$

Thus the chiral components have <u>identical</u> gauge transformations. In the next section we shall construct a chiral $SU(2)_L \times U(1)$ gauge theory of weak and electromagnetic interactions. Crucially e_L and e_R will have <u>different</u> gauge transformations.

2. Glashow's Model $SU(2)_L \times U(1)_Y$

We begin by defining a weak isospin doublet containing a left-handed electron and electron neutrino,

$$\chi_L = \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} \equiv \begin{bmatrix} \nu \\ e \end{bmatrix}_L ,$$

with an adjoint

$$\bar{\chi}_L = (\bar{\nu}_L \bar{e}_L) .$$

These row and column vectors are acted on by isospin generators in the form of 2×2 Pauli matrices

$$\tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \qquad \tau^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

The generators $\frac{1}{2}\tau^i$ satisfy the SU(2) algebra

$$\left[\tfrac{1}{2}\tau^i , \tfrac{1}{2}\tau^j\right] = i\, \epsilon_{ijk}\, \tfrac{1}{2}\tau^k .$$

The isospin raising and lowering operators are $\tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau^2)$.

One can then write an isospin triplet of weak currents

$$J^i_\mu = \bar{\chi}_L \gamma_\mu \tfrac{1}{2}\tau^i \chi_L . \qquad i = 1,2,3$$

Putting in the row vectors, column vectors and matrices, we have explicitly on multiplying out (Check!)

$$J^1_\mu = \frac{1}{2}(\bar{e}_L\gamma_\mu\nu_L + \bar{\nu}_L\gamma_\mu e_L)$$

$$J^2_\mu = \frac{i}{2}(\bar{e}_L\gamma_\mu\nu_L - \bar{\nu}_L\gamma_\mu e_L)$$

$$J^3_\mu = \frac{1}{2}(\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L) .$$

The charge raising and lowering V-A currents can be written in terms of J^1_μ and J^2_μ.

$$J^\pm_\mu = \bar{\chi}_L\gamma_\mu \tau^\pm \chi_L = J^1_\mu \pm iJ^2_\mu .$$

The isospin triplet of currents have corresponding charges

$$T^i = \int J^i_0(x)\, d^3x$$

and these satisfy an SU(2) algebra,

$$[T^i, T^j] = i\,\epsilon_{ijk}\,T^k .$$

To construct a combined weak and electromagnetic theory we will also require the electromagnetic current

$$J_\mu^{em} = Q(\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R)$$

where Q denotes the charge of the particle (in this case an electron) in units of e ≃ .303. So Q = -1 for e^-. In terms of the net charge of interacting particles J_μ^3 and J_μ^{em} are neutral currents, whereas J_μ^1 and J_μ^2 are charged currents. J_μ^3 does not involve e_R whereas electromagnetism does and so to have a gauge theory involving both weak and electromagnetic interactions we must add an extra current J_μ^Y to J_μ^3. The simplest approach is simply to write

$$J_\mu^{em} = J_\mu^3 + \tfrac{1}{2}J_\mu^Y$$

then putting in the expressions for J_μ^{em} and J_μ^3 we have

$$J_\mu^Y = -\bar{\chi}_L\gamma_\mu\chi_L - 2\bar{e}_R\gamma_\mu e_R$$

$$= -\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L - 2\bar{e}_R\gamma_\mu e_R .$$

In virtue of the above identity between J_μ^{em}, J_μ^3 and J_μ^Y the corresponding charges Q (electric charge in units of e), T^3 (third component of weak isospin) and Y (termed hypercharge) satisfy

$$Q = T^3 + \frac{Y}{2} .$$

This is identical to the Gell-Mann Nishijima relation obtained in the quark model of hadrons. The $\frac{1}{2}$ coefficient in front of J^Y_μ is conventional. T^3, Q, Y may be read off from the coefficients of the $\bar{\nu}_L\gamma_\mu\nu_L$, $\bar{e}_L\gamma_\mu e_L$, $\bar{e}_R\gamma_\mu e_R$ terms in J^3_μ, J^{em}_μ, J^Y_μ above. They are summarised in the Table.

Lepton	T	T^3	Q	Y
ν_L	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
e_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_R	0	0	-1	-2

Each generation of leptons will have a similar weak isospin doublet with the same quantum numbers,

$$\begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L , \quad \begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L , \quad \begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L .$$

We have an $SU(2)_L \times U(1)_Y$ structure where the generators of $U(1)_Y$ commute with those of $SU(2)_L$. This implies that members of an isospin doublet must have the <u>same</u> hypercharge.

We have the following commutation relations for the generators T^i, Q, Y $(i = 1,2,3)$,

$$[T^i, Y] = 0; \qquad [Q, Y] = 0; \qquad [Q, T^i] = i\epsilon_{3ij} T^j$$

so Q, T^3, Y form a mutually commuting set of generators, but only <u>two</u> are independent because of the relation $Q = T^3 + \frac{Y}{2}$. The maximum number of independent mutually commuting generators of a Lie group defines the <u>rank</u> of the group. $SU(2)_L \times U(1)_Y$ has rank 2. This concept will be of importance much later when we attempt to find candidate unification groups in which to embed the Standard Model.

Notice that $U(1)_Y$ is chiral since e_L^- and e_R^- have <u>different</u> hypercharges whereas the electromagnetic charges are the same.

To complete the specification of an $SU(2)_L \times U(1)_Y$ gauge theory, invariant under local gauge transformations, we need to introduce suitable vector fields to couple with these currents.

QED is based on the interaction $-eJ^{em\mu}A_\mu$ of the electromagnetic current $Q\bar{\psi}\gamma^\mu\psi$ with the photon field A_μ. This leads to a term in the Lagrangian $\bar{\psi}\gamma^\mu(i\partial_\mu + eA_\mu)\psi$. Analogously we introduce an isotriplet of vector gauge bosons V_μ^i, i = 1,2,3, to gauge the $SU(2)_L$ symmetry with coupling g, and a vector boson B_μ to gauge the $U(1)_Y$ symmetry with coupling g′/2. The interaction (analogous to QED) will be $-gJ^{i\mu}V_\mu^i - \frac{g'}{2}J^{Y\mu}B_\mu$, leading to the lepton-gauge boson portion of $\mathscr{L}$

$$\mathscr{L}(e) = \bar{\chi}_L\gamma^\mu\left[i\partial_\mu - g(\tfrac{1}{2})\vec{\tau}\cdot\vec{V}_\mu - \frac{g'}{2}(-1)B_\mu\right]\chi_L$$

$$+ \bar{e}_R\gamma^\mu\left[i\partial_\mu - \frac{g'}{2}(-2)B_\mu\right]e_R .$$

The $(\frac{1}{2})$, (-1), (-2) in brackets are, respectively, the weak isospin of the doublet χ_L, $Y(e_L)$ and $Y(e_R)$. The notation $\vec{\tau}\cdot\vec{V}_\mu$ is shorthand for $\tau^1V_\mu^1 + \tau^2V_\mu^2 + \tau^3V_\mu^3$. The full lepton-gauge boson Lagrangian will contain $\sum_{\ell=e,\mu,\tau} \mathscr{L}(\ell)$.

The $SU(2)_L$ and $U(1)_Y$ gauge transformations under which $\mathscr{L}(\ell)$ is invariant (see Exercise 2) are (cf. Ian's notes)

$$\chi_L \rightarrow \chi_L' = \exp\left[-ig\,\frac{\vec{\tau}}{2}\cdot\vec{\Lambda} + i\,\frac{g'}{2}\Lambda\right]\chi_L$$

$$e_R \longrightarrow e'_R = \exp(ig'\Lambda)e_R$$

$$\vec{V}_\mu \longrightarrow \vec{V}'_\mu = \vec{V}_\mu + g\vec{\Delta} \times \vec{V}_\mu + \partial_\mu\vec{\Delta}$$

$$B_\mu \longrightarrow B'_\mu = B_\mu + \partial_\mu\Lambda .$$

Where $\Lambda(x)$ specifies the local $U(1)_Y$ gauge transformations and $\vec{\Delta}(x)$ $(\Delta^1(x),\Delta^2(x),\Delta^3(x))$, the $SU(2)_L$ local gauge transformations. Explicitly $V^{i'}_\mu = V^i_\mu + g\epsilon_{ijk}\Delta^j V^k_\mu + \partial_\mu\Delta^i$.

Separating off the interaction piece of $\mathscr{L}(\ell)$ we have

$$\mathscr{L}_I = \bar{\chi}_L\gamma^\mu[-g\,\tfrac{1}{2}\,\vec{\tau}\cdot\vec{V}_\mu + \tfrac{1}{2}\,g'B_\mu]\chi_L$$

$$+ \bar{e}_R\gamma^\mu g'B_\mu e_R .$$

We want to decompose this into a charged current (exchange of electrically charged $W^\pm$), and neutral current (exchange of electrically neutral Z^0).

$$\mathscr{L}_I = \mathscr{L}_{CC} + \mathscr{L}_{NC} .$$

Consider the $\vec{\tau}\cdot\vec{V}_\mu$ term in $\mathscr{L}_I$. We have

$$\tfrac{1}{2}(\vec{\tau}\cdot\vec{V}_\mu) = \frac{\tau^1}{2}V^1_\mu + \frac{\tau^2}{2}V^2_\mu + \frac{\tau^3}{2}V^3_\mu$$

$$= \frac{1}{\sqrt{2}}\,(\tau^+V^+_\mu + \tau^-V^-_\mu) + \frac{\tau^3}{2}V^3_\mu .$$

Here we have defined the charged vector fields $V^\pm_\mu = \frac{1}{\sqrt{2}}\,(V^1_\mu \mp iV^2_\mu)$. The V^3_μ

term is neutral and so belongs in $\mathscr{L}_{NC}$. We therefore have

$$\mathscr{L}_{CC} = \bar{\chi}_L \gamma^\mu \left[-\frac{g}{\sqrt{2}} (\tau^+ W^+_\mu + \tau^- W^-_\mu) \right] \chi_L$$

$$= -\frac{g}{\sqrt{2}} [J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu}] \ .$$

So our V-A charge raising and charge lowering currents couple to the charged $W^\pm_\mu$ fields.

The rest of $\mathscr{L}_I$ gives us

$$\mathscr{L}_{NC} = \bar{\chi}_L \gamma^\mu \left[-\frac{g}{2} \tau^3 W^3_\mu + \frac{g'}{2} B_\mu \right] \chi_L + \bar{e}_R \gamma^\mu g' B_\mu e_R$$

$$= -g \ J^3_\mu W^{3\mu} - \frac{g'}{2} J^Y_\mu B^\mu \ .$$

The next step is to identify the physical neutral vector fields Z_μ and A_μ corresponding to the Z^0 and the photon. We therefore write W^3_μ and B_μ as an orthogonal mixture of Z_μ and A_μ.

$$\begin{bmatrix} W^3_\mu \\ B_\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{bmatrix} \begin{bmatrix} Z_\mu \\ A_\mu \end{bmatrix} \ .$$

The angle θ_w is the weak mixing angle (w stands for Glasho<u>w</u> or weak, not Weinberg!).

So in terms of Z_μ and A_μ

$$\mathscr{L}_{NC} = -g \ J^3_\mu [\cos\theta_w \ Z^\mu + \sin\theta_w \ A^\mu] - \frac{g'}{2} J^Y_\mu [-\sin\theta_w \ Z^\mu + \cos\theta_w \ A^\mu] \ .$$

We must have that $J^{em}_\mu = J^3_\mu + \frac{1}{2}J^Y_\mu$ is coupled to A_μ with strength e, so we need

$$\mathscr{L}_{NC} = -eA^\mu(J^3_\mu + \tfrac{1}{2}J^Y_\mu) + \ldots$$

So both $J^3_\mu A^\mu$ and $\frac{1}{2}J^Y_\mu A^\mu$ terms must have coefficient -e, implying

$$g \sin\theta_W = g' \cos\theta_W = e ,$$

or equivalently

$$\frac{1}{g^2} + \frac{1}{g'^2} = \frac{1}{e^2} .$$

We then have

$$\mathscr{L}_{NC} = -e\, J^{em}_\mu A^\mu$$

$$+ Z^\mu[-g \cos\theta_W\, J^3_\mu - g' \sin\theta_W J^3_\mu + g' \sin\theta_W\, J^{em}_\mu]$$

where J^Y_μ has been eliminated using $J^Y_\mu = 2(J^{em}_\mu - J^3_\mu)$. The terms in the square bracket coefficient of Z^μ can then be written

$$\left[-g \frac{\cos^2\theta_W}{\cos\theta_W} J^3_\mu - g \frac{\sin^2\theta_W}{\cos\theta_W} J^3_\mu + g \frac{\sin^2\theta_W}{\cos\theta_W} J^{em}_\mu\right]$$

where $g' = g \sin\theta_W/\cos\theta_W$ has been used. Then setting $\cos^2 + \sin^2 = 1$ we get

$$\mathscr{L}_{NC} = -e\, J^{em}_\mu A^\mu - \frac{g}{\cos\theta_W}[J^3_\mu - \sin^2\theta_W\, J^{em}_\mu]Z^\mu .$$

So the neutral current piece comprises the standard QED electromagnetic interaction, strength e, and a weak neutral current interaction with a Z^0 with strength $\frac{g}{\cos\theta_W}$.

So finally we have

$$\mathscr{L}_I = -\frac{g}{\sqrt{2}}\left[J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu}\right]$$

$$- e\, J^{em}_\mu A^\mu - \frac{g}{\cos\theta_w}\left[J^3_\mu - \sin^2\theta_w\, J^{em}_\mu\right] Z^\mu \ .$$

Expressing the currents in terms of the full fermion fields ν,e we obtain

$$\mathscr{L}_I = -\frac{g}{\sqrt{2}}[\bar{\nu}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)e\, W^{+\mu}$$

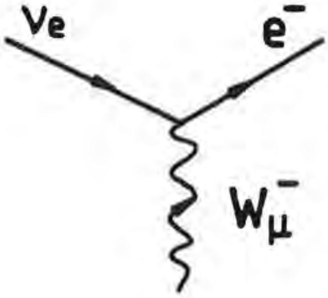
$$+ \bar{e}\, \gamma_\mu \tfrac{1}{2}(1-\gamma_5)\nu\, W^{-\mu}]$$

$$+ e(\bar{e}\, \gamma_\mu\, e\, A^\mu)$$

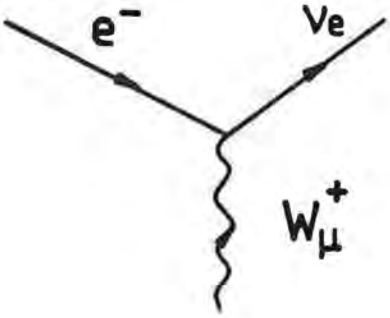
$$- \frac{g}{2\cos\theta_w}\left\{\bar{\nu}\, \gamma_\mu \tfrac{1}{2}(1-\gamma_5)\nu - \bar{e}\, \gamma_\mu \tfrac{1}{2}(1-\gamma_5)e\right.$$

$$\left.+ 2\sin^2\theta_w\, \bar{e}\, \gamma_\mu\, e\right\} Z^\mu \ .$$

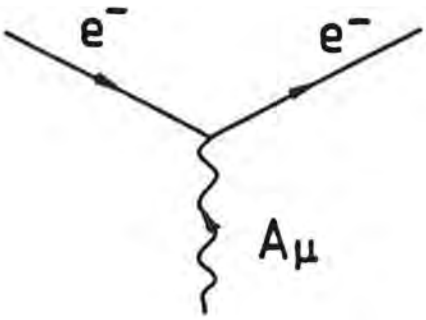
From the coefficients of the $\ell\bar{\ell}V$ terms (ℓ = e,ν, $V = A(\gamma), W^\pm, Z$) multiplied by i, we obtain the vertex factors listed below.



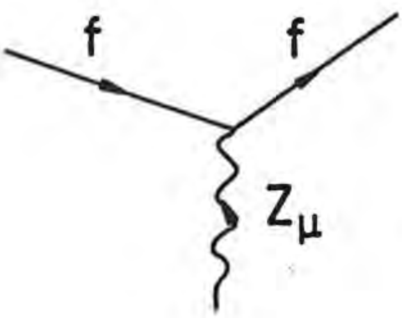
$$-\frac{ig}{\sqrt{2}}\ \gamma_\mu \tfrac{1}{2}\,(1-\gamma_5)$$



$$-\frac{ig}{\sqrt{2}}\ \gamma_\mu \tfrac{1}{2}\,(1-\gamma_5)$$



$$ie\ \ \gamma_\mu(-ie\,Q\gamma_\mu)$$



$$\frac{-ig}{2\cos\theta_w}\gamma_\mu\,[\,c_V^f - c_A^f\gamma_5\,]$$

For $f = \nu_e$ $\quad c_V^\nu = c_A^\nu = \frac{1}{2}$ (ν_L only!)

For $f = e^-$ $\quad c_V^e = -\frac{1}{2} + 2\sin^2\theta_W$, $\quad c_A^e = -\frac{1}{2}$

(e_L, e_R have different couplings)

In general $c_V^f = T_f^3 - 2Q_f\sin^2\theta_W$, $\ c_A^f = T_f^3$.

To complete the Glashow model Lagrangian we need $SU(2)_L \times U(1)_Y$ gauge invariant self-interaction and kinetic energy terms for the vector boson fields. In QED we have the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As discussed in Ian's notes the relevant terms for the V_μ^i fields ($\mathscr{L}_V$) and B_μ ($\mathscr{L}_B$) are

$$\mathscr{L}_V = -\frac{1}{4}\vec{V}_{\mu\nu}\cdot\vec{V}^{\mu\nu} = -\frac{1}{4}\sum_i (\vec{V}_{\mu\nu})^i(\vec{V}^{\mu\nu})^i ,$$

where

$$\vec{V}_{\mu\nu} = \partial_\mu\vec{V}_\nu - \partial_\nu\vec{V}_\mu - g\,\vec{V}_\mu \times \vec{V}_\nu ,$$

and

$$(\vec{V}_{\mu\nu})^i = \partial_\mu V_\nu^i - \partial_\nu V_\mu^i - g\,V_\mu^k\,V_\nu^\ell\,\epsilon_{ik\ell} .$$

Explicitly in terms of the fields V_μ^i $\ i = 1,2,3$ which gauge $SU(2)_L$. For the $U(1)_Y$ gauge fields B_μ, $\mathscr{L}_B = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$ with $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

These can of course be rewritten in terms of the physical fields $W_\mu^+, W_\mu^-, Z_\mu, A_\mu$.

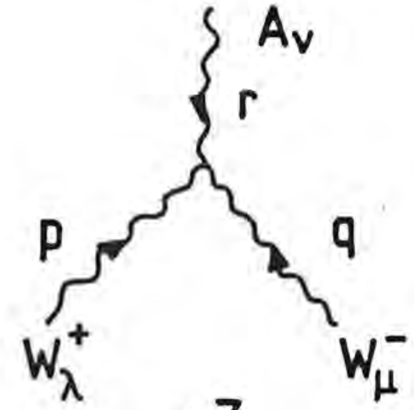
$$V_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-)$$

$$V_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^- - W_\mu^+)$$

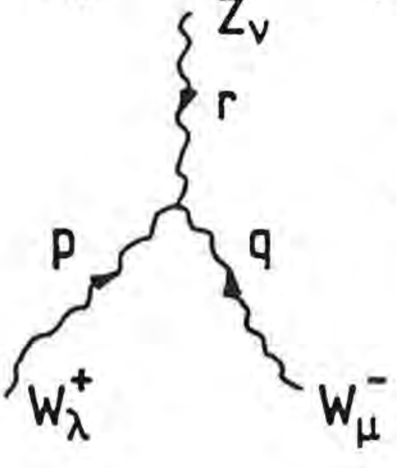
$$V_\mu^3 = \cos\theta_W\,Z_\mu + \sin\theta_W\,A_\mu$$

$$B_\mu = \cos\theta_W\,A_\mu - \sin\theta_W\,Z_\mu .$$

Having rewritten $\mathscr{L}_V$ and $\mathscr{L}_B$ we can pick out the $(\partial_\mu V)VV$ and VVVV cross-terms in $\mathscr{L}_V$ and $\mathscr{L}_B$ which will correspond to VVV and VVVV interactions. The Feynman rules are in momentum space so $i\partial_\mu V$ should be replaced by $p_\mu V$, where p_μ is the momentum of the vector boson V. The non-vanishing VVV interactions are then

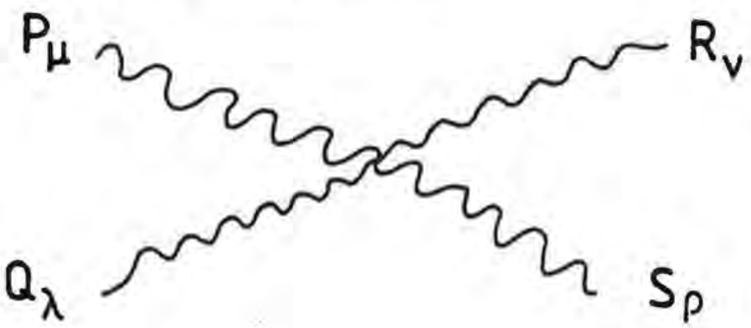


$$e[(r-q)_\lambda g_{\mu\nu} + (q-p)_\nu g_{\lambda\mu} + (p-r)_\mu g_{\lambda\nu}]$$



$$g\cos\theta_w [(r-q)_\lambda g_{\mu\nu} + (q-p)_\nu g_{\lambda\mu} + (p-r)_\mu g_{\lambda\nu}]$$

We will also have VVVV interactions



where P,Q,R,S are vector bosons. The vertex factor is then $K_{PQRS}\ T_{\mu\nu\lambda\rho}$ where

$$T_{\mu\nu\lambda\rho} = 2g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\ ,$$

and the coefficients K_{PQRS} are summarised for the non-vanishing vertices below.

P	Q	R	S	K_{PQRS}
V^+	V^-	V^+	V^-	ig^2
A	V^+	A	V^-	$-ie^2$
Z	V^+	Z	V^-	$-ig^2\cos^2\theta_W$
A	V^+	Z	V^-	$-ieg\cos\theta_W$

We now have all the Feynman rules for the Glashow model Lagrangian

$$\mathscr{L} = \sum_{\ell=e,\mu,\tau} \mathscr{L}(\ell) + \mathscr{L}_V + \mathscr{L}_B\ .$$

Notice that there are <u>no</u> mass terms. If we want to have an $SU(2)_L \times U(1)_Y$ gauge invariant theory we cannot have them! For instance a mass term for B_μ would be $\frac{1}{2}M_B^2 B_\mu B^\mu$. Under a $U(1)_Y$ gauge transformation $B_\mu \rightarrow B'_\mu = B_\mu + \partial_\mu \Lambda$ and obviously $\frac{1}{2}M_B^2 B_\mu B^\mu \neq \frac{1}{2}M_B^2 B'_\mu B'^\mu$. Similarly for a term involving $M_V^2 \vec{V}_\mu \cdot \vec{V}^\mu$ under an $SU(2)_L$ gauge transformation. A Dirac mass term for the leptons is also disallowed since $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$. This is gauge invariant in QED which is $L \leftrightarrow R$ symmetric, but in the chiral $SU(2)_L \times U(1)_Y$ theory ψ_R and ψ_L have <u>different</u> gauge transformations.

We could of course take the attitude that there is nothing sacrosanct about the gauge principle and simply add mass terms by brute force, including an extra gauge non-invariant term

$$\mathscr{L}_M = - \sum_{\ell=e,\mu,\tau} m_\ell \bar{\ell}\ell + M_V^2 V_\mu^+ V^{-\mu} + \tfrac{1}{2} M_Z^2 Z_\mu Z^\mu .$$

The resulting theory is however not renormalisable. This means that the results of calculating Feynman diagrams containing loops are uncontrollably divergent. In field theory we are used to the idea that there will be loop divergences. In renormalisable theories such as QED and the massless Yang-Mills-Shaw-Glashow theory above one can absorb these infinities into a redefinition of the physical charges, masses, couplings of the theory in such a way that finite results are obtained. In abandoning gauge invariance this renormalisation programme breaks down. Interestingly, as we shall mention later, supersymmetric field theories can yield finite results without the necessity for renormalisation. Dirac went to his grave believing strongly that the need to renormalise was indicative of our incomplete understanding of field theory. Even at the tree level the Glashow model with the ad hoc mass term is unsatisfactory. The propagator for a massive vector boson of virtuality q^2 involves $(g_{\mu\nu} - q_\mu q_\nu / M_V^2)/(q^2 - M_V^2)$. Longitudinally polarised V bosons are described by polarisation ve tors with $\epsilon_\mu^L \rightarrow \frac{q_\mu}{M_V}$ as $q^2 \rightarrow \infty$, so the propagator approaches a constant at large q^2. This means that the longitudinally polarized $V_L^+ V_L^- \rightarrow V_L^+ V_L^-$ scattering cross section grows like the square of the c.m. energy and unitarity is violated. In QED γ^L virtual photons do not contribute since $U(1)_{em}$ gauge invariance implies that amplitudes are invariant under $\epsilon_\mu \rightarrow \epsilon_\mu + \lambda q_\mu$ so pieces proportional to q_μ do not contribute to physical processes.

We therefore need to generate masses more subtly. One possibility is to exploit the so-called Higgs mechanism suggested by Peter Higgs in 1964 and motivated by the generation of the masses of Cooper pairs in superconductivity, and obtain the masses by spontaneous breaking of a symmetry.

3. The Higgs Mechanism — Weinberg's Model

We begin by defining the $SU(2)_L \times U(1)_Y$ covariant derivative

$$D_\mu = \partial_\mu + \frac{ig}{2} \vec{\tau}\cdot\vec{W}_\mu + ig' \frac{Y}{2} B_\mu .$$

We introduce an $SU(2)_L$ doublet of complex scalar Higgs fields

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} .$$

The doublet has weak isospin $T = \frac{1}{2}$ and hypercharge $Y = 1$ leading to electromagnetic charges +1,0 for the $T^3 = \pm\frac{1}{2}$ upper and lower members of the doublet ($Q = T^3 + \frac{Y}{2}$).

In terms of real scalar fields ϕ_i

$$\phi^+ = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \text{and} \quad \phi^0 = \frac{\phi_3 + i\phi_4}{\sqrt{2}} .$$

We then add to the massless Glashow model Lagrangian the scalar contribution

$$\mathscr{L}_\Phi = (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi) .$$

The conjugate $\Phi^\dagger$ contains the anti-particles ($\phi^-\bar{\phi}^0$).

The most general $SU(2)_L$ invariant and renormalisable $V(\Phi)$ (mass dimension ≤ 4 where $[\Phi] = 1$, $[\psi] = \frac{3}{2}$, for a scalar and fermion field) is

$$V(\Phi) = -\mu^2(\Phi^\dagger\Phi) + \lambda(\Phi^\dagger\Phi)^2 .$$

We arrange that $\mathscr{L}_\Phi$ contains a $+\mu^2\Phi^\dagger\Phi$ term. Notice that an ordinary scalar mass term would be $-\mu^2\Phi^\dagger\Phi$, but we want $V(\Phi)$ to be bounded below so that there will be an $SU(2)_L$ invariant manifold of minima. $\mathscr{L}_\Phi$ is invariant under the $SU(2)_L \times U(1)_Y$ gauge transformations

$$\Phi \longrightarrow \Phi' = \exp\left[-ig\,\frac{\vec{\tau}}{2}\cdot\vec{\Lambda} - i\frac{g'}{2}\Lambda\right]\Phi .$$

$V(\Phi)$ has minima specified by

$$\frac{dV}{d(\Phi^\dagger\Phi)} = 0 \Rightarrow -\mu^2 + 2\lambda(\Phi^\dagger\Phi) = 0$$

so

$$\Phi^\dagger\Phi\Big|_{min} = \frac{\mu^2}{2\lambda} ,$$

or in terms of real scalar fields ϕ_i

$$\frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \frac{\mu^2}{2\lambda} .$$

We need to spontaneously break $SU(2)_L \times U(1)_Y$ by picking the vacuum from the set of minima of the potential V. We shall choose the vacuum expectation values (VEV's)

$$\phi_1 = \phi_2 = \phi_4 = 0 , \qquad \phi_3^2 = \frac{\mu^2}{\lambda} = v^2 .$$

Letting the neutral field ϕ_3 acquire a VEV will, as we shall see, allow the photon to remain massless as it must.

So we choose $\langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\v\end{bmatrix}$.

We now expand Φ around this vacuum, setting $\phi_3 = H + v$, where H is the neutral scalar Higgs field. It is possible to choose a special gauge, the unitary gauge, in which

$$\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\H+v\end{bmatrix} .$$

That is the 'Goldstone' fields with zero VEV, ϕ_1,ϕ_2,ϕ_4 can be eliminated. To see this apply the local gauge transformation $\exp(i\vec{\tau}\cdot\vec{\theta}(x)/v)$ to Φ, to obtain

$$\Phi' = \frac{1}{\sqrt{2}}\exp\left[\frac{i\vec{\tau}\cdot\vec{\theta}(x)}{v}\right]\begin{bmatrix}0\\H+v\end{bmatrix} .$$

Expanding the exponential to $O(\theta)$

$$\Phi' = \frac{1}{\sqrt{2}}\begin{bmatrix}1+i\theta_3/v & i(\theta_1-i\theta_2)/v\\ i(\theta_1+i\theta_2)/v & 1-i\theta_3/v\end{bmatrix}\begin{bmatrix}0\\v+H\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}\theta_2+i\theta_1\\ v+H-i\theta_3\end{bmatrix} .$$

So we see that $\frac{1}{\sqrt{2}}\begin{bmatrix}0\\H+v\end{bmatrix}$ is a gauge transformation of a general Φ with four independent scalar fields. The idea is that the three originally massless gauge fields $W^{\pm},Z^0$ will become massive and acquire three longitudinal polarization degrees of freedom by eating the three Goldstone bosons. Notice the above gauge transformation accordingly uses only three of the four possible parameters $\Lambda = 0$, $\vec{\Lambda} = -\frac{2\vec{\theta}}{v}$.

We can now write out $\mathscr{L}_\Phi$ in the unitary gauge explicitly and exhibit the spontaneously generated mass terms for $W^{\pm}$ and Z^0. Setting $\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\H+v\end{bmatrix}$

and recalling the form of the covariant derivative D_μ we have

$$D_\mu\Phi = \begin{bmatrix} \partial_\mu + i\frac{g}{2} V_\mu^3 + i\frac{g'}{2} B_\mu & \frac{ig}{2}(V_\mu^1 - iV_\mu^2) \\ i\frac{g}{2} V_\mu^1 + iV_\mu^2 & \partial_\mu - \frac{ig}{2} V_\mu^3 + \frac{ig'}{2} B_\mu \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ H+v \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{ig}{2} (V_\mu^1 - iV_\mu^2)(H+v) \\ (\partial_\mu - \frac{ig}{2} V_\mu^3 + \frac{ig'}{2} B_\mu)(H+v) \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{ig}{\sqrt{2}} V_\mu^+ (H+v) \\ (\partial_\mu - \frac{i}{2} (g \cos\theta_W + g' \sin\theta_W) Z_\mu) (H+v) \end{bmatrix}$$

Notice that there is no A_μ involved, only $V_\mu^\pm$ and Z_μ. The photon cannot acquire a mass term $A_\mu A^\mu$, therefore. The masslessness of the photon is guaranteed by the $U(1)_{em}$ gauge invariance of the vacuum. Denoting $\Phi_0 \equiv \langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ v \end{bmatrix}$ we have

$$Q\Phi_0 \equiv \left[\frac{T^3}{2} + \frac{Y}{2}\right]\Phi_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix} = 0 .$$

Evidently therefore

$$e^{i\alpha(x)Q}\,\Phi_0 = \Phi_0$$

for any local $U(1)_{em}$ gauge transformation $\alpha(x)$. $U(1)_{em}$ is a residual symmetry which keeps the photon massless. $SU(2)_L \times U(1)_Y$ has been spontaneously broken to $U(1)_{em}$, and the originally massless $V^\pm$,Z gauge bosons have acquired masses in the process.

We finally obtain in the unitary gauge

$$\mathscr{L}_\Phi = (D_\mu\Phi)^\dagger D^\mu\Phi + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$

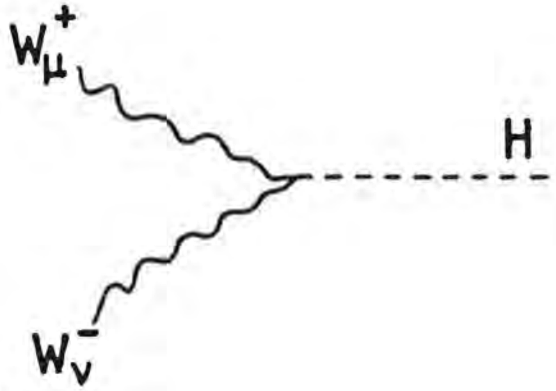
$$= \frac{1}{2}\,\partial_\mu H\partial^\mu H + \frac{1}{4}\,g^2(H^2+2vH+v^2)V^+_\mu V^{-\mu}$$

$$+ \frac{1}{8}\,(g^2+g'^2)(H^2+2vH+v^2)Z_\mu Z^\mu$$

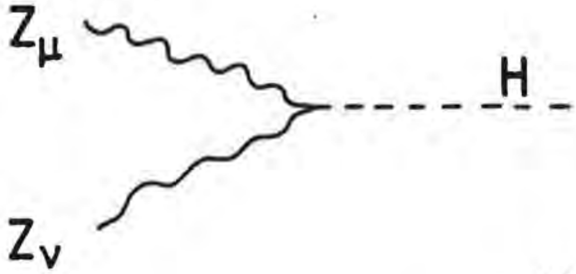
$$- \mu^2H^2 - \frac{\lambda}{4}(H^4+4vH^3)\ .$$

Notice that $(g\cos\theta_W + g'\sin\theta_W)^2 = g^2 + g'^2$. The masses of $V^\pm$ and Z can then be read off directly by identifying the terms $M_V^2V^+_\mu V^{-\mu}$ and $\frac{1}{2}M_Z^2Z_\mu Z^\mu$ in the above expression. We find $M_V = \frac{1}{2}gv$, $M_Z = \frac{1}{2}(g^2+g'^2)^{1/2}v = \frac{1}{2}\,\frac{g}{\cos\theta_W}\,v$. For the Higgs scalar mass we identify the overall H^2 term $(\frac{1}{2}\mu^2 - \frac{3}{2}\lambda v^2)H^2$ coming from $\frac{\mu^2}{2}(H+v)^2 - \frac{\lambda}{4}(H+v)^4$, and recalling that $\mu^2 = \lambda v^2$ we obtain the H^2 coefficient $-\frac{1}{2}M_H^2 = -\mu^2$ or $M_H = \sqrt{2}\mu$.

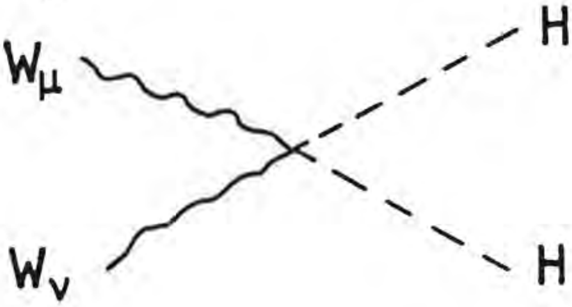
There are also VVH, VVHH interactions and HHH, HHHH Higgs self-interactions. The vertex factors are listed below.



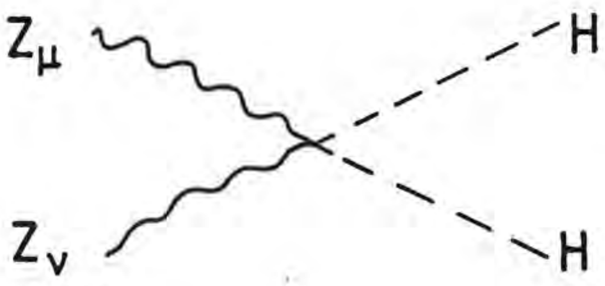
$$i g \; M_W \; g_{\mu\nu}$$



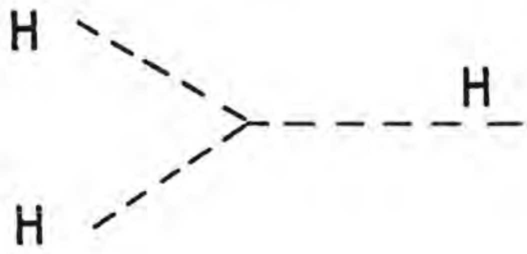
$$i g \; \frac{M_Z^2}{M_W} \; g_{\mu\nu}$$



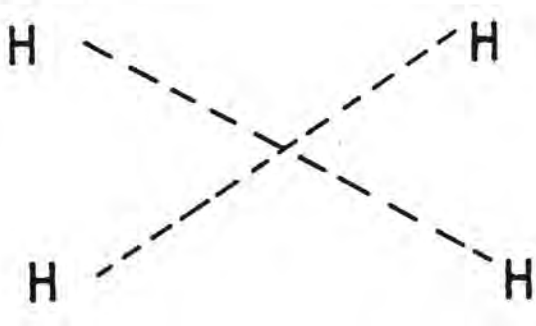
$$\frac{i}{2} \; g^2 \; g_{\mu\nu}$$



$$\frac{i}{2} \; g^2 \; \frac{M_Z^2}{M_W^2} \; g_{\mu\nu}$$



$$-i \; \frac{3}{4} \; g^2 \; \frac{M_H^2}{M_W^2}$$



$$-i \; \frac{3}{2} \; g \; \frac{M_H^2}{M_W}$$

An immediate consequence of the above vector boson masses is that

$$\frac{M_W}{M_Z} = \cos\theta_W .$$

This result is often referred to as the "weak $\Delta I = \frac{1}{2}$ rule" and is connected with our choice of a Higgs doublet to perform the spontaneous symmetry breaking.

Notice that from the measured fine structure constant α $\left[= \frac{e^2}{4\pi}\right]$ and the vector boson masses, M_W and M_Z, we can determine $\sin^2\theta_w$, v and g, but <u>not</u> μ. This means that the Higgs mass M_H is not determined directly by other experimentally measured parameters. We shall return a little later to a discussion of the number of independent Standard Model parameters.

To complete the specification of the Weinberg-Salam Standard Model we need to give masses to the charged leptons and to include massive quarks.

To give charged leptons a mass one adds a so-called Yukawa term to the Lagrangian

$$\mathcal{L}_Y(\ell), \quad \ell = e,\mu,\tau, \qquad \mathcal{L}_Y(e) = -G_e[\bar{\chi}_L\Phi e_R + \bar{e}_R\Phi^\dagger\chi_L] .$$

This is $SU(2)_L \times U(1)_Y$ invariant. On spontaneous symmetry breaking we have in the unitary gauge $\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ H+v \end{bmatrix}$, substituting this into $\mathcal{L}_Y(e)$

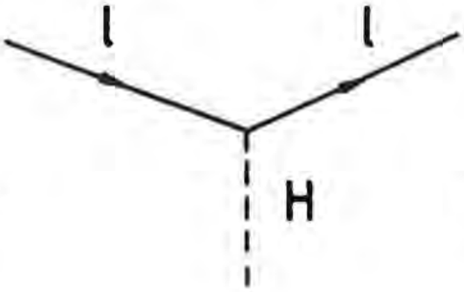
$$\mathcal{L}_Y(e) = -\frac{G_e}{\sqrt{2}}(v+H)(\bar{e}_Le_R + \bar{e}_Re_L)$$

$$= -\frac{G_e}{\sqrt{2}}(H+v)\bar{e}e = -\frac{G_ev}{\sqrt{2}}(\bar{e}e) - \frac{G_e}{\sqrt{2}}(\bar{e}eH) .$$

From which we can identify the electron mass $m_e = \frac{G_e v}{\sqrt{2}}$ and the lepton-Higgs coupling $g(He\bar{e}) = \frac{m_e}{v} = \frac{g m_e}{2M_W}$ (using $M_W = \frac{1}{2}gv$).

The lepton-Higgs Yukawa vertex factor is

$l = e, \mu, \tau$



$$-\frac{i}{2}\, g\, \frac{m_l}{M_W}$$

To include quarks we first need to add a suitable term $\mathscr{L}(q)$ to the massless Glashow model. We have the six quarks (three generations) u,d,s,c,b,t. $Q_u = Q_c = Q_t = \frac{2}{3}$, and $Q_d = Q_s = Q_b = -\frac{1}{3}$. We can construct $SU(2)_L$ doublets as for e, μ, τ

$$\chi_L^f = \begin{bmatrix} U_f \\ D_f \end{bmatrix}_L \qquad f = 1,2,3$$

where $U_1 = u$, $U_2 = c$, $U_3 = t$ and $D_1 = d$, $D_2 = s$, $D_3 = b$. However, experimentally one observes $n \rightarrow pe^-\bar{\nu}_e$ and also $\Lambda \rightarrow pe^-\bar{\nu}_e$, corresponding to $d \rightarrow u$ and $s \rightarrow u$ transitions. This implies that the weak interaction eigenstates are mixtures of the flavour eigenstates. We therefore replace the above χ_L^f by

$$\chi_L^f = \begin{bmatrix} U_f \\ D'_f \end{bmatrix}_L,$$

with $D'_f = \sum_{f'=1,2,3} V_{ff'} D_{f'}$.

Here V is a 3 × 3 unitary matrix called the Cabibbo-Kobayashi-Maskawa

(CKM) matrix. For two generations we have the Cabibbo model

$$D_1' = \cos\theta_c d + \sin\theta_c s$$

$$D_2' = -\sin\theta_c d + \cos\theta_c s \ .$$

Here θ_c is the Cabibbo angle, and experimentally one finds $\theta_c \simeq 13^0$, $\cos\theta_c \simeq .97$. As Dick will discuss, the three generation CKM matrix has the following $|V_{ij}|$ structure.

$$V = \begin{bmatrix} |V_{ud}| = .973 & |V_{us}| = .23 & |V_{ub}| = 0 \\ |V_{cd}| = .24 & |V_{cs}| = .97 & |V_{cb}| = .06 \\ |V_{td}| \simeq 0 & |V_{ts}| \simeq 0 & |V_{tb}| \simeq 1 \end{bmatrix} .$$

The matrix involves 4 parameters - 3 angles and 1 complex phase.

In analogy with the leptonic charge raising and lowering currents one defines

$$J_\mu^{f+} = \bar{U}_f \tfrac{1}{2}\gamma_\mu(1-\gamma_5)\, D_f' = \bar{U}_f \tfrac{1}{2}\gamma_\mu(1-\gamma_5) V_{ff'} D_{f'}$$

$$= \bar{U}_{fL}\gamma_\mu V_{ff'} D_{f'L}$$

$$J_\mu^{f-} = \bar{D}_f' \tfrac{1}{2}\gamma_\mu(1-\gamma_5)\, U_f = \bar{D}_f \tfrac{1}{2}\gamma_\mu(1-\gamma_5) V_{ff'}^\dagger U_f'$$

$$= \bar{D}_{fL}\gamma_\mu V_{ff'}^\dagger U_{f'L}$$

One has an isotriplet of weak quark currents,

$$J_\mu^{fi} = \bar{\chi}_L^f \gamma_\mu \tfrac{1}{2} \tau^i \chi_L^f \ ,$$

so that

$$J_\mu^{f\pm} = J_\mu^{f1} \pm i\, J_\mu^{f2} ,$$

$$J_\mu^{f3} = \tfrac{1}{2}(\bar{U}_{fL}\gamma_\mu U_{fL} - \bar{D}'_{fL}\gamma_\mu D'_{fL})$$

$$= \tfrac{1}{2}(\bar{U}_{fL}\gamma_\mu U_{fL} - \bar{D}_{fL}\gamma_\mu D_{fL}) .$$

The change from D′ to D in the last term above uses the unitarity of V ($VV^\dagger = 1$). One writes

$$J_\mu^{fem} = \left[\frac{2}{3}\right]\bar{U}_f\gamma_\mu U_f + \left[-\frac{1}{3}\right]\bar{D}_f\gamma_\mu D_f$$

$$= J_\mu^{f3} + \frac{1}{2}\, J_\mu^{fY}$$

leading to the quark hypercharge current

$$J_\mu^{fY} = \left[\frac{1}{3}\right](\bar{U}_{fL}\gamma_\mu U_{fL} + \bar{D}_{fL}\gamma_\mu D_{fL})$$

$$+ \left[\frac{4}{3}\right]\bar{U}_{fR}\gamma_\mu U_{fR} + \left[-\frac{2}{3}\right]\bar{D}_{fR}\gamma_\mu D_{fR} .$$

The T^3,Q,Y quantum numbers for the quarks are tabulated below.

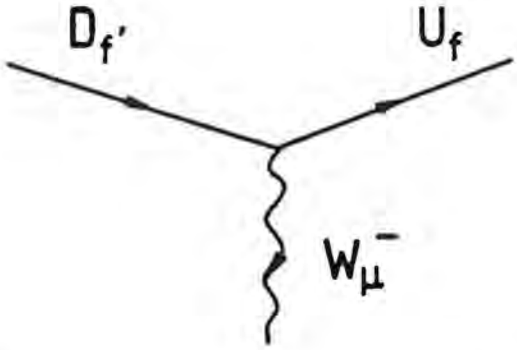
Quark	T	T^3	Q	Y
u_L	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
d_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
u_R	0	0	$\frac{2}{3}$	$\frac{4}{3}$
d_R	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$

So finally analogous to $\mathscr{L}(\ell)$ we obtain

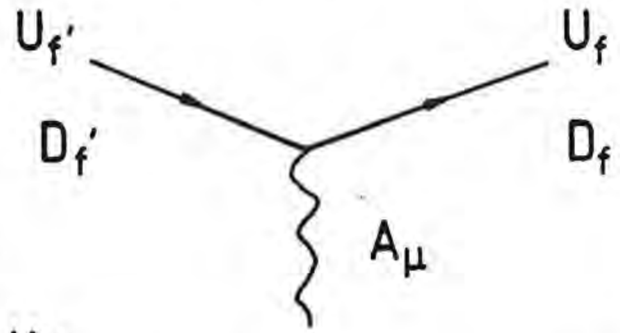
$$\mathcal{L}(q) = \sum_f \left\{ \bar{\chi}_L^f \gamma^\mu \left(i\partial_\mu - g\tfrac{1}{2}\,\vec{\tau}\cdot\vec{V}_\mu - \tfrac{g'}{2}\left[\tfrac{1}{3}\right]B_\mu\right)\chi_L^f \right.$$

$$\left. + \bar{U}_{fR}\gamma^\mu\left(i\partial_\mu - \tfrac{g'}{2}\left[\tfrac{4}{3}\right]B_\mu\right)U_{fR} + \bar{D}_{fR}\gamma^\mu\left(i\partial_\mu - \tfrac{g'}{2}\left[-\tfrac{2}{3}\right]B_\mu\right)D_{fR}\right\} .$$

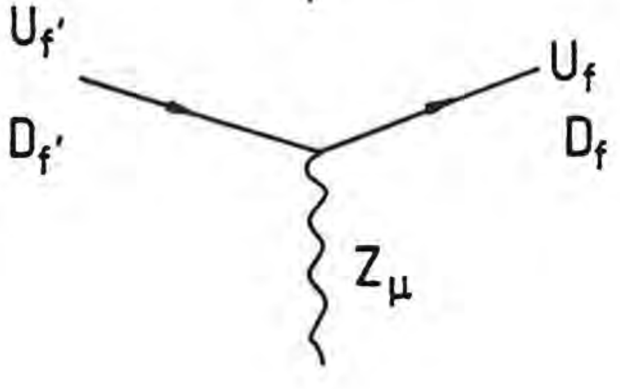
The qqV interaction vertices contained in the above are listed below.



$$-\frac{ig}{2\sqrt{2}}\,\gamma_\mu\,(1-\gamma_5)\,V_{ff'}$$



$$-ie\;Q_{\substack{U\\D}}\;\gamma_\mu\;\delta_{ff'}$$



$$\frac{-ig}{2\cos\theta}\,\gamma_\mu\left[c_V^{U/D} - c_A^{U/D}\gamma_5\right]\delta_{ff'}$$

$$c_V^U = \frac{1}{2} - \frac{4}{3}\sin^2\theta_W , \qquad c_A^U = \frac{1}{2}$$

$$c_V^D = -\frac{1}{2} + \frac{2}{3}\sin^2\theta_W , \qquad c_A^D = -\frac{1}{2}$$

The general expressions, as given earlier, are

$$c_V^f = T_f^3 - 2Q_f\sin^2\theta_W , \qquad c_A^f = T_f^3 .$$

To give the quarks masses we introduce a quark Yukawa coupling Lagrangian

$$\mathscr{L}_Y(q) = -[\bar{\chi}_L^f G_{ff'}^D \Phi D_{f'R} + \bar{\chi}_L^f G_{ff'}^u \Phi^c U_{f'R} + \text{h.c.}] .$$

Here the G^D and G^U are Yukawa couplings. $\Phi = \begin{bmatrix}\phi^+\\ \phi^0\end{bmatrix}$ as before, $Y(\Phi) = +1$, transforming as

$$\Phi \to \Phi' = (1 - ig\,\frac{1}{2}\,\vec{\tau}\cdot\vec{\Lambda} - \frac{ig'}{2}\Lambda)\Phi$$

under an $SU(2)_L \times U(1)_Y$ gauge transformation. To make $\mathscr{L}_Y(q)$ invariant we need

$$\Phi^c \to \Phi^{c\prime} = (1 - ig\,\frac{1}{2}\,\vec{\tau}\cdot\vec{\Lambda} + \frac{ig'}{2}\Lambda)\Phi^c ,$$

so $Y(\Phi^c) = -1$. We also require that after spontaneous symmetry breaking we generate masses for the upper as well as the lower member of each quark doublet in order that both acquire a mass (in the lepton case the neutrino remains massless!). So we require that in the unitary gauge

$$\Phi^c = \frac{1}{\sqrt{2}}\begin{bmatrix}H+v\\ 0\end{bmatrix} .$$

A suitable choice with $Y(\Phi^C) = -1$ is

$$\Phi^C = \begin{bmatrix} \bar{\phi}^0 \\ -\phi^- \end{bmatrix} = i\tau^2\Phi^* .$$

In the unitary gauge we then have

$$\mathcal{L}_Y(q) = -\frac{1}{\sqrt{2}} [\bar{D}'_{fL}\, G^D_{ff'}\, D_{f'R}\, (H+v)$$

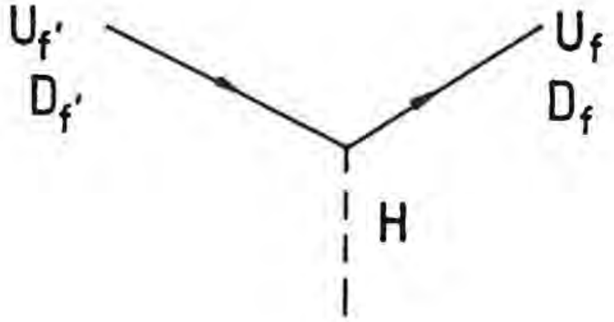
$$+ \bar{U}_{fL}\, G^U_{ff'}\, U_{f'R}\, (H+v) + \text{h.c.}]$$

where $\bar{D}'_{fL} = V^\dagger_{f'L}\bar{D}_{f'L}$. For suitable diagonalizing choices of G^D, G^U we then have the mass terms

$$\frac{v}{\sqrt{2}}(V^\dagger G^D)_{ff'} = \text{diag}(m_d, m_s, m_b) \equiv m(D)$$

$$\frac{v}{\sqrt{2}}\, G^U_{ff'} = \text{diag}(m_u, m_c, m_t) \equiv m(U)$$

The qqH vertex factor is



$$-\frac{ig}{2M_W}\, [m\binom{U}{D}]_{ff'}$$

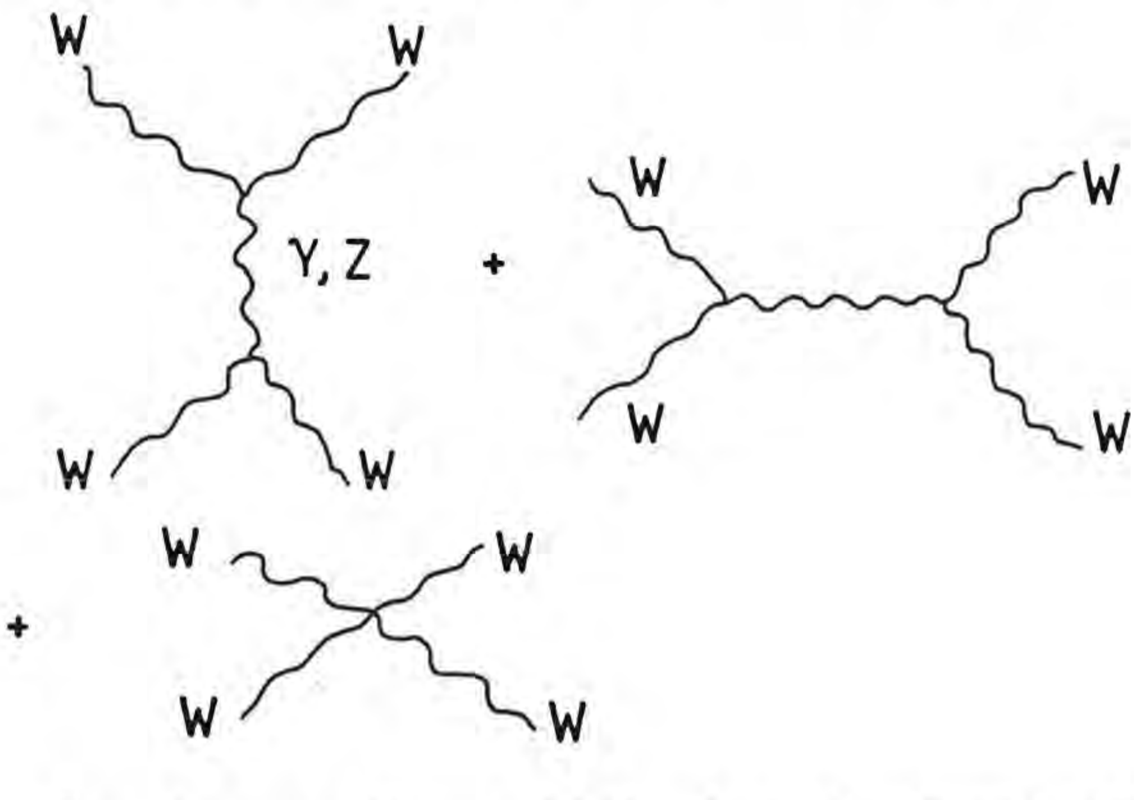
Putting all these pieces together and adding QCD we finally arrive at the Glashow-Weinberg-Salam Standard Model Lagrangian

$$\mathcal{L}_{SM} = \mathcal{L}_W + \mathcal{L}_B + \sum_{\ell=e,\mu,\tau} \mathcal{L}(\ell) + \sum_{f=1,2,3} \mathcal{L}(q)$$

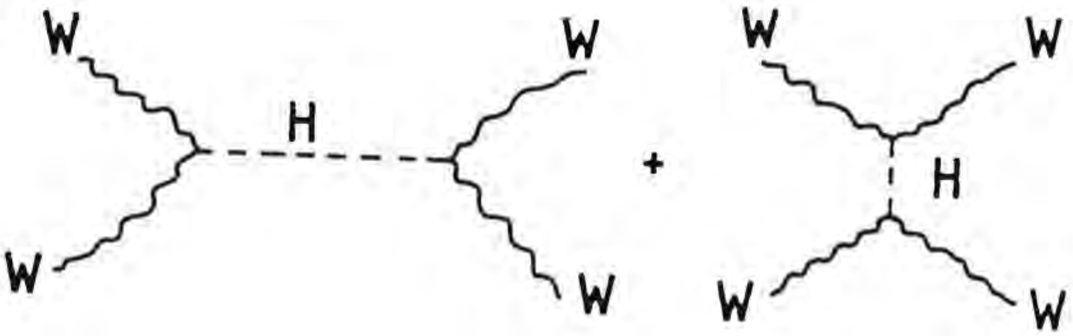
$$+ \mathcal{L}_\phi + \sum_{\ell=e,\mu,\tau} \mathcal{L}_Y(\ell) + \sum_{f=1,2,3} \mathcal{L}_Y(q) + \mathcal{L}_{QCD} \, .$$

By adding $\mathcal{L}_{QCD}$ from Tim's course we obtain the full $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge invariant theory.

The Standard Model as specified by the above Lagrangian has been shown to be renormalizable ('t Hooft, Veltman). The tree-level unitarity problem suffered by Glashow's model has also been cured. If we consider $W_L^+W_L^- \rightarrow W_L^+W_L^-$ we will have to consider the Glashow Model Feynman diagrams



These give $\sigma \sim s$ and hence a violation of unitarity. However, in the Standard Model the Higgs sector of the Lagrangian will give rise to extra diagrams



When these are included tree-level unitarity is restored providing that $M_H < \sqrt{\frac{16\pi}{3}}\, v \sim 1$ TeV. This gives a crude upper bound on the Higgs mass which ultimately comes from the fact that the Higgs self coupling $\propto M_H^2$, and if the mass is too large one cannot apply perturbation theory. However, this is not a fundamental bound, just an admission of our inability to calculate anything if the Higgs mass were to be much larger than ~ 1 TeV. Unfortunately not only is it hard to constrain the Higgs mass but the sorts of Higgs production processes which will be relevant depend on what the mass is, and are usually swamped in any case by severe backgrounds, leaving almost no 'gold-plated' ways of seeing the Higgs experimentally. Higgs phenomenology is to this extent rather dispiriting. There is also a significant body of theoretical opinion which regards the rather ad hoc addition of $\mathscr{L}_\Phi$ with considerable suspicion, whereas the massless Glashow model to which it is tacked on seems elegant and inevitable. There are several alternative suggestions for mass generation (technicolour, Nambu model, ...) which we shall not discuss.

It is of interest to count the independent parameters of the Standard Model. There are overall fifteen parameters (ignoring quarks and QCD) which we may divide up as

<u>Couplings</u> $e(\alpha), g, g', G_e, G_\mu, G_\tau$ (expt: $\cdot 303, 0.65, 0.34, 3.1\times 10^{-6}, 6.4\times 10^{-4}, 0.0$

<u>Masses</u> $M_W, M_Z \cdot M_H, m_e, m_\mu, m_\tau$

(expt: 82 GeV, 91.2 GeV, ? , .51 MeV, 105.7 MeV, 1.8 GeV)

<u>Higgs sector</u> μ^2, λ ($v^2 = \frac{\mu^2}{\lambda}$) (expt: v = 235 GeV)

<u>mixing angle</u> $\sin^2\theta_W$ (expt: .23)

Of course these fifteen are not independent since, for example, $\frac{M_W}{M_Z} = \cos\theta_W$, $M_W = \frac{1}{2}gv$, $e = g\sin\theta_W$, etc. There are in fact seven independent parameters which must be input from experiment, and from which all fifteen above then follow. This set of seven can be chosen in various ways:

$g, g', G_e, G_\mu, G_\tau, \mu^2, \lambda$,

OR $\alpha, M_W, M_Z, M_H, m_e, m_\mu, m_\tau$

OR $\alpha, \sin^2\theta_W, M_H, v, G_e, G_\mu, G_\tau$,

are all possibilities. Including the electroweak quark sector adds the CKM matrix V (three angles and one complex phase) and mass matrices m(U), m(D) ($m_u, m_c, m_t, m_d, m_s, m_b$) making 4 + 3 + 3 = 10 extra parameters. Including QCD we have in addition Λ_{QCD} and the QCD θ-parameter involved in the strong CP problem. So overall there are 19 independent free parameters in $SU(3)_C \times SU(2)_L \times U(1)_Y$, at least, since in addition we may have neutrino masses and mixings.

A model with at least 19 undetermined parameters, in which the particular representations containing fermions and scalars are not compellingly motivated, and with a mysterious replication of three generations, does not seem a likely candidate for a complete theory of everything, even though it has so far proved consistent with experiment in every detail checked!

Although we have achieved a partial unification of weak and electromagnetic forces the gauge group is a product of simple groups each with its own coupling constant, g and g'. The weak mixing angle which relates these is not determined by the theory. One way forward would be to try and embed the $SU(3)_C \times SU(2)_L \times U(1)_Y$ inside a single larger simple group. All the particles would then lie inside bigger irreducible representations of this unifying group, and there should be some resulting constraints reducing the number of independent parameters. We shall explore Grand Unified theories, in particular SU(5), in some detail in the next section.

We end this section with some further comments about the Standard Model, or minimal Standard Model, to be more precise. We have assumed a simple structure with one Higgs doublet but there is nothing to prevent us adding an arbitrary number of scalar doublets Φ_i with hypercharges $Y_i = +1$. If the neutral members acquire VEV's one can check that

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1 ,$$

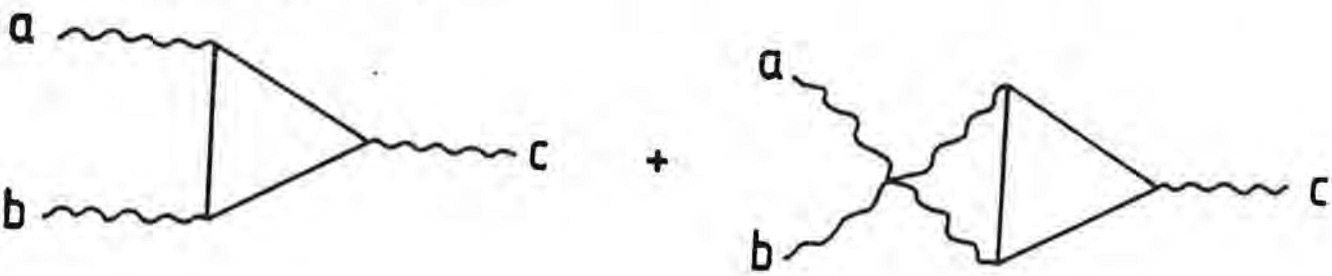
irrespective of the number of such Higgs doublets (see Exercise 5).

The renormalizability of the Standard Model hangs by a thread since chiral theories have technical problems arising from the so-called triangle anomaly. In a chiral theory there are loop diagrams arising at the quantum level which can potentially violate gauge invariance and destroy the renormalization program. Such contributions must cancel for a consistent renormalizable chiral field theory.

Consider a chiral theory with fermion currents which couple to gauge bosons

$$J_\mu^a = \bar{\psi}_R\gamma_\mu T_R^a\psi_R + \bar{\psi}_L\gamma_\mu T_L^a\psi_L .$$

In the Standard Model one has $J_\mu^{1,2,3,Y}$ with the T_L^a Pauli matrices for instance. The 'a' labels the different matrix representations $T_{R,L}^a$ for right and left-handed fermions. We have triangle diagrams which must vanish for consistency. So we need



$$\sim \sum_{\text{Reps}} \mathrm{Tr}\left\{(T_L^aT_L^b + T_L^bT_L^a)T_L^c - (T_R^aT_R^b + T_R^bT_R^a)T_R^c\right\} = 0 .$$

The trace sums over all fermions in each representation, and one then sums over the representations.

There are various possibilities for a,b,c in the Standard Model.

(i) a,b,c are all SU(2) generators. Only left-handed fermions in the doublet representation contribute and one requires

$$\mathrm{Tr}\left\{(T_L^aT_L^b + T_L^bT_L^a)T_L^c\right\}$$

$$= \mathrm{Tr}\left\{(\tau^a\tau^b + \tau^b\tau^a)\tau^c\right\} = 2\delta^{ab}\mathrm{Tr}(\tau^c) = 0 .$$

(ii) a SU(2); b,c U(1). Then we need

$$\mathrm{Tr}(T_L^a Y_L^2) \;\propto\; \mathrm{Tr}\tau^a \;=\; 0\ .$$

So far the traces have vanished automatically.

(iii) a,b SU(2); C U(1). Then we need

$$\mathrm{Tr}(Y_L) \;=\; 0 \;\Rightarrow\; \mathrm{Tr}Q \;=\; 0 \;\Rightarrow\; \sum_f Q_f \;=\; 0$$

where we have used $Y_L = 2(Q-T_L^3)$, $\sum_f T_L^3 = 0$.

(iv) a,b,c all U(1). We require

$$\mathrm{Tr}\left\{(Y_L)^3 - (Y_R)^3\right\} \;=\; 0 \;\Rightarrow \mathrm{Tr}Q \;=\; 0 \;\Rightarrow\; \sum_f Q_f \;=\; 0$$

$\frac{1}{2}Y_L = Q - T_L^3$, $\frac{1}{2}Y_R = Q$ (R all $SU(2)_L$ singlets).

So the Standard Model anomaly vanishes provided that $\mathrm{Tr}Q = \sum_f Q_f = 0$.

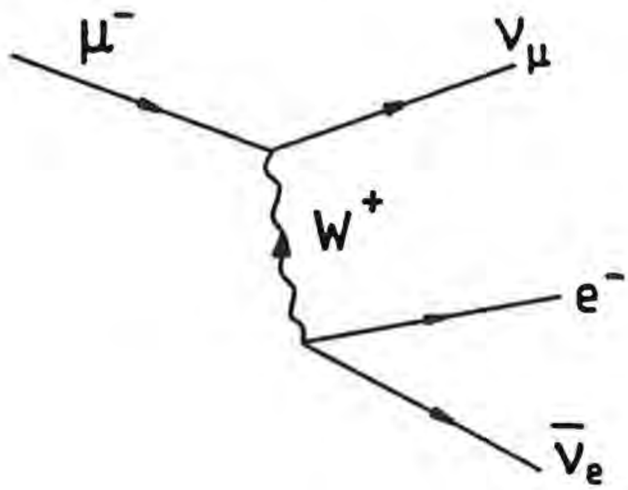
Summing the ν_e,e^-,u,d charges over a generation we find $\sum_f Q_f = 0 - 1 + N_c\left[\frac{2}{3} - \frac{1}{3}\right]$, where N_c denotes the number of colours. This vanishes provided that $N_c = 3$, so the number of colours is related to the quark changes!

4. Grand Unified Theories — Basic ideas

As mentioned in the last section the idea is to seek a simple group G large enough to contain $SU(3)_C \times SU(2)_L \times U(1)_Y$ as a subgroup. G would then be spontaneously broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ at an energy scale M_X ($\gg M_W$), and then the spontaneous electroweak symmetry breaking at energy M_W occurs leaving a residual $SU(3)_C \times U(1)_{em}$ symmetry.

How big should the unification scale M_X be? Associated with the spontaneous symmetry breaking there will be new, originally massless, gauge bosons which will acquire masses $\sim M_X$. In a GUT these will link quarks and leptons living in extended representations of G, hence leading to lepton and baryon number violating processes. Proton decay in particular will become possible. By using the experimental results on the proton lifetime, $\tau_p \gtrsim 10^{32}$ years, we can make a crude dimensional estimate of M_X.

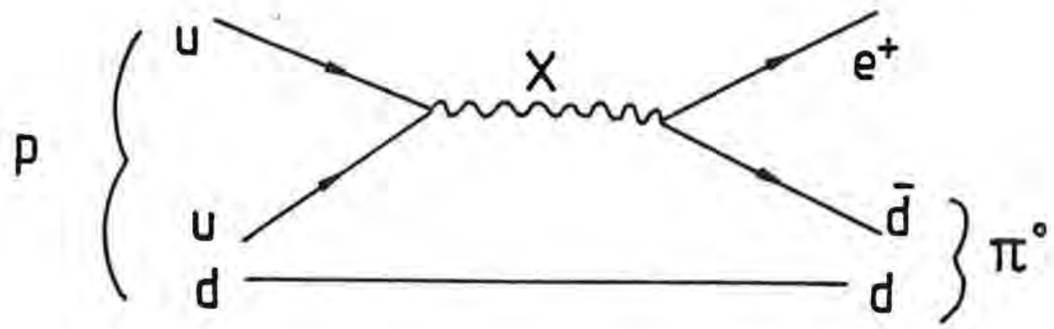
We begin by considering muon decay



Up to a constant factor we have (see Exercise 3)

$$\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) \sim \left[\frac{g^2}{M_W^2}\right]^2 m_\mu^5 .$$

This low energy process is therefore sensitive to the mass of the heavy W boson via virtual particle exchange. In just the same way proton decay will be a low energy manifestation of exchange of virtual heavy leptoquarks in the GUT. For instance one will have $p \longrightarrow e^+\pi^0$ via exchange of a massive charge $\frac{4}{3}$ leptoquark.



$$\Gamma(p \to e^+\pi^0) \sim \left[\frac{g_G^2}{M_X^2}\right]^2 m_p^5$$

where g_G is the single unified GUT coupling constant. Assuming $g_G \sim g \sim e$ and explicitly exhibiting $\hbar$ and c factors we have an estimated proton lifetime

$$\tau_p \sim \frac{1}{\alpha^2} \left[\frac{\hbar}{c^2}\right] \frac{M_X^4}{m_p^5} .$$

Experimentally one has $\tau_p \gtrsim 10^{32}$ years and so we require $M_X \gtrsim 10^{14}\text{-}10^{15}$ GeV/c^2.

So the unification scale is expected to be at least $\sim 10^{15}$ GeV. It is to be hoped that it is less than the Planck energy $\left[\frac{\hbar c^5}{G}\right]^{1/2} \approx 10^{19}$ GeV where gravity becomes strong, otherwise one would be forced to unify gravity with the other forces, a formidable task.

For the Standard Model to be embedded in a grand unified theory one requires that the three renormalized coupling constants of $SU(3)_C \times SU(2)_L \times U(1)_Y$, should approach a single unified coupling as the energy approaches the unification scale. Let us denote the three couplings by $g_3 = g_S$, $g_2 = g$, $g_1 = g'$. We shall define $\alpha_i = \frac{g_i^2}{4\pi}$, i = 1,2,3. The Standard Model is renormalizable and so we can define renormalized couplings $\alpha_i(\mu)$, where μ is the renormalization scale, which with a physical renormalization scheme can be chosen to correspond to the energy. The μ dependence of the $\alpha_i(\mu)$ is given by integrating the so-called beta function equation

$$\frac{\partial \alpha_i}{\partial \ln\mu} = -\frac{b_i}{2\pi}\alpha_i^2 + O(\alpha_i^3) .$$

One has that for two scales (energies) μ_1, μ_2

$$\frac{1}{\alpha_i(\mu_1)} - \frac{1}{\alpha_i(\mu_2)} = \frac{b_i}{2\pi} \ln\left[\frac{\mu_1}{\mu_2}\right] .$$

The beta function coefficients are

$$b_1 = -\frac{4}{3}N_f - \frac{1}{10}N_H$$

$$b_2 = \frac{22}{3} - \frac{4}{3}N_f - \frac{1}{6}N_H$$

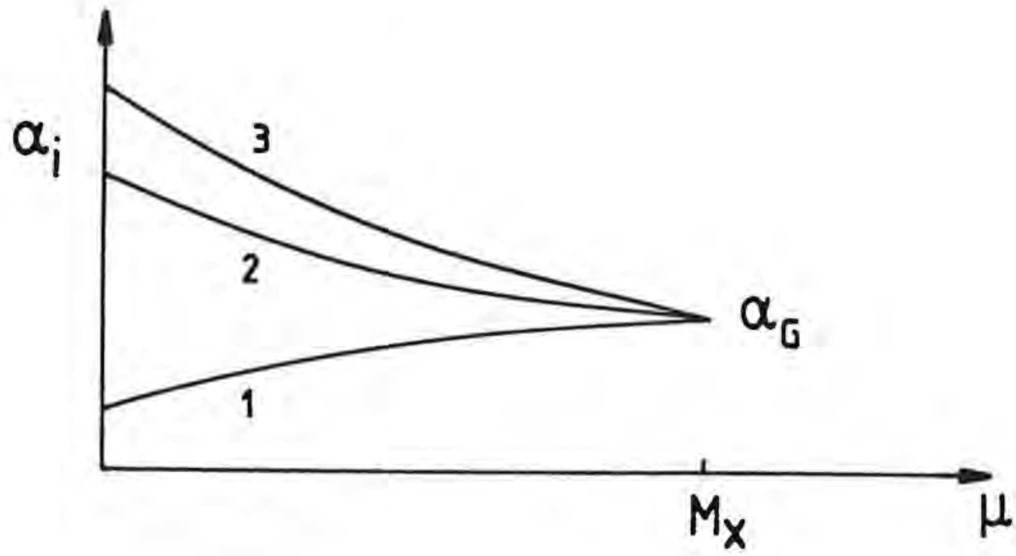
$$b_3 = 11 - \frac{4}{3}N_f$$

where N_f denotes the number of families or generations, N_H the number of Higgs doublets in the electroweak sector.

For $N_f = 3$ and $N_H = 1$ we have $b_2, b_3 > 0$ and $b_1 < 0$ so that α_2, α_3

decrease as μ increases, and α_1 increases. b_1, b_2, b_3 are rather insensitive to N_H. b_3 remains positive and b_1 negative irrespective of the number of Higgs doublets, but b_2 becomes negative for $N_H > 20$.

The idea will be that the renormalized couplings run together as μ increases to M_X, $\alpha_1(M_X) = \alpha_2(M_X) = \alpha_3(M_X) = \alpha_G$.



Before going further we need to be a little more careful about the definitions of the α_i. Let us assume that the fifteen fermions of the Standard Model completely fill a (possibly reducible) representation of G. We stress that we are assuming that there are no new particles to be encountered between existing energies of M_W and the unification scale of $\sim 10^{15}$ GeV. This assumption is often referred to as the 'desert hypothesis'.

In terms of their $SU(3)_C \times SU(2)_L$ representations we have the fifteen particles

$$(1,2) + (3,2) + (1,1) + (\bar{3},1) + (\bar{3},1)$$

$$\begin{bmatrix}\nu_e\\ e^-\end{bmatrix}_L \quad \begin{bmatrix}u_i\\ d_i\end{bmatrix}_L \quad e^c_L \quad (u^c_i)_L \quad (d^c_i)_L$$

The index i runs over the $N_c = 3$ colours. Notice that we have used conjugate left-handed fields instead of e_R, u_R, d_R, so that the whole extended representation is left-handed. In the specific example of G = SU(5), which we shall turn to later, the fifteen completely fill a $\underline{\bar{5}}$ + $\underline{10}$ of SU(5).

Since all the particles lie in a single representation of a simple group G we must have that for any generators T_i, $\mathrm{Tr}(T_i) = 0$ and $\mathrm{Tr}(T_iT_j) = k\,\delta_{ij}$, where the constant k depends only on the representation of G over which the trace is taken. So for the generators of $U(1)_Y$ and $SU(2)_L$ we require $\mathrm{Tr}((\tfrac{1}{2}Y)^2) = \mathrm{Tr}((T^3)^2)$, over the fifteen fermions. However we have instead,

$$\frac{\mathrm{Tr}(\tfrac{1}{4}Y^2)}{\mathrm{Tr}((T^3)^2)} = \frac{\tfrac{1}{4}[2(-1)^2 + 6(\tfrac{1}{3})^2 + (2)^2 + 3(-\tfrac{4}{3})^2 + 3(\tfrac{2}{3})^2]}{(1+3)\ [(\tfrac{1}{2})^2+(-\tfrac{1}{2})^2]} = \frac{5}{3}\,.$$

This mismatch in the normalization of the generator $\tfrac{1}{2}Y$ can be absorbed into a rescaling of the associated coupling g'. So g_1 should be redefined as $g_1{}^2 = \frac{5}{3}\,g'^2$.

Assuming coupling constant unification one has an automatic prediction for $\sin^2\theta_W$ at the scale M_X. For renormalization scale μ,

$$\sin^2\theta_W(\mu) = \frac{g'^2(\mu)}{g^2(\mu)+g'^2(\mu)} = \frac{\tfrac{3}{5}\alpha_1(\mu)}{\alpha_2(\mu)+\tfrac{3}{5}\alpha_1(\mu)}\ .$$

At $\mu = M_X$, $\alpha_1(M_X) = \alpha_2(M_X)$ and so

$$\sin^2\theta_W(M_X) = \frac{\frac{3}{5}\alpha_1(M_X)}{\alpha_1(M_X)+\frac{3}{5}\alpha_1(M_X)} = \frac{3}{8}.$$

We know from lower energy experimental data that $\sin^2\theta_W(M_Z) \simeq 0.2336 \pm .0018$, the electromagnetic coupling $\alpha(M_Z) \simeq \frac{1}{128.8}$ (compared with $\alpha(m_e) \simeq \frac{1}{137}$), and the strong coupling $\alpha_s(M_Z) \simeq 0.108 \pm 0.007$. From these values we deduce the central values

$$\alpha_1^{-1}(M_Z) = \frac{3}{5}\frac{\cos^2\theta_W}{\alpha(M_W)} = 59.23$$

$$\alpha_2^{-1}(M_Z) = \frac{\sin^2\theta_W}{\alpha(M_W)} = 30.08$$

$$\alpha_3^{-1}(M_Z) = 9.26 .$$

We now want to evolve these to larger μ and see if they intersect at a single point.

Suppose that $\alpha_1(M_X) = \alpha_2(M_X) = \alpha_3(M_X) = \alpha_G$, then using our earlier relation between $\alpha(\mu_1)$ and $\alpha(\mu_2)$ we have

$$\frac{1}{\alpha_G} - \frac{1}{\alpha_i(\mu)} = \frac{b_i}{2\pi}\,\ell n\left[\frac{M_X}{\mu}\right].$$

This implies that the quantities

$$\Delta_{ij} = \frac{\alpha_i^{-1}(\mu)-\alpha_j^{-1}(\mu)}{(b_j-b_i)} = \frac{1}{2\pi}\,\ell n\left[\frac{M_X}{\mu}\right]$$

should be independent of i,j for any μ. So for unification we require $\Delta_{12} = \Delta_{13} = \Delta_{23} = \Delta$, and the unification scale is then $M_X = \exp(2\pi\Delta)\mu$. $\exp(2\pi\Delta_{ij})\mu$ gives the scale at which α_i and α_j intersect. These intersection points must coincide for unification. The above unification condition can also be reduced to a compact necessary and sufficient condition on the sum of the α_i^{-1}. For $N_H = 0$ and independent of N_f we have the condition that (independent of μ) $\alpha_1^{-1}(\mu) - 3\alpha_2^{-1}(\mu) + 2\alpha_3^{-1}(\mu) = 0$. For the minimal Standard Model with $N_H = 1$ this becomes $115\alpha_1^{-1}(\mu) - 333\alpha_2^{-1}(\mu) + 218\alpha_3^{-1}(\mu) = 0$, again independent of N_f.

From the experimental data on $\alpha_i(M_Z)$ we obtain (with $N_H = 1$), $\Delta_{12} = 4.01$, $\Delta_{13} = 4.50$, $\Delta_{23} = 5.43$, corresponding, respectively, to unification scales of 8×10^{12}, 1.7×10^{14}, 6×10^{16} GeV (taking $\mu = M_Z = 91$ GeV). We see that α_1,α_2 intersect first, then α_1,α_3, and finally α_2,α_3. Unification does not work particularly well. The situation is not changed greatly by including two-loop corrections in the beta functions.

We shall see somewhat later that Grand Unified theories suffer from some insuperable problems, and supersymmetry seems to offer a solution. The idea, which we will explore in Section 6, is to add extra 'sparticles' to partner the leptons and quarks. One then arrives at a supersymmetric extension of the Standard Model including squarks, sleptons, gluinos, photinos. In such a SUSY extension the beta function coefficients become

$$b_1 = -2N_f - \frac{3}{10} N_H$$

$$b_2 = 6 - 2N_f - \frac{1}{2} N_H$$

$$b_3 = 9 - 2N_f .$$

Using these b_i we now obtain from the $\alpha_i(M_Z)$ experimental data, $\Delta_{12} = 5.20$, $\Delta_{13} = 5.20$, $\Delta_{23} = 5.20$. We have taken $N_H = 2$, which is the number of Higgs doublets required in the minimal SUSY extension. The unification scale is 1.4×10^{16} GeV, and the couplings cross beautifully at a point with $\alpha_G \simeq \frac{1}{25}$. The compact unification condition becomes for minimal SUSY, $5\alpha_1^{-1}(\mu) - 12\alpha_2^{-1}(\mu) + 7\alpha_3^{-1}(\mu) = 0$. With $\mu = M_Z$ the central values of the data give $296.15 - 360.96 + 64.82 = 0.01$, a level of agreement which is surely fortuitous since the relation is only valid at the one-loop level! More sophisticated analyses including two-loop corrections and threshold effects have been undertaken with encouraging results.

5. Minimal SU(5)-Georgi-Glashow GUT

We now return to the construction of a Grand Unified model. The first question concerns the choice of simple group G in which the Standard Model is to be embedded. We can identify three requirements that G must satisfy:

(1) $G \supset SU(3) \times SU(2) \times U(1)$. This implies that G must be a Lie group with rank ≥ 4. Recall that the rank of a group corresponds to the number of generators that can be simultaneously diagonalized (number of generators which can take simultaneous eigenvalues). U(1) is rank 1, and so is SU(2). SU(3) has rank 2. The rank of a direct product of groups is additive so $SU(3) \times SU(2) \times U(1)$ has rank 4, implying rank ≥ 4 for the group G.

(2) G must have complex representations. For instance for SU(3) the $\underline{3}$ transforms differently to $\bar{\underline{3}} = \underline{3}^*$. Parity violation requires that L and R fermions belong to different representations of G. ψ_L and ψ_L^c (ψ_R) have opposite helicity and live in conjugate representations so that they

transform differently under chiral gauge transformations.

(3) G should have a single unified gauge coupling which implies that it should be a <u>simple</u> group or a product of <u>identical</u> simple groups whose couplings are required to be equal by some discrete symmetry.

Thanks to Eli Cartan we have a complete classification of all possible simple Lie groups, and they are tabulated below. The order of the group is the number of generators.

Classical name	Rank	Order	Complex reps.
SU(n+1)	n	n(n+2)	$n \geq 2$
SO(2n+1)	n	n(2n+1)	None
Sp(2n)	n	n(2n+1)	None
SO(2n)	n	n(2n-1)	n = 5,7,9,...
G_2	2	14	None
F_4	4	52	None
E_6	6	78	Yes
E_7	7	133	None
E_8	8	248	None

The <u>only</u> possibility for a simple group of rank 4 admitting complex representations is SU(5), so this is the minimal choice for G, as first pointed out by Georgi and Glashow in 1974. Using the last clause of (3) we might also try SU(3) × SU(3) which also has rank 4. However, one factor would have to be $SU(3)_c$, the leptons do not carry colour and must therefore lie in different representations of SU(3) × SU(3) to the quarks. However we must have TrQ = 0 for each representation, but for quarks and leptons separately $\mathrm{Tr}Q \neq 0$. We can therefore rule out $SU(3) \times SU(3)$. Higher rank non-minimal possibilities which have also been considered are $SO(10)$ (rank 5) and E_6 (rank 6).

Let us now see how to fit the fifteen Standard Model fermions inside $SU(5)$. The fundamental $\underline{5}$ representation of $SU(5)$ can be decomposed in terms of $(SU(3),SU(2))$ representations as $\underline{5} = (3,1) + (1,2)$, and $\bar{\underline{5}} = (\bar{3},1) + (1,2)$. Recalling the $(SU(3),SU(2))$ decomposition of the fifteen exhibited in the last section we see that the $\bar{\underline{5}}$ can contain $(u_i^c)_L$ <u>or</u> $(d_i^c)_L$ for $(\bar{3},1)$, and $\begin{bmatrix}\nu_e \\ e^-\end{bmatrix}_L$ for $(1,2)$. In fact we must have $(d_i^c)_L$ since we require $\mathrm{Tr}Y = 0$, $\mathrm{Tr}Q = 0$ over the $\bar{\underline{5}}$ representation. For $(d_i^c)_L$ we have $3\left[\frac{2}{3}\right] - 2 = 0$ and $3\left[\frac{1}{3}\right] - 1 = 0$, a non-zero trace is obtained for $(u_i^c)_L$. The tracelessness requirement imposes $Q(e) = N_c Q(d)$, and ultimately provides an explanation for the relation between the proton and electron charge, $Q(e^-) = -Q(p)$.

So we have the $\bar{\underline{5}},\underline{5}$ representations, respectively

$$\psi_L = \begin{bmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu \end{bmatrix}_L , \qquad \psi_R^c = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\bar{\nu} \end{bmatrix}_R .$$

The first three indices are $SU(3)_C$, the remaining two are $SU(2)$.

What about the remaining ten fermions?

$$(3,2) + (\bar{3},1) + (1,1)$$

$$\begin{bmatrix} u_i \\ d_i \end{bmatrix}_L \qquad (u_i^c)_L \qquad e_L^c$$

Now, the SU(5) product representation $\underline{5} \times \underline{5}$ decomposes into (SU(3),SU(2)) representations as

$$\begin{aligned}\underline{5} \times \underline{5} &= [(3,1)+(1,2)] \times [(3,1)+(1,2)] \\ &= [(6,1) + (3,2) + (1,3)]_S \\ &+ [(\bar{3},1) + (3,2) + (1,1)]_A \\ &= \underline{15}_S + \underline{10}_A .\end{aligned}$$

This is comparable to adding ordinary isospins - $\underline{2} \times \underline{2} = \underline{1}_A + \underline{3}_S$. So the remaining ten fermions fit into a $\underline{10}_A$ of SU(5). So the fifteen fermions completely fill a $\underline{5} + \underline{10}$ of SU(5).

The SU(5) Lagrangian $\mathscr{L}$ will contain interactions of the $\psi_R^C(\underline{5})$ multiplet with 24 gauge bosons A_j^μ lying in the adjoint representation of SU(5). $\underline{5} \times \underline{5} = \underline{24} + \underline{1}$, to be compared with the 8 gluons of SU(3) QCD - $\underline{3} \times \underline{\bar{3}} = \underline{8} + \underline{1}$.

$$\mathscr{L} = i\, \bar{\psi}_R^C\, \gamma_\mu(\partial_\mu + ig \sum_{j=1}^{24} \tfrac{1}{2}\lambda_j\, A_j^\mu)\psi_R^C + \ldots$$

The $\frac{\lambda_i}{2}$ $i = 1,2,\ldots,24$ are 5^2-1 traceless hermitean generators of SU(5), satisfying

$$\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}$$

$$[\lambda_i,\lambda_j] = 2\, i\, c_{ijk}\, \lambda_k ,$$

where the c_{ijk} are structure constants of SU(5). In terms of (SU(3),SU(2)) representations we have

$$\underline{24} = (8,1) + (1,3) + (1,1) + (\bar{3},2) + (3,2)$$

$$\text{gluons} \quad \underbrace{W^i_\mu \quad B_\mu}_{W^\pm, Z^0, \gamma} \quad \underbrace{X,Y \quad \bar{X},\bar{Y}}_{\text{leptoquarks}}$$

So the 12 Standard Model gauge bosons are augmented by an additional 12 leptoquarks, X,Y, and their antiparticles $\bar{X},\bar{Y}$ lying in the fundamental representation of SU(3) and SU(2). $Q(X) = \frac{4}{3}$, $Q(Y) = \frac{1}{3}$. These additional particles carry colour and are responsible for lepton and baryon number violating processes such as proton decay, as already discussed in the last section.

The λ_j, A_j are labelled so that the structure of the 5 × 5 matrix of vector boson fields is

$$\sum_{j=1}^{24} \frac{1}{2} \lambda_j A_j = \begin{bmatrix} & & & \bar{X}_1 & \bar{Y}_1 \\ & g & & \bar{X}_2 & \bar{Y}_2 \\ & & & \bar{X}_3 & \bar{Y}_3 \\ X_1 & X_2 & X_3 & & \\ Y_1 & Y_2 & Y_3 & & W \end{bmatrix} + \frac{1}{2} \lambda_{24} B .$$

λ_{24} corresponds to $U(1)_Y$ and must commute with the other λ's. For j = 1,2,...,8 the A_j are taken to be the eight gluon fields and

$$\lambda_{1-8} = \begin{bmatrix} "\lambda_{1-8}" & 0 \\ 0 & 0 \end{bmatrix} ,$$

with "λ_{1-8}" the eight 3 × 3 SU(3) Gell-Mann λ-matrices. For j = 9,10,...,20 the A_j are taken to be the fields of the twelve leptoquarks with, for instance

$$\lambda_9 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

For j = 21,22,23 the A_j are the three SU(2) fields $V^1_\mu, V^2_\mu, V^3_\mu$, and

$$\lambda_{21-23} = \begin{bmatrix} 0 & 0 \\ 0 & \tau^{1-3} \end{bmatrix} ,$$

where the τ^i are the SU(2) Pauli matrices.

Since λ_{24} must commute with these it will have the structure diag$(\alpha,\alpha,\alpha,\beta,\beta)$. Tr $\lambda_{24} = 0 \Rightarrow 3\alpha + 2\beta = 0$, so that we must have $\lambda_{24} \propto$ diag(-2.-2.-2,3,3). Using the other condition $\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}$ we must have $\mathrm{Tr}(\lambda_{24}^2) = 2$ which determines $\lambda_{24} = \frac{1}{\sqrt{15}}$ diag(-2.-2.-2.3.3). The $\underline{10}_A$ is represented by the antisymmetric 5 × 5 array

$$\chi_{ab} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^+ \\ d_1 & d_2 & d_3 & e^+ & 0 \end{bmatrix}$$

The a,b (a,b = 1,2,...,5) are SU(5) indices.

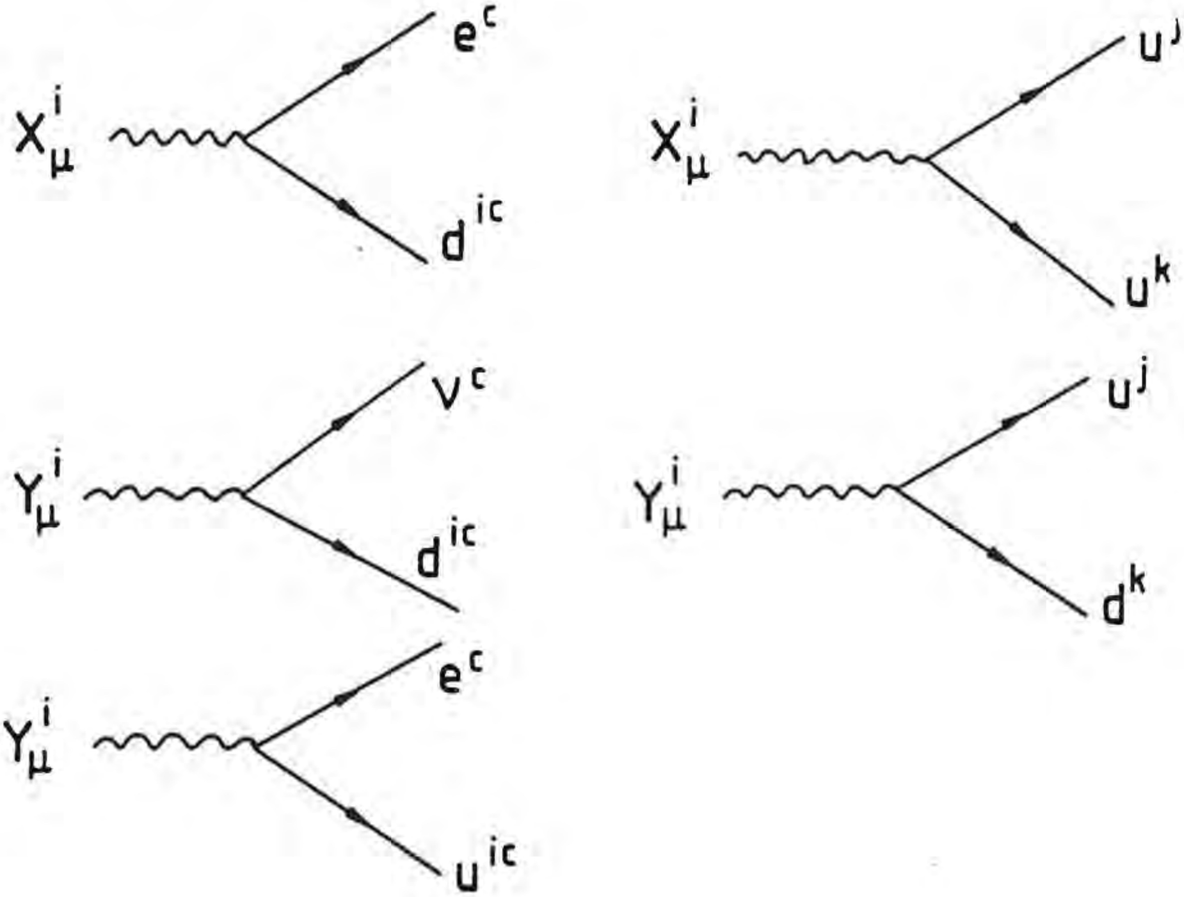
The piece of the SU(5) invariant Lagrangian containing the interaction of the 24 gauge bosons with the $\underline{10}$ and $\underline{5}$ of fermions is then

$$\mathcal{L}_{int} = -2g(\bar{\chi})_{ac}\,\gamma^\mu(\tfrac{1}{2}\vec{\lambda}\cdot\vec{A}_\mu)_{ab}\,\chi_{bc} - g(\bar{\psi}^c_R)_a\,\gamma^\mu\,(\tfrac{1}{2}\vec{\lambda}\cdot\vec{A}_\mu)_{ab}(\psi^c_R)_b \,.$$

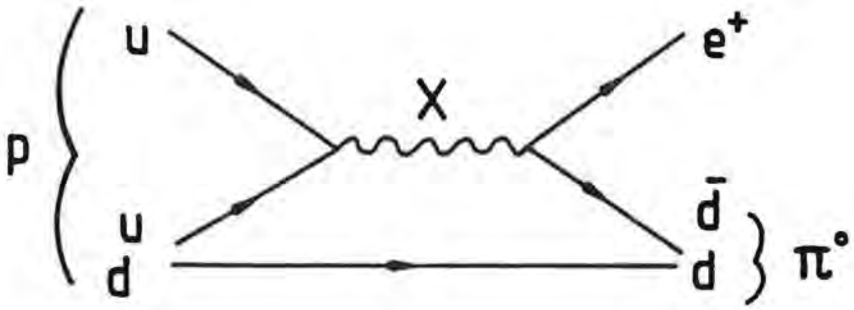
$\frac{1}{2}\vec{\lambda}\cdot\vec{A}_\mu$ is just shorthand for the sum over j. A summation convention is assumed on repeated SU(5) indices. Restricting a,b,c to {1,2,3} one obtains the QCD Lagrangian, restricting a,b,c to {4,5} one obtains the $SU(2)_L \times U(1)_Y$ Lagrangian. g is the unified coupling constant (g_G).

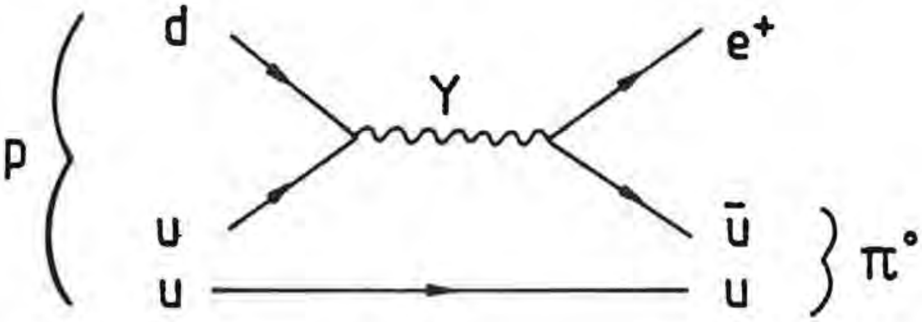
Just as for the Standard Model it is not possible to add mass terms to the Lagrangian whilst preserving SU(5) invariance, and so spontaneous symmetry breaking will have to be used once again to generate masses.

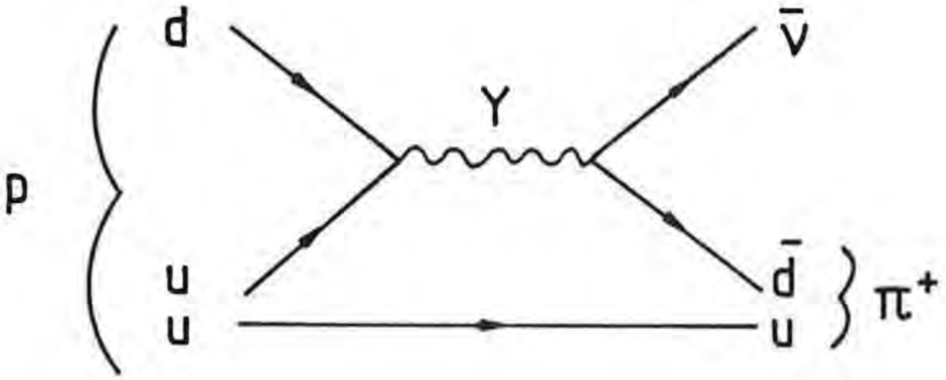
The lepton-quark-leptoquark vertices are drawn below. i = 1,2,3 is a colour index.



Proton decay via modes such as $p \longrightarrow \pi^0 e^+$ and $p \longrightarrow \bar{\nu}\pi^+$ is then possible, as pictured below.







Clearly B and L are separately violated by these vertices. However one can consistently assign $(B-L)_{\substack{X\\Y}} = \frac{2}{3}$, and this is conserved.

The overlap between outgoing and incoming <u>hadronic</u> states requires modelling and this leads to uncertainties in the calculation of the proton lifetime. Using lower energy data to fix M_X, as previously discussed, and including all uncertainties one finds the theoretical prediction

$$[\Gamma(p \to e^+\pi^0)]^{-1}_{Th} = 4.5 \times 10^{29\pm1.7} \text{ yrs} ,$$

whereas experimentally for this mode one has the bound $\geq 10^{32}$ yrs. So minimal SU(5) seems to be ruled out.

We note in passing that the condition for anomaly cancellation in SU(5) is that

$$\sum_{\underline{5},\underline{10}} \mathrm{Tr}(Q^3) = 0$$

We have

$$\underbrace{\left[3\left[\frac{1}{3}\right]^3 + (-1)^3\right]}_{\underline{5}} + \underbrace{\left[3\left[\frac{2}{3}\right]^3 + 3\left[-\frac{1}{3}\right]^3 + 3\left[-\frac{2}{3}\right]^3 + 1^3\right]}_{\underline{10}}$$

$$= -\frac{8}{9} + \frac{8}{9} = 0 .$$

So minimal SU(5) is anomaly free.

We shall conclude this section by discussing the generation of masses by spontaneous symmetry breaking for the SU(5) GUT. It will turn out that GUTs suffer from a fatal generic problem — the hierarchy problem — which can be cured if we have a supersymmetric theory. We shall briefly introduce supersymmetry in the final section.

The spontaneous symmetry breaking of SU(5) proceeds in two stages involving a 24 of massive Higgs ($M \sim M_X$), and a $\bar{5}$ of light Higgs ($M \sim M_W$).

$$SU(5) \xrightarrow[\langle\Phi\rangle \sim 10^{14}\text{ GeV}]{24} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow[\langle H\rangle \sim 10^{2}\text{ GeV}]{5} SU(3)_C \times U(1)_{em}$$

We shall label the 24 Φ_a, they transform under a real adjoint representation of SU(5). The 5 will be labelled H_a a = 1,5. They transform as a complex five dimensional representation of SU(5) and comprise a colour triplet H_a $a = 1,2,3$, and a colour singlet doublet, which is essentially just the usual Standard Model Higgs doublet.

Let us consider the 24 first. Writing

$$\Phi = \sum_{j=1}^{24} \Phi_j \frac{\lambda_j}{2}$$

we add a term to the Lagrangian,

$$\mathscr{L}_\Phi = \frac{1}{2} \mathrm{Tr}\{[D_\mu \Phi][D_\mu \Phi]^\dagger\} - \mathscr{V}(\Phi) .$$

The covariant derivative for the adjoint representation is

$$D_\mu \Phi = \partial_\mu \Phi + ig\Big[(\vec{A}^\mu \cdot \frac{\lambda}{2})\Phi - \Phi(\vec{A}_\mu \cdot \frac{\vec{\lambda}}{2})\Big] .$$

The most general renormalizable (dimension ≤ 4) $\mathscr{V}(\Phi)$ is

$$\mathscr{V}(\Phi) = -\frac{1}{2} M^2 \mathrm{Tr}(\Phi)^2 + \frac{1}{4} a[\mathrm{Tr}(\Phi^2)]^2 + \frac{1}{2} b\, \mathrm{Tr}(\Phi^4) .$$

One could also have a cubic term, but this can be eliminated by imposing a discrete $\Phi \longrightarrow -\Phi$ symmetry.

For $M^2 > 0$ spontaneous symmetry breaking can occur. For $\mathscr{V}(\Phi)$ to be bounded from below the quartic terms must also be positive which implies $15a + 7b > 0$. If $b > 0$ one can diagonalize Φ and solving $\frac{d\mathscr{V}}{d\Phi} = 0$ obtain a set of minima. For residual $SU(3) \times SU(2) \times U(1)$ invariance we must select a vacuum expectation value proportional to λ_{24}, so

$$\langle 0|\Phi|0\rangle = \Phi_0 = V\,\mathrm{diag}(1,1,1,-\tfrac{3}{2},-\tfrac{3}{2}) \ .$$

Then

$$\mathscr{V}(\Phi_0) = \frac{15}{4} V^2 [-M^2+(15a+7b)\frac{V^2}{4}]$$

which is minimized when $V^2 = \frac{2M^2}{15a+7b}$.

Writing $D_\mu\Phi$ out explicitly in $\mathscr{L}_\Phi$ with $\Phi = \Phi_0$ the leptoquark masses can be extracted from

$$\frac{g^2}{2}\,\mathrm{Tr}\left[((\vec{A}\cdot\frac{\lambda}{2})^2\Phi_0 - \Phi_0(\vec{A}\cdot\frac{\lambda}{2}))^2\right]$$

with the result

$$M_X^2 = M_Y^2 = \frac{25}{8} g^2V^2 \ .$$

Counting degrees of freedom we see that in breaking $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ there are $24 - 8 - 3 - 1 = 12$ broken generators. This implies 12 massless Higgs fields which are eaten by the 12 originally massless leptoquarks X,Y to provide their longitudinal polarization degree of freedom when they acquire masses of order V. The 12 remaining Higgs fields get masses of order $V \simeq M_X$ but do not couple to fermions, and so are of no further interest. It is just as well that they do not since otherwise one would have fermion masses $\sim M_X$!

We now turn to the lower energy symmetry breaking $SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)_{em}$ involving the 5 of Higgs. Writing

$$H = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \phi^+ \\ \phi^0 \end{bmatrix}$$

we introduce the potential

$$\bar{V}(H) = -\mu^2 H^\dagger H + \lambda(H^\dagger H)^2 .$$

With $\mu^2 > 0$, $\lambda > 0$ we have a potential bounded below and can select the vacuum expectation value

$$\langle 0|H|0\rangle = H_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \end{bmatrix} .$$

Notice that the three upper components cannot be gauged away. Just as for the SU(2) Higgs sector we obtain $\mu^2 = v^2\lambda$ and $M_W^2 = M_Z^2 \cos^2\theta_W = \frac{1}{4}g^2v^2$. The colour triplet H_1, H_2, H_3 remain <u>massless</u>, however. As we shall see when we discuss the Yukawa sector these can then mediate ultra-fast proton decay which is disastrous. This problem in fact fixes itself when we are more careful about the renormalizability of the model, but we then inevitably introduce yet another, finally disastrous problem, the hierarchy problem alluded to earlier.

The $\psi_L(\underline{\bar{5}})$ and $\chi(\underline{10})$ fermion representations in SU(5) imply Higgs representation decompositions of $\underline{\bar{5}} \times \underline{\bar{5}} = \underline{10} + \underline{15}$, $\underline{\bar{5}} \times \underline{10} = \underline{5} + \underline{45}$ or $\underline{10} \times \underline{10} = \underline{5} + \underline{45} + \underline{50}$. None of these contain $\underline{24}$ and so, fortunately, the heavy $\underline{24}$ of Higgs is not involved in the Yukawa sector, only the $\underline{5}$.

One introduces two Yukawa couplings G_D and G_U and arrives at the SU(5) invariant term in the Lagrangian

$$\mathscr{L}_Y = \left[G_D(\bar{\psi}_R^c)_a(\chi_L)_{ab}\, H_b^\dagger + \tfrac{1}{4}G_U\, \epsilon_{abcde}(\bar{\chi}_R^c)_{ab}(\chi_L)_{cd}\, H_e + \text{h.c.} \right] .$$

Then replacing H by the VEV

$$H_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \end{bmatrix}$$

one obtains

$$\mathscr{L}_Y = -\frac{1}{2} v \, G_D \, (\bar{e}e + \bar{d}_\alpha d_\alpha) - \frac{v \, G_u}{\sqrt{2}} (\bar{u}_\alpha u_\alpha) ,$$

from which one can read off the fermions masses, $m_e = m_d = \frac{1}{4} v \, G_D$, $m_u = \frac{v \, G_u}{\sqrt{2}}$. For the other families one will have for instance $m_\mu = m_s$, $m_\tau = m_b$. The mass terms are part of an SU(5) invariant Lagrangian and so should apply at $\mu = M_X$. The fermion masses in renormalizable field theory are running quantities dependent on the renormalization scale. One has

$$m_f(\mu) = m_f(M_X) \prod_{i=1,2,3} \left[\frac{\alpha_i(\mu)}{\alpha_i(M_X)}\right]^{\gamma_f^i/b_i}$$

where the $\alpha_1, \alpha_2, \alpha_3$ are the $U(1)_Y$, $SU(2)_L$, SU(3) running couplings discussed when considering unification, and the γ_f^i are so-called anomalous dimensions. From this we can obtain

$$\frac{m_\ell(\mu)}{m_q(\mu)} = \prod_{i=1}^{3} \left[\frac{\alpha_i(\mu)}{\alpha_i(M_X)}\right]^{(\gamma_\ell^i - \gamma_q^i)/b_i} .$$

Now we can run the masses down from $\mu = M_X$ to obtain, for instance $\frac{m_\tau}{m_b}$ for the actual physical particles, i.e. with $\mu \simeq 10$ GeV say.

$\gamma_q^3 = 4$, $\gamma_\ell^q = 0$ (no QCD lepton coupling to gluons), $\gamma_\ell^2 = \gamma_q^2$ (same SU(2) interactions), $\gamma_\ell^1 - \gamma_q^1 = -1$. So we have

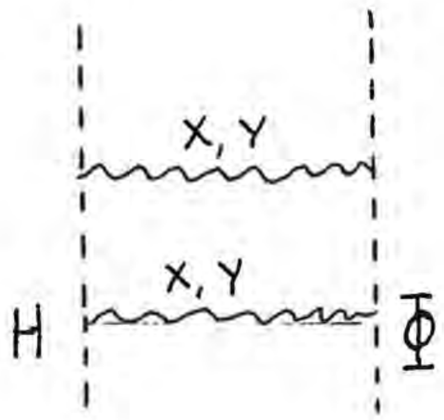
$$\frac{m_\ell(\mu)}{m_q(\mu)} = \left[\frac{\alpha_3(M_X)}{\alpha_3(\mu)}\right]^{\frac{4}{(11-\frac{2}{3}N_f)}} \left[\frac{\alpha_1(M_X)}{\alpha_1(\mu)}\right]^{-\frac{1}{(-\frac{4}{3}N_f-\frac{N_H}{10})}}$$

Taking $N_f = 3$, $N_H = 1$, $\alpha_3(M_X) = \alpha_1(M_X) = \alpha_G \simeq \frac{1}{42}$, and $\mu = 10$ GeV, together with the $\mu = M_Z$ α_1, α_3 data values evolved down, we find $m_b \simeq 3\, m_\tau$ which is in reasonable agreement with actual values — $m_\tau = 1.78$ GeV, $m_b \simeq \frac{1}{2} m_\Upsilon$ $= 5$ GeV.

The corresponding ratios for the first two families have μ too small for the coupling constant evolution to be believable. We would expect $\frac{m_s(\mu)}{m_d(\mu)}$ approximately μ-independent, so we should have $\frac{m_s(\mu)}{m_d(\mu)} = \frac{m_\mu}{m_e} = 200$. This is to be compared with the current algebra prediction $\frac{m_s}{m_d} \simeq 20$, so these mass relations are problematic for GUTs with minimal Higgs.

Setting $H_i \to v_i + H_i$ in $\mathscr{L}_Y$ generates interactions $H_i e^c u_i^c$, $H_i H_j d_k$, which mediate fast proton decay, for instance $p \to e^+\pi^0$ from the massless Higgs exchange.

A Lagrangian containing $\mathscr{V}(\Phi) + \tilde{V}(H)$ is <u>not</u> renormalizable. Extra mixing terms are necessary and will arise in any case from radiative corrections



which will introduce an additional term

$$V(\Phi,H) = \alpha H^\dagger H\,\mathrm{Tr}(\Phi^2) + \beta H^\dagger \Phi^2 H\ .$$

We can arrange that $\mathcal{V}(\Phi) + \tilde{V}(H) + V(\Phi,H)$ has a minimum at H_0,Φ_0 as before. H_0 is $SU(3) \times U(1)_{em}$ invariant, not $SU(2)$ invariant and so there must then be a small $SU(2)$ breaking term in Φ_0, which can no longer be proportional to λ_{24}. We will have

$$\Phi_0 = V\,\mathrm{diag}\left[1,1,1,-\frac{3}{2} - \frac{\epsilon}{2},-\frac{3}{2} + \frac{\epsilon}{2}\right]$$

with

$$\epsilon \simeq \frac{3\beta}{20b}\left[\frac{v^2}{V^2}\right] \simeq \left[\frac{M_W^2}{M_X^2}\right] \sim 10^{-24}.$$

This is unobservably small but would in principle modify the "weak $\Delta I = \frac{1}{2}$ rule" $M_W^2 = M_Z^2 \cos^2\theta_W$. The potential minimization equations get modified to

$$M^2 = \frac{1}{2}(15a + 17b)V^2 + \underline{\left[\alpha + \frac{3}{10}\right]v^2}$$

$$\mu^2 = \lambda v^2 + \underline{\frac{3}{2}\left[5\alpha + \frac{3}{2}\beta - \epsilon\beta\right]V^2}$$

where the extra terms introduced by the mixing $V(\Phi,H)$ are underlined. The $\mathrm{Tr}(\Phi^2)H^\dagger H$ in $V(\Phi,H)$ will generate contributions to the H_i masses $O(V)$ so H_1,H_2,H_3 acquire masses $\sim M_X$, and the fast proton decay is avoided.

However, as mentioned earlier, a new and finally disastrous Hierarchy problem is now evident. In order that the underlined corrections to the above M^2,μ^2 equations do not give masses $\sim M_X$ to all the particles they must be small. This is automatic for the M^2 equation since $v^2 \ll V^2$, but for the

μ^2 equation we have the requirement that $\frac{3}{2}\left[5\alpha + \frac{3}{2}\beta - \epsilon\beta\right]V^2 \sim v^2$. So we must arrange the parameters α,β in $V(\Phi,H)$ so that $\frac{3}{2}\left[5\alpha + \frac{3}{2}\beta - \epsilon\beta\right] \sim \frac{v^2}{V^2} \sim 10^{-24}$. Thus the parameters must be fine tuned to incredible precision. If so adjusted in lowest order higher order radiative corrections detune the cancellation. This is the so-called fine tuning or hierarchy problem. The moral of the tale is that spontaneous symmetry breaking and a big mass hierarchy don't mix!

The essence of the problem is that some (Higgs) scalars must have zero (small) mass. Chiral symmetry, e.g. $SU(2)_L$ forbids a term $m\bar{\psi}_R\psi_L$ and hence ensures zero fermion masses, the scalar mass term $m\phi\phi^*$ always respects this symmetry however. In a supersymmetric model $m_\psi = m_\phi$ for the scalar ϕ and its associated fermionic superpartner ψ. Chiral symmetry protects the fermion mass and the supersymmetry mass relation then prevents the scalar from acquiring a mass.

6. Supersymmetry (SUSY)

The idea is to introduce fermionic spinor generators of SUSY transformations, Q, which change fermions into bosons and vice versa. Schematically

$$Q|F\rangle = |B\rangle \ , \qquad Q|B\rangle = |F\rangle \ .$$

To illustrate these ideas let us consider a simple harmonic oscillator with bosonic and fermionic degrees of freedom. The creation and annihilation operators are $a^\dagger,a$ for the bosons and $b^\dagger,b$ for the fermions. With $\hbar = 1$ we have the commutation and anticommutation relations

$$[a,a^\dagger] = 1 \ , \qquad [a,a] = [a^\dagger,a^\dagger] = 0$$

$$\{b,b^\dagger\} = 1 \ , \qquad \{b,b\} = \{b^\dagger,b^\dagger\} = 0 \ .$$

$b^\dagger b^\dagger|0\rangle = 0$ so two fermions cannot occupy the same state, as required by the Pauli exclusion principle. The Hamiltonian will be

$$H = \frac{1}{2}\,\omega_B\{a^\dagger,a\} + \frac{1}{2}\,\omega_F[b^\dagger,b]$$

where ω_B,ω_F are the classical frequencies of the bosonic (B) and fermionic (F) oscillators. With $\hbar = 1$ the energy will be $E = \omega_B(n_B + \frac{1}{2}) + \omega_F(n_F - \frac{1}{2})$. In the supersymmetric limit $\omega_B = \omega_F = \omega$ one has $E = \omega(n_B+n_F)$. For the fermions we can only have $n_F = 0,1$. Each energy level $|n\rangle$ is doubly degenerate therefore since we can make $n = n_B+n_F$ in two ways: $n = (n-1) + 1$ (fermionic) and $n = n + 0$ (bosonic). An exception is the ground state $n_B = 0$, $n_F = 0$. So there are equal numbers of fermionic and bosonic states with degenerate energies, and a single bosonic ground state.

We can introduce a fermionic supersymmetry operator

$$Q = \sqrt{2\omega}\ a^\dagger b \ , \qquad Q^\dagger = \sqrt{2\omega}\ b^\dagger a \ .$$

Q replaces a fermion by a boson so $n_B \rightarrow n_B + 1$, $n_F \rightarrow n_F - 1$, and $Q^\dagger$ replaces a boson by a fermion so $n_B \rightarrow n_B - 1$, $n_F \rightarrow n_F + 1$. Since in both cases $n_B + n_F$ is unchanged, the total energy is preserved and Q must commute with the Hamiltonian,

$$[Q,H] = [Q^\dagger,H] = 0 \ .$$

Crucially, in addition, we have $\{Q,Q^\dagger\} = 2H$.

More generally we add to the spacetime transformations of the Poincaré Group a spinorial (fermionic) object Q_α ($\alpha = 1,2,3,4$) which is a Majorana spinor ($Q = Q^c = CQ^T$. We can always split any Dirac spinor into two independent Majorana spinors ψ_1, ψ_2. $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$ or $\psi_1 = \frac{1}{\sqrt{2}}(\psi + \psi^c)$, $\psi_2 = \frac{-i}{\sqrt{2}}(\psi - \psi^c)$.)

The Poincaré Group has generators of translations P_μ, and the angular momentum tensor $M_{\mu\nu}$. M_{ij} generates rotations about the k-axis, i,j,k = {1,2,3}, M_{oi} generates boosts along the i axis. These generators satisfy the commutation relations

$$[P_\mu, P_\nu] = 0 , \qquad [M_{\mu\nu}, P_\rho] = i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) ,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}) .$$

We add to these Poincaré relations the additional commutators involving the Q_α supersymmetry generators,

$$[P_\mu, Q_\alpha] = 0 , \qquad [M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta}\, Q_\beta ,$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, with γ_μ the familiar 4 × 4 Dirac Gamma matrices. In order to close the algebra we need to add a single <u>anticommutation</u> relation

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}P_\mu .$$

This structure involving commutation and anticommutation relations defines a <u>graded</u> Lie algebra. One can show that, analogous to the simple toy example,

$$\sum_\alpha \{Q_\alpha, Q_\alpha^\dagger\} = 8H .$$

This has profound consequences since it ensures that

$$8\langle 0|H|0\rangle \;=\; \sum_{\alpha} |Q_\alpha|0\rangle|^2 + \sum_{\alpha} |Q_\alpha^\dagger|0\rangle|^2 \geq 0$$

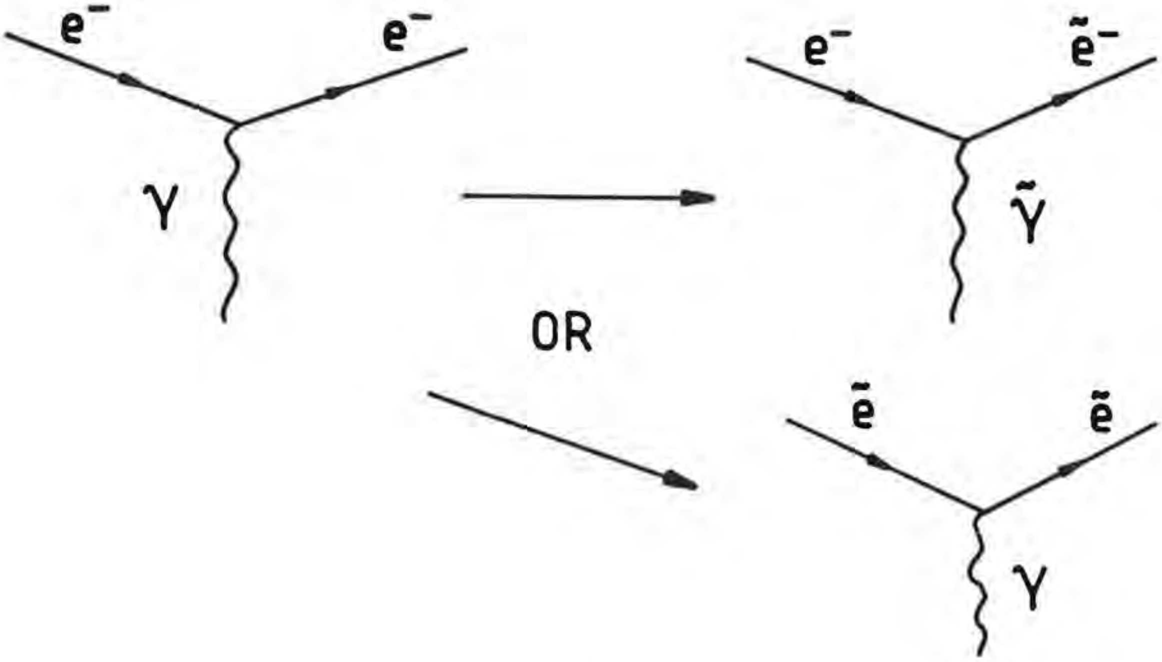
ensuring that the SUSY vacuum energy cannot be negative. This has important implications in discussions of the cosmological constant.

One consequence of a theory constructed on the basis of such transformations is, as in the simple toy example, the inevitable pairing up of fermions and bosons. We will have to introduce a whole heap of new <u>sparticles</u> to partner the particles, and with

$$J_{sparticle} = |J_{particle} - \tfrac{1}{2}| \;.$$

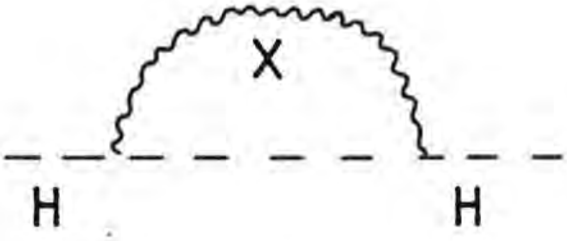
In the unbroken theory $m_{sparticle} = m_{particle}$ so one should expect to see electrons e ($J = \frac{1}{2}$) and selectrons $\tilde{e}$ ($J = 0$), neutrinos ν ($J = \frac{1}{2}$) and sneutrinos $\tilde{\nu}$ ($J = 0$), similarly quarks and squarks. Gauge bosons will have fermionic partners called gauginos. So J = 1 photon (γ) and $J = \frac{1}{2}$ photino ($\tilde{\gamma}$), similarly for W's and winos $\tilde{W}$, Z's and zinos $\tilde{Z}$, gluons and gluinos. The scalar Higgs (H) will be partnered by a fermionic sHiggs ($\tilde{H}$).

Since the existing catalogue of particles is not doubled we know that supersymmetry must be broken. It is natural to expect the 'ordinary' fermions to be lighter because their masses are protected by chiral gauge invariance as discussed at the end of the last section. If the theory is based on existing particle interactions one cannot just replace a particle by a sparticle since this will violate J by $\frac{1}{2}$. One can, however, replace two particles by sparticles. So from the standard QED $ee\gamma$ vertex we can generate $e\tilde{e}\tilde{\gamma}$ or $\tilde{e}\tilde{e}\gamma$ vertices as illustrated.



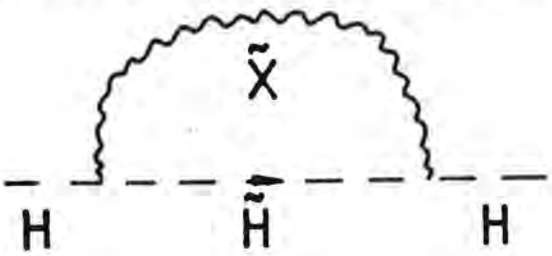
This leads to the idea of a multiplicatively conserved R parity (R = 1 for conventional particles, R = -1 for sparticles).

As discussed at the end of the last section the fine tuning of Higgs parameters necessary to achieve a large mass hierarchy in a GUT is detuned by (among other things) loop corrections such as



$\sim g^2 M_X^2$

In an extended SUSY GUT, however R parity allows the additional term



$\sim -g^2 M_X^2$

These terms cancel exactly in the unbroken SUSY theory, and the Higgs mass is protected. If SUSY is to solve the hierarchy problem the energy scale at which SUSY is broken cannot be much larger than that of the electroweak Higgs sector which is to have its masses protected. We therefore need $m_{sparticle} \lesssim 1$ TeV. So far these heavier sparticles have not been detected experimentally. Their phenomenology depends on various model-dependent assumptions. If R parity is conserved then the lightest sparticle must be stable. The likely signatures then depend on what the lightest sparticle is. If the photino is lightest one can expect $\tilde{q}\tilde{\bar{q}}$ production at colliders with subsequent decay into $q\tilde{\gamma} + \bar{q}\tilde{\gamma}$, appearing as two hadronic jets and missing transverse energy. Other signatures will follow for other choices of lightest sparticle, for instance the gluino.

Assuming that these sparticles are seen at upcoming colliders such as LHC, SSC then SUSY GUT seems an excellent solution to the hierarchy problem. As we saw at the end of section 4 lower energy data seem to be in excellent agreement with a unification scale of order 10^{16} GeV for minimal SUSY GUT. This larger unification scale reduces the proton decay rate and in fact the dominant decay mode is now $p \longrightarrow K^{+}\nu_{\tau}$, so there is no longer any inconsistency with the experimental bound on the proton lifetime.

Do your bit and help to fill in the blanks in the Sparticle Data Review!

Acknowledgements

It is a pleasure to thank Roger Barlow for his excellent organisation of this enjoyable School, all the students, lecturers and tutors for being such an entertaining bunch, and the Cosener's House staff, particularly those in the kitchen, for their fine service. Penny Carse is thanked for uncomplainingly word processing a difficult manuscript in record time.

Exercises

1. Defining $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ show that $\gamma_5^2 = 1$, $\gamma_5^\dagger = \gamma_5$ and $\{\gamma_5, \gamma_\mu\} = 0$.

Show that $P_{\substack{R\\L}} = \frac{1}{2}(1\pm\gamma_5)$ satisfy the projection operator properties $P_R + P_L = 1$, $P_L^2 = P_L$, $P_R^2 = P_R$, $P_LP_R = P_RP_L = 0$.

Consider a massless fermion with momentum p along the z direction, $p_\mu = (E,0,0,E)$. Show that $P_Ru(p)$ and $P_Lu(p)$ are eigenstates of helicity $h = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = -\frac{1}{2}\,\frac{\gamma^0\gamma_5\vec{\gamma}\cdot\vec{p}}{E}$, with eigenvalues $\pm\frac{1}{2}$.

2. Show that, as claimed, $\mathscr{L}(e)$ is $SU(2)_L \times U(1)_Y$ invariant, by checking explicitly that the $\chi_L, e_R, \vec{V}_\mu, B_\mu$ infinitesimal transformations given in the lecture leave it invariant.

3. There is one diagram in lowest order electroweak theory for μ^- decay,

$$\mu^-(p) \rightarrow \nu_\mu(k) + e^-(p') + \bar{\nu}_e(k') \ .$$

Draw this diagram and use the electroweak Feynman rules to calculate the spin averaged $|\bar{M}|^2$ for this decay. To simplify the calculation retain m_μ but set $m_e = 0$. Also, evaluate in the effective "Fermi theory" where you leave out the V propagator (set it to $g_{\mu\nu}$) and replace g at the vertices by g/M_V. Why is this a very good approximation for μ^- decay? Does setting $m_\mu = 0$ make any difference? [You are given (I'm very merciful!!)

$$\mathrm{Tr}[\gamma^\mu(1-\gamma_5)\not{p}_1\gamma^\nu(1-\gamma_5)\not{p}_2]\mathrm{Tr}[\gamma_\mu(1-\gamma_5)\not{p}_3\gamma_\nu(1-\gamma_5)\not{p}_4] = 256\ (p_1\cdot p_3)(p_2\cdot p_4)\ .]$$

Using the formulae in Tim's course write out an expression for $d\Gamma$ (the differential decay rate) in terms of $|\bar{M}|^2$ and phase space.

A tedious phase space integration which you need not attempt then leads to the total μ^- decay rate,

$$\Gamma(\mu^-) = \frac{g^4 m_\mu^5}{(6144)\pi^3 M_V^4}\ .$$

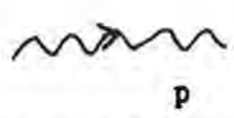
Given $m_\mu = 105.66$ MeV and the μ^- lifetime

$$\tau(\mu^-)^{exp} = \frac{1}{\Gamma(\mu^-)} = (2.197138 \pm .000065) \times 10^{-6}\ \text{sec}$$

estimate 'v' the Higgs VEV in the minimal Standard Model. [In natural units 1 sec = 1.52×10^{24} GeV^{-1}.]

4. Use the electroweak Feynman rules to calculate the polarization averaged decay width for Z^0 decay, $\Gamma(Z^0 \to f\bar{f})$, $f = e,\nu,q\ldots$ Take f massless.

[For an external massive spin 1 vector boson with mass M_V you need the Feynman rule



$: \ \epsilon_\mu^{(\lambda)}$

where the completeness sum over the polarizations is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M_V^2}\ .$$

Suitable choices are ($\vec{p}$ along z-axis)

$$\epsilon^{(\lambda = \pm 1)} = \mp(0,1,\pm i,0)/\sqrt{2} \text{ and}$$

$$\epsilon^{(\lambda = 0)} = (|\vec{p}|,0,0,E)/M_V .$$

From Tim's notes can infer (Z^0 rest frame)

$$\Gamma(Z^0 \to f\bar{f}) = \frac{1}{64\pi^2 M_Z} \int \overline{|M|^2} \, d\Omega .\Big]$$

Estimate the total Z^0 decay width (take $M_Z = 91$ GeV, $g = 0.65$, $\sin^2\theta_W = .23$) which should have been observed at LEP. Don't forget three colours for each quark flavour!

5. Consider a model with several representations of Higgs Φ_i, $i = 1,2,\ldots,N$, and such that the neutral members ϕ_i^0 acquire VEV's v_i. Then show that

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{\sum_{i=1}^{N} v_i^2[T_i(T_i+1)-\frac{1}{4}Y_i^2]}{\sum_{i=1}^{N} \frac{1}{2} v_i^2 Y_i^2}$$

where T_i, Y_i are the weak isospin and hypercharge of the representations.

Show in particular that any number of conventional Higgs doublets with $T_i = \frac{1}{2}$, $Y_i = \pm 1$ will result in $\rho = 1$.

[Hint: Rewrite $D_\mu\Phi$ in terms of τ^+,τ^- raising and lowering ops.

Extract M_W^2, M_Z^2 from $(D_\mu\Phi)(D^\mu\Phi)^\dagger$ and use

$$\tau^\pm \Phi_0^i(T_i, T_3^i) = \sqrt{T_i(T_i+1) - (T_i^3)^2 \mp T_i^3}\; \Phi_0^i(T_i, T_i^3 \pm 1)$$

Note that for neutral VEV's $T_i^3 = -\frac{Y_i}{2}$.]

THE PHYSICS OF STRUCTURE

By Prof F E Close
Rutherford Appleton Laboratory

Lectures delivered at the School for Young High Energy Physicists
Rutherford Appleton Laboratory, September 1992

Notes related to Frank Close's lectures on

The Physics of Structure

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1 The Physics of Structure

There are two independent hints that hadrons have internal structure
a) spectra - states with large spin J decay into lighter states with low J, reminiscent of the excitation spectra in molecules, atoms or even nuclei.
b) scattering - highly inelastic scattering reveal the presence of underlying constituents.

The inelastic scattering of leptons (e, μ, ν) from protons and neutron show that the latter contain $J = 1/2$, pointlike fermions with fractional charges (quarks). The phenomenology underlying this is described in chapter 11 of ref (1). The kinematics, deviations of cross-sections, definitions of structure functions, which apply basic ideas of the Dirac equation, are in chapter 9. Start with pages 171-173. Then try sections 9.3 and 9.4 and chapter 11.

I will cover this in the latter part of these lectures and refer to the book for the formulae.

The bulk of the lectures deals with spectra and how they reveal details on QCD dynamics.

Essentially all hadrons are described, in spectroscopic terms, as either baryons (qqq) or mesons $(q\bar{q})$. I use q = quark; $\bar{q}$ = antiquark. Let's look at mesons.

Generic features of $q\bar{q}$ spectrum

q and $\bar{q}$ each have spin $S = 1/2$; hence $q\bar{q}$ have total $S = 0$ or 1.

Suppose that they are in some potential and have relative orbital angular momentum $L = 0, 1, 2$ etc. The total $\vec{J} = \vec{L} + \vec{S}$.

Consider the case of a single variety of quark (e.g. charm, c). The $c\bar{c}$ states have various values of J, and their wavefunctions have $\pm$ eigenvalues for parity (P) and charge conjugation (c). We label the states J^{PC}.

Quantum theory implies that for a bound state of fermion - antifermion (e.g. $c\bar{c}$)

$$P = (-1)^{L+1} \quad ; \quad C = (-1)^{L+S}$$

(this is the algebraic $L+S$ not vector $\vec{L}+\vec{S}$).
(see page 86 of ref 1)

Exercise 1: Look up $c\bar{c}$ states in Particle Data Tables and compare with your spectrum. Also include radial excitations (the $\psi(3685)$ is radial excitation of $\psi(3095)$ - see chapter 16 of ref 1).

Exercise 2: Do the same for $b\bar{b}$. Compare the relative masses of the excitations compared to the lightest $b\bar{b}$ with those for $c\bar{c}$.

Exercise 3: Given $P = (-)^{L+1}; C = (-)^{L+S}$, show that $q\bar{q}$ cannot make all possible J^{PC} combinations. Which ones are impossible?

The meson spectroscopy is well described as $q_1, \bar{q}_2$ bound states, where $q_i =$ any of b, c, s, u, d.

A rule of thumb is to give these flavours effective masses (in GeV)

$$\begin{aligned} b &\simeq 4.7 \\ c &\simeq 1.5 \\ s &\simeq 0.5 \\ u \simeq d &\simeq 0.3 \end{aligned}$$

(don't take these seriously; they are just a ball-park guide). Combine these in all possible pairs and you will find $J^P = 1^-$ mesons within 100 or 200 MeV (above) the total (e.g. $s\bar{s} = 1GeV; \phi(1020)$). The excitation spectroscopy arises if each unit or orbital excitation "costs" $\simeq$ 400 MeV (for b, c), slightly more for the lighter flavours.

Exercise 4: Construct the $0^+1^+2^+$ states. See which ones have been found (see Particle Data Tables). Give estimates for masses of the "missing" ones with these quantum numbers.

The c, b are much heavier than s, u, d and their spectra are consequently rather clean. The s, u, d have similar masses and the electrically neutral particles tend to be mixtures of $u\bar{u}, d\bar{d}$ and $s\bar{s}$. Thus it is usual to consider these three light flavours together. The electrical charges are $u = 2/3, d = s$=-1/3. The antiquarks have opposite charges: $\bar{u} = -2/3, \bar{d} = \bar{s} = 1/3$. The strange quark s, has strangeness -1; the $\bar{s}$ has strangeness +1.

Exercise 5: Construct nine combinations of meson $q_i\bar{q}_j i, j = u, d, s$ and tabulate their electrical charge and strangeness. Identify $J^P = 0^-$ and 1^- states from the Particle Data Tables.

Exercise 5 reveals the existence of a hexagonal pattern with three states at the centre. Similar hexagons arise for baryons. This similarity of pattern is what first gave the clue to the presence of quarks. However the patterns arise for mesons and baryons in different ways.

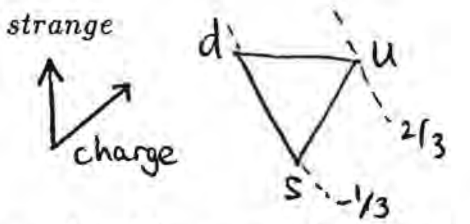
If you collect together the proton and other spin 1/2 baryons and plot them on a diagram according to their strangeness and electrical charge you obtain the familiar patterns of 1, 2. The same pattern qualitatively emerges for the pseudoscalar ($J^p = 0^-$) mesons including the pions. Although this similarity proved suggestive in 1964, we shall see that the pattern arise for rather different reasons. A corresponding pattern emerges for the vector mesons (ρ, K^*, ω, ϕ) while the $J = 3/2$ baryons, ($\Delta, \Sigma^*\Xi^*, \Omega$) form an enlarged triangle.

These patterns emerge naturally once the idea of quarks is accepted.

These particular hadrons are built from three varieties, or flavours, of quark; the **up, down** and **strange**. There are other varieties known today which build up other hadrons and we will meet these later.

The quarks have charge and strangeness as follows

	charge	*strange*
u	$2/3$	0
d	$-1/3$	0
s	$-1/3$	-1



and so form a triangle on a strangeness-charge plot. The antiquarks have the opposite strangeness and charge to their quark counterparts and form an inverted triangle.

	charge	*strange*
$\bar{u}$	$-2/3$	0
$\bar{d}$	$+1/3$	0
$\bar{s}$	$+1/3$	$+1$

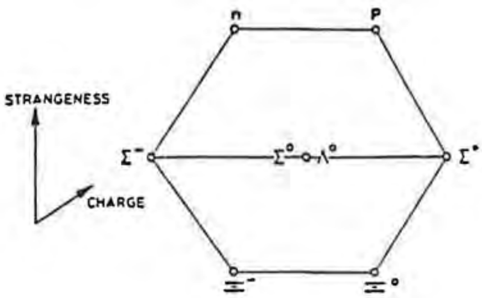


Fig 1

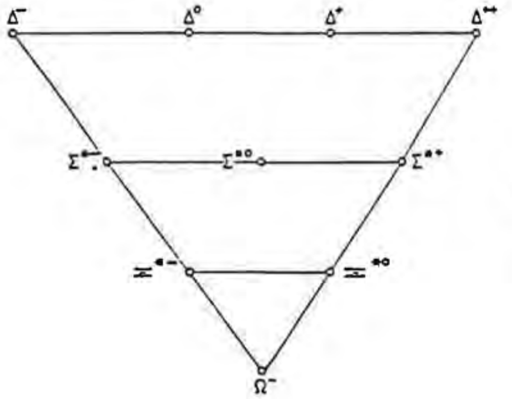


Fig2

The quarks and antiquarks all have spin 1/2. So three quarks clustered together in $S-$ wave have $J = 1/2$ or $3/2$. A pair, in particular quark and antiquark, will form $J = 0$ or 1. Now let's see how the patterns emerge.

Two quarks form a total of six possible combinations as shown in fig **3**. All these states have fractional electric charge and none are seen in nature. But put three together and you get the large triangle of states in fig **3**. These have integer charges and strangeness from 0 to -3. They correspond precisely with the ten members of the family including the delta and omega-minus. If we lop off the corners (the meaning of this will emerge shortly) we obtain the pattern of the J=1/2 states including the proton and neutron. As an exercise verify that the charges and strangenesses do agree with the empirically observed states.

When we combined two quarks we obtained fractionally charged, unseen, states. But if we combine a quark and an antiquark we obtain integer charged, "real" states corresponding to the known mesons. This is shown in fig **4**. Compare this with fig **3** and verify that the quantum numbers all match.

The pattern looks to be the same as for baryons but closer examination shows that they are not. There are octets and decuplets (tens) of baryons whereas mesons come in nonets. This basic pattern has been verified over and over as more hadrons with higher spins have been uncovered during the last 25 years. In all cases, hadrons containing strange quarks are slightly heavier than their nonstrange counterparts. A strange quark has about 150 MeV more mass than do the (nearly) degenerate up and down quarks.

All of these hadrons can be explained if we regard the quarks and antiquarks as dynamical objects that interact much as electrons and nuclei in atoms. So there will be a set of excited states with the quark spins coupled to their relative orbital angular momenta. First we illustrate this for mesons; a quark orbiting an antiquark.

In the data tables you will see the mesons classified by properties such as parity P, behaviour under spatial reflection) and charge conjugation, C (particle-antiparticle symmetry). For fermion and antifermion in angular momentum L the parity is

$$P = (-)^{L+1}$$

In deriving this note that fermion and antifermion have opposite intrinsic parity. The charge conjugation of such a pair in orbital state L and net spin S is

$$C = (-)^{L+S}$$

It is therefore a straightforward task to work out the set of allowed J^{PC} for a quark and antiquark in net spin 0 or 1 coupled in orbital angular momentum L. The lowest few are in the table

$L = 3$	3^{+-}	$2^{++}3^{++}4^{++}$
$L = 2$	2^{-+}	$1^{--}2^{--}3^{--}$
$L = 1$	1^{+-}	$0^{++}1^{++}2^{++}$
$L = 0$	0^{-+}	1^{--}
	$S = 0$	$S = 1$

Note that there are no states (odd)$^{-+}$ nor even (even)$^{+-}$. Such states (in particular 1^{-+}) are exotics within the framework of the quark model, where $Q\bar{Q}$ are the states. If such an exotic were found it would show that states exist beyond simply $Q\bar{Q}$. None have yet been seen. This supports the quark model of mesons.

Some candidates for these multiplets and their masses are shown in fig **5**. The states with $S = 1$ are ideally mixed (with the exception of the 0^{++} which is a puzzle to be discussed later). These multiplets each contain two states that are almost degenerate in mass and so presumably are the orthogonal isostates made

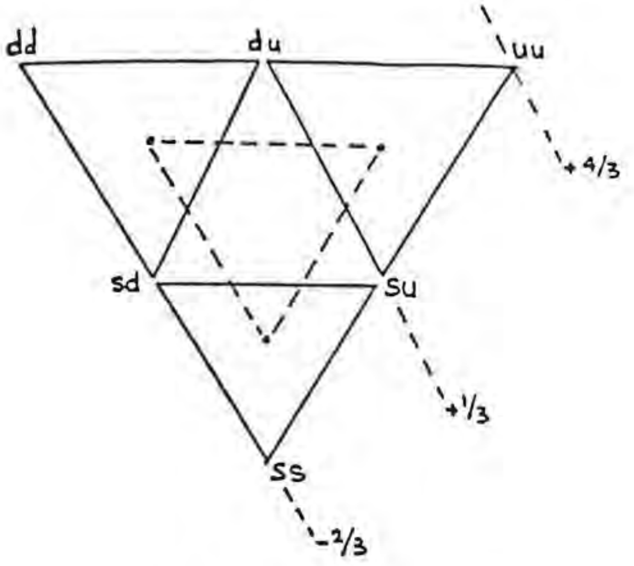


FIG 4

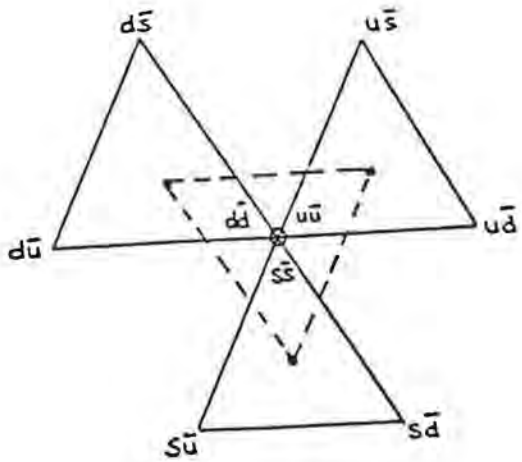


FIG 3

from up and down quarks.

$2^{-+}\left\{\begin{array}{c}\text{------}\\ L(1770)\\ \text{------}\\ A(1680)\end{array}\right.$	$3^{--}\left\{\begin{array}{c}\phi(1850)\\ K^*(1780)\\ \omega(1670)\\ g(1690)\end{array}\right.$ $2^{--}\left\{\begin{array}{c}\text{------}\\ \text{------}\\ \text{------}\\ \text{------}\end{array}\right.$ $1^{--}\left\{\begin{array}{c}\phi'(1680)\\ \text{------}\\ \text{------}\\ \rho'(1600)\end{array}\right.$
$1^{+-}\left\{\begin{array}{c}\text{------}\\ Q_B(1280)\\ H(1190)\\ B(1235)\end{array}\right.$	$2^{++}\left\{\begin{array}{c}f_2'(1514)\\ K^*(1430)\\ f_2(1270)\\ A_2(1320)\end{array}\right.$ $1^{++}\left\{\begin{array}{c}f_1(1440)\\ Q_A(1400)\\ f_1(1285)\\ A_1(1270)\end{array}\right.$ $0^{++}\left\{\begin{array}{c}f_0(1520)\\ \kappa(1350)\\ a_0(1320?)\\ f_0(1000)\end{array}\right.$
$0^{-+}\left\{\begin{array}{c}\eta'(960)\\ K(495)\\ \eta(550)\\ \pi(140)\end{array}\right.$	$1^{--}\left\{\begin{array}{c}\phi(1020)\\ K^*(890)\\ \omega(780)\\ \rho(770)\end{array}\right.$

Fig 5
Some members of meson multiplets

This is supported by the fact that the strange states K^*, with an up (or down) and a strange quark is midway between these ρ and ω the hidden strange $\phi(s\bar{s})$

member. The radiative decays of these states also support this pattern of mixing. It appears that the up and down quarks are (nearly) degenerate and that the strange quark is some 150 MeV heavier. Being heavier its excitation is less than the up and down. Hence the mass separations are slightly less in the D-wave than S-wave multiplets.

This pattern emerges naturally from a mass matrix if there is no mixing between the $u\bar{u}$ and $s\bar{s}$ states (e.g. by annihilation through gluon intermediate states). The pattern in the 0^{-+} multiplet is quite different from these. Here the pion is light and the K and η have similar masses to one another. The remaining member is yet heavier. This sort of pattern emerges if there is an important mixing between the $q\bar{q}$ flavours.

Colour

Evidence for Colour

Quarks have electric charge and any of three colours. The following gives some of the evidence for colour.

(i) Baryons exist

The most dramatic evidence for colour is that hadrons have no net colour (?!) and that we exist.

In QED the U(1) electrical charge attracts positive and negative to form neutral atoms with excitation energies of the order of eV. Quarks and antiquarks carry positive and negative of some "strong charge" which attracts to form neutral mesons. The excitation energies are of the order of hundreds of MeV. In part this is because the hadrons are smaller than atoms and so the associated energy scales are larger. However, this still requires an effective interaction that is 10 to 100 times stronger than electromagnetic $\alpha = 1/137$.

Apart from the fact of ionisation, the mesons are quite similar to ordinary atoms. The real mystery is that baryons exist: the nuclei of all our atoms contain neutrons and protons after all. One quark feels an attraction to two more quarks. In some sense a pair of quarks can act like a single antiquark. This turns out to be a natural consequence of colour forces. A pair of quarks can have the same effective colour charge as a single antiquark, and hence attract, and neutralise, a neighbouring quark.

A second paradox for baryons is that the $\Omega^-(sss)$, among others, exists with three identical flavours and spin states for its quarks, in apparent defiance of the Pauli principle. However, if quarks have any of three varieties of colour and each one in a baryon has a different colour, then they are distinguishable and Pauli is satisfied.

(ii) $\sigma(e^+e^- \to$ hadrons)

In the quark model the ratio of the production of hadrons relative to muon pairs in electron positron annihilation is given by the squared charges of all possible quarks. This entails a sum over both the flavours and the three colours of quarks. The data require threefold colour. Below charm production threshold the ratio (R) will be given by the sum of u, d, s flavours (in three colours) and then above charm production threshold the charmed quark contribution is included.

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q e_q^2 =^{3x2/3(u,d,s)}_{3x4/9(u,d,s,c)}$$

(iii) Heavy Lepton Decay

This is like e^+e^- annihilation "turned on its side". We have

$$\text{"}R\text{"} = \frac{\tau^+ \to \nu + \text{hadrons}}{\tau^+ \to \nu(e\nu)} = N_c = 3$$

and hence a branching ratio into $\nu e\nu$ of 20%. QCD enhances the hadron (quarks) slightly at the expense of leptons and agrees with the data which have

$$\begin{aligned} B.R.(\tau \to e\nu\nu) &= 16.5 \pm 1\% \\ B.R.(\tau \to \mu\nu\nu) &= 18.5 \pm 1\% \end{aligned}$$

(iv) $\Gamma(\pi^0 \to \gamma\gamma)$

There is an exact theorem (6) that the rate $\Gamma(\pi^0 \to \gamma\gamma)$ is proportional to the square of N_c with a calculable coefficient.

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.87eV \times (N_c/3)^2$$

The data give 7.95 $\pm$ 0.55 eV which clearly require 3 colours.

(v) Fractionally charged quarks

To construct a satisfactory renormalisable theory of leptons and quarks appears to require that their electric charges neutralise overall. The third fractional charges of quarks are then correlated with their threefold colour.

(vi) QCD

The presence of colour naturally leads to a theory of quark interactions which continues to confront data, and predict phenomena, with success.

Colour: A Non-abelian SU(3) Theory

Quarks are hypothesised to belong to the fundamental 3 representation of SU(3)-colour. Antiquarks then below to the $\bar{3}$. Leptons are singlets: they have no colour at all. Hadrons are singlets in that they have no net colour.

If we hypothesise that only colour singlets exist free in Nature, then what possibilities arise?

First let's cluster two or three quarks and antiquarks together. The dimensionalities of the resulting SU(3) representations are listed alongside (to construct these you need to know the rules for Young tableaux which are given for instance in ref 1).

$$\begin{array}{rrl} QQ & 3\times 3 = & 6+\bar{3} \\ QQQ & 3\times 3\times 3 = & (10+8)+(8+1) \\ Q\bar{Q} & 3\times \bar{3} = & 8+1 \\ QQ\bar{Q} & 3\times 3\times \bar{3} = & 15+3+\bar{6}+3 \end{array}$$

Notice that two quarks can form a $\bar{3}$, hence behaving like an antiquark under colour. The only singlets occur for $q\bar{q}$ and qqq, as in mesons and baryons. If we consider larger numbers of quarks and antiquarks we find that colour singlets occur only for

$$(QQQ)^N; (Q\bar{Q})^N$$

The former can be associated with nuclei. The latter are interesting because they might contain more than trivial decay products as in

$$\rho(Q\bar{Q})_1 \to \rho(Q\bar{Q}Q\bar{Q})_1 \to \pi(Q\bar{Q})_1 + \pi(Q\bar{Q})_1$$

It is likely that such $q\bar{q}q\bar{q}$ will fall apart into two $q\bar{q} + q\bar{q}$ except when they lie below dissociation threshold. There may be one or two cases where this happens, such as the scalar mesons S^* (960) and δ (980), interpreted as $q\bar{q}q\bar{q}$ states.

All this discussion has revolved around non-Abelian SU(3). Now consider the additional complexity if we build a non-Abelian gauge theory (NAGT) of quark interactions based on this threefold colour. Just as QED contains photons, so will QCD contain "gluons". It is a property of NAGT that the radiation quanta belong to the "regular" representation of the group. For SU(N) this contains N-1 members. So, for SU(3) we have 8 gluons.

We can construct colour singlet hadrons involving gluons. These can consist of gluons along ("gluonia or glueballs") or contain quarks and glue ("hermaphrodites, (8) meiktons (9) or hybrids").

$$(8 \times 8)1 \leftrightarrow GG$$
$$(8 \times 8 \times 8)1 \leftrightarrow GGG$$
$$(8 \times (3 \times \bar{3}))1 \leftrightarrow GQ\bar{Q}$$
$$(8 \times (3 \times 3 \times 3))1 \leftrightarrow GQQQ$$

The empirical status of glueballs and hybrid is open at present. This is an area of active research.

Colour and the Pauli Principle at Work in Hadrons

There are six ways that we can make a symmetric combination of two colours (R, B, G) and three (actually an antitriplet) that form an antisymmetric combination. These are

$3(anti)$	$6(sym)$
$RB - BR$	$RB + BR$
$RG - GR$	$RG + GR$
$BG - GB$	$BG + GB$
	RR
	GG
	BB

If we cluster three quarks together there is one totally antisymmetric combination of colours.

$$\underset{\sim}{1} = ((RB - BR)G + (GR - RG)B + (BG - GB)R)/\sqrt{6}$$

This "colour singlet" consists of a triplet combined with an antitriplet (the bracketed quantities). Note well that it is antisymmetric in any pair of colours.

The Pauli principle requires total antisymmetry in the wave function. As this colour configuration is already antisymmetric, the three quark wavefunction must be **symmetric** in all else (where "all else" means "apart from colour").

This is quite different from nuclear clusters where nucleons have no net colour (and s are trivially symmetric in colour!). Hence Pauli may be summarised as follows:

If we forget about colour nucleons are antisymmetric in pairs but quarks are symmetric.

Two quarks can couple their spins as follows S=1 Symmetric; S=0: Antisymmetric.

Similarly, two u, d flavours form isospin states as follows I=1 Symmetric; I=0: Antisymmetric.

When all three quarks are in the ground state, S wave, their spatial wavefunction is trivially symmetric. Hence for pairs in $L = 0$ we have a correlation of spin and isospin:

S=1 and I=1 correlate; S=0 and I=0 correlate.

This has interesting consequences as I shall now show that the sigma and lambda, which are distinguished 0 respectively, also have the pair in spin 1 or 0.

Spin Flavour Correlations & Magnetic Moments

The up and down quarks combine in threes to make the neutron and proton. In turn the neutron and proton combine in threes to make ^3H and ^3He nuclei. The magnetic moment relations in these similar looking systems reflect the deep role of colour in the quark as against nuclear examples.

The data are

$$\begin{Bmatrix} n & ; & p \\ ddu & ; & uud \end{Bmatrix} \quad \frac{\mu_p}{\mu_n} = -\frac{3}{2}$$
$$\begin{Bmatrix} 3H & ; & ^3He \\ nnp & ; & ppn \end{Bmatrix} \quad \frac{\mu(^3He)}{\mu(eH)} = -\frac{2}{3}$$

A quick way of seeing the nuclear result is to use Pauli for the alpha particle. The nucleons' spins and flavours saturate the totality of quantum states and so the alpha particle has no magnetic moment.

$$^4He = \mu(^4He; p^\uparrow p^\downarrow n^\uparrow n^\downarrow) = 0.$$

Then, up to mass

$$\mu(^3He) = \mu(^4He) - \mu_n = -\mu_n$$
$$\mu(^3H) = \mu(^4He) - \mu_p = -\mu_p$$

and so

$$\frac{\mu(^3He)}{\mu(^3H)} = \frac{\mu_n}{\mu_p} = -\frac{2}{3}$$

The comparison with the quark example is clearest if we derive this result for ^3He directly.

$^3He = ppn$; the pp are flavour symmetric and hence spin antisymmetric i.e. S=0. Hence the ^3He magnetic moment is given entirely by the spare neutron; the proton pair do not contribute to the magnetic moment.

$$[^3He]_\uparrow = (pp)_0 n^\uparrow$$
$$\mu[^3He] = 0 + \mu_n$$

Similarly ^3H has a magnetic moment given by the spare proton.

$$\mu[^3H] = 0 + \mu_p$$

Now we can study the nucleons in a similar way.

The proton contains uu flavour symmetric and colour antisymmetric; thus the spin of the "like pair" is symmetric (S= 1) in contrast to the nuclear example where this pair had S=0. We must couple spin 1 and 1/2 together to make J=1/2. The Clebsches give

$$p^\uparrow = \frac{1}{\sqrt{3}}(uu)_0 d^\uparrow + \sqrt{\frac{2}{3}}(uu)_1 d^\downarrow$$

Then, up to mass factors, we have

$$\mu(p) = \frac{1}{3}(0 + d) + \frac{2}{3}(2u - d)$$

Suppose that the magnetic moments are proportional to the quark charges, so

$$\mu_u = -2\mu_d$$

and hence the magnetic moments of the nucleons are in ratio

$$\frac{\mu_p}{\mu_n} = \frac{4u - d}{4d - u} = -\frac{3}{2}$$

(the neutron and proton are related by replacing u with d and vice versa). This discussion can be extended to the whole octet of baryons.

Inelastic Scattering: Scaling; its Violation and Significance

Atoms

In 1911 the scattering of low energy alpha particles on atoms revealed the atomic nucleus. The beam was powerful enough to resolve subatomic dimensions but not able to penetrate the nucleus itself. The nucleus appeared to be a point charge.

Nucleus

More powerful beams, of electrons in particular, reveal the inner structure of nuclei. If we fix the energy of an incident electron beam and count events at some fixed angle, the target will recoil. The count rate looks like fig **6**. There are peaks in the scattering from carbon for different values of the scattered electron's energy. We illustrate what this means by showing underneath how light scattering centres take up more recoil than heavy ones. So, the peaks are due to coherent scattering from the bulky nucleus (extreme right), from alpha particles within it and finally, at the left, from the protons that comprise the nucleus.

We can change the violence of the impact by changing the electron energy or scattering angle. The beams here had energies of about 200 MeV and as we change the scattering angle from 80^0 (low violence) to 135^0 (violent) we see the coherent peak die away (see figs in ref 1).

The nucleus is breaking up. The elastic form factor of the nucleus dies but "quasi elastic" scattering from the constituents survives.

The modern convention is to plot the data against $x = Q^2/2M_\nu$ instead of E'. The kinematic variables here are $\nu = E - E', Q^2 = \nu^2 - \vec{q}^2$. Elastic scattering from the target occurs when x=1; quasi-elastic scattering from the constituents occurs when $x = \frac{1}{A}$ if there are A weakly bound constituents in the target.

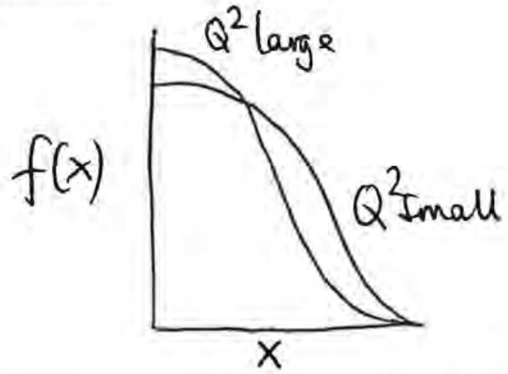
Consider the dimensionless quantity $Q^4 d\sigma/\underline{d}Q^2$

The elastic scattering peak dies off as $(Q^2R^2)^{-N}$ where R is a dimensional scale related to the size of the target. When $Q^2 < R^{-2}$ no structure is resolved; the target appears pointlike. The dimensionless $Q^4 d\sigma/sQ^2$ scales: i.e. it is invariant under changes of Q^2. This is indeed what we see for the quasi elastic peak: its inner structure is not resolved on the range $Q^2 < 0.1$ GeV^2. But when we attain values of $Q^2 \simeq$ 1-10 GeV^2 the proton scattering is dying out and the quark constituents show up - as pointlike particles. Notice the gradual leftward shift of the data as more structure is resolved.

In the bottom figure I have shown two alternative definitions of x in terms of nuclear or nucleon mass. These have different ranges as shown. Let's use x_{Bj} from here on and so recast the data from 0 to 1. This brings us to nucleons.

Nucleons

In 1968-72 the classic experiments at SLAC and CERN showed the quark substructure of nucleons. These first experiments showed data that scaled - $Q^4 d\sigma/dQ^2$ appeared to be independent of Q^2 at fixed x. Today we have data over a much greater kinematic range. As we increase Q^2 up to 100 GeV^2 we see a leftwards shift in the data.



Quark structure is being resolved. This is not showing that quarks are made of discrete subquarks: rather, we see a continuous shift as the quarks are resolved into quarks and gluons and a sea of quark antiquark pairs as the resolution improves, Q^2 increases).

This behaviour is expected in QCD. The quarks are quasi free to zeroth order ("parton model") due to the quark-gluon coupling tending to small values at high momentum. So we can apply perturbation theory and investigate the effects at first order in QCD (or higher order if we have enough motivation). I will summarise up to first order here.

Structure Functions

The use of $Q^4 d\sigma/sQ^2$ is rather cavalier. Really we have a double differential cross section $d\sigma/dQ^2 d\nu$

From this we can form the dimensionless quantity

$$\mu^2 d\sigma / d(Q^2/2M_\nu) d(\nu/E)$$

where μ is some mass scale. If we define $x = Q^2/2M_\nu$ and $y = \nu/E$ then we can consider

$$\mu^2 d\sigma / dx dy$$

The cross section for electron scattering is then

$$\frac{d\sigma}{dxdy} = 4\pi\alpha^2 \frac{2ME}{Q^4}[(1-y)F_2(x,Q^2) + y^2 x F_1(X,Q^2)]$$

The Q^{-4} comes from the propagator of the exchanged photon. The 2ME is the invariant energy incident. The F_1, F_2 are dimensionless "structure functions" that summarise the dynamics and are in general functions of two kinematic variables. For convenience we will choose the variables to be x and Q^2. If the data scale then the structure functions will be functions only of the dimensionless quantity x; there will be no dependence on Q^2.

There are two structure functions because there are two essential degrees of freedom. The incident photon can be transversely or longitudinally polarised. F_1 is essentially transverse, and F_2 is related to the sum of transverse and longitudinal (for detailed kinematics see ref 1).

If parity were violated then there would be a third degree of freedom, namely the relative importance of left and right handed interaction. This would require a third structure function F_3. But in neutrino interactions, parity is violated and hence there is a third F_3. The cross section in this case reads

$$\frac{d\sigma}{dxdy} = \frac{G_F^2}{2\pi} 2ME[(1-y)F_2 + y^2 x F_1 \mp y(1-\frac{y}{2})xF_3]$$

The relative size of the structure functions tells you about the constituents at work. Spin 1/2 quarks contribute little to the longitudinal and dominantly to the transverse scattering (this is a consequence of helicity conservation at high energies). Thus

$$F_2 = 2xF_1$$

or equivalently

$$R \equiv \frac{\sigma_L}{\sigma_T} \simeq 0.$$

Data are in excellent agreement with this (the ratio in principle could have been infinity after all!).

The structure function F_3 distinguishes quarks from antiquarks (left handed versus right handed scattering). Use $F_2 = 2xF_1$ and rewrite the cross section as

$$\frac{d\sigma}{dxdy} = \frac{G^2 2ME}{2\pi} F_2(x, Q^2)[\frac{1+(1-y)^2}{2} \mp \frac{1-(1-y)^2}{2}\frac{xF_3}{F_2}]$$

At the fundamental quark level we have (ref 1)

$$\frac{d\sigma^\nu}{dxdy} = \frac{G^2 2ME}{2\pi} x[q(x,Q^2) + (1-y)^2 \bar{q}(x,Q^2)]$$
$$\frac{d\sigma^{\bar{\nu}}}{dxdy} = \frac{G^2 2ME}{2\pi} x[\bar{q}(x,Q^2) + (1-y)^2 q(x,Q^2)]$$

and so we see that xF_3/F_2 separates quark and antiquark distributions.

$$\frac{xF_3(x)}{F_2(x)} = \frac{q(x) - \bar{q}(x)}{q(x) + \bar{q}(x)}$$

When $x > 0.2$, $xF_3 \simeq F_2$ which means that antiquarks are small here. When $x \to 0$, by contrast, $xF_3 \to 0$ which means that quarks and antiquarks are equally likely here. This fits with the heuristic picture of the proton containing three "valence" quarks and a soft sea of quarks and antiquarks.

Sum Rules and Evidence for Glue

The inelastic cross sections directly measure the quantum numbers of quarks.

When $Q >>$ inverse proton size (≥ 1 GeV) and the energy, ν, is much greater than that needed to excite resonances ($>> 1$ GeV), the scattering is incoherent from quasi free quarks. The structure functions are then given by the following expressions. I have retained only the first generation flavours but these expressions can be generalised as an exercise for the reader

$$2xF_1^e \equiv F_2^e(x) = \sum_f e_f^2 x f(x) \quad ; \quad \nu(\frac{d}{u}) \to \mu^-(\frac{u}{d})$$

So for electromagnetic scattering we have; for weak interactions we have

$$\frac{1}{x}F^{ep}(x) = \frac{4}{9}(u+\bar{u}) + \frac{1}{9}(d+\bar{d}) \quad ; \quad \frac{1}{x}F_2^{\nu p} = 2[d+\bar{u}]$$
$$\frac{1}{x}F^{en}(x) = \frac{4}{9}(d+\bar{d}) + \frac{1}{9}(u+\bar{u}) \quad ; \quad \frac{1}{x}F_2^{\nu n} = 2[u+\bar{d}]$$

The probability to find a u in a proton is the same as to find d in a neutron. So I have rewritten

$$d^n \equiv u^p = u$$

and dropped the superscripts, thereby referring to the proton distribution functions always. For neutrino charged current interactions I have assumed that the vector and axial currents have equal strengths, hence the factor of 2.

The structure function xF_3 is given by the corresponding difference of quark and antiquark distribution functions, e.g.

$$\frac{1}{x}F^{\nu p} = 2(d - \bar{u})$$

There is a nontrivial relation between the electromagnetic and weak structure functions which is a measure of the quark charges.

$$\frac{F^{en+ep}}{F^{\nu n+\nu p}} = \frac{5(u+\bar{u}+d+\bar{d})+2(s+\bar{s})}{18(u+\bar{u}+d+\bar{d})} \geq \frac{5}{18}$$

This is well satisfied by the data, showing that the scattering centres are indeed fractionally charged quarks.

The fact that there are net 2, 1, 0 up, down and strange quarks in a proton gives sum rules for the distribution.

$$\begin{aligned} 2 &= \int_0^1 dx(u - \bar{u}) \\ 1 &= \int_0^1 dx(d - \bar{d}) \\ 0 &= \int_0^1 dx(s - \bar{s}) \end{aligned}$$

These can be converted into sum rules for the structure functions. There is a net excess of three quarks, which gives the Gross Llewellyn Smith sum rules.

$$\begin{aligned} 3 \equiv N_q - N_{\bar{q}} &= \int_0^1 dx[u+d+s\ldots) - (\bar{u}+\bar{d}+\bar{s}\ldots)] \\ &\equiv \frac{1}{2}\int_0^1 dx F_3^{\nu p+\nu n} \quad Data = 3.2 \pm 0.6 \end{aligned}$$

If we assume that $\bar{u}$ and $\bar{d}$ have the same sea distributions then we get a sum rule from the squared charges of the quarks

$$\int_0^1 (F_2^{ep} - F_2^{en})\frac{dx}{x} = \int_0^1 \frac{1}{3}(u-d)dx = \frac{1}{3} \quad ; \quad Data = 0.28 \pm ?$$

which is reasonably consistent with data.

All of the data point towards quarks being the constituents of hadrons. Nothing yet is sensitive to neutrals, such as glue. The glue shows up when we balance momentum

$$p_{u+d} = \int_0^1 dx(u+\bar{u}+d+\bar{d}) \equiv \frac{1}{2}\int_0^1 dx F_2^{\nu n+\nu p}(x, Q^2)$$

The data are too small; the integrand is only about 50%. The remaining momentum must be carried by neutrals, such as gluons.

The QCD theory of quark interactions naturally leads us to expect that gluons will also be present in the nucleon. It also predicts that quarks are quasi-free when probed in high momentum transfer processes, such as those that we have been studying here, hence our ability to analyse the data in this very simple way. The theory also requires that quarks and gluons interact and that characteristic patterns of violation of scaling should occur in the data as a result. This aspect of the data is what we turn to next.

QCD Scaling Violations in Inelastic Scattering

Suppose we have measured some distribution at moderate Q^2 and then increase Q^2 and try again. We resolve better than before and may find that a quark with momentum x when seen at Q^2 turns out to be a slower quark, y, and a gluon. In generally we will see more slow quarks at the expense of fast ones when we improve the resolution. The structure functions will qualitatively shift as indicated already. Data show this behaviour too. How does this compare quantitatively with QCD?

Perhaps the cleanest way to study this is with the structure function xF_3 as gluons do not contribute to this directly. We compute the moments of xF_3 as

$$M_n(Q^2) = \int_0^1 dx\ x^{n-2} xF_3(x, Q^2)$$

Large n weighs high x where the structure function dies out with Q^2. So M_n dies out with increasing Q^2. If we plot the log of one moment against the log of another we will get the following[13]. At Q_1^2 there will be a value for each moment and so a single point on the plot. At $Q_2^2 > Q_1^2$ the point will have moved down as both moments are smaller than before. The trajectory is predicted to be a straight line in any field theory. Its slope is the ratio of two numbers that are calculable given the tensor property of the field theory (i.e. vector for QCD). Some slopes for the ratios of moments in QCD are compared with the data in the table

QCD	
1.29	1.29 ± 0.06
1.46	1.50 ± 0.08
1.76	1.84 ± 0.20

Why Moments?

For free quarks we have

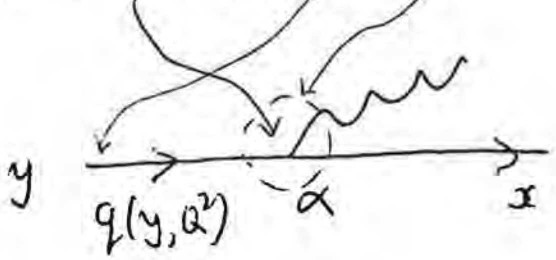
$$q(x, Q_1^2) = q(x, Q_2^2)$$

and so the dimensionless measure vanishes:-

$$\frac{Q^2 \delta q(x, Q^2)}{\delta Q^2} \equiv \frac{\delta q(x, Q^2)}{\delta \log Q^2} = 0$$

Now let's consider interacting quarks and gluons. Increasing Q^2 improves the resolution by fractional amount $\Delta \ln Q^2$. There is an increased likelihood of finding slow quarks $q(x, Q^2)$ at the expense of fast ones $q(y > x, Q^2)$. The dimensionless measure is no longer zero:-

$$\frac{\delta q(x, Q^2)}{\delta \ln Q^2} = \alpha_s(Q^2) \int_x^1 \frac{dy}{y} q(y, Q^2) p(x/y)$$



and we have shown the origin of the various terms. The quantity $P(x/y)$ is a calculable function which depends on the nature of the quark-gluon vertex. In vector theory (as QCD) this is

$$p_{qq}(z) = \frac{4}{3}\ \frac{1 + z^2}{1 - z}$$

(Notice that as $z \to 1$, soft gluon emission, we have an infra red divergence). The above equation immediately gives us the behaviour of the structure function as Q^2 changes

$$\frac{\delta F_3(x, Q^2)}{\delta \ln Q^2} = \alpha_s(Q^2) \int_x^1 \frac{dy}{y} F_3(y, Q^2) p(x/y)$$

This is the master equation for explicit formulation of scaling violation.

This expression can be used to analyse data but is not yet in the most useful form. If we integrate over x and write $z = x/y$ then we have,

$$\frac{\delta}{\delta \ln Q^2} = \int_0^1 F(x, Q^2) dx = \alpha \int_0^1 dy F(y, Q^2) \int_0^1 dz p(z)$$

or for any n

$$\frac{\delta}{\delta \ln Q^2} = \int_0^1 x^n F(x,Q^2)dx = \alpha \int_0^1 dy y^n F(y,Q^2) \int_0^1 dz z^n p(z)$$

Defining the moment as

$$M_n(Q^2) \equiv \int_0^1 dx x^n F(x,Q^2)$$

we have the useful equation that describes the change in the moment with Q^2

$$\frac{\delta \ln M_n}{\delta \ln Q^2} = \alpha d_n (\text{where } d_n \equiv \int_0^1 dz z^n p(z))$$

It is because this is so direct that we analyse in terms of moments. The relative Q^2 dependence of two moments is then

$$\frac{d \ln M_n}{d \ln M_m} = \frac{d_n}{d_m}$$

and hence the straight line, with slope given by d_N/d_M, that we met above.

In QCD the coupling is Q^2 dependent with a form summarised as

$$\alpha(Q^2) = c/\ln(Q^2/\Lambda^2)$$

We can now solve the equation and obtain an explicit Q^2 dependence for the moments

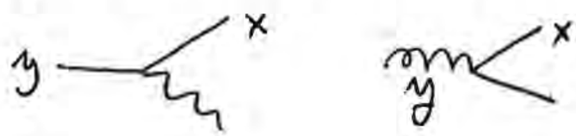
$$M_n(Q^2) \sim [\ln Q^2/\Lambda^2]^{-d_n} \sim [\alpha(Q^2)]^{d_n}$$

or

$$M_n^{-1/d_n} \sim \ln Q^2 - \ln \Lambda^2$$

If we analyse structure functions where gluons can directly contribute then things are more complicated. We have to solve coupled equations for the quark and gluon distributions separately. These are, with origin again illustrated.

$$\frac{\delta q}{\delta \ln Q^2} = \alpha \int_x^1 \frac{dy}{y} [q(y,Q^2)p_{qq}(x/y) + g(y,Q^2)p_{gq}(x/y)]$$



$$\frac{\delta g}{\delta \ln Q^2} = \alpha \int_x^1 \frac{dy}{y} [\sum q(y, Q^2) p_{gq}(x/y) + g(y, Q^2) p_{gg}(x/y)]$$

The role of QCD evolution and of gluons is very marked in the data now that we can probe to very large Q^2. The resulting information is used in planning the next generation of accelerators.

References

[1] F.E. Close, "Introduction to Quarks and Partons", Academic Press 1979.

[2] Particle Data Group: Phys. Rev. **D45**, June 1992.

A second paradox for baryons is that the $\Omega^-(sss)$, among others, exists with three identical flavours and spin states for its quarks, in apparent defiance of the Pauli principle. However, if quarks have any of three varieties of colour and each one in a baryon has a different colour, then they are distinguishable and Pauli is satisfied.

(ii) $\sigma(e^+e^- \to$ hadrons)

In the quark model the ratio of the production of hadrons relative to muon pairs in electron positron annihilation is given by the squared charges of all possible quarks. This entails a sum over both the flavours and the three colours of quarks. The data require threefold colour. Below charm production threshold the ratio (R) will be given by the sum of u, d, s flavours (in three colours) and then above charm production threshold the charmed quark contribution is included.

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q e_q^2 =^{3x2/3(u,d,s)}_{3x4/9(u,d,s,c)}$$

(iii) Heavy Lepton Decay

This is like e^+e^- annihilation "turned on its side". We have

$$\text{“}R\text{”} = \frac{\tau^+ \to \nu + \text{hadrons}}{\tau^+ \to \nu(e\nu)} = N_c = 3$$

and hence a branching ratio into $\nu e \nu$ of 20%. QCD enhances the hadron (quarks) slightly at the expense of leptons and agrees with the data which have

$$\begin{aligned} B.R.(\tau \to e\nu\nu) &= 16.5 \pm 1\% \\ B.R.(\tau \to \mu\nu\nu) &= 18.5 \pm 1\% \end{aligned}$$

(iv) $\Gamma(\pi^0 \to \gamma\gamma)$

There is an exact theorem (6) that the rate $\Gamma(\pi^0 \to \gamma\gamma)$ is proportional to the square of N_c with a calculable coefficient.

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.87eV \times (N_c/3)^2$$

The data give 7.95 $\pm$ 0.55 eV which clearly require 3 colours.

(v) Fractionally charged quarks

To construct a satisfactory renormalisable theory of leptons and quarks appears to require that their electric charges neutralise overall. The third fractional charges of quarks are then correlated with their threefold colour.

(vi) QCD

The presence of colour naturally leads to a theory of quark interactions which continues to confront data, and predict phenomena, with success.

Colour: A Non-abelian SU(3) Theory

Quarks are hypothesised to belong to the fundamental 3 representation of SU(3)-colour. Antiquarks then below to the $\bar{3}$. Leptons are singlets: they have no colour at all. Hadrons are singlets in that they have no net colour.

If we hypothesise that only colour singlets exist free in Nature, then what possibilities arise?

First let's cluster two or three quarks and antiquarks together. The dimensionalities of the resulting SU(3) representations are listed alongside (to construct these you need to know the rules for Young tableaux which are given for instance in ref 1).

$$\begin{array}{rrl} QQ & 3\times 3 = & 6+\bar{3} \\ QQQ & 3\times 3\times 3 = & (10+8)+(8+1) \\ Q\bar{Q} & 3\times\bar{3} = & 8+1 \\ QQ\bar{Q} & 3\times 3\times\bar{3} = & 15+3+\bar{6}+3 \end{array}$$

Notice that two quarks can form a $\bar{3}$, hence behaving like an antiquark under colour. The only singlets occur for $q\bar{q}$ and qqq, as in mesons and baryons. If we consider larger numbers of quarks and antiquarks we find that colour singlets occur only for

$$(QQQ)^N; (Q\bar{Q})^N$$

The former can be associated with nuclei. The latter are interesting because they might contain more than trivial decay products as in

$$\rho(Q\bar{Q})_1 \rightarrow \rho(Q\bar{Q}Q\bar{Q})_1 \rightarrow \pi(Q\bar{Q})_1 + \pi(Q\bar{Q})_1$$

It is likely that such $q\bar{q}q\bar{q}$ will fall apart into two $q\bar{q} + q\bar{q}$ except when they lie below dissociation threshold. There may be one or two cases where this happens, such as the scalar mesons S^* (960) and δ (980), interpreted as $q\bar{q}q\bar{q}$ states.

All this discussion has revolved around non-Abelian SU(3). Now consider the additional complexity if we build a non-Abelian gauge theory (NAGT) of quark interactions based on this threefold colour. Just as QED contains photons, so will QCD contain "gluons". It is a property of NAGT that the radiation quanta belong to the "regular" representation of the group. For SU(N) this contains N-1 members. So, for SU(3) we have 8 gluons.

We can construct colour singlet hadrons involving gluons. These can consist of gluons along ("gluonia or glueballs") or contain quarks and glue ("hermaphrodites, (8) meiktons (9) or hybrids").

$$\begin{aligned}(8 \times 8)1 &\leftrightarrow GG \\ (8 \times 8 \times 8)1 &\leftrightarrow GGG \\ (8 \times (3 \times \bar{3}))1 &\leftrightarrow GQ\bar{Q} \\ (8 \times (3 \times 3 \times 3))1 &\leftrightarrow GQQQ\end{aligned}$$

The empirical status of glueballs and hybrid is open at present. This is an area of active research.

Colour and the Pauli Principle at Work in Hadrons

There are six ways that we can make a symmetric combination of two colours (R, B, G) and three (actually an antitriplet) that form an antisymmetric combination. These are

$3(anti)$	$6(sym)$
$RB - BR$	$RB + BR$
$RG - GR$	$RG + GR$
$BG - GB$	$BG + GB$
	RR
	GG
	BB

If we cluster three quarks together there is one totally antisymmetric combination of colours.

$$\underset{\sim}{1} = ((RB - BR)G + (GR - RG)B + (BG - GB)R)/\sqrt{6}$$

This "colour singlet" consists of a triplet combined with an antitriplet (the bracketed quantities). Note well that it is antisymmetric in any pair of colours.

The Pauli principle requires total antisymmetry in the wave function. As this colour configuration is already antisymmetric, the three quark wavefunction must be **symmetric** in all else (where "all else" means "apart from colour").

This is quite different from nuclear clusters where nucleons have no net colour (and s are trivially symmetric in colour!). Hence Pauli may be summarised as follows:

If we forget about colour nucleons are antisymmetric in pairs but quarks are symmetric.

Two quarks can couple their spins as follows S=1 Symmetric; S=0: Antisymmetric.

Similarly, two u, d flavours form isospin states as follows I=1 Symmetric; I=0: Antisymmetric.

When all three quarks are in the ground state, S wave, their spatial wavefunction is trivially symmetric. Hence for pairs in $L = 0$ we have a correlation of spin and isospin:

S=1 and I=1 correlate; S=0 and I=0 correlate.

This has interesting consequences as I shall now show that the sigma and lambda, which are distinguished 0 respectively, also have the pair in spin 1 or 0.

Spin Flavour Correlations & Magnetic Moments

The up and down quarks combine in threes to make the neutron and proton. In turn the neutron and proton combine in threes to make ^3H and ^3He nuclei. The magnetic moment relations in these similar looking systems reflect the deep role of colour in the quark as against nuclear examples.

The data are

$$\begin{Bmatrix} n & ; & p \\ ddu & ; & uud \end{Bmatrix} \quad \frac{\mu_p}{\mu_n} = -\frac{3}{2}$$

$$\begin{Bmatrix} 3H & ; & {}^3He \\ nnp & ; & ppn \end{Bmatrix} \quad \frac{\mu({}^3He)}{\mu(eH)} = -\frac{2}{3}$$

A quick way of seeing the nuclear result is to use Pauli for the alpha particle. The nucleons' spins and flavours saturate the totality of quantum states and so the alpha particle has no magnetic moment.

$$^4He = \mu(^4He; p^\uparrow p^\downarrow n^\uparrow n^\downarrow) = 0.$$

Then, up to mass

$$\mu(^3He) = \mu(^4He) - \mu_n = -\mu_n$$
$$\mu(^3H) = \mu(^4He) - \mu_p = -\mu_p$$

and so

$$\frac{\mu(^3He)}{\mu(^3H)} = \frac{\mu_n}{\mu_p} = -\frac{2}{3}$$

The comparison with the quark example is clearest if we derive this result for ^3He directly.

$^3He = ppn$; the pp are flavour symmetric and hence spin antisymmetric i.e. S=0. Hence the ^3He magnetic moment is given entirely by the spare neutron; the proton pair do not contribute to the magnetic moment.

$$[^3He]_\uparrow = (pp)_0 n^\uparrow$$
$$\mu[^3He] = 0 + \mu_n$$

Similarly ^3H has a magnetic moment given by the spare proton.

$$\mu[^3H] = 0 + \mu_p$$

Now we can study the nucleons in a similar way.

The proton contains uu flavour symmetric and colour antisymmetric; thus the spin of the "like pair" is symmetric (S= 1) in contrast to the nuclear example where this pair had S=0. We must couple spin 1 and 1/2 together to make J=1/2. The Clebsches give

$$p^\uparrow = \frac{1}{\sqrt{3}}(uu)_0 d^\uparrow + \sqrt{\frac{2}{3}}(uu)_1 d^\downarrow$$

Then, up to mass factors, we have

$$\mu(p) = \frac{1}{3}(0 + d) + \frac{2}{3}(2u - d)$$

Suppose that the magnetic moments are proportional to the quark charges, so

$$\mu_u = -2\mu_d$$

and hence the magnetic moments of the nucleons are in ratio

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The modern convention is to plot the data against $x = Q^2/2M_\nu$ instead of E'. The kinematic variables here are $\nu = E - E', Q^2 = \nu^2 - \vec{q}^2$. Elastic scattering from the target occurs when x=1; quasi-elastic scattering from the constituents occurs when $x = \frac{1}{A}$ if there are A weakly bound constituents in the target.

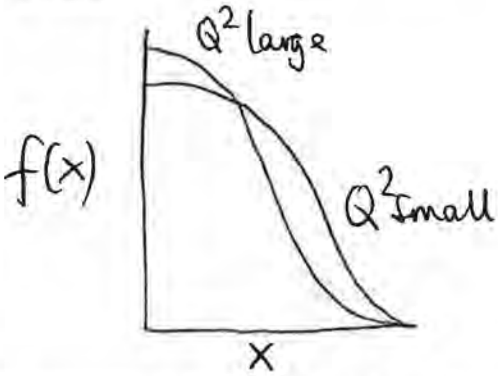
Consider the dimensionless quantity $Q^4 d\sigma/\underline{d}Q^2$

The elastic scattering peak dies off as $(Q^2R^2)^{-N}$ where R is a dimensional scale related to the size of the target. When $Q^2 < R^{-2}$ no structure is resolved; the target appears pointlike. The dimensionless $Q^4 d\sigma/sQ^2$ scales: i.e. it is invariant under changes of Q^2. This is indeed what we see for the quasi elastic peak: its inner structure is not resolved on the range $Q^2 < 0.1$ GeV^2. But when we attain values of $Q^2 \simeq$ 1-10 GeV^2 the proton scattering is dying out and the quark constituents show up - as pointlike particles. Notice the gradual leftward shift of the data as more structure is resolved.

In the bottom figure I have shown two alternative definitions of x in terms of nuclear or nucleon mass. These have different ranges as shown. Let's use x_{Bj} from here on and so recast the data from 0 to 1. This brings us to nucleons.

Nucleons

In 1968-72 the classic experiments at SLAC and CERN showed the quark substructure of nucleons. These first experiments showed data that scaled - $Q^4 d\sigma/dQ^2$ appeared to be independent of Q^2 at fixed x. Today we have data over a much greater kinematic range. As we increase Q^2 up to 100 GeV^2 we see a leftwards shift in the data.



Quark structure is being resolved. This is not showing that quarks are made of discrete subquarks: rather, we see a continuous shift as the quarks are resolved into quarks and gluons and a sea of quark antiquark pairs as the resolution improves, Q^2 increases).

This behaviour is expected in QCD. The quarks are quasi free to zeroth order ("parton model") due to the quark-gluon coupling tending to small values at high momentum. So we can apply perturbation theory and investigate the effects at first order in QCD (or higher order if we have enough motivation). I will summarise up to first order here.

Structure Functions

The use of $Q^4 d\sigma/sQ^2$ is rather cavalier. Really we have a double differential cross section $d\sigma/dQ^2 d\nu$

From this we can form the dimensionless quantity

$$\mu^2 d\sigma / d(Q^2/2M_\nu) d(\nu/E)$$

where μ is some mass scale. If we define $x = Q^2/2M_\nu$ and $y = \nu/E$ then we can consider

$$\mu^2 d\sigma / dx dy$$

The cross section for electron scattering is then

$$\frac{d\sigma}{dxdy} = 4\pi\alpha^2 \frac{2ME}{Q^4}[(1-y)F_2(x,Q^2) + y^2 x F_1(X,Q^2)]$$

The Q^{-4} comes from the propagator of the exchanged photon. The 2ME is the invariant energy incident. The F_1, F_2 are dimensionless "structure functions" that summarise the dynamics and are in general functions of two kinematic variables. For convenience we will choose the variables to be x and Q^2. If the data scale then the structure functions will be functions only of the dimensionless quantity x; there will be no dependence on Q^2.

There are two structure functions because there are two essential degrees of freedom. The incident photon can be transversely or longitudinally polarised. F_1 is essentially transverse, and F_2 is related to the sum of transverse and longitudinal (for detailed kinematics see ref 1).

If parity were violated then there would be a third degree of freedom, namely the relative importance of left and right handed interaction. This would require a third structure function F_3. But in neutrino interactions, parity is violated and hence there is a third F_3. The cross section in this case reads

$$\frac{d\sigma}{dxdy} = \frac{G_F^2}{2\pi} 2ME[(1-y)F_2 + y^2 x F_1 \mp y(1-\frac{y}{2})xF_3]$$

The relative size of the structure functions tells you about the constituents at work. Spin 1/2 quarks contribute little to the longitudinal and dominantly to the transverse scattering (this is a consequence of helicity conservation at high energies). Thus

$$F_2 = 2xF_1$$

or equivalently

$$R \equiv \frac{\sigma_L}{\sigma_T} \simeq 0.$$

Data are in excellent agreement with this (the ratio in principle could have been infinity after all!).

The structure function F_3 distinguishes quarks from antiquarks (left handed versus right handed scattering). Use $F_2 = 2xF_1$ and rewrite the cross section as

$$\frac{d\sigma}{dxdy} = \frac{G^2 2ME}{2\pi} F_2(x, Q^2)[\frac{1+(1-y)^2}{2} \mp \frac{1-(1-y)^2}{2}\frac{xF_3}{F_2}]$$

At the fundamental quark level we have (ref 1)

$$\frac{d\sigma^\nu}{dxdy} = \frac{G^2 2ME}{2\pi} x[q(x,Q^2) + (1-y)^2 \bar{q}(x,Q^2)]$$
$$\frac{d\sigma^{\bar{\nu}}}{dxdy} = \frac{G^2 2ME}{2\pi} x[\bar{q}(x,Q^2) + (1-y)^2 q(x,Q^2)]$$

and so we see that xF_3/F_2 separates quark and antiquark distributions.

$$\frac{xF_3(x)}{F_2(x)} = \frac{q(x) - \bar{q}(x)}{q(x) + \bar{q}(x)}$$

When $x > 0.2$, $xF_3 \simeq F_2$ which means that antiquarks are small here. When $x \to 0$, by contrast, $xF_3 \to 0$ which means that quarks and antiquarks are equally likely here. This fits with the heuristic picture of the proton containing three "valence" quarks and a soft sea of quarks and antiquarks.

Sum Rules and Evidence for Glue

The inelastic cross sections directly measure the quantum numbers of quarks.

When $Q >>$ inverse proton size (≥ 1 GeV) and the energy, ν, is much greater than that needed to excite resonances ($>> 1$ GeV), the scattering is incoherent from quasi free quarks. The structure functions are then given by the following expressions. I have retained only the first generation flavours but these expressions can be generalised as an exercise for the reader

$$2xF_1^e \equiv F_2^e(x) = \sum_f e_f^2 x f(x) \quad ; \quad \nu(\frac{d}{u}) \to \mu^-(\frac{u}{d})$$

So for electromagnetic scattering we have; for weak interactions we have

$$\frac{1}{x}F^{ep}(x) = \frac{4}{9}(u+\bar{u}) + \frac{1}{9}(d+\bar{d}) \quad ; \quad \frac{1}{x}F_2^{\nu p} = 2[d+\bar{u}]$$
$$\frac{1}{x}F^{en}(x) = \frac{4}{9}(d+\bar{d}) + \frac{1}{9}(u+\bar{u}) \quad ; \quad \frac{1}{x}F_2^{\nu n} = 2[u+\bar{d}]$$

The probability to find a u in a proton is the same as to find d in a neutron. So I have rewritten

$$d^n \equiv u^p = u$$

and dropped the superscripts, thereby referring to the proton distribution functions always. For neutrino charged current interactions I have assumed that the vector and axial currents have equal strengths, hence the factor of 2.

The structure function xF_3 is given by the corresponding difference of quark and antiquark distribution functions, e.g.

$$\frac{1}{x}F^{\nu p} = 2(d - \bar{u})$$

There is a nontrivial relation between the electromagnetic and weak structure functions which is a measure of the quark charges.

$$\frac{F^{en+ep}}{F^{\nu n+\nu p}} = \frac{5(u+\bar{u}+d+\bar{d})+2(s+\bar{s})}{18(u+\bar{u}+d+\bar{d})} \geq \frac{5}{18}$$

This is well satisfied by the data, showing that the scattering centres are indeed fractionally charged quarks.

The fact that there are net 2, 1, 0 up, down and strange quarks in a proton gives sum rules for the distribution.

$$\begin{aligned} 2 &= \int_0^1 dx(u-\bar{u}) \\ 1 &= \int_0^1 dx(d-\bar{d}) \\ 0 &= \int_0^1 dx(s-\bar{s}) \end{aligned}$$

These can be converted into sum rules for the structure functions. There is a net excess of three quarks, which gives the Gross Llewellyn Smith sum rules.

$$\begin{aligned} 3 \equiv N_q - N_{\bar{q}} &= \int_0^1 dx[u+d+s\ldots)-(\bar{u}+\bar{d}+\bar{s}\ldots)] \\ &\equiv \frac{1}{2}\int_0^1 dxF_3^{\nu p+\nu n} \quad Data = 3.2 \pm 0.6 \end{aligned}$$

If we assume that $\bar{u}$ and $\bar{d}$ have the same sea distributions then we get a sum rule from the squared charges of the quarks

$$\int_0^1 (F_2^{ep} - F_2^{en})\frac{dx}{x} = \int_0^1 \frac{1}{3}(u-d)dx = \frac{1}{3} \quad ; \quad Data = 0.28 \pm ?$$

which is reasonably consistent with data.

All of the data point towards quarks being the constituents of hadrons. Nothing yet is sensitive to neutrals, such as glue. The glue shows up when we balance momentum

$$p_{u+d} = \int_0^1 dx(u+\bar{u}+d+\bar{d}) \equiv \frac{1}{2}\int_0^1 dxF_2^{\nu n+\nu p}(x,Q^2)$$

The data are too small; the integrand is only about 50%. The remaining momentum must be carried by neutrals, such as gluons.

The QCD theory of quark interactions naturally leads us to expect that gluons will also be present in the nucleon. It also predicts that quarks are quasi-free when probed in high momentum transfer processes, such as those that we have been studying here, hence our ability to analyse the data in this very simple way. The theory also requires that quarks and gluons interact and that characteristic patterns of violation of scaling should occur in the data as a result. This aspect of the data is what we turn to next.

QCD Scaling Violations in Inelastic Scattering

Suppose we have measured some distribution at moderate Q^2 and then increase Q^2 and try again. We resolve better than before and may find that a quark with momentum x when seen at Q^2 turns out to be a slower quark, y, and a gluon. In generally we will see more slow quarks at the expense of fast ones when we improve the resolution. The structure functions will qualitatively shift as indicated already. Data show this behaviour too. How does this compare quantitatively with QCD?

Perhaps the cleanest way to study this is with the structure function xF_3 as gluons do not contribute to this directly. We compute the moments of xF_3 as

$$M_n(Q^2) = \int_0^1 dx\; x^{n-2} x F_3(x, Q^2)$$

Large n weighs high x where the structure function dies out with Q^2. So M_n dies out with increasing Q^2. If we plot the log of one moment against the log of another we will get the following[13]. At Q_1^2 there will be a value for each moment and so a single point on the plot. At $Q_2^2 > Q_1^2$ the point will have moved down as both moments are smaller than before. The trajectory is predicted to be a straight line in any field theory. Its slope is the ratio of two numbers that are calculable given the tensor property of the field theory (i.e. vector for QCD). Some slopes for the ratios of moments in QCD are compared with the data in the table

QCD	
1.29	1.29 ± 0.06
1.46	1.50 ± 0.08
1.76	1.84 ± 0.20

Why Moments?

For free quarks we have

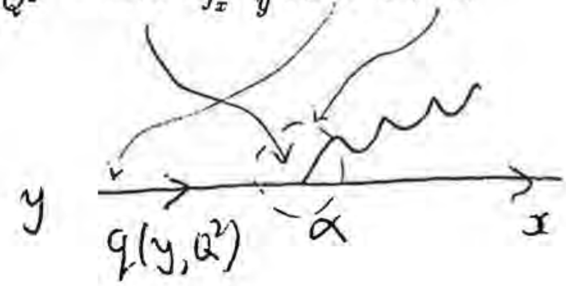
$$q(x, Q_1^2) = q(x, Q_2^2)$$

and so the dimensionless measure vanishes:-

$$\frac{Q^2 \delta q(x, Q^2)}{\delta Q^2} \equiv \frac{\delta q(x, Q^2)}{\delta \log Q^2} = 0$$

Now let's consider interacting quarks and gluons. Increasing Q^2 improves the resolution by fractional amount $\Delta \ln Q^2$. There is an increased likelihood of finding slow quarks $q(x, Q^2)$ at the expense of fast ones $q(y > x, Q^2)$. The dimensionless measure is no longer zero:-

$$\frac{\delta q(x, Q^2)}{\delta \ln Q^2} = \alpha_s(Q^2) \int_x^1 \frac{dy}{y} q(y, Q^2) p(x/y)$$



and we have shown the origin of the various terms. The quantity $P(x/y)$ is a calculable function which depends on the nature of the quark-gluon vertex. In vector theory (as QCD) this is

$$p_{qq}(z) = \frac{4}{3} \; \frac{1 + z^2}{1 - z}$$

(Notice that as $z \to 1$, soft gluon emission, we have an infra red divergence). The above equation immediately gives us the behaviour of the structure function as Q^2 changes

$$\frac{\delta F_3(x, Q^2)}{\delta \ln Q^2} = \alpha_s(Q^2) \int_x^1 \frac{dy}{y} F_3(y, Q^2) p(x/y)$$

This is the master equation for explicit formulation of scaling violation.

This expression can be used to analyse data but is not yet in the most useful form. If we integrate over x and write $z = x/y$ then we have,

$$\frac{\delta}{\delta \ln Q^2} = \int_0^1 F(x, Q^2) dx = \alpha \int_0^1 dy F(y, Q^2) \int_0^1 dz p(z)$$

or for any n

$$\frac{\delta}{\delta \ln Q^2} = \int_0^1 x^n F(x, Q^2) dx = \alpha \int_0^1 dy y^n F(y, Q^2) \int_0^1 dz z^n p(z)$$

Defining the moment as

$$M_n(Q^2) \equiv \int_0^1 dx x^n F(x, Q^2)$$

we have the useful equation that describes the change in the moment with Q^2

$$\frac{\delta \ln M_n}{\delta \ln Q^2} = \alpha d_n (\text{where } d_n \equiv \int_0^1 dz z^n p(z))$$

It is because this is so direct that we analyse in terms of moments. The relative Q^2 dependence of two moments is then

$$\frac{d \ln M_n}{d \ln M_m} = \frac{d_n}{d_m}$$

and hence the straight line, with slope given by d_N/d_M, that we met above.

In QCD the coupling is Q^2 dependent with a form summarised as

$$\alpha(Q^2) = c/\ln(Q^2/\Lambda^2)$$

We can now solve the equation and obtain an explicit Q^2 dependence for the moments

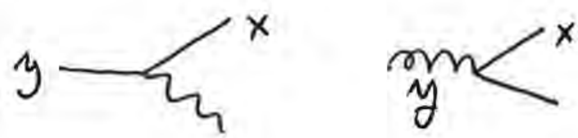
$$M_n(Q^2) \sim [\ln Q^2/\Lambda^2]^{-d_n} \sim [\alpha(Q^2)]^{d_n}$$

or

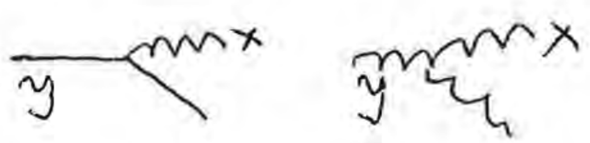
$$M_n^{-1/d_n} \sim \ln Q^2 - \ln \Lambda^2$$

If we analyse structure functions where gluons can directly contribute then things are more complicated. We have to solve coupled equations for the quark and gluon distributions separately. These are, with origin again illustrated.

$$\frac{\delta q}{\delta \ln Q^2} = \alpha \int_x^1 \frac{dy}{y} [q(y, Q^2) p_{qq}(x/y) + g(y, Q^2) p_{gq}(x/y)]$$



$$\frac{\delta g}{\delta \ln Q^2} = \alpha \int_x^1 \frac{dy}{y}[\sum q(y,Q^2)p_{gq}(x/y) + g(y,Q^2)p_{gg}(x/y)]$$



The role of QCD evolution and of gluons is very marked in the data now that we can probe to very large Q^2. The resulting information is used in planning the next generation of accelerators.

References

[1] F.E. Close, "Introduction to Quarks and Partons", Academic Press 1979.

[2] Particle Data Group: Phys. Rev. **D45**, June 1992.