Theory of underdense laser-plasma interactions with photon kinetic theory

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Abstract We review recent developments in the theory of laser-plasma interactions, with a focus on generalizations of the theory of parametric instabilities driven by lasers in underdense plasmas to include the effects of broadband or partially incoherent radiation via generalized photon kinetic theory. After an introduction addressing the fundamental concepts underlying parametric instabilities, the key concepts and techniques of photon kinetic theory are presented, along with the steps required to obtain the generalized dispersion relations for the different parametric instabilities. The main details of generalized photon kinetic theory are presented such that this chapter can also be used as reference for future work on generalized photon kinetic theory. As a particular example of the application of this theoretical approach, focus will be given to the derivation of the dispersion relation for Stimulated Brillouin Scattering. The main results for Stimulated Raman Scattering by a broadband or partially coherent radiation pump field will also be reviewed.

1 Introduction

All material substances interact nonlinearly with intense electromagnetic radiation and plasma is no exception. Such nonlinearity leads to so-called parametric excitation or parametric instabilities. Parametric excitation may be defined as an amplification of an oscillation due to a periodic modulation of a parameter that characterizes the oscillation. Physically, parametric excitation can be looked upon as a nonlinear instability of two waves (an idler and a signal) by a modulating wave (a pump) due to a mode coupling or wave-wave interaction. The simplest example is the three-wave interaction subject to the frequency and wavenumber matching

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conditions known as the Manley-Rowe relations

$$\begin{aligned}\omega_0 &= \omega_1 + \omega_2 \\ \mathbf{k}_0 &= \mathbf{k}_1 + \mathbf{k}_2\end{aligned} \tag{1}$$

where ω_0 is the pump frequency, $4\omega_{1,2}$ are the decay waves.

The concept of parametric excitation dates back to Lord Rayleigh and has subsequently found extensive application in electronic devices and nonlinear optics. Its more recent application to laser produced plasmas has led to the prediction of a large number of possible plasma instabilities. Some lead to anomalous electron and ion heating, others to scattering of electromagnetic energy out of the plasma. These collective phenomena are therefore of paramount importance in laser driven fusion and laser plasma accelerators. Radiation intensities of the order of $10^{14}\mathrm{W/cm}^2$ and greater are involved here. Other applications are the heating of magnetically confined plasmas by intense radio frequency radiation, the heating of the ionosphere by intense radar and acceleration of particles by the intense electromagnetic fields surrounding a pulsar. The description of nonlinear effects is therefore one of the central problems in modern plasma physics. Nonlinear theory of waves in plasmas, or of any other nontrivial phenomena, does not consist of a sweeping treatment of the subject which contains linear theory as a simple case; rather, it constitutes a number of cautious excursions from the familiar and comparatively safe territory of linear theory into the regime of nonlinearity. Computer simulations of the problems are helpful, but this can be expensive if complicated situations are to be modeled realistically, and it is easy for the physics to be obscured. What is required is a formalism which will simplify the analysis to the greatest possible degree.

Plasmas can support a number of normal modes of collective excitations, which can co-exist independently for sufficiently weak perturbations from the equilibrium. If the characteristic parameters of the plasma, like density, are modulated periodically in time and space by some large amplitude (pump) wave propagating as $\exp i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)$, the dielectric properties of the plasma are likewise changed. As a result beat waves (or forced oscillations), with frequency $\omega_0 \pm \omega_1$ and wavenumber $\mathbf{k}_0 \pm \mathbf{k}_1$, can be excited in the plasma, where $\omega_1, \mathbf{k}$ represents the frequency and wavenumber of one of the normal modes. If the following resonance conditions are satisfied $\omega_0 \pm \omega_1 = \omega_2, \mathbf{k}_0 \pm \mathbf{k}_1 = \mathbf{k}_2$ where $\omega_2, \mathbf{k}_2$ are the frequency and wavenumber of another normal mode of the plasma, the amplitudes of the two normal modes $(\omega_1, \mathbf{k},)$ and $(\omega_2, \mathbf{k}_2)$ can grow in time leading to instability, and an exchange of energy and momentum will take place among the three waves. This is not the only significant nonlinear process; the beat wave can also interact with charged particles moving at its phase velocity such that $\omega_0 - \omega_1 = (\mathbf{k}_0 - \mathbf{k}_1).\underline{v}$ where $\underline{v}$ is the velocity of a bunch of particles. Nonlinear wave-particle interaction then occurs, sometimes referred to as nonlinear Landau damping or stimulated Compton scattering. In the presence of a magnetic field the number of normal modes increases and hence the number of possible coupling processes increases. Parametric excitation in a magnetic field is of interest to fusion scientists because of the possibility of heating magnetically confined plasmas by high frequency electric fields. In laser fusion stimulated scattering, filamentation and modulational instabilities are seen as detrimental to the coupling of the laser energy to the plasma. Processes such as stimulated Raman and Brillouin scattering can result in a large function of the laser energy being scattered back out of the plasma while filamentation of the laser beams creates hot spots in the plasma. Stimulated Brillouin scattering (SBS) is a serious instability since it can scatter most of the incident laser light out of the plasma before it reaches the critical surface. Stimulated Brillouin scattering describes the decay of an incident electromagnetic wave into a scattered electromagnetic wave and an ion acoustic wave. Although seen to be an instability to be avoided in direct drive laser fusion indirect drive experiments using hohlraums have taken advantage of SBS by using it to transfer energy between beams. This has been demonstrated at NIF where energy transfer between outer and inner cores of laser beams can be controlled. This is not the case in direct drive where potentially a large fraction of the laser energy can be scattered out of the SBS arises when an incident laser beam with electric field E_0 couples to a low frequency ion acoustic density perturbation δn producing a transverse current $\propto \delta n E_0$ producing a scattered wave with field E_s. The ponderomotive force $< E_o E_s >$, where $<>$ denotes time average, set up by the heating of the incident and scattered electromagnetic wavve enhances the original density perturbation δn, thus producing a feedback mechanism that results in exponential growth of stimulated Brillouin scattering provided that the frequency and wavenumber of the three waves satisfy the conditions

$$\omega_0 = \omega_s + \omega_{ia}, \mathbf{k}_0 = \mathbf{k}_s + \mathbf{k}_{ia} \tag{2}$$

where subscripts $0, s$ represent pump and scattered waves and ia is the ion acoustic wave.

For underdense plasmas where $\omega_o \gg \omega_{pe} (\omega_{pe} = n_o e^2 / m_e E_o$ is the plasma frequency) and $\mid \mathbf{k}_0 \mid \simeq \mid \mathbf{k}_s \mid$, the matching conditions imply

$$\mathbf{k}_s = -\mathbf{k}_0 \tag{3}$$

$$\mathbf{k}_{ia} = 2\mathbf{k}_0 \simeq 2\omega_0 / c \tag{4}$$

The growth rate and threshold of stimulated Brillouin scattering can easily be derived from the three wave equations that describe stimulated Brillouin scattering. For a comprehensive treatment of the linear phase of the instability [1]. SBS can be responsible for significant loss of photon energy that is the scattered out of the plasma. The nonlinear evolution of SBS at high laser intensities where radiation pressure is greater than thermal pressure, $2I_L/c > n_e T_e$, where I_L is laser intensity, c is the speed of light, n_e is plasma density and T_e is the electron temperature, momentum coupling steepens the density profile. This causes the effectiveness of SBS to decrease and reduce the reflectivity. For the opposite case with the radiation pressure $2I_L/c < n_e T_e$ and in a long underdense plasma SBS growth rapidly producing a strong instability, with a large amount of radiation backscattered out of the plasma. In this case the ion acoustic wave can reach large amplitudes and the SBS saturates by a number of processes such as particle trapping, wavebreaking or shock formation, nonlinear ion heating and wave mixing due to light reflected from critical. A discussion of these saturation processes can be found in [2]. Saturation limits the amplitude of the ion acoustic wave density perturbation $\delta n_i/n_o$. Wave breaking and particle trapping occur at high laser intensities and can lead to ion acoustic wave amplitude levels of up to 20%. The fraction of the incident to laser beam that goes into the ion wave during SBS is given by $\omega_{ia}/\omega_0 = 2c_s/c$, where c_s is the ion acoustic speed. The ion wave can produce a tail on the ion distribution function forming high energy ions. In reality, laser plasmas are usually far from being homogenous and gradients in density and velocity controls the threshold and growth rate [2]. In a plasma with a velocity gradient dv/dx the frequency of the ion acoustic wave $\omega_{ia} = k_{ia}(c_s - v(x))$ changes with position this limits the region that satisfies the frequency and wavenumber matching conditions to a small region around their resonant position. SBS can be controlled by several processes such as broad laser bandwidth where the spectral width of the laser light is larger than the effective SBS gain width. Density and velocity of expansion irregularities can reduce stimulated Brillouin scattering significantly. Irregularities destroys the coherence of the waves involved and thus reduces convective amplification significantly.

Stimulated Raman scattering results in the incident laser beam decaying into an electron plasma wave and a scattered electromagnetic wave. Since the frequency of the electron plasma wave sometimes referred to as a Langmuir wave is given by,

$$\omega_{pw}^2 = \omega_{pe}^2 + 3k_{pw}^2 v_{Te}^2$$

the wavenumber and frequency matching conditions can only be satisfied for densities less than quarter critical i.e $n_e < n_c/4$. Stimulated Raman Scattering (SRS) therefore occurs for densities up to $n_c/4$. At the quarter critical another instability namely the two plasmon decay suitability competes with the Raman instability. In two plasmon decay the incident laser beam decays onto two Langmuir waves, the wavenumber matching conditions require that the Langmuir waves propagate in almost opposite directions at angles near 45^0 to the incident laser wavenumber. Raman backscatter at quarter critical is an absolute instability, the backscatter radiation group speed is zero, at the same time the two plasmon decay instability can also be an absolute instability, with the result that the Raman and two plasmon instabilities thresholds due to inhomogeneity are relatively low near quarter critical. These instabilities have therefore strong non-linear effects near $n_c/4$. The ponderomotive force due to the different plasma waves is strong enough to create density structures and in some cases the wave can be trapped in the density cavities. At the same time, large ion density fluctuations are formed that propagate down the density gradient. Profile steepening can also be responsible for increasing the inhomogeneity threshold switching off the instabilities. A consequence of Raman and two plasmon instabilities is the generation of a high energy electron tail. These heated electrons are a major concern since they preheat the fuel in laser fusion capsules. At intermediate densities $n_e < 0.2n_c$ stimulated Raman matter is less strong but it can still result in the backscatter of a large fraction of the incident laser energy and contribute to the formation of high energy electron tails.

As with SBS, a broad laser bandwidth can reduce the growth rate of stimulated Raman scattering. At the critical surface it is possible for the incident laser beam to excite the parametric decay instability where the resultant waves are a Langmuir wave and an ion acoustic wave. The parametric decay instability has maximum growth rate for the decay waves to propagate almost parallel to the laser electric field. The parametric decay instability results in wave absorption of the laser energy at the critical density. A variation of the parametric decay instability is the oscillating two stream which is basically a four wave instability where the ion acoustic mode is purely growing. The oscillating two stream instability also occurs at the critical density and results in the laser beam absorption. It can also ripple the critical density surface resulting in non-uniform absorption.

An instability that is also four wave is the filamentation instability. The filamentation instability occurs for densities less than the critical density where the laser beam couples to an ion acoustic perturbation that is purely growing producing density ripples. Filamentation produces an intensity modulation across the laser beam. This modulation grows and results in the laser beam breaking up into filaments that become more pronounced as the beam propagates through the plasma towards the critical density. Filamentation is caused by variations in intensity across the beam regions of initially higher intensity push the plasma aside due to the ponderomotive force. As a consequence this reduces the density locally and increases the index of refraction of the plasma in the higher intensity region bending the wave fronts in such a way that the curvature of the wave fronts produces a focusing effect increasing the intensity still further. Filamentation is a convective instability amplifying any intensity variation initially present in the beam or plasma. Since the density fluctuation in filamentation is purely growing and does not correspond to a resonant mode, the instability is not as sensitive to plasma inhomogeneity as the three wave instabilities. In addition to ponderomotive driven filamentation and self-focusing they can also be driven by thermal or relativistic effects In laser fusion parametric instabilities are normally seen as being harmful to successful coupling into fusion pellet. However, there is a great deal of research into controlling both simulated Brillouin and Raman scattering with a view to applications. In particular SBS is employed to control the energy distribution between the inner and outer laser cones in hohlraum targets and SRS in being studied as a possible future amplification technique where energy is transferred from a long pump beam to a much shorter probe beam [3]. This has the potential to reach laser intensities of $10^{25}\mathrm{W/cm^2}$.

This overview demonstrates the wide range of scenarios and phenomenology associated with parametric instabilities in plasmas and their roles in many applications. There is a tremendous body of theoretical and numerical work on these instabilities, with a good starting point being refs. [1, 2]. Here we will focus on novel theoretical approaches to study parametric instabilities in the presence of broadband radiation fields of arbitrary intensity. This is

2 Motivation for a Generalized Photon Kinetic Theory

The study of parametric instabilities is important in many fields of science [2, 4, 5, 6]. Standard methods use a coherent wave description to study this problem, but the externally induced incoherence or the partial coherence of most systems render this method incomplete. A plan to describe the instabilities of broadband radiation must therefore include an alternative (but formally equivalent) representation of the full nonlinear wave equation for electromagnetic waves in plasmas. A statistical description of the photons in a phase space $(\mathbf{r},\mathbf{k})$, with the corresponding distribution function of photons in this phase space to represent the radiation field, would therefore meet the requirements for a fully self-consistent description of parametric instabilities driven by broadband radiation of arbitrary intensity.

The Wigner-Moyal statistical theory provides the toolbox to study parametric instabilities, as first explored in nonlinear optics. With a derivation of a statistical description of a partially incoherent electromagnetic wave propagating in a nonlinear medium [7], it became clear that a stabilization of the modulational instability is possible as a result of an effect similar to Landau damping and caused by random phase fluctuations of the propagating wave, equivalent to the broadening of the Wigner spectrum. Similar studies [8, 9] focused on the onset of the transverse instability in nonlinear media in the presence of a partially incoherent light. The Wigner-Moyal theory applied to electromagnetic waves in plasmas or photon kinetics also provides an alternative approach to numerical modeling of laser propagation in plasmas via the photon-in-cell paradigm [10]. In nonlinear optics, the standard Wigner-Moyal theory is perfectly adjusted, without any required generalizations, since the paraxial wave equation is usually sufficient to describe the main physical processes, thus justifying a forward propagating *ansatz* for the evolution of electromagnetic waves in dispersive nonlinear media. In this context the standard Wigner-Moyal formalism, which is formally equivalent to the Schrödingerequation, can be used directly. In plasma physics, this is clearly a limitation, as many critical aspects in ICF, fast ignition and several applications in laser-plasma and astrophysical scenarios demand a detailed analysis and the inclusion of the backscattered radiation.

The inclusion of bandwidth or incoherence effects in laser driven parametric instabilities has also been studied extensively. The addition of small random deflections to the phase of a plane wave was shown to significantly suppress the three-wave decay instability [11], which was one of the first mechanisms where the introduction of some degree of incoherence in the laser was proposed as a way to avoid the deleterious effects of the instability. The threshold values for some electrostatic instabilities can also be effectively increased either by applying a random amplitude modulation to the laser or by the inclusion of a finite bandwidth of the pump wave [12, 13]. A new method for the inclusion of finite bandwidth effects on parametric instabilities, allowing arbitrary fluctuations of any group velocity, has also been developed in [14, 15]. As far as the Stimulated Raman Scattering instability is concerned, it became clear that, although it may seriously decollimate a coherent laser beam, the increase of the laser bandwidth is an effective way to suppress the instability [16].

A statistical description of light can be achieved through the Wigner-Moyal formalism of quantum mechanics, which provides, in its original formulation, a one-mode description of systems ruled by Schrödinger-like equations. In order to address other processes where side or backscattering can be important, a generalization of the Photon Kinetics theory (GPK) was recently developed by J. E. Santos and L. O. Silva [17]. This new formulation is completely equivalent to the full Klein-Gordon equation and was readily employed to derive a general dispersion relation for stimulated Raman scattering driven by white light [18]. This is the basis for the discussion in the next chapters.

3 Generalized photon kinetic theory

Let us first consider the propagation of a linearly polarized electromagnetic wave (polarized along the y direction) in a plasma. The wave equation describing the evolution of the vector potential of the electromagnetic wave A_y can be written as

$$\frac{1}{c^2}\partial_t^2 A_y - \partial_x^2 A_y \simeq -\frac{\omega_{p0}^2}{c^2}\left(1+\frac{\delta n}{n_0}-\frac{1}{2}\frac{e^2A_y^2}{m^2c^4}\right)A_y \qquad (5)$$

For now we will not discuss the particular plasma response *i.e.* the dynamics of the plasma in the presence of the light wave, since this will be different depending on the particular regime of interest (coupling with electron plasma waves as in Stimulated Raman Scattering, or coupling with ion acoustic waves as in Stimulated Brillouin Scattering). Using normalized units, where length is normalized to c/ω_{p0}, with c the velocity of light in vacuum and $\omega_{p0} = (4\pi e^2 n_{e0}/m_e c^2)^{1/2}$ the electron plasma frequency, time to $1/\omega_{p0}$, mass and absolute charge to those of the electron, respectively, m_e and e, with $e > 0$, and the vector potential A_y is normalized to $m_e c^2$, and neglecting the relativistic mass correction term associated with A_y^2, Eq. (5) reduces to

$$\left(\partial_t^2 - \partial_x^2\right) a_y + (1+\delta n) a_y = 0 \qquad (6)$$

Using the standard methods [2, 24] it is not feasible to consider a broadband or partially incoherent radiation field associated with a_y, and determine the properties of the parametric instabilities from Eqs.(5,6). To achieve this it is critical to provide a statistical description of the field. This is the main goal of generalized photon kinetic theory (GPK) [17, 18].

As detailed in Ref. [18], instead of performing the calculations with respect to linear polarization we will focus our discussion in circularly polarized light, being straightforward the modification for linearly polarized radiation. We will use $\mathbf{a}_p(\mathbf{r},t) = 2^{-1/2}(\hat{z}+i\hat{y})a_0\int d\mathbf{k}A(\mathbf{k})\exp[i(\mathbf{k}.\mathbf{r}-(\mathbf{k}^2+1)^{1/2}t)]$ as the normalized vector potential of the circularly polarized pump field, $\mathbf{a}_p = e\mathbf{A}_p/m_e c^2$, where $(\mathbf{k}^2+1)^{1/2} \equiv \omega(\mathbf{k})$ is the monochromatic dispersion relation in a uniform plasma, where n_{e0} and n_{i0} are the equilibrium (zeroth order) particle densities of the electrons and ions, respectively, and the densities are normalized to the equilibrium electron density, such that $n_{e0} = 1$ and $n_{i0} = 1/Z$, where Z is the electric charge of the ions in units of e. We also allow for a stochastic component in the phase of the vector potential $A(\mathbf{k}) = \hat{A}(\mathbf{k})\exp[i\psi(\mathbf{r},t)]$ such that $\langle \mathbf{a}_p^*(\mathbf{r}+\mathbf{y}/2,t).\mathbf{a}_p(\mathbf{r}-\mathbf{y}/2)\rangle = a_0^2 m(\mathbf{y})$ is independent of $\mathbf{r}$ with $m(0) = 1$ and $|m(\mathbf{y})|$ is bounded between 0 and 1, which means that the field is spatially stationary.

Instead of using the field $\mathbf{a}$, GPK replaces the radiation field $\mathbf{a}$ by two auxiliary fields, ϕ and χ, such that

$$\phi, \chi = (\mathbf{a} \pm i\partial_t \mathbf{a})/2 \tag{7}$$

With these fields it is possible to easily demonstrate that the full wave equation (e.g. Eq.(eq:waveeq)) is formally equivalent to two coupled Schrödinger equations for the auxiliary fields [17]. This prescription is due to Feschbach and Villars [23].

With the introduction of four real phase-space densities:

$$W_0 = W_{\phi\phi} - W_{\chi\chi} \tag{8}$$

$$W_1 = 2\mathrm{Re}[W_{\phi\chi}] \tag{9}$$

$$W_2 = 2\mathrm{Im}[W_{\phi\chi}] \tag{10}$$

$$W_3 = W_{\phi\phi} + W_{\chi\chi} \tag{11}$$

with the usual definition for the Wigner transform

$$W_{\mathbf{f}.\mathbf{g}}(\mathbf{k},\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}.\mathbf{y}} \mathbf{f}^* \left(\mathbf{r}+\frac{\mathbf{y}}{2}\right).\mathbf{g}\left(\mathbf{r}-\frac{\mathbf{y}}{2}\right) d\mathbf{y} \tag{12}$$

as in Refs. [19, 20, 21, 22], the coupled equations for ϕ,χ (and, therefore, the complete Klein-Gordon equation corresponding to Eqs.(5,6)) are shown [17] to be equivalent to the following set of transport equations for the W_i, $i = 0,...,3$

$$\partial_t W_0 + \hat{\mathscr{L}}(W_2 + W_3) = 0 \tag{13}$$

$$\partial_t W_1 - \hat{\mathscr{G}}(W_2 + W_3) - 2W_2 = 0 \tag{14}$$

$$\partial_t W_2 - \hat{\mathscr{L}} W_0 + \hat{\mathscr{G}} W_1 + 2W_1 = 0 \tag{15}$$

$$\partial_t W_3 + \hat{\mathscr{L}} W_0 - \hat{\mathscr{G}} W_1 = 0 \tag{16}$$

with the following definition for the operators $\hat{\mathscr{L}}$ and $\hat{\mathscr{G}}$

$$\hat{\mathscr{L}} \equiv \mathbf{k}.\nabla_{\mathbf{r}} - n\sin\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{r}}.\overrightarrow{\nabla}_{\mathbf{k}}\right) \tag{17}$$

$$\hat{\mathscr{G}} \equiv \left(\mathbf{k}^2 - \frac{\nabla_{\mathbf{r}}^2}{4}\right) + n\cos\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{r}}.\overrightarrow{\nabla}_{\mathbf{k}}\right) \tag{18}$$

where the arrows denote the direction of the operator and the trigonometric functions represent the equivalent series expansion of the operators.

The transport equations (13,14,15,16) are formally equivalent to the full wave equation that describes the propagation of an arbitrarily intense electromagnetic wave in a plasma (e.g. Eq.(5)) and describe the evolution of the radiation field and therefore are the field equations in GPK. Even if formally more complex, it is now possible to describe arbitrary distributions of photons in the phase space $(\mathbf{r},\mathbf{k})$ and the perturbation techniques over distribution functions, common in plasma physics, can also be used over the transport equations of GPK.

For instance it is illustrative to evaluate the zeroth order terms of each W_i, $i = 0,\ldots,3$, so we use $\mathbf{a}=\mathbf{a}_p$. It can be easily shown that

$$W^{(0)}_{\phi\phi} = \frac{\rho_0(\mathbf{k})}{4}\left[1+\omega^2(\mathbf{k})+2\omega(\mathbf{k})\right] \tag{19}$$

$$W^{(0)}_{\chi\chi} = \frac{\rho_0(\mathbf{k})}{4}\left[1+\omega^2(\mathbf{k})-2\omega(\mathbf{k})\right] \tag{20}$$

$$W^{(0)}_{\phi\chi} = \frac{\rho_0(\mathbf{k})}{4}\left[1-\omega^2(\mathbf{k})\right] = -\frac{\rho_0(\mathbf{k})}{4}\mathbf{k}^2 \tag{21}$$

where $\rho_0(\mathbf{k}) \equiv W_{\mathbf{a}_p,\mathbf{a}_p}$ can be interpreted as the equilibrium distribution function of the photons. We can also immediately write

$$W^{(0)}_0 = W^{(0)}_{\phi\phi} - W^{(0)}_{\chi\chi} = \rho_0(\mathbf{k})\omega(\mathbf{k}) \tag{22}$$

$$W^{(0)}_1 = 2\mathrm{Im}\left[W^{(0)}_{\phi\chi}\right] = 0 \tag{23}$$

$$W^{(0)}_2 = 2\mathrm{Re}\left[W^{(0)}_{\phi\chi}\right] = -\frac{\rho_0(\mathbf{k})}{2}\mathbf{k}^2 \tag{24}$$

$$W^{(0)}_3 = W^{(0)}_{\phi\phi} + W^{(0)}_{\chi\chi} = \rho_0(\mathbf{k})\left(1+\frac{\mathbf{k}^2}{2}\right) \tag{25}$$

where we have taken into account the Wigner function can take only real values [19, 20, 21, 22].

The first order perturbative term of the transport equations are critical to understand coupling with the plasma density perturbation. From the first transport equation (13 we obtain, in first order,

$$\partial_t \tilde{W}_0 + \mathbf{k}.\nabla_{\mathbf{r}}(\tilde{W}_2+\tilde{W}_3) - \tilde{n}\sin\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{r}}.\overrightarrow{\nabla}_{\mathbf{k}}\right)\rho_0(\mathbf{k}) = 0 \tag{26}$$

where $\tilde{}$ describes the first order perturbed quantities. Performing time and space Fourier transforms ($\partial t \to i\omega_L, \nabla_{\mathbf{r}} \to -i\mathbf{k}_L$), leads to

$$i\omega_L \tilde{W}_0 - i\mathbf{k}.\mathbf{k}_L(\tilde{W}_2+\tilde{W}_3) + \tilde{n}\sin\left(\frac{i}{2}\mathbf{k}_L.\nabla_k\right)\rho_0(\mathbf{k}) = 0 \tag{27}$$

We note that we can write $\sin\hat{\mathscr{A}} = \frac{e^{i\hat{\mathscr{A}}}-e^{-i\hat{\mathscr{A}}}}{2i}$, for any operator $\hat{\mathscr{A}}$. Similarly, $\cos\hat{\mathscr{A}} = \frac{e^{i\hat{\mathscr{A}}}+e^{-i\hat{\mathscr{A}}}}{2}$. Making use of these relations, we have

$$e^{\mathbf{A}.\nabla_k} f(\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(\mathbf{A}.\nabla_k)^n}{n!} f(\mathbf{k}) = f(\mathbf{k}+\mathbf{A}) \tag{28}$$

The first transport equation can then be reduced to

$$\omega_L \tilde{W}_0 - \mathbf{k}.\mathbf{k}_L(\tilde{W}_2+\tilde{W}_3) - \tilde{n}\frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)-\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{2} = 0 \tag{29}$$

We proceed analogously with the other three transport equations, leading to a system of four independent first order equations for the four variables $\tilde{W}_i$. We also note that

$$W_2 + W_3 = W_{\phi\phi} + W_{\chi\chi} + 2\mathrm{Re}[W_{\phi\chi}] = W_{\mathbf{a}.\mathbf{a}} \tag{30}$$

In zeroth order, as expected,

$$W_2^{(0)} + W_3^{(0)} = W_{\mathbf{a}_p.\mathbf{a}_p} = \rho_0(\mathbf{k}) \tag{31}$$

while in first order

$$\tilde{W}_2 + \tilde{W}_3 = W_{\mathbf{a}_p.\tilde{\mathbf{a}}} + W_{\tilde{\mathbf{a}}.\mathbf{a}_p} = 2W_{\mathbf{a}_p.\tilde{\mathbf{a}}} \tag{32}$$

where we have used the simmetry property of the Wigner distribution function that can be immediately derived from its realness ($W_{\mathbf{f}.\mathbf{g}} = W_{\mathbf{g}.\mathbf{f}}$).

Since the plasma response is proportional to the beating of the pump wave $\mathbf{a}_p$ and the scattered wave $\tilde{\mathbf{a}}$ (and the real part of this beating) it is important to obtain an equation for $W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]}$. Taking the real part of Eq.(32) and solving this equation together with the four independent equations for each $\tilde{W}_i$ yields, after some lengthy but straightforward calculations,

$$W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]} = \frac{1}{2}\tilde{n}\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right] \tag{33}$$

with

$$\frac{1}{D_s^{\mp}} = \frac{1 \pm \frac{2\mathbf{k}.\mathbf{k}_L}{\omega_L^2} \pm \frac{2\omega\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{\omega_L}}{\omega_L^2 - 4\mathbf{k}_L^2 - \mathbf{k}_L^2 + 4\frac{(\mathbf{k}.\mathbf{k}_L)^2}{\omega_L^2} - 4} \tag{34}$$

The expression for $D_s^{\mp}$ can simplified to

$$D_s^{\pm} = \frac{(\omega_L^2 \mp 2\mathbf{k}\mathbf{k}_L)^2 - \left[2\omega_L \omega \left(\mathbf{k} \mp \frac{\mathbf{k}_L}{2}\right)\right]^2}{\omega_L^2 \mp 2\mathbf{k}.\mathbf{k}_L \mp 2\omega_L \omega \left(\mathbf{k} + \frac{\mathbf{k}_L}{2}\right)}, \tag{35}$$

providing the driving term of the parametric instability as

$$W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]} = \frac{1}{2}\tilde{n}\left[\frac{\rho_0\left(\mathbf{k} + \frac{\mathbf{k}_L}{2}\right)}{D^-} + \frac{\rho_0\left(\mathbf{k} - \frac{\mathbf{k}_L}{2}\right)}{D^+}\right], \tag{36}$$

where $D^{\pm} = \omega_L^2 \mp \left[\mathbf{k}.\mathbf{k}_L - \omega_L \omega \left(\mathbf{k} \mp \frac{\mathbf{k}_L}{2}\right)\right]$ and ω_L ($\mathbf{k}_L$) represents the instability frequency (wave vector).

As can be observed from Eq.(36), an arbitrary distribution function of photons ρ_0 can be considered for the pump field, thus allowing for the inclusion of a broadband or a partially coherent radiation pump field. Eq. (36) connects the propagation of the pump and the scattered fields with the plasma response (associated with $\tilde{n}$).

4 Derivation of the dispersion relation for Stimulated Brillouin Scacttering

We now consider the coupling of the intense radiation field with the plasma, when the radiation field is described by the transport equations derived in the previous section. In our plan to describe parametric instabilities, we should now analyze the plasma response to the presence of the pump and scattered fields. For illustration purposes, we will analyze coupling with ion acoustic waves *i.e.* Stimulated Brillouin Scattering) [28, 29]. We consider the plasma as an interpenetrating fluid of both electrons and ions, with n_{e0} and n_{i0} their equilibrium (zeroth order) particle densities, respectively.

To obtain a dispersion relation for SBS we must couple the typical plasma response to the independently derived driving term, obtained within the GPK framework, and given by Eq.(drivingterm).

Combining the continuity and force equations for each species and closing the system with an isothermal equation of state, we can readily obtain the plasma response to the propagation of a light wave $\mathbf{a}_p$, beating with its scattered component $\tilde{\mathbf{a}}$, to produce the ponderomotive force of the laser [2]

$$\left(\frac{\partial^2}{\partial t^2} - 2\tilde{\nu}\partial t - c_S^2 \nabla^2\right)\tilde{n} = \frac{Z}{M}\nabla^2 \mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}], \tag{37}$$

with $c_S \equiv \sqrt{\frac{Z\theta_e}{M}}$ being the ion-sound velocity, M the mass of the ions, θ_e the electron temperature and $\tilde{\nu}$ an integral operator whose Fourier transform is $\nu|\mathbf{k}_L|c_S$, accounting for the damping of the ion acoustic waves (e.g. via Landau damping).

Performing time and space Fourier transforms ($\partial t \to i\omega_L, \nabla_{\mathbf{r}} \to -i\mathbf{k}_L$) on the plasma response (37) gives

$$\mathscr{F}[\tilde{n}] = \frac{Z}{M} \frac{k_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2} \mathscr{F}\left[\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]\right], \tag{38}$$

and the same operations on the driving term Eq.(36) yield

$$\mathscr{F}\left[W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]}\right] = \frac{1}{2}\mathscr{F}[\tilde{n}] \left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D^+}\right], \tag{39}$$

with $D^{\pm} = \omega_L^2 \mp \left[\mathbf{k}.\mathbf{k}_L - \omega_L\omega\left(\mathbf{k} \mp \frac{\mathbf{k}_L}{2}\right)\right]$, as before, and $c_S \equiv \sqrt{\frac{Z\theta_e}{M}}$.

The general dispersion relation can now be obtained using one of the properties of the Wigner function [19, 20, 21, 22]

$$\int W_{f.g} d\mathbf{k} = f^* g \Rightarrow \int \frac{W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]}}{\mathrm{Re}\left[\mathbf{a}_p.\tilde{\mathbf{a}}\right]} d\mathbf{k} = 1 \tag{40}$$

as

$$1 = \frac{\omega_{pi}^2}{2} \frac{\mathbf{k}_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2} \int \left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D^+}\right] d\mathbf{k}, \tag{41}$$

with $\omega_{pi} = \sqrt{Z/M}$ the ion plasma frequency in normalized units.

By making an appropriate change of variables, our general dispersion relation can then be written in a more compact way

$$1 = \frac{\omega_{pi}^2}{2} \frac{\mathbf{k}_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2} \int \rho_0(\mathbf{k}) \left(\frac{1}{D^+} + \frac{1}{D^-}\right) d\mathbf{k}, \tag{42}$$

with $D^{\pm}(\mathbf{k}) = [\omega(\mathbf{k}) \pm \omega_L]^2 - (\mathbf{k} \pm \mathbf{k}_L)^2 - 1$.

We first apply our general dispersion relation (42) to the simple and common case of a pump plane wave of wave vector $\mathbf{k}_0$, which means that $\rho_0(\mathbf{k}) = a_0^2\delta(\mathbf{k}-\mathbf{k}_0)$, and we drop the Landau damping contribution. The dispersion relation then becomes

$$1 = \frac{\omega_{pi}^2}{2} \frac{\mathbf{k}_L^2}{\omega_L^2 - c_S^2\mathbf{k}_L^2} a_0^2 \left\{ \frac{1}{[\omega(\mathbf{k}_0) + \omega_L]^2 - (\mathbf{k}_0 + \mathbf{k}_L)^2 - 1} + \right. \tag{43}$$

$$\left. + \frac{1}{[\omega(\mathbf{k}_0) - \omega_L]^2 - (\mathbf{k}_0 - \mathbf{k}_L)^2 - 1} \right\}. \tag{44}$$

This result recovers that of Ref. [2], which studies the case of a pump wave $\mathbf{A}_L = \mathbf{A}_{L0}\cos(\mathbf{k}_0.\mathbf{r} - \omega_0 t)$, if we account for the difference in polarization and use $\omega_0 =$

$\omega(\mathbf{k}_0)$. All the conclusions derived in Ref. [2], based on this dispersion relation, are then consistent with the predictions of GPK.

A more in depth analysis of the dispersion relation is beyond the scope of this chapter. The interested reader can find additional details on the consequences of Eq.(42)) in [28, 29]. We observe that to obtain the dispersion relation for Stimulated Raman Scattering [18] it would suffice to replace the equation for the plasma response (37) that in the case discussed here describes coupling with the (low frequency) ion acoustic waves, with the plasma response corresponding to the (high frequency) electron plasma waves viz.

$$\left(\partial_t^2+\frac{\omega_{p0}^2}{\gamma_0}\right)\tilde{n}=\frac{n_0c^2}{\gamma_0^2}\nabla_{\mathbf{r}}{}^2\mathrm{Re}\left[\mathbf{a}_p\cdot\tilde{\mathbf{a}}^*\right] \tag{45}$$

A summary of the results obtained with generalized photon kinetic theory for Stimulated Raman Scattering, Stimulated Brillouin Scattering, and the (relativistic) modulation instability is presented in the next section.

5 Main results obtained with generalized photon kinetic theory

Generalized photon kinetic theory has been applied to the study of several parametric instabilities, in particular SRS [18], SBS [28, 29], and the modulational instability [30].

For the monochromatic pump field described in the previous section $\rho_0(\mathbf{k})=a_0^2\delta(\mathbf{k}-\mathbf{k}_0)$ the standard results were recovered for all these instabilities, as expected, but the introduction of broadband effects have demonstrated novel features. The main theoretical results have been obtained for a waterbag distribution function of photons, such that analytical results could be obtained. The key parameters in this distribution are the width of the distribution Δ, and the central wavenumber of the photon distribution function $\mathbf{k}_0$.

The analysis of the effects of Stimulated Raman Scattering as shown two important results [18]. First of all, and for Raman backscattering, the dependence of the growth rate on the bandwidth was found to scale as $\propto 1/\Delta$ as previously works have demonstrated and as expected from a three-wave resonant process. On the other hand, for Raman Forward Scattering the growth rate scales as $\propto 1/\sqrt{\Delta}$, thus decaying much slower than what was expected and predicted in previous works. This is associated with the fact that Raman Forward Scattering is, in general, a four-wave nonresonant process, which means that it is less sensitive to broadband effects. This slow decay with Δ also indicates that Raman Forward Scattering will be significantly more difficult to shutdown by increasing the bandwidth of the pump laser field, unlike what happens in

The role of the bandwidth of the pump radiation is Stimulated Brillouin Scattering bears some resemblance with Stimulated Raman Bacscattering since it is also a three-wave resonant process, showing a growth rate dependence with the bandwidth scaling as $\propto 1/\Delta$. The general dispersion relation for Stimulated Brillouin Scattering in Eq.(42) has been compared with other models for the effects of the bandwidth on the instability and the analysis in refs. [28, 29] has revealed that the range of validity of GPK is significantly larger than previous models. This work has relied not only on the analysis of the more academic waterbag distribution function but also on the exploration of realistic broadband distribution functions such as those relevant for ICF experiments.

The dispersion relation for the the modulational instability can be solved for a Lorentian distribution function of the transverse wavenumbers k_z, with a width Δ ($f(k_z) = \frac{\Delta}{\pi}\frac{1}{k_z^2+\Delta^2}$) [30]. It was found that the range of unstable wavenumbers has an upper bound at

$$k_{\max} = \frac{\omega_{p0}}{c}\sqrt{2\frac{a_0^2}{\gamma_0^3} - 4\Delta^2} \tag{46}$$

which corresponds to an instability threshold given by

$$\frac{a_0^2}{\Delta^2} > 2\gamma_0^3 \tag{47}$$

This threshold presents a similar dependence as the threshold for filamentation/self-focusing of a Gaussian laser pulse ($a_0^2 W_0^2 \omega_{p0}^2/c^2 > 32$), which is even more clear if we consider that the spread of the transverse wavenumbers of a Gaussian laser pulse is $\Delta \approx 1/W_0$ where W_0 is the laser spot size. Moreover, the presence of broadband radiation can shutdown the modulational instability even at relativistic intensities in the long wavelength limit.

6 Summary

In this chapter we have presented the fundamentals aspects of generalized photon kinetic theory. Particular emphasis was given to the material that is not generally present in the litterature and that will allow the reader to use this technique in a broader range of physical conditions. The focus of GPK has been on the parametric instabilities driven by intense lasers in plasmas but the GPK toolbox can be easily used in other nonlinear systems, with the most impact in scenarios where the backscattered radiation/waves play an important role on the overall dynamics of the parametric instabilities. Further generalizations of GPK should address the the coupling of the different parametric instabilities, and the spatio-temporal theory of the parametric instabilities driven by broadband or partially coherent radiation.

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