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A Complete Leading-Order, Renormalization-Scheme-Consistent Calculation of Small-x Structure Functions, Including Leading-$\ln(1/x)$ Terms

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A Complete Leading–Order, Renormalization–Scheme–Consistent Calculation of Small–x Structure Functions, Including Leading–$\ln(1/x)$ Terms.

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Abstract

We present calculations of the structure functions $\mathcal{F}_2(x,Q^2)$ and $\mathcal{F}_L(x,Q^2)$, concentrating on small x. After discussing the standard expansion of the structure functions in powers of $\alpha_s(Q^2)$ we consider a leading–order expansion in $\ln(1/x)$ and finally an expansion which is leading order in both $\ln(1/x)$ and $\alpha_s(Q^2)$, and which we argue is the only really correct expansion scheme. Ordering the calculation in a renormalization–scheme–consistent manner, there is no factorization scheme dependence, as there should not be in calculations of physical quantities. The calculational method naturally leads to the "physical anomalous dimensions" of Catani, but imposes stronger constraints than just the use of these effective anomalous dimensions. In particular, a relationship between the small–x forms of the inputs $\mathcal{F}_2(x,Q_0^2)$ and $\mathcal{F}_L(x,Q_0^2)$ is predicted. Analysis of a wide range of data for $\mathcal{F}_2(x,Q^2)$ is performed, and a very good global fit obtained, particularly for data at small x. The fit allows a prediction for $\mathcal{F}_L(x,Q^2)$ to be produced, which is smaller than those produced by the usual NLO–in–$\alpha_s(Q^2)$ fits to $\mathcal{F}_2(x,Q^2)$ and different in shape.

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1. Introduction.

The recent measurements of $\mathcal{F}_2(x,Q^2)$ at HERA have provided data on a structure function at far lower values of x than any previous experiments, and show that there is a marked rise in $\mathcal{F}_2(x,Q^2)$ at very small x down to rather low values of Q^2 [1][2]. Indeed, the most recent measurements have demonstrated that the rise persists for values of Q^2 as low as 1.5 GeV^2.

The qualitative result of a steep rise at small x initially led to surprise in many quarters. The main reason for this was that standard methods used to fit the data were based on the solution of the Altarelli–Parisi evolution equation [3] (along with convolution with coefficient functions) at the two–loop level, using flat input parton distributions at starting scales of $Q_0^2 \sim 4\text{GeV}^2$ (e.g. [4]). This method followed the reasoning that steep behaviour can only come about from perturbative physics (the Donnachie–Landshoff pomeron used to describe soft physics has behaviour $x^{-0.08}$ [5], and we will take steep to mean any powerlike behaviour steeper than this), and that a starting scale for perturbative evolution should be high enough for $\alpha_s(Q^2)$ to be fairly small ($\lesssim 0.3$) and to avoid any significant corrections from "higher twist" (Λ^2_{QCD}/Q^2) corrections. This procedure results in an effectively steep[1] behaviour at small x [6], but only after a long evolution length, and therefore at values of $Q^2 \gg 4\text{GeV}^2$.

Thus, the data led to a degree of optimism amongst those advocating an alternative description of small–x structure functions, i.e. using the BFKL equation [7]. This equation provides the unintegrated gluon Green's function which includes the leading power of $\ln(1/x)$ for any power of α_s (where α_s is taken to be fixed). It was traditionally solved analytically in the asymptotic limit $x \to 0$, or numerically for finite x, and predicted a powerlike behaviour of $x^{-1-\lambda}$ for the gluon distribution function, where $\lambda = 4\ln 2\bar{\alpha}_s$ and $\bar{\alpha}_s = (3/\pi)\alpha_s$, i.e. $\lambda \sim 0.5$ if $\alpha_s \sim 0.2$. This was assumed to lead to $\mathcal{F}_2(x,Q^2)$ behaving like $x^{-\lambda}$ and could be claimed to be in rough qualitative agreement with the data, even if λ was somewhat high. It could also be seen as some justification for choosing powerlike inputs (with $\lambda \sim 0.2 - 0.3$) for the parton distributions (e.g. [8]), which could then enable a good fit to the data using the Altarelli–Parisi equation.

However, it was also convincingly demonstrated that it was possible to generate the observed steep behaviour by being a little less conservative concerning the region in which perturbative evolution could be applied. Glück, Reya and Vogt had in fact predicted a sharp rise in $F_2(x,Q^2)$ at small x, even for $Q^2 \sim 1\text{GeV}^2$, by using two–loop evolution from roughly valence–like parton distributions at a starting scale of $Q_0^2 = 0.34\text{GeV}^2$ [9]. This attracted criticism not only on the grounds that higher twist corrections should be important in the region of evolution, but also because $\alpha_s(Q^2)$ was as high as ~ 0.5 at the lower end of the range, and therefore perturbation

[1] We use the term effectively steep since the slope at any x and Q^2 is a function of these variables rather than a constant, i.e. $\mathcal{F}_2(x,Q^2) \sim x^{-\lambda}$ where $\lambda \sim 0.7\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)^{\frac{1}{2}} \ln^{-\frac{1}{2}}(0.1/x)$, and is smaller than any power of x as $x \to 0$.

theory itself should be questionable. Starting at a higher scale, $Q_0^2 = 1\text{GeV}^2$, Ball and Forte were able to fit the small–x data using their double asymptotic scaling (DAS) formula [10], which is a simple, but very accurate, approximation to the solution of the one–loop evolution equation with flat[2] inputs and which is valid in the region of small x ($x \lesssim 0.01$). They also showed that two–loop evolution with flat inputs starting at $Q_0^2 \approx 2\text{GeV}^2$ could fit the data available at that time very well [11], and had a similar shape to the DAS result (as indeed it must, since both fit the data).

This state of affairs clearly left scope for argument about the real underlying physics describing the small–x behaviour of $\mathcal{F}_2(x, Q^2)$. Those who used Altarelli–Parisi evolution from small scales could be accused firstly of working in regions where perturbation theory was questionable and, perhaps more importantly, of ignoring terms of higher order in α_s (but also higher order in $\ln(1/x)$), which seemed from the BFKL equation to have very important effects. Conversely, those who used the BFKL approach could be accused of ignoring all but the leading-$\ln(1/x)$ terms (and hence ignoring the large–x data) and also of working in a less well–defined theoretical framework than the renormalization group approach based on the factorization of collinear singularities [12]. Starting from an input for the parton distributions with $\lambda \sim 0.25$ at values of $Q_0^2 \sim 4\text{GeV}^2$ was taking the best of both worlds. However, this lacked a real justification for the choice of input, which was significantly steeper than that expected from non–perturbative physics, but rather smaller than that from the BFKL equation, and also ignored potentially important $\ln(1/x)$ terms in the evolution.

A significant step forward in the investigation of small–x structure functions was the development of the k_T–factorization theorem [13][14]. This is the prescription for the way in which an off–shell photon–gluon scattering amplitude can be convoluted with the unintegrated gluon Green's function calculated using the BFKL equation to provide the small–x structure functions themselves (once convoluted with a bare, off–shell gluon density). Hence, it enables one to find effective moment–space coefficient functions (or in the case of $\mathcal{F}_2(x, Q^2)$ and massless quarks, where $d\mathcal{F}_2(x, Q^2)/d\ln Q^2$ is calculated, a mixture of coefficient function and anomalous dimensions) within the BFKL framework. Numerical calculations performed using this method were able to match the available data in a qualitative manner [15], as did similar calculations [16] using a modification of the BFKL equation, i.e. the CCFM equation [17].

However, the k_T–factorization formula was also shown to be very important if one insisted on working within the rigorous framework of the traditional renormalization group approach [18]. In this approach the infrared poles in the calculated coefficient functions, or photon–parton scattering amplitudes, are removed order by order in α_s and absorbed into the essentially nonperturbative parton distribution functions. In order to guarantee the independence of physical quantities on the

[2] Parton inputs behaving like x^{-1}, as opposed to $x^{-1-\lambda}$, are referred to as flat, i.e. the parton density rescaled by x is flat.

factorization scale μ_F, the latter then evolve in $\ln(\mu_F^2)$ according to the renormalization group equations governed by calculable perturbative anomalous dimensions. By showing how k_T–factorization fits within the collinear factorization framework, Catani and Hautmann were able to calculate all the renormalization group anomalous dimensions to lowest nontrivial order in α_s for each power of $\ln(1/x)$, and similarly for a number of coefficient functions.

In order to explain this properly we digress for a moment. We may write the Altarelli–Parisi splitting functions as

$$P(x,\alpha_s(Q^2)) = \frac{1}{x}\sum_{n=1}^{\infty}\alpha_s^n(Q^2)\Big(\sum_{m=0}^{n-1}a_{nm}\frac{\ln^m(1/x)}{m!} + \text{terms less singular as } x\to 0\Big), \tag{1.1}$$

or taking the Mellin transform of the splitting function (weighted by x)

$$\gamma(N,\alpha_s(Q^2)) = \int_0^1 x^N P(x,\alpha_s(Q^2))dx, \tag{1.2}$$

and working in moment space,

$$\gamma(N,\alpha_s(Q^2)) = \sum_{n=1}^{\infty}\alpha_s^n(Q^2)\Big(\sum_{m=1}^{n}a_{nm}N^{-m} + a_n(N)\Big), \tag{1.3}$$

where $A(N)$ is regular as $N\to 0$. In the normal loop expansion one solves the evolution equations order by order in α_s. We may however write

$$P(x,\alpha_s(Q^2)) = \frac{1}{x}\sum_{n=0}^{\infty}\alpha_s^n(Q^2)\Big(\sum_{m=1}^{\infty}a_{nm}\alpha_s^m(Q^2)\frac{\ln^{m-1}(1/x)}{m!}\Big) + \text{terms less singular as } x\to 0, \tag{1.4}$$

or in moment space,

$$\gamma(N,\alpha_s(Q^2)) = \sum_{n=0}^{\infty}\alpha_s^n(Q^2)\sum_{m=1}^{\infty}\tilde{a}_{nm}\alpha_s^m(Q^2)N^{-m} + \sum_{n=1}^{\infty}\alpha_s^n(Q^2)a_n(N), \tag{1.5}$$

where the $a_n(N)$ are regular as $N\to 0$, but have singularities at negative integer values of N. When one calculates the structure function by converting back to x space (i.e. performing the inverse Mellin transform), these singularities lead to contributions suppressed by powers of x, and so are negligible at small x. Therefore, ignoring these contributions and expanding the $a_n(N)$ about $N=0$ we get

$$\gamma(N,\alpha_s(Q^2)) = \sum_{n=0}^{\infty}\alpha_s^n(Q^2)\sum_{m=1-n}^{\infty}\tilde{a}_{nm}\alpha_s^m(Q^2)N^{-m} \equiv \sum_{n=0}^{\infty}\alpha_s^n(Q^2)\gamma^n(\alpha_s(Q^2)/N), \tag{1.6}$$

where the expressions are strictly convergent only for $|N|<1$. Thus, in order to include the leading powers in $\ln(1/x)$ for the splitting function, we have to take the $n=0$ part of the expression (1.6) for the anomalous dimension. Next to leading order (NLO) in $\ln(1/x)$ is the $n=1$ part, and so on.

Using the k_T–factorization formula Catani and Hautmann demonstrated that within the renormalization group framework $\gamma^0_{gg}(N,Q^2)$ and $\gamma^0_{gf}(N,Q^2)$ were the same renormalization–scheme–independent expressions as the effective anomalous dimensions [19] given by the BFKL equation. i.e.

$$\gamma^0_{gg}(\alpha_s(Q^2)/N) = \sum_{m=1}^{\infty} a_{0,m}\left(\frac{\bar{\alpha}_s(Q^2)}{N}\right)^m, \qquad \gamma^0_{gf}(\alpha_s(Q^2)/N) = \frac{4}{9}\gamma^0_{gg}(\alpha_s(Q^2)/N), \quad (1.7)$$

where $\gamma^0_{gg}(\alpha_s(Q^2)/N)$ is given by the iterative solution of

$$1 = \frac{\bar{\alpha}_s(Q^2)}{N}\chi(\gamma^0_{gg}), \quad (1.8)$$

and,

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma). \quad (1.9)$$

This solution as a power series in $\bar{\alpha}_s(Q^2)/N$ exists only for $|N| \geq \lambda(\bar{\alpha}_s(Q^2))$. The anomalous dimension develops a branch point at $N = \lambda(\bar{\alpha}_s(Q^2))$, but the series in (1.7) is analytic and convergent outside the circle $|N| = \lambda(Q^2)$. In fact, this anomalous dimension leads to the gluon distribution function having an asymptotic powerlike behaviour of $x^{-\lambda(Q_0^2)}$, so it does not make sense to talk about its Mellin transform for $\Re eN \leq \lambda(Q_0^2)$, it simply does not exist in this region of moment space. However, the inverse Mellin transform may be performed by analytic continuation into the region $\Re eN < \lambda(Q_0^2)$ (but $|N| \geq \lambda(Q_0^2)$) from $\Re eN \geq \lambda(Q_0^2)$, if so desired.

Catani and Hautmann also derived expressions for $\gamma^1_{ff}(\alpha_s(Q^2)/N)$ and $\gamma^1_{fg}(\alpha_s(Q^2)/N)$ in certain factorization schemes ($\gamma^0_{ff}(\alpha_s(Q^2)/N)$ and $\gamma^0_{fg}(\alpha_s(Q^2)/N)$ being zero), and also for the coefficient functions $C^g_{L,1}(\alpha_s(Q^2)/N)$, $C^f_{L,1}(\alpha_s(Q^2)/N)$, $C^g_{2,1}(\alpha_s(Q^2)/N)$ and $C^f_{2,1}(\alpha_s(Q^2)/N)$ (all zeroth–order quantities being zero except $C^f_{2,0}$, which is unity). This facilitated calculations of structure functions within the normal renormalization group framework, but including much of what is often called the BFKL physics (i.e. the leading–$\ln(1/x)$ terms), and indeed, a number of calculations were performed [20]–[23], and in most cases comparisons with data made. These calculations used different methods of solution, made rather different assumptions and used different ansätze for unknown terms. Consequently different results were obtained. The conclusions which could be drawn regarding the inclusion of the leading–$\ln(1/x)$ terms depended very much on which of the approaches was taken. However, it seemed that by including these terms it was not possible to improve upon the best fits for the small–x data using one– or two–loop evolution from soft inputs [20][22]. Indeed, many ways of including them made the fits significantly worse, and this seemed to be universally true if the fits were more global, i.e. constrained by large–x data [20]. Also, it seemed that there was a very strong dependence on the factorization scheme used to perform the calculations when including the leading–$\ln(1/x)$ terms [20][22][24][25], and a number of new factorization schemes were invented, e.g. the SDIS scheme [24] and the Q_0 scheme [25].

The high precision of the most recent HERA data constrains theory far more than previously, and has changed the above picture somewhat. The best recent global fits seem to come from those intermediate approaches which use NLO perturbation theory with a quite steep (and completely unexplained) input for the singlet quark with $\lambda \sim 0.2$ and a similar form of small–x input for the gluon [26] (unless Q_0^2 is less than $\sim 4\text{GeV}^2$, in which case the gluon must be flatter or even valence–like). Fixed order perturbation theory using flat or valence–like inputs and low Q_0^2 fails at the lowest x values, and for fits to the small–x data alone relatively steep inputs for the singlet quark, i.e. $\lambda \gtrsim 0.2$, seem to be absolutely necessary [27]. Moreover, approaches including the leading–$\ln(1/x)$ terms now seem to fail [28][29] in practically all factorization schemes.

In this paper we will take issue with the above conclusions. In particular we will demonstrate that the apparent failure of approaches using the leading–$\ln(1/x)$ terms, and certainly the factorization scheme dependence, is due to incorrect methods of incorporating these terms. Indeed, Catani has already shown how to obtain factorization–scheme–invariant results in the small–x expansion by writing evolution equations in terms of the physical quantities, the structure functions and "physical anomalous dimensions", rather than in terms of parton densities and of the usual anomalous dimensions [30]. In this paper we will go further and show that the correct leading–order, renormalization–scheme–consistent (RSC) calculation of the structure functions must naturally include some leading–$\ln(1/x)$ terms in the form of these "physical anomalous dimensions" . It also provides limited predictive power at small x, giving justification for the input $\mathcal{F}_2(x, Q_0^2)$. We will discuss this method of calculation, then make detailed comparisons to data, and demonstrate with the aid of the new HERA data that it leads to a very good global fit to all $\mathcal{F}_2(x, Q^2)$ data. Indeed, the complete RSC calculation, including leading–$\ln(1/x)$ terms, is clearly preferred by the latest data, particularly that at small x.

We note that a very brief presentation of the complete RSC calculation of structure functions has already appeared in [31].[3] However, this current paper gives a complete, and very detailed discussion of the correct calculation of structure functions, as well as examples of commonly encountered pitfalls, whereas there is only the barest outline of the full calculation in [31]. Moreover, there is a far more comprehensive presentation of the comparison with experimental data, and with alternative approaches, in the current paper.

This paper will be structured as follows: after giving a brief outline of the different possible types of scheme dependence in the calculation of structure functions, we also quickly review the work of Catani, illustrating that it is indeed very easy to guarantee factorization–scheme–invariant results. We then give a very detailed description of the calculation, within moment space, of structure functions in various expansion schemes using the normal parton language. The first

[3] As in this shorter presentation, the present paper deals with the heavy quark thresholds in a rather naive manner, i.e. the quarks are taken to be massless, with a particular flavour becoming active only above a certain Q^2.

part of this, regarding the normal loop expansion, will be largely a review, but will highlight points usually not discussed in detail, particularly the role played by the scale at which the parton distributions are input and the form of the inputs. The second part, discussing the leading-$\ln(1/x)$ expansion, will present the only correct way to perform this expansion. One finds that by remedying the factorization scheme dependence one is forced towards an alternative derivation of Catani's factorization-scheme-independent anomalous dimensions. However, we will also show that, in order to make an ordered calculation of quantities, more care is needed than simply working with a factorization-scheme-independent set of variables, and demonstrate that there is a certain degree of predictive power for the form of the inputs for the structure functions at small x. We also explain why the standard solutions using the small-x expansions are strongly factorization scheme dependent. To conclude this section we present the argument that there is a unique renormalization-scheme-consistent calculation of structure functions, which applies to both large and small x. We present this calculation for both the currently academic case of the nonsinglet structure functions and for the phenomenologically important case of the singlet structure functions. We then discuss how we move from moment space and obtain our x-space solutions, and the qualitative form these solutions must take, i.e. our best attempt at predictions.. After this long theoretical presentation we consider the comparison with experiment. We fit the data for $\mathcal{F}_2(x,Q^2)$ using the renormalization-scheme-consistent solutions, and compare to global fits at NLO using the normal loop expansion. We also comment on alternative fits to the data made by other groups, and on determinations of α_s. We conclude that the full renormalization-scheme-consistent calculation gives the best global fit to structure function data, particularly at small x. Finally we investigate the phenomenological consequences for $\mathcal{F}_L(x,Q^2)$, and also preliminary indications for the charm structure function. Good measurements of either (but preferably both) of these quantities, particularly at small x, would help determine whether the approach developed in this paper, and hence the inclusion of leading-$\ln(1/x)$ terms in the calculation of structure functions, is indeed correct.

2. Scheme and Scale Choices.

For simplicity we work in moment-space for much of this paper, i.e. define the moment-space structure functions by the Mellin transform, i.e.

$$F(N,Q^2) = \int_0^1 x^{N-1}\mathcal{F}(x,Q^2)dx. \tag{2.1}$$

The moment-space coefficient function is defined similarly but, as with the definition of the anomalous dimension, we define the moment space expression for the parton distribution as the Mellin transform of a rescaled parton density i.e

$$f(N,Q^2) = \int_0^1 x^N \mathrm{f}(x,Q^2)dx. \tag{2.2}$$

Let us consider the most general moment–space expression for a structure function, i.e. the sum of the products of the expressions for hard scattering with a certain parton (the coefficient functions) with the corresponding, intrinsically nonperturbative parton distributions.

$$F(N,Q^2) = \sum_a C^a(N,\alpha_s(\mu_R^2),Q^2/\mu_F^2,\mu_R^2/\mu_F^2) f_a(N,\alpha_s(\mu_R^2),\mu_R^2/\mu_F^2), \tag{2.3}$$

μ_F is a factorization scale separating the ultraviolet physics from the soft infrared physics, and on which the left–hand side of (2.3) is independent. μ_R is the renormalization scale, i.e. the appropriate scale at which to define the coupling constant, and on which $F(N,Q^2)$ is again independent.

Despite being intrinsically nonperturbative, the parton distributions evolve according to the perturbative renormalization group equation

$$\frac{d\, f_a(\alpha_s(\mu_R^2),\mu_R^2/\mu_F^2)}{d\ln\mu_F^2} = \sum_b \gamma_{ab}(\alpha_s(\mu_R^2),\mu_R^2/\mu_F^2) f_b(\alpha_s(\mu_R^2),\mu_R^2/\mu_F^2). \tag{2.4}$$

In principle there are many choices to be made when performing a perturbative calculation of structure functions to a finite order. However, two of the choices left open in the above expressions are common to all perturbative calculations in quantum field theory: the choice of renormalization scheme and subsequently the choice of renormalization scale. When one removes ultraviolet divergences from perturbative calculations, there is always a freedom in what type of regularization is used and/or how much of the finite part of a calculation is removed at the same time as the divergent parts. Although the all orders calculation of a physical quantity is independent of the convention, the renormalization scheme used, the perturbative expansion of the quantity is not. Hence, neither is the definition of the expansion parameter, the running coupling constant. All couplings in quantum field theory satisfy a renormalization group equation,

$$\frac{d\,\alpha(\mu^2)}{d\ln(\mu^2)} = -\sum_{n=0}^{\infty} b_n \alpha^{n+2}(\mu^2) \equiv -\beta(\alpha(\mu^2)), \tag{2.5}$$

where μ is the scale at which the coupling is defined, and the convention of the minus sign is introduced in order to make the β–function for QCD positive. The solution to this equation depends on the coefficients b_n and on a scale Λ. This scale, and the value of the coefficients beyond $n = 1$ are renormalization scheme dependent ($b_0 = (11 - 2N_f/3)/4\pi$ and $b_1 = (102 - 38N_f/3)/16\pi^2$ in all renormalization schemes which use dimensional regularization, where N_f is the number of quark flavours). Hence, a choice of renormalization scheme amounts to a choice of which expansion parameter is to be used, and consequently the form of the perturbative expansion. One would hope the choice would be such as to make the series converge as quickly as possible, but this is difficult to guarantee. Conventionally the $\overline{\rm MS}$ renormalization scheme is used. Also the choice of the appropriate scale to be used in the coupling for a given process must be made. Sometimes this is relatively clear, but in deep inelastic scattering it is not entirely obvious. Traditionally, the

simple choice $\mu_R = \mu_F$ is taken, and indeed, the data seem to favour this choice [32]. Hence, we will assume this to be the case from now on.

Having settled for a particular choice of renormalization scheme and renormalization scale the ambiguities due to ultraviolet regularization have been dealt with. Our fundamental equations become

$$F(N,Q^2) = \sum_a C^a(N,\alpha_s(\mu_F^2),Q^2/\mu_F^2) f_a(N,\mu_F^2), \tag{2.6}$$

and

$$\frac{d\, f_a(\mu_F^2)}{d\ln\mu_F^2} = \sum_b \gamma_{ab}(\alpha_s(\mu_F^2)) f_b(\mu_F^2), \tag{2.7}$$

where α_s is defined in a particular renormalization scheme. The remaining ambiguities are due to the particular problems in calculating quantities in QCD, i.e separating the physical quantity into the perturbative coefficient function and the intrinsically nonperturbative parton distribution. To begin with we have the freedom of choosing the factorization scale μ_F. As with renormalization scheme dependence this does not affect the all–orders calculation, but does affect the form of the perturbative expansion, since it affects the scale at which the coupling in the coefficient function is evaluated. One might imagine that it is desirable to choose μ_F^2 to be large in order to make the expansion parameter $\alpha_s(\mu_F^2)$ as small as possible, and indeed μ_F^2 is nearly always chosen to be equal to the hard scattering scale Q^2. We shall also make this simple choice.

This leaves us with our defining equations[4]

$$F(N,Q^2) = \sum_a C^a(N,\alpha_s(Q^2)) f_a(N,Q^2), \tag{2.8}$$

and

$$\frac{d\, f_a(Q^2)}{d\ln Q^2} = \sum_b \gamma_{ab}(\alpha_s(Q^2)) f_b(Q^2). \tag{2.9}$$

In principle there are now choices to be made for the scale at which to begin the evolution of the parton densities, Q_0^2, and for the inputs for the partons at this scale. This question will be dealt with in detail later in this paper, so we will leave it for the moment. This still leaves us one more freedom in our calculation, i.e. how we choose to remove the infrared divergences from the bare coefficient functions and hence how we define our parton distributions. Starting from any particular choice for the definition of parton distributions one may always choose a new set of parton distributions by an invertible transformation

$$\breve{f}_a(N,Q^2) = \sum_b U_{ab}(N,\alpha_s) f_b(N,Q^2), \tag{2.10}$$

[4] Because we consider only massless quarks, once we set $Q^2 = \mu_F^2$ the only scale dependence in the coefficient functions comes from $\alpha_s(Q^2)$.

where $U_{ab}(N,\alpha_s)$ has a power series expansion in α_s such that $U_{ab}(N,\alpha_s) = \delta_{ab} + \mathcal{O}(\alpha_s)$. The structure functions will clearly be unchanged as long as the coefficient functions obey the transformation rule

$$\check{C}^a(N,\alpha_s) = (U^T)^{-1}_{ab}(N,\alpha_s) C^b(N,\alpha_s). \tag{2.11}$$

By substituting $f_a(N,Q^2) = \sum_b U^{-1}_{ab}(N,\alpha_s)\check{f}_b(N,Q^2)$ into (2.9) we easily find that the new parton densities evolve according to the standard evolution equations but with the new anomalous dimensions

$$\check{\gamma}_{ab}(N,\alpha_s) = \sum_c \sum_d U_{a,c}(N,\alpha_s)\gamma_{cd}(N,\alpha_s)U^{-1}_{db}(N,\alpha_s) + \sum_c \beta(\alpha_s)\frac{\partial U_{ac}(N,\alpha_s)}{\partial \alpha_s} U^{-1}_{cb}(N,\alpha_s). \tag{2.12}$$

The matrix U must obey a number of conditions in order that physical requirements on the parton distributions are maintained, e.g. flavour and charge conjugation invariance, fermion number conservation and longitudinal momentum conservation (see for example the second of [18]). However, none of these needs to be satisfied simply in order to keep the structure functions unchanged.

The transformation defined above is called a change of factorization scheme and is constructed precisely so that the physical structure functions are left invariant. However, it is important to realize that, unlike the changes in renormalization scheme, a change in factorization scheme leaves the expression for the structure functions unchanged not only to all orders, but order by order in α_s, and calculations performed carefully at a given well–defined order in one scheme will lead to precisely the same results for the structure functions as those in another scheme. Also, the coupling constant to be used depends only on the ultraviolet renormalization.[5] We will illustrate this point in §4, where we will calculate structure functions to a well–defined order in a variety of expansion schemes and demonstrate factorization scheme independence of our expressions. However, we will first illustrate Catani's recent proposal for the construction of factorization–scheme–independent structure functions.

3. Evolution Equations for Structure Functions.

It is, as Catani noticed [30], very simple to obtain factorization–scheme–independent expressions for the structure functions, or more precisely, factorization–scheme–independent effective anomalous dimensions governing the evolution of the structure functions (reflecting the fact that we cannot make a concrete prediction of the structure functions at a given scale using QCD, but only how they change with Q^2). In order to obtain these effective anomalous dimensions all one has to do is eliminate the parton densities from the equations (2.8) and (2.9).

[5] It is perfectly possible (if somewhat perverse) to choose the $\overline{\rm MS}$ scheme to remove ultraviolet divergences, but the MS scheme to remove infrared divergences from the bare coefficient functions. In this case it is the $\overline{\rm MS}$ scheme coupling constant that must be used.

In order to demonstrate this, let us now be a little more careful in our definition of the relevant structure functions. There are two independent structure functions $F_2(N,Q^2)$ and $F_L(N,Q^2)$. In general we may write

$$F_i(N,Q^2) = \frac{1}{N_f}\left(\sum_{j=1}^{N_f} e_j^2\right) F_i^S(N,Q^2) + F_i^{NS}(N,Q^2) \qquad (i=2,L), \tag{3.1}$$

where the singlet and nonsinglet structure functions are defined by

$$F_i^S(N,Q^2) = C_i^f(N,\alpha_s) f^S(N,Q^2) + C_i^g(N,\alpha_s) g(N,Q^2), \tag{3.2}$$

and

$$F_i^{NS}(N,Q^2) = C_i^{NS}(N,\alpha_s) \sum_{j=1}^{N_f} e_j^2 f_{q_j}^{NS}(N,Q^2), \tag{3.3}$$

where N_f is the number of active quark flavours, $f^S(N,Q^2)$ and $f_{q_j}^{NS}(N,Q^2)$ are the singlet and nonsinglet quark distribution functions respectively, and $g(N,Q^2)$ is the gluon distribution. In any factorization scheme obeying the requirements of flavour and charge conjugation invariance the renormalization group equations for the singlet and nonsinglet sectors will be decoupled. The equations for the nonsinglet distributions will be ordinary differential equations,

$$\frac{d\, f_{q_j}^{NS}(N,Q^2)}{d\ln Q^2} = \gamma_{NS}(N,\alpha_s) f_{q_j}^{NS}(N,Q^2), \tag{3.4}$$

while those for the singlet sector are coupled

$$\frac{d}{d\ln Q^2}\begin{pmatrix} f^S(N,Q^2) \\ g(N,Q^2) \end{pmatrix} = \begin{pmatrix} \gamma_{ff}(N,\alpha_s) & \gamma_{fg}(N,\alpha_s) \\ \gamma_{gf}(N,\alpha_s) & \gamma_{gg}(N,\alpha_s) \end{pmatrix} \begin{pmatrix} f^S(N,Q^2) \\ g(N,Q^2) \end{pmatrix}. \tag{3.5}$$

Let us first consider the simple case of the nonsinglet structure function $F_2^{NS}(N,Q^2)$. Multiplying both sides of (3.4) by $\sum_{j=1}^{N_f} e_j^2$ we can clearly write

$$\frac{d}{d\ln Q^2}\left(\frac{F_2^{NS}(N,Q^2)}{C_2^{NS}(N,\alpha_s)}\right) = \gamma_{NS}(N,\alpha_s)\frac{F_2^{NS}(N,Q^2)}{C_2^{NS}(N,\alpha_s)}, \tag{3.6}$$

which becomes the factorization–scheme–independent equation

$$\frac{d\, F_2^{NS}(N,Q^2)}{d\ln Q^2} = \Gamma_{2,NS}(N,\alpha_s) F_2^{NS}(N,Q^2), \tag{3.7}$$

where $\Gamma_{2,NS}(N,\alpha_s) = \gamma_{NS}(N,\alpha_s) + d\ln(C_2^{NS}(N,\alpha_s))/d\ln Q^2$. Therefore, we have an effective anomalous dimension governing the evolution of each of the nonsinglet structure functions, and clearly $\Gamma_i^{NS}(N,\alpha_s)$ must be a factorization–scheme–independent quantity (and is in principle measurable). The solution to this equation is trivial:

$$F_2^{NS}(N,Q^2) = F_2^{NS}(N,Q_0^2)\exp\left[\int_{\ln Q_0^2}^{\ln Q^2} \Gamma_{2,NS}(N,\alpha_s) d\ln q^2\right]. \tag{3.8}$$

The situation for the singlet structure functions is more complicated. As we see from (3.5) the evolution equations for the singlet quark density and gluon density are coupled. However, using (3.2) for $i = 2, L$ we may solve for the parton densities in terms of the two structure functions and the coefficient functions. Substituting these into (3.5) we then obtain the coupled evolution equations

$$\frac{d}{d\ln Q^2}\begin{pmatrix} F_2^S(N,Q^2) \\ F_L^S(N,Q^2)\end{pmatrix} = \begin{pmatrix} \check{\Gamma}_{22}(N,\alpha_s) & \check{\Gamma}_{2L}(N,\alpha_s) \\ \check{\Gamma}_{L2}(N,\alpha_s) & \check{\Gamma}_{LL}(N,\alpha_s)\end{pmatrix}\begin{pmatrix} F_2^S(N,Q^2) \\ F_L^S(N,Q^2)\end{pmatrix}. \tag{3.9}$$

The expressions for the physical anomalous dimensions, $\check{\Gamma}_{22}(N,\alpha_s)$, $\check{\Gamma}_{2L}(N,\alpha_s)$, etc., are straightforward to derive in terms of anomalous dimensions and coefficient functions in any particular factorization scheme using the above procedure, but result in rather cumbersome expressions. It is simplest first to define a factorization scheme such that $F_2^S(N,Q^2) = f^S(N,Q^2)$, i.e. $C_2^f(N,\alpha_s) = 1$, $C_2^g(N,\alpha_s) = 0$ (this is generally known as a DIS type scheme [33][6]). In terms of the coefficient functions and anomalous dimensions in this type of scheme we have

$$\begin{aligned}
\check{\Gamma}_{22}(N,\alpha_s) &= \gamma_{ff}(N,\alpha_s) - \frac{C_L^f(N,\alpha_s)}{C_L^g(N,\alpha_s)}\gamma_{fg}(N,\alpha_s), \\
\check{\Gamma}_{2L}(N,\alpha_s) &= \frac{\gamma_{fg}(N,\alpha_s)}{C_L^g(N,\alpha_s)}, \\
\check{\Gamma}_{L,2}(N,\alpha_s) &= C_L^g(N,\alpha_s)\gamma_{gf}(N,\alpha_s) - C_2^f(N,\alpha_s)\gamma_{gg}(N,\alpha_s) + \frac{dC_L^f(N,\alpha_s)}{d\ln Q^2} + C_L^f(N,\alpha_s)\gamma_{ff}(N,\alpha_s) \\
&\quad - C_L^f(N,\alpha_s)\frac{d\ln(C_L^g(N,\alpha_s))}{d\ln Q^2} - \frac{(C_L^f(N,\alpha_s))^2}{C_L^g(N,\alpha_s)}\gamma_{fg}(N,\alpha_s), \\
\check{\Gamma}_{LL}(N,\alpha_s) &= \gamma_{gg}(N,\alpha_s) + \frac{d\ln(C_L^g(N,\alpha_s))}{d\ln Q^2} + \frac{C_L^f(N,\alpha_s)}{C_L^g(N,\alpha_s)}\gamma_{fg}(N,\alpha_s).
\end{aligned} \tag{3.10}$$

Thus, we have a factorization–scheme–invariant set of anomalous dimensions governing the evolution of the structure functions. Before going any further let us remark that we believe there is a (purely technical) problem with the above expression. As is well known (and as we will discuss in the next section), $F_L(N,Q^2)$ starts at an order of α_s higher than $F_2(N,Q^2)$. Because of this there is an intrinsic asymmetry in the above definitions, with $\check{\Gamma}_{2L}(N,\alpha_s)$ beginning at zeroth order in α_s, $\check{\Gamma}_{22}(N,\alpha_s)$ and $\check{\Gamma}_{LL}(N,\alpha_s)$ beginning at first order, and $\Gamma_{L2}(N,\alpha_s)$ beginning at second order. The fact that one of the physical anomalous dimensions has a part at zeroth order in α_s seems against the spirit of the perturbative approach. In practice, the result is that, if one solves the evolution equations order by order in α_s the solutions are very different from those obtained from an order–by–order solution using the parton densities, which we know work very well for all but possibly the lowest values of x yet probed. (We will discuss this point rather more in §4.)

[6] We call it a DIS "type" scheme because satisfying the above requirement still leaves freedom in how we may define the gluon density, and thus we are still considering a family of schemes.

A trivial modification of Catani's approach is therefore to accept that $F_L(N,Q^2)$ contains an extra power of α_s, and to define the new structure function $\hat{F}_L(N,Q^2) = F_L(N,Q^2)/(\alpha_s/(2\pi))$. The longitudinal coefficient functions are likewise changed to $\hat{C}^a_L(N,\alpha_s) = C^a_L(N,\alpha_s)/(\alpha_s/(2\pi))$, and the singlet evolution equations become

$$\frac{d}{d\ln Q^2}\begin{pmatrix} F_2^S(N,Q^2) \\ \hat{F}_L^S(N,Q^2)\end{pmatrix} = \begin{pmatrix} \Gamma_{22}(N,\alpha_s) & \Gamma_{2L}(N,\alpha_s) \\ \Gamma_{L2}(N,\alpha_s) & \Gamma_{LL}(N,\alpha_s)\end{pmatrix}\begin{pmatrix} F_2^S(N,Q^2) \\ \hat{F}_L^S(N,Q^2)\end{pmatrix}, \tag{3.11}$$

where the $\Gamma(N,\alpha_s)$'s are defined precisely as in (3.10), but in terms of of $\hat{C}^a_L(N,\alpha_s)$ rather than $C^a_L(N,\alpha_s)$. This procedure restores the symmetry between the physical anomalous dimensions, and makes the order–by–order–in–α_s calculations essentially the same as when using evolution of parton distributions. It is, of course, trivial to obtain the physical $F_L(N,Q^2)$ from $\hat{F}_L(N,Q^2)$.[7]

Having made our redefinition of the quantities with which we work, we now have a direct relationship between possible calculations using the evolution equations for structure functions and the solutions using the parton densities. At present the parton anomalous dimensions and coefficient functions are known to order α_s^2. It is easy to see that this allows us to derive each of the Γ's to order α_s^2. For example, at first order in α_s they are

$$\begin{aligned}
\Gamma^{0,l}_{22}(N,\alpha_s) &= \gamma^{0,l}_{ff}(N,\alpha_s) - \frac{\hat{C}^S_{L,0,l}(N,\alpha_s)}{\hat{C}^g_{L,0,l}(N,\alpha_s)}\gamma^{0,l}_{fg}(N,\alpha_s), \\
\Gamma^{0,l}_{2L}(N,\alpha_s) &= \frac{\gamma^{0,l}_{fg}(N,\alpha_s)}{\hat{C}^g_{L,0,l}(N,\alpha_s)}, \\
\Gamma^{0,l}_{L,2}(N,\alpha_s) &= \hat{C}^g_{L,0,l}(N,\alpha_s)\gamma^{0,l}_{gf}(N,\alpha_s) - \hat{C}^S_{2,0,l}(N,\alpha_s)\gamma^{0,l}_{gg}(N,\alpha_s) + \hat{C}^f_{L,0,l}(N,\alpha_s)\gamma^{0,l}_{ff}(N,\alpha_s) \\
&\quad - \frac{(\hat{C}^S_{L,0,l}(N,\alpha_s))^2}{\hat{C}^g_{L,0,l}(N,\alpha_s)}\gamma^{0,l}_{fg}(N,\alpha_s), \\
\Gamma^{0,l}_{LL}(N,\alpha_s) &= \gamma^{0,l}_{gg}(N,\alpha_s) + \frac{\hat{C}^S_{L,0,l}(N,\alpha_s)}{\hat{C}^g_{L,0,l}(N,\alpha_s)}\gamma^{0,l}_{fg}(N,\alpha_s),
\end{aligned} \tag{3.12}$$

where the super–subscript n,l denotes the $(n+1)$–loop quantity. The $\mathcal{O}(\alpha_s^2)$ expressions for the physical anomalous dimensions are straightforward to derive, but are rather complicated. Similarly, from the known expansions of the parton anomalous dimensions and coefficient functions in the form $\alpha_s^n \sum_{m=1-n}^{\infty} a_m(\alpha_s/N)^m$, we can calculate $\Gamma^0_{LL}(N,\alpha_s)$ and $\Gamma^0_{L2}(N,\alpha_s)$, $\Gamma^0_{2L}(N,\alpha_s)$ and $\Gamma^0_{22}(N,\alpha_s)$ (where both are trivially zero), and $\Gamma^1_{2L}(N,\alpha_s)$ and $\Gamma^1_{22}(N,\alpha_s)$. This is the same order as for the parton anomalous dimensions, with the longitudinal anomalous dimensions having similar structure to the gluon anomalous dimensions and the $\Gamma_{2a}(N,\alpha_s)$'s having similar structure to the quark

[7] Similarly, it is also best to work with the rescaled nonsinglet structure function $\hat{F}_L^{NS}(N,Q^2)$. Doing this, the analogous procedure to (3.7) and (3.8) leads to $\hat{F}_L^{NS}(N,Q_0^2)$ beginning at zeroth order in $\alpha_s(Q_0^2)$, as does $F_2^{NS}(N,Q_0^2)$.

anomalous dimensions. The exact form of these physical anomalous dimensions may be calculated using the expressions for the parton anomalous dimensions and coefficient functions in the standard DIS scheme in [18]. The resulting expressions are relatively simple, being

$$\begin{aligned}
\Gamma^1_{22}(\alpha_s/N) &= -\frac{1}{(2\pi)}(\hat{C}^f_{L,1,0} - \tfrac{4}{9}\hat{C}^g_{L,1,0})(\tfrac{3}{2}\gamma^0_{gg}(\alpha_s/N) + \sum_{n=0}^{\infty}(\gamma^0_{gg}(\alpha_s/N))^n) \\
&\quad - \tfrac{4}{9}\gamma^{1,0}_{fg}(N,\alpha_s), \\
\Gamma^1_{2L}(\alpha_s/N) &= \frac{1}{(2\pi)}(\tfrac{3}{2}\gamma^0_{gg}(\alpha_s/N) + \sum_{n=0}^{\infty}(\gamma^0_{gg}(\alpha_s/N))^n), \\
\Gamma^0_{L,2}(N,\alpha_s) &= -(\hat{C}^f_{L,1,0} - \tfrac{4}{9}\hat{C}^{1,0}_{L,g})\gamma^0_{gg}(\alpha_s/N), \\
\Gamma^0_{LL}(\alpha_s/N) &= \gamma^0_{gg}(\alpha_s/N),
\end{aligned} \tag{3.13}$$

where $\gamma^{1,0}_{fg}$, $C^f_{L,1,0}$ and $C^g_{L,1,0}$ are the one–loop contributions to $\gamma^1_{fg}(\alpha_s/N)$, $C^f_{L,1}(\alpha_s/N)$ and $C^g_{L,1}(\alpha_s/N)$ respectively. We note that each of these anomalous dimensions is renormalization scheme invariant as well as factorization scheme invariant, as we would expect for leading–order physical quantities.

Solving the evolution equations for the structure functions using any subset of the currently known physical anomalous dimensions guarantees a result which is factorization scheme independent. However, since the evolution equations are coupled, a simple expression for the solutions, such as (3.8), is impossible. There is, in fact, considerable freedom in how we may solve the equations. We could, for example, simply put all of the anomalous dimensions currently known into (3.11) and then find the whole solution. Alternatively, we could solve using just the order α_s anomalous dimensions and then try to perturb about this solution in an ordered manner. These two approaches would lead to rather different answers, but both would be factorization scheme independent. The problem of obtaining a correctly ordered solution for the structure function will be discussed in detail in the next section. We will initially use the familiar parton distributions and coefficient functions, and show that, even when using this approach, if we solve producing a well–defined expansion for the structure functions, we automatically avoid the problem of factorization scheme dependence.

4. Ordered Calculations of Structure Functions.

There are in principle many different expansion methods one may use when obtaining solutions for the structure functions. The standard one is simply solving order by order in α_s. But there is also the expansion in leading powers of $\ln(1/x)$ for given powers in α_s (or equivalently in powers of N^{-1} in moment space), as we have already discussed. One can also combine the two expansion methods, and indeed, we will later argue that this is the correct thing to do. Nevertheless, we will begin by outlining the procedure for making a well–ordered calculation of structure functions

using the standard loop expansion. Although this is well known, we feel it is worth presenting it pedagogically, and making some points which are not usually highlighted, especially concerning the role of the starting scale. We may then discuss the more complicated cases of the expansion in leading powers of $\ln(1/x)$ and the combined expansion.

4.1. Loop Expansion.

We begin by introducing some new notation. In order to solve the evolution equations for the parton densities order by order in α_s and hence obtain expressions for $F_i^{NS}(N,Q^2)$ and $F_i^S(N,Q^2)$ we make use of equation (2.5) to rewrite the evolution equation for the nonsinglet parton density as

$$\alpha_s^2(Q^2)\frac{d\,f_{q_j}^{NS}(N,Q^2)}{d\alpha_s(Q^2)} = -\tilde{\gamma}_{NS}(N,\alpha_s)f_{q_j}^{NS}(N,Q^2), \tag{4.1}$$

where $\tilde{\gamma}_{NS}(N,\alpha_s) = \alpha_s^2(Q^2)\gamma_{NS}(N,\alpha_s)/\beta(\alpha_s)$. Similarly

$$\alpha_s^2(Q^2)\frac{d}{d\alpha_s(Q^2)}\begin{pmatrix} f^S(N,Q^2) \\ g(N,Q^2)\end{pmatrix} = -\begin{pmatrix} \tilde{\gamma}_{ff}(N,\alpha_s) & \tilde{\gamma}_{fg}(N,\alpha_s) \\ \tilde{\gamma}_{gf}(N,\alpha_s) & \tilde{\gamma}_{gg}(N,\alpha_s)\end{pmatrix}\begin{pmatrix} f^S(N,Q^2) \\ g(N,Q^2)\end{pmatrix}, \tag{4.2}$$

with similar definitions for $\tilde{\gamma}_{ff}(N,\alpha_s)$ etc. as for $\tilde{\gamma}_{NS}(N,\alpha_s)$. Each of the $\tilde{\gamma}$'s may now be written as

$$\tilde{\gamma}(N,\alpha_s) = \sum_{n=0}^{\infty}\tilde{\gamma}^{n,l}(N)\,\alpha_s^{n+1}, \tag{4.3}$$

where we also have the analogous definition

$$\gamma(N,\alpha_s) = \sum_{n=0}^{\infty}\gamma^{n,l}(N)\,\alpha_s^{n+1}, \tag{4.4}$$

for the normal anomalous dimensions.[8] Thus, at zeroth order the $\tilde{\gamma}(N)$'s only differ from the $\gamma(N)$'s by the normalization factor b_0. Beyond this order the differences are more complicated.

Using our definition, the evolution equations may be solved order by order in α_s. When doing this it is necessary to choose a starting scale Q_0^2 for the perturbative evolution of the parton distributions (or equivalently a starting value of the coupling $\alpha_s(Q_0^2)$), and specify input parton distributions at this scale. Let us discuss the choice of this scale briefly. Q_0^2 must clearly be chosen to be large enough that perturbative evolution should be reliable and also such that higher twist corrections should be very small. Traditionally evolution only takes place up from this starting scale. This has been both for the simple reason of convenience, and also because some form of the inputs has been expected at low starting scales. Until a few years ago the requirements described above led to a choice of $Q_0^2 \approx 4\text{GeV}^2$. In the past couple of years this value has often been chosen

[8] We use the superscript l in order to denote that this is the standard α_s expansion of the anomalous dimensions rather than the expansion in (1.6), i.e. the leading-$\ln(1/x)$ expansion, which will be used more frequently in this paper.

to be rather lower, due to the apparent success of evolution from lower starting scales, and also because much of the interesting small–x data is now at $Q^2 \leq 4\mathrm{GeV}^2$. These choices have been accompanied by guesses for the form of the inputs at low starting scales, e.g valence–like [9], or flat [4][10]. However, with the quality of the most recent data, these guesses for the form of the inputs no longer seem to lead to good quantitative agreement with data.

We make no assumptions of the above sort about the value of Q_0^2. We do require it to be high enough to be in the perturbative regime and to avoid higher twists, but acknowledge that there is no reason why Q_0^2 cannot be chosen to be quite large, and evolution away from the starting scale performed both up and down in Q^2. Taking this open–minded approach we then simply assume that only perturbative effects can lead to deviations from soft behaviour of the structure functions, and also demand that the form of our well–ordered expressions for the structure functions is as insensitive as possible to this choice, thus making the choice of Q_0^2 as open as possible. We will see the consequences of this unusual approach to the input scale and the inputs for the structure functions as we progress.

We begin by solving for the nonsinglet parton distributions, which are an easily understandable model. In this case the solution is particularly simple. Integrating both sides of (4.1) we obtain

$$f_{q_j}^{NS}(N,Q^2) = \left[\sum_{k=0}^{\infty} \alpha_s^k(Q_0^2) f_{q_j,k}^{NS}(N,Q_0^2)\right] \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\gamma_{NS}^{0,l}(N)} \exp\left[\sum_{n=1}^{\infty} \frac{(\alpha_s^n(Q_0^2) - \alpha_s^n(Q^2))}{n} \tilde{\gamma}_{NS}^{n,l}(N)\right]. \tag{4.5}$$

Perhaps unconventionally, we explicitly express the input $f_{q_j}^{NS}(N,Q_0^2)$ as a power series in $\alpha_s(Q_0^2)$. There are two reasons why this is necessary. Firstly, changes in the starting scale Q_0^2 lead to $\alpha_s(Q_0^2)$–dependent changes in the expression for the evolution term, which must be compensated for by $\alpha_s(Q_0^2)$–dependent changes in the starting distribution in order to leave the whole expression for the parton distributions unchanged, as required.[9] Let us examine this briefly by looking at the change of the lowest–order piece of (4.5), i.e.

$$f_{q_j,0}^{NS}(N,Q^2) = f_{q_j,0}^{NS}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\gamma_{NS}^{0,l}(N)}, \tag{4.6}$$

under a change in starting scale, $Q_0^2 \to (1+\delta)Q_0^2$, where δ is some constant. Each of the terms in our expressions for the structure functions would now be written in the forms shown above, but as functions of $(1+\delta)Q_0^2$. We may regain expressions in terms of Q_0^2 by expanding the coupling constant $\alpha_s((1+\delta)Q_0^2)$ in the form

$$\alpha_s((1+\delta)Q_0^2) = \alpha_s(Q_0^2) - \delta b_0 \alpha_s^2(Q_0^2) + \mathcal{O}(\alpha_s^3(Q_0^2)). \tag{4.7}$$

[9] Since the structure function is obtained by multiplying the parton distributions by the Q_0^2–independent coefficient functions, the parton distributions must be Q_0^2-independent to make the structure functions so.

Under this change in the input coupling constant, the evolution term in (4.6) undergoes a change

$$\Delta\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\gamma_{NS}^{0,l}(N)} = -\alpha_s(Q_0^2)\delta b_0 \tilde{\gamma}_{NS}^{0,l}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\gamma_{NS}^{0,l}(N)} + \text{higher order in } \alpha_s(Q_0^2). \quad (4.8)$$

This change in the parton distribution due to the variation in the leading term can be countered, up to higher orders, by a change in the order $\alpha_s(Q_0^2)$ input of the form

$$\Delta f_{q_j,1}^{NS}(N,Q_0^2) = \delta b_0 \tilde{\gamma}_{NS}^{0,l}(N) f_{q_j,0}^{NS}(N,Q_0^2). \quad (4.9)$$

Because changes in the evolution term due to a change in Q_0^2 begin at first order in $\alpha_s(Q_0^2)$, and are therefore absorbed by terms in the input at first order and beyond, the zeroth–order input must be insensitive to such changes and is Q_0^2–independent: $f_{q_j,0}^{NS}(N,Q_0^2) \equiv f_{q_j,0}^{NS}(N)$. It is not difficult to see that higher–order changes in the parton distributions due to changes in Q_0^2 can all be accounted for by changes in the higher–order inputs and that in order to satisfy this constraint alone, it is consistent for the higher–order inputs to be equal to functions of N dependent only on the anomalous dimensions. Therefore, these higher–order inputs are in some sense perturbative, consisting of perturbative parts multiplying the fundamentally nonperturbative $f_{q_j,0}^{NS}(N)$. As we will soon discuss, there are other constraints to be satisfied, e.g. factorization scheme independence of the input for the structure function, and this slightly complicates the above picture, but does not change the main conclusions.

Thus, we see that it is necessary to express the input as a power series in $\alpha_s(Q_0^2)$ in order to make the parton distribution Q_0^2–independent, but that only one intrinsically nonperturbative input which is Q_0^2-independent is needed. Usually in analyses of structure functions the parton inputs are taken to be a single $\alpha_s(Q_0^2)$–independent function which is implicitly allowed to be Q_0^2–dependent. Phenomenologically, this is normally much the same, but we stress the formally correct expression for the input here since it is rather important when constructing properly ordered solutions, and leads to some predictive power, especially in the small x limit.

Another reason for explicitly writing the input as a power series in $\alpha_s(Q_0^2)$ is that it makes little sense to demand that the starting distribution should not have a perturbative expansion, unless one insists that there is something special about a particular factorization scheme. If there is not, then any transformation from a scheme in which the starting parton distribution is purely zeroth order in $\alpha_s(Q_0^2)$ will lead to a starting distribution with a power series expansion in $\alpha_s(Q_0^2)$, but again, where there seems to be some underlying nonperturbative input $f_{q_j,0}^{NS}(N)$ which is unaltered by the change in factorization scheme.

Accepting that the parton inputs should be a power series in $\alpha_s(Q_0^2)$, and substituting our solution for the parton distribution into (3.3), we obtain the general expression for the nonsinglet

structure functions,

$$F_i^{NS}(N,Q^2) = \left[\left(\delta_{i,2} + \sum_{m=1}^{\infty} C_{i,m,l}^{NS}(N)\alpha_s^m(Q^2)\right)\sum_{j=1}^{N_f} e_j^2 \sum_{k=0}^{\infty} \alpha_s^k(Q_0^2) f_{q_j,k}^{NS}(N,Q_0^2)\right] \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_{NS}^{0,l}(N)} \exp\left[\sum_{n=1}^{\infty} \frac{(\alpha_s^n(Q_0^2) - \alpha_s^n(Q^2))}{n} \tilde{\gamma}_{NS}^{n,l}(N)\right]. \tag{4.10}$$

It is clear that this may be written as

$$F_i^{NS}(N,Q^2) = \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_{NS}^{0,l}(N)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{i,nm}^{NS}(N,Q_0^2)\alpha_s^{n-m}(Q_0^2)\alpha_s^m(Q^2), \tag{4.11}$$

and that, once a choice of renormalization scheme and starting scale have been made, each of the $F_{nm}^{NS}(N)$ must be invariant quantities under changes of factorization scheme in order to guarantee the scheme independence of the whole structure function. However, it is also clear from (4.10) that each of the $F_{nm}^{NS}(N)$ is potentially a function of the starting distribution, the anomalous dimension and the coefficient function. Any well–ordered calculation of the structure function should include all complete terms in (4.11) up to a given order in n and m, and no partial terms. In practice it is possible to work to a given order in n including all $m \leq n$, i.e. to expand to a given order in powers of $\alpha_s(Q^2)$ plus powers of $\alpha_s(Q_0^2)$, if the $\tilde{\gamma}$'s and C_i's are known to this order. (It is impossible to work to a given value of m including higher values of $n-m$ without knowledge of $\tilde{\gamma}$ to order $n-m$, i.e. it is impossible to expand just in powers of $\alpha_s(Q^2)$.)

We shall briefly describe how to construct this ordered solution for the structure functions, working up from zeroth order, so that at each order the solution is factorization scheme dependent. Consider first calculating $F_2^{NS}(N,Q^2)$ by working to zeroth order in $C_2^{NS}(N,\alpha_s)$, $\tilde{\gamma}_{NS}(N,\alpha_s)$ and the starting distribution (remembering that this is Q_0^2–independent). To this order

$$f_{q_j,0}^{NS}(N,Q^2) = f_{q_j,0}^{NS}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_{NS}^{0,l}(N)}, \tag{4.12}$$

and therefore

$$F_{2,0}^{NS}(N,Q^2) = \sum_{j=1}^{N_f} e_j^2 f_{q_j,0}^{NS}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_{NS}^{0,l}(N)}. \tag{4.13}$$

Using the one–loop expression for the running coupling, as is appropriate for a lowest–order calculation, each of the quantities in this expression is factorization scheme independent and indeed, also renormalization scheme independent. Therefore we have a consistent leading–order (LO) expression.

If we calculate $F_L^{NS}(N,Q^2)$, again by combining the coefficient functions with the solutions for the parton evolution, then working to the order $n=0$ in all quantities leads to $F_{L,0}^{NS}(N,Q^2) = 0$,

since the zeroth–order coefficient function is zero. However, looking at the expressions for the general solutions, (4.10) and (4.11), one sees that the only contribution for $n = 1$ in (4.11) comes from working to first order in $C_L^{NS}(N, \alpha_s)$ and to zeroth order in $\tilde{\gamma}_{NS}(N, \alpha_s)$ and the starting distribution. This leads to the LO expression for $F_L^{NS}(N, Q^2)$ of

$$F_{L,1}^{NS}(N, Q^2) = \alpha_s(Q^2) C_{L,1,l}^{NS}(N) \sum_{j=1}^{N_f} e_j^2 f_{q_j,0}^{NS}(N) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\tilde{\gamma}_{NS}^{0,l}(N)}. \tag{4.14}$$

Again, using the one–loop expression for the running coupling, every term in this expression is both factorization scheme and renormalization scheme independent, giving a well–defined LO expression.

We now consider the first correction to these expressions. The first–order expression for the renormalization group equation is

$$\alpha_s(Q^2) \frac{d f_{q_j,1}^{NS}(N, Q^2)}{d\alpha_s(Q^2)} = -\tilde{\gamma}_{NS}^{0,l}(N, \alpha, s) f_{q_j,1}^{NS}(N, Q^2) - \alpha_s(Q^2) \tilde{\gamma}_{NS}^{1,l}(N) f_{q_j,0}^{NS}(N, Q^2), \tag{4.15}$$

with solution

$$f_{q_j,1}^{NS}(N, Q^2) = \left[(\alpha_s(Q_0^2) - \alpha_s(Q^2)) \tilde{\gamma}_{NS}^{1,l}(N) f_{q_j,0}^{NS}(N) + \alpha_s(Q_0^2) f_{q_j,1}^{NS}(N, Q_0^2) \right] \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\tilde{\gamma}_{NS}^{0,l}(N)}. \tag{4.16}$$

Multiplying $f_{q_j,0}^{NS}$ by $\alpha_s(Q^2) C_{2,1,l}^{NS}(N)$ and adding to $f_{q_j,1}^{NS}$, we clearly obtain all terms in the expression (4.11) for $F_2^{NS}(N, Q^2)$) at $n = 1$. Adding this to (4.13) we obtain the factorization–scheme–independent expression for $F_2^{NS}(N, Q^2)$ up to $n = 1$.

We note that this is not the same as finding the complete solution to the renormalization group equation including all terms in the anomalous dimension up to first order in $\tilde{\gamma}$ and multiplying the solution by the coefficient function up to first order. This procedure would involve the exponentiation of the anomalous dimension, and thus would include incomplete parts of the $F_{nm}^{NS}(N)$'s for $n \geq 1$, and would be a factorization–scheme–dependent, and hence physically ambiguous quantity.

Similarly to $F_2^{NS}(N, Q^2)$ we can obtain the NLO factorization–scheme–independent expression for $F_L^{NS}(N, Q^2)$. The expression at $n = 2$ is obtained by adding the product of the first–order coefficient function and $f_{q_j,1}^{NS}(N, Q^2)$ to the product of the second–order coefficient function and $f_{q_j,0}^{NS}(N, Q^2)$. The NLO $F_L^{NS}(N, Q^2)$ is the sum of this and (4.14). When working to NLO for either structure function we now have expressions which are renormalization scheme dependent. This scheme dependence compensates for the renormalization scheme dependence of the two–loop coupling constant (which has a renormalization–scheme–dependent value for Λ_{QCD}), and it is this expression for the coupling that we should use at this level. Doing so guarantees the renormalization scheme independence of the structure functions up to corrections of higher order in α_s, i.e. $\mathcal{O}(\alpha_s^2 F_{2(L),0(1)}^{NS})$.

It is now simple to see how to construct factorization–scheme–independent structure functions order by order. Defining the $n_{\rm th}$ order renormalization group equation by

$$\alpha_s(Q^2)\frac{d\,f^{NS}_{q_j,n}(N,Q^2)}{d\alpha_s(Q^2)} = -\sum_{m=0}^{n}\tilde{\gamma}^{m,l}_{NS}(N,\alpha,s)f^{NS}_{q_j,n-m}(N,Q^2), \tag{4.17}$$

it is easy to prove by induction that the solution contains all terms in the full solution with a given sum of powers of $\alpha_s(Q^2)$ and $\alpha_s(Q_0^2)$ multiplying the everpresent $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\tilde{\gamma}^{0,l}_{NS}(N)}$ factor. Thus, defining part of (4.11) by

$$F^{NS}_{i,n}(N,Q^2) = \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}^{0,l}_{NS}(N)}\sum_{m=0}^{n}F^{NS}_{i,nm}(N)\alpha_s^{n-m}(Q_0^2)\alpha_s^m(Q^2), \tag{4.18}$$

$F^{NS}_{i,n}(N,Q^2)$ is given by

$$F^{NS}_{i,n}(N,Q^2) = \sum_{j}^{Nf}e_j^2\sum_{m=0}^{n}C^{NS}_{i,m,l}\alpha_s^m(Q^2)f^{NS}_{q_j,n-m}(N,Q^2). \tag{4.19}$$

The $n_{\rm th}$–order scheme–independent term in the expression for the structure functions is given by the product of the $m_{\rm th}$–order coefficient functions and the $(n-m)_{\rm th}$–order solutions to the renormalization group equation summed over m. The $n_{\rm th}$–order scheme–independent structure function is then the sum of these terms up to order n. We must however remember that including all $F^{NS}_{2,m}(N,Q^2)$ up to order n is working to $(n+1)_{\rm th}$ nontrivial order, and requires the $(n+1)$–loop coupling in order to make the expression renormalization scheme invariant up to higher orders in α_s. Similarly, including all $F^{NS}_{L,m}(N,Q^2)$ up to order n is working to $n_{\rm th}$ nontrivial order, and requires the n–loop coupling.

This procedure clearly provides factorization scheme independence and renormalization scheme independence for this method of expansion. We can also see how it relates to the discussion of the factorization–scheme–invariant evolution equations in terms of the structure functions. In order to do this let us consider the solution for the non–singlet structure function $F_2^{NS}(N,Q^2)$ again. We may rewrite our general solution (4.10) in the form

$$F_2^{NS}(N,Q^2) = \left[\left(1+\sum_{m=1}^{\infty}C^{NS}_{2,m,l}(N)\alpha_s^m(Q_0^2)\right)\sum_{j=1}^{N_f}e_j^2\sum_{k=0}^{\infty}\alpha_s^k(Q_0^2)f^{NS}_{q_j,k}(N,Q_0^2)\right]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}^{0,l}_{NS}(N)}\times$$
$$\exp\left[\sum_{n=1}^{\infty}\frac{(\alpha_s^n(Q_0^2)-\alpha_s^n(Q^2))}{n}\tilde{\gamma}^{n,l}_{NS}(N)+\int_{\ln Q_0^2}^{\ln Q^2}\frac{d}{d\ln q^2}\ln\Big(1+\sum_{m=1}^{\infty}C^{NS}_{2,m,l}(N)\alpha_s^m(q^2)\Big)d\ln q^2\right]. \tag{4.20}$$

This way of writing $F_2^{NS}(N,Q^2)$ is particularly useful since it separates the solution into the value of the structure function at Q_0^2 (the term in square brackets), and the ratio of its value at a different Q^2 with this initial value (the rest of the expression). Clearly these two quantities must

be separately factorization scheme independent. Also, it is clear that this solution is of exactly the same form as (3.8), and we can express it simply in terms of an input for $F_2^{NS}(N,Q^2)$ at Q_0^2 and a physical anomalous dimension which governs the evolution, i.e. our input and our prediction. Since these are both physical quantities they must be separately renormalization scheme independent.

The input and the evolution will mix with each other if we make a change in the starting scale, however, as with the parton distribution. Examining the effects of such a change for the full physical quantity gives us information about the form of the input. As we have already seen, at lowest order the nonsinglet structure function is just the sum of the charge weighted nonsinglet parton distributions. So the lowest–order input is just

$$F_{2,0}^{NS}(N) = \sum_{j=1}^{N_f} e_j^2 f_{q_j,0}^{NS}(N). \tag{4.21}$$

Making the change of starting scale and consequently of $\alpha_s(Q_0^2)$ already considered, the change in the lowest–order evolution is as in (4.8), and this leads to a change in the lowest–order structure function which is of higher order, and which can be absorbed by a change in the NLO input for the structure function of

$$\Delta F_{2,1}^{NS}(N,Q_0^2) = \delta b_0 \tilde{\gamma}_{NS}^{0,l}(N) F_{2,0}^{NS}(N). \tag{4.22}$$

In terms of parton distributions and coefficient functions

$$F_{2,1}^{NS}(N,Q_0^2) = \sum_{j=1}^{N_f} e_j^2 (f_{q_j,1}^{NS}(N,Q_0^2) + C_{2,1,l}^{NS}(N) f_{q_j,0}^{NS}(N)). \tag{4.23}$$

We chose the change in $f_{q_j,1}^{NS}(N,Q_0^2)$ in (4.9) so that the structure function would be independent of Q_0^2, and it is clear that that is consistent with (4.22) and (4.23). However, we can now say more about the form of $f_{q_j,1}^{NS}(N,Q_0^2)$. Because it is the leading term in the expression for the input which depends on $\alpha_s(Q_0^2)$, $F_{2,1}^{NS}(N,Q_0^2)$ must be renormalization scheme independent. However, $C_{2,1,l}^{NS}(N)$ is renormalization scheme dependent, so $f_{q_j,1}^{NS}(N,Q_0^2)$ must also be renormalization scheme dependent in a way such as to cancel this. Hence, $f_{q_j,1}^{NS}(N,Q_0^2)$ must not only have a part like $\ln(Q_0^2)\gamma_{NS}^{0,l}(N) f_{q_j,0}^{NS}(N)$ in order to maintain Q_0^2–independence of the structure function, but also a part like $-C_{2,1,l}^{NS}(N) f_{q_j,0}^{NS}(N)$ in order to maintain renormalization scheme independence, i.e.

$$f_{q_j,1}^{NS}(N,Q_0^2) = (\ln(Q_0^2/A_{NS})\gamma_{NS}^{0,l}(N) - C_{2,1,l}^{NS}(N)) f_{q_j,0}^{NS}(N), \tag{4.24}$$

where A_{NS} is some unknown scale parameter. So we see that

$$F_{2,1}^{NS}(N,Q_0^2) = \ln(Q_0^2/A_{NS})\gamma_{NS}^{0,l}(N) F_{2,0}^{NS}(N) \equiv \ln(Q_0^2/A_{NS})\Gamma_{NS}^{0,l}(N) F_{2,0}^{NS}(N). \tag{4.25}$$

It is clear that this does not spoil our argument that $f_{q_j,1}^{NS}(N,Q_0^2)$ consists of perturbatively calculable quantities multiplying the fundamentally nonperturbative $f_{q_j,0}^{NS}(N)$. It is also clear that all the

higher–order inputs may be chosen to be perturbative functions multiplying this nonperturbative input, and therefore that the input for the structure function is a perturbative power series (depending on the physical anomalous dimension) multiplying the single nonperturbative factor $f^{NS}_{q_i,0}(N)$, which from (4.21) may be interpreted as a fundamentally nonperturbative input for the structure function $F^{NS}_{2,0}(N)$. Hence, demanding invariance of our expression for the structure function under changes in Q^2_0 leads us to a power series expression for the input, but with only one (for each quark) really nonperturbative factor for this input. We also see that if $Q^2_0 = A_{NS}$, the first–order perturbative correction to $F^{NS}_2(N, Q^2_0)$ vanishes. Hence, we might expect A_{NS} to be some scale typical of the transition between perturbative and nonperturbative physics, i.e $A_{NS} \lesssim 1\text{GeV}$.

Separating the expression for the structure function into a definite input and evolution part also enables us to view the loop expansion in an alternative manner. We see that when expanding out to n_{th} order in the loop expansion we are including all terms where the order of the input part for the structure function added to the order of the evolution part of the structure function is less than or equal to n. This is clearly the same as including all powers of α_s in the expression for the structure function up to n_{th} order, but gives us some additional physical interpretation.

Writing the solution as in (4.20) does, however, also illustrate that demanding factorization scheme invariance does not on its own force us into the strictly defined loop expansion. It is clear that we could, if we wished, expand the input and evolution term out to different orders in α_s, still maintaining factorization scheme independence. However, this is not a sensible approach for reasons of renormalization scheme dependence. If we were to expand out the input and evolution terms to different orders in α_s we should really use α_s itself calculated to a different order in each case, surely a perverse thing to do. Also, when using the resulting expression to evaluate the structure function at some Q^2 away from Q^2_0 we would only have a subset of the terms at some given power of α_s (where α_s may represent $\alpha_s(Q^2_0)$ or $\alpha_s(Q^2)$). Each of these terms is presumably of similar magnitude, which is rather importantly equal to, or greater than, the magnitude of the uncertainty due to renormalization scheme dependence. This is even the case even if we simply use the product of the input specified to n_{th} order and the evolution calculated to n_{th} order. In this case there is no problem in what definition of α_s to use: we simply use the n–loop expression for the input and evolution and hence the n–loop coupling for both. However, the resulting expression contains, for example, a term like $\alpha_s^{n-1}(Q^2_0)(\alpha_s^{n-1}(Q^2_0) - \alpha_s^{n-1}(Q^2))F^{NS}_{2,0}(N, Q^2)$, but none like $\alpha_s^{n-1-m}(Q^2_0)(\alpha_s^{n-1+m}(Q^2_0) - \alpha_s^{n-1+m}(Q^2))F^{NS}_{2,0}(N, Q^2)$ which should be of the same size. One only has a complete set of terms of order α_s^k up to $k = n - 1$. Moreover, this type of term is higher order than the overall uncertainty in the expression due to the renormalization scheme dependence, which is of order $\alpha_s^n F^{NS}_{2,0}(N, Q^2)$. Exponentiating the solution for the evolution part, once it is found to a given order, is also redundant, since that would introduce only a subset of the terms at higher orders in α_s for the evolution, and these terms would be renormalization scheme dependent.

Thus, all ways of obtaining factorization–scheme–invariant expressions other than the loop expansion contain terms which are essentially redundant using this expansion method, i.e. those beyond the order where all complete parts of the loop expansion exist. These other terms in the expansion are the same order as uncertainties due to renormalization scheme dependence. The loop expansion gives the minimum expression which is totally factorization scheme independent, and moreover, renormalization scheme consistent (forgetting complications due to powers of N^{-1} for the present) to a given order. If solving using the factorization–scheme–invariant equations in terms of physical variables it is useful to bear this in mind.[10]

The solution for the longitudinal structure function is much the same as for $F_{2,0}^{NS}(N,Q^2)$. We may write $F_L^{NS}(N,Q^2)$ as

$$F_L^{NS}(N,Q^2) = \left[\left(\sum_{m=1}^{\infty} C_{L,m,l}^{NS}(N)\alpha_s^m(Q_0^2)\right)\sum_{j=1}^{N_f} e_j^2 \sum_{k=0}^{\infty}\alpha_s^k(Q_0^2) f_{q_j,k}^{NS}(N,Q_0^2)\right]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_{NS}^{0,l}(N)-1} \times$$
$$\exp\left[\sum_{n=1}^{\infty}\frac{(\alpha_s^n(Q_0^2)-\alpha_s^n(Q^2))}{n}\tilde{\gamma}_{NS}^{n,l}(N) + \int_{\ln Q_0^2}^{\ln Q^2}\frac{d}{d\ln q^2}\ln\Big(\sum_{m=1}^{\infty} C_{L,m,l}^{NS}(N)\alpha_s^{m-1}(q^2)\Big)d\ln q^2\right]. \tag{4.26}$$

So as with $F_2^{NS}(N,Q^2)$ the full solution in terms of parton densities, anomalous dimensions and coefficient functions is easily separated into its input[11] and evolution parts and therefore directly compared with the solution using the effective physical anomalous dimensions (where if we solve using the $\tilde{\Gamma}$s, the solution is expressed in terms of an input and evolution part once we write the overall power of $\alpha_s(Q^2)$ multiplying $\hat{F}_L^{NS}(N,Q^2)$ in the form $\alpha_s(Q_0^2)\times(\alpha_s(Q^2)/\alpha_s(Q_0^2))$.) Once again, from (4.26) we see that the loop expansion multiplies orders in the input by orders in the evolution in a systematic manner.

The situation for the singlet structure function is similar to that for the nonsinglet, but is complicated by the fact that we now have coupled evolution equations for the quark and gluon distributions. This makes it impossible to write a closed form for the solution to the renormalization group equations in the way we did for the nonsinglet case in (4.20) and (4.26), but the equations may be solved order by order in the same way.

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[10] It is also true that, when solving using the parton model, other ways of obtaining factorization–scheme–invariant expressions other than the loop expansion would need, in comparison, more complicated prescriptions.

[11] Comparing (4.26) with (4.20) we see that there are no new fundamentally nonperturbative parts in the input. Hence, the complete expression may be interpreted as a perturbative part multiplying the nonperturbative factor which is $F_{2,0}^{NS}(N)$. It is only the perturbative factor which is different in the two nonsinglet structure functions.

The lowest–order solution to (4.2) is

$$f_0^S(N,Q^2) = f_0^{S,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_+^{0,l}(N)} + f_0^{S,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_-^{0,l}(N)},$$
$$g_0(N,Q^2) = g_0^+(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_+^{0,l}(N)} + g_0^-(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_-^{0,l}(N)}, \tag{4.27}$$

where $\tilde{\gamma}_+^{0,l}(N)$ and $\tilde{\gamma}_-^{0,l}(N)$ are the eigenvalues of the zeroth–order matrix for the $\tilde{\gamma}$'s, and $f_0^{S,+}(N) + f_0^{S,-}(N) = f_0^S(N)$ and $g_0^+(N) + g_0^-(N) = g_0(N)$. Similarly to the nonsinglet case, the lowest–order factorization–scheme– and renormalization–scheme–independent expressions for the structure functions are

$$F_2^S(N,Q^2) = f_0^S(N,Q^2), \tag{4.28}$$

and

$$F_L^S(N,Q^2) = \alpha_s(Q^2)(C_{L,1}^f(N) f_0^S(N,Q^2) + C_{L,1}^g(N) g_0(N,Q^2)). \tag{4.29}$$

Under a change in starting scale, the change of the input for $F_2^S(N,Q^2)$ begins at order $\alpha_s(Q_0^2)$, and that for $F_L^S(N,Q^2)$ begins at order $\alpha_s^2(Q_0^2)$. Therefore, as with the nonsinglet quark distributions, the lowest–order inputs for the partons are Q_0^2–independent. Also, if we expect any powerlike behaviour to come only from perturbative effects, then these Q_0^2–independent inputs for the quark and gluon are analytic for $N > 0$.

Again in clear analogy to the nonsinglet case, the first–order expression for the renormalization group equations is

$$\alpha_s(Q^2)\frac{d}{d\alpha_s(Q^2)}\begin{pmatrix} f_1^S(N,Q^2) \\ g_1(N,Q^2)\end{pmatrix} = -\begin{pmatrix} \tilde{\gamma}_{ff}^{0,l}(N,\alpha_s) & \tilde{\gamma}_{fg}^{0,l}(N,\alpha_s) \\ \tilde{\gamma}_{gf}^{0,l}(N,\alpha_s) & \tilde{\gamma}_{gg}^{0,l}(N,\alpha_s)\end{pmatrix}\begin{pmatrix} f_1^S(N,Q^2) \\ g_1(N,Q^2)\end{pmatrix}$$
$$-\alpha_s(Q^2)\begin{pmatrix} \tilde{\gamma}_{ff}^{1,l}(N,\alpha_s) & \tilde{\gamma}_{fg}^{1,l}(N,\alpha_s) \\ \tilde{\gamma}_{gf}^{1,l}(N,\alpha_s) & \tilde{\gamma}_{gg}^{1,l}(N,\alpha_s)\end{pmatrix}\begin{pmatrix} f_0^S(N,Q^2) \\ g_0(N,Q^2)\end{pmatrix} \tag{4.30}$$

and the solution is

$$f_1^S(N,Q^2) = (\alpha_s(Q_0^2) - \alpha_s(Q^2))\Bigg(e_1^{S,+}(f_0^S, g_0, \tilde{\gamma}^0, \tilde{\gamma}^1)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_+^{0,l}(N)}$$
$$+ e_1^{S,-}(f_0^S, g_0, \tilde{\gamma}^0, \tilde{\gamma}^1)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_-^{0,l}(N)}\Bigg) + \alpha_s(Q_0^2)\Bigg(f_1^{S,+}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_+^{0,l}(N)}$$
$$+ f_1^{S,-}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\gamma}_-^{0,l}(N)}\Bigg), \tag{4.31}$$

and similarly for $g_1(N,Q^2)$, where $f_1^{S,+}(N,Q_0^2) + f_1^{S,-}(N,Q_0^2) = f_1^S(N,Q_0^2)$.

This is clearly of the same form as (4.16) and, as in the nonsinglet case, the $n = 1$ term in the factorization–scheme–independent expression for $F_2^S(N,Q^2)$, including all single powers of $\alpha_s(Q^2)$ or $\alpha_s(Q_0^2)$ multiplying $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}_{+,-}^{0,l}(N)}$, is

$$f_1^S(N,Q^2) + \alpha_s(Q^2)(C_{2,1}^f(N) f_0^S(N,Q^2) + C_{2,1}^g(N) g_0(N,Q^2)). \tag{4.32}$$

In the same manner the $n = 2$ term in the factorization–scheme–independent expression for the longitudinal structure function is

$$\alpha_s(Q^2)(C_{L,1}^f(N) f_1^S(N,Q^2) + C_{2,1}^g(N) g_1(N,Q^2)) + \alpha_s^2(Q^2)(C_{L,2}^f(N) f_0^S(N,Q^2) + C_{L,2}^g(N) g_0(N,Q^2)). \tag{4.33}$$

As for the singlet structure function, there is a simple prescription for calculating $n_{\rm th}$–order factorization–scheme–independent structure functions. Defining the $m_{\rm th}$–order solution to the renormalization group equation as the solution to

$$\alpha_s(Q^2) \frac{d}{d\alpha_s(Q^2)} \begin{pmatrix} f_m^S(N,Q^2) \\ g_m(N,Q^2) \end{pmatrix} = -\sum_{k=0}^{m} \alpha_s^k(Q^2) \begin{pmatrix} \tilde{\gamma}_{ff}^k(N,\alpha_s) & \tilde{\gamma}_{fg}^k(N,\alpha_s) \\ \tilde{\gamma}_{gf}^k(N,\alpha_s) & \tilde{\gamma}_{gg}^k(N,\alpha_s) \end{pmatrix} \begin{pmatrix} f_{m-k}^S(N,Q^2) \\ g_{m-k}(N,Q^2) \end{pmatrix}, \tag{4.34}$$

the $n_{\rm th}$–order term in the expression for the structure functions is given by

$$F_{i,n}^S(N,Q^2) = \sum_{m=0}^{n} \alpha_s^m(Q^2)(C_{i,m}^f(N) f_{n-m}^S(N,Q^2) + C_{i,m}^g(N) g_{n-m}(N,Q^2),) \tag{4.35}$$

and the $n_{\rm th}$–order structure function is the sum of all such terms up to n. The comments concerning the order of the renormalization scheme independence and the order of the coupling constant to be used for $F_2^S(N,Q^2)$ and $F_L^S(N,Q^2)$ for the singlet case apply in exactly the same way as for the nonsinglet case. Once again, the loop expansion is a well–ordered way to expand the structure functions.

We may now discuss the relationship to the solutions using the evolution equations for the structure functions. First, we can explain in more detail why we feel it is appropriate to take out the power of $\alpha_s(Q^2)$ from the longitudinal structure function when using this approach. Using the parton model the lowest–order expression for $F_2^S(N,Q^2)$ consists of two Q^2–independent factors, the inputs for the structure function, multiplying $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}_{+,-}^{0,l}}$. Similarly, we may think of the lowest–order expression for $F_L^S(N,Q^2)$ as being two inputs for the structure function multiplying the Q^2–dependent factors $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}_{+,-}^{0,l}-1}$, with the extra power of $\alpha_s(Q^2)$ coming from the coefficient function. So, at lowest non–trivial order the Q^2 evolution of the two structure functions is different. If we were to solve the evolution equations for the structure functions themselves at lowest order, then we would do it in the same way as for the parton distributions: obtaining the Q^2–dependence from the eigenvalues of the lowest–order anomalous dimension matrix, and the inputs multiplying the two evolution terms from $F_2^S(N,Q_0^2)$ and $F_L^S(N,Q_0^2)$ and the eigenvectors of

the matrix. Thus, the terms governing Q^2 evolution would have to be the same for both structure functions, which is clearly in contradiction with our lowest–order result using the parton model (also, the lowest–order input for $F_L^S(N,Q^2)$ would be zero). This leads to the solutions for the structure functions being built up in a very different way using this method of solution from that using the parton model.

Hence, we choose not to forget the success of the parton model, and let the longitudinal structure function be multiplied by an overall power of $\alpha_s(Q^2)$, as discussed earlier. Working with the $\Gamma(N,\alpha_s)$'s in (3.12) and solving the lowest–order evolution equations, one then obtains precisely the same lowest–order solution for the structure functions as above. It is not difficult to see that, although the $\Gamma^{0,l}$'s are not identical to the $\gamma^{0,l}$'s, the eigenvalues of the anomalous dimension matrix are the same. Thus, we obtain the factors of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}^{0,l}_{+,-}}$ in these solutions as we would hope, and get the extra factor of $(\alpha_s(Q^2)/\alpha_s(Q_0^2))$ in the expression for $F_L^S(N,Q^2)$ from the overall power of $\alpha_s(Q^2)$ multiplying the solution (and simultaneously get the extra power of $\alpha_s(Q_0^2)$ in the input). The eigenvectors are different from those in the parton model, of course, but this takes account of the fact that the inputs are now those for the structure functions themselves, rather than, in the case of $F_L^S(N,Q^2)$, the parton densities weighted by coefficient functions. Working beyond this lowest order and finding the $n_{\rm th}$–order solutions to the evolution equations for the structure functions, in the same way as those for the parton densities, as outlined above, leads to all powers of α_s up to power n multiplying the factors of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}^{0,l}_{+,-}}$ for $F_2^S(N,Q^2)$, and powers of α_s up to power $n+1$ multiplying $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}^{0,l}_{+,-}-1}$ for $F_L^S(N,Q^2)$. Thus, it produces the same result as the $n_{\rm th}$–order solution using the parton distributions and the loop expansion.

Unlike the nonsinglet case it is rather difficult to see how to express the solution in terms of factorized inputs and evolution parts; the evolution parts will not be simple exponentials, as in (4.20). However, the solution for the structure functions can be written as a series of terms each of which can be factored into a part dependent only on $\alpha_s(Q_0^2)$ and one depending on both $\alpha_s(Q_0^2)$ and $\alpha_s(Q^2)$, the latter either vanishing at $Q^2=Q_0^2$, or being equal to $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\gamma}^{0,l}_{+,-}(-1)}$ and thus equal to unity at $Q^2=Q_0^2$. For each of these terms the former part may be interpreted as an input and the latter part may be interpreted as the evolution. Within the loop expansion we then include all terms where the order of the input part plus the order of the evolution part sums to less than or equal to some integer n.

By examining the form of the inputs under a change in starting scale, as with the nonsinglet structure functions, we find that the only fundamentally nonperturbative parts of the inputs are the zeroth–order parts, $F_{2,0}^S(N)$ and $F_{L,0}^S(N)$, with all other inputs being in principle expressed as perturbative functions of the physical anomalous dimensions multiplying one of these nonperturbative components. If we wished to stay within the language of partons then we would equivalently have the two nonperturbative parton inputs $f_0^S(N)$ and $g_0(N)$, where the link between the two pairs of nonperturbative inputs is

$$F_{2,0}^S(N) = f_0^S(N), \tag{4.36}$$

and

$$F_{L,0}^S(N) = C_{L,1}^f(N) f_0^S(N) + C_{L,1}^g(N) g_0(N). \tag{4.37}$$

We choose to think of the expressions for the structure functions as the real nonperturbative functions, since they are, of course, the physically relevant quantities.[12] This is consistent with the reasoning that $g_0(N)$ is Q_0^2–independent because the invariance of the physical quantity $F_L^S(N, Q^2)$ under a change in starting scale leads to $F_{L,0}^S(N)$ being Q_0^2–independent, which then, using (4.37), leads to $g_0(N)$ being Q_0^2–independent.

The extra complexity of the solution in the singlet case, when compared to the nonsinglet case, also means it is far from obvious how to express the physical anomalous dimensions in terms of the parton anomalous dimensions and coefficient functions simply by comparing the forms of the solutions in the partonic language and purely in terms of structure functions. However, the expressions for the physical anomalous dimensions can be found at each order using Catani's expressions (3.10). One could then solve for the structure functions in a manner different from the loop expansion. An example would be to calculate the solution to the whole evolution equation for the structure functions using the physical anomalous dimensions up to a given order. This would be by definition factorization scheme invariant, but would only be renormalization scheme invariant up to the same order in the solution as the order of the anomalous dimensions. The rest of the solution would contain only a subset of the possible terms of a given form obtained from the full calculation, and would therefore still have no real significance. Hence, we conclude that the evolution equations in terms of structure functions do nothing to alter the strict ordering of the solution using this method of expansion, but do make finding this ordered solution somewhat easier.

This whole discussion of the order–by–order–in–α_s expansion scheme is perhaps a little academic since the errors invoked in performing the calculations without paying strict heed to the formally correct procedures are rather small. For example, let us consider factorization scheme dependence. When using this standard method of expansion only two different factorization schemes are generally considered, the $\overline{\rm MS}$ scheme and the DIS scheme. The first of these simply involves calculating the coefficient functions using standard techniques and using the $\overline{\rm MS}$ scheme to remove both the the infrared and ultraviolet divergences and hence provide the anomalous dimensions. The second makes a change of parton distributions, so that the singlet quark distribution is equal to the singlet structure function. It is clear from the above discussion that, if a calculation is made in one scheme to a well–defined order (usually next to leading order), and then a transformation to the other scheme made correctly, precisely the same result will be obtained. In practice, small

[12] We could, if we wished, make a change in factorization scheme where the zeroth–order part of the matrix U was not the unit matrix, but a function of N. This changes no physical result, but does change the zeroth–order parton inputs.

differences are sometimes noticed between calculations using different schemes within the loop expansion to NLO, but these come from well–understood sources. One common source is that the starting distributions in both schemes are described by a simple functional form, e.g.

$$xf(x,Q_0^2) = A(Q_0^2)x^{-\lambda(Q_0^2)}(1-x)^{\eta(Q_0^2)}(1+\epsilon(Q_0^2)x^{1/2}+\gamma(Q_0^2)x), \tag{4.38}$$

rather than the formally correct expression of a power series in $\alpha_s(Q_0^2)$ with essentially perturbative coefficients convoluted with a Q_0^2–independent nonperturbative function of x. If the starting distribution is of the form above in one scheme, then the starting distribution in the other scheme will not be modelled precisely by a function of the same form. However, the error is in general small. Alternatively, if the calculations are not done in a well–ordered manner, e.g. simply solving the whole evolution equation using the two–loop anomalous dimensions, then differences between the calculation done in the two schemes, or between this type of calculation and the correct NLO calculation, will be of NNLO. Again this usually results in only small differences.[13] Similarly, small differences would be obtained by working in two different renormalization schemes, and of course, there is nothing which can be done about this.

There are a couple of points we wish to make here. Firstly concerning the form of the input in (4.38) we note that, if one considers the input to be consistent with the loop expansion, e.g. NLO–order evolution should be accompanied by a NLO input, the power of $\lambda(Q_0^2)$ should not correspond to parton distributions much steeper than flat for the singlet quark or gluon at this order. This is because the first–order–in–$\alpha_s(Q_0^2)$ input should be accompanied by no more than a single power of $\ln(1/x)$, this being all that is required to absorb the change in the zeroth–order evolution under a change in starting scale at this order in $\alpha_s(Q_0^2)$. In practice restricting $\lambda(Q_0^2)$ in this way is rather important for the fits to the low x data, and would mean that NLO fits to small–x data would be very poor. The only way to avoid this is to let $\lambda(Q_0^2)$ be an artificial free parameter, in which case, if (4.38) describes the singlet quark density, it must be $\sim 0.2-0.3$ for practically any Q_0^2. This value is totally unjustified, and the need for this steepness in the input for the quark is a clear sign of the limited usefulness of the NLO–in–α_s calculation at small x. We will see that the situation is more more satisfactory when using the correct expansion method.

Ignoring this problem with the inputs, there is another reason for being concerned about the validity of the loop expansion at small x. The main reason for the smallness of the differences between inaccurately performed NLO calculations noted above, even at small x, is that the differences between these calculations do not contain terms which are any more leading in $\ln(1/x)$ than

[13] For a comparison of calculations done at NLO using different methods see [34]. One sees here that starting with the same input parton distributions the different calculations agree to better than 5% except for very small x and large Q^2, where the discrepancy can approach 10%. The differences observed can be compensated for almost completely by small changes in inputs.

the NLO calculation itself. Hence they are genuinely an order of α_s down on the NLO calculation, with no small–x enhancement. This is also true for calculations done in different renormalization schemes (or at different renormalization scales). The real NNLO contribution will be higher order in α_s but will also contain terms at higher order in $\ln(1/x)$, and will therefore be potentially large at small x. We therefore stress that the relative insensitivity of structure functions to changes in renormalization or factorization scheme for calculations which are not carefully ordered is no guarantee that genuine higher–order corrections will be small at small x when using the loop expansion. Indeed we would naively expect them to be large. This is a point to which we shall return.

In contrast to the insensitivity when using the loop expansion, when using the leading–$\ln(1/x)$ expansion very large differences between calculations done in a large number of different factorization schemes have been noted. This is an obvious sign that the calculations are not being done in a well–ordered manner, and that the ambiguity introduced by lack of care in the calculations is greater in this method of expansion than the standard loop expansion. We will now demonstrate that this is indeed the case.

4.2. The Leading–$\ln(1/x)$ Expansion: Parton Distributions.

It should clearly be possible to define a well–ordered, and hence, factorization scheme independent expansion in leading powers of $\ln(1/x)$, or equivalently, in leading powers of $1/N$ in moment space. We will now demonstrate that this is indeed the case. As in the loop expansion, we will first work in terms of the traditional parton distribution functions and coefficient functions, and see how this results in expressions containing the physical anomalous dimensions. Doing this enables us to see how large factorization scheme dependence can arise when calculating less carefully within this expansion scheme.

First we must set up our notation and qualify the statements made in the introduction concerning the leading–$\ln(1/x)$ expansion. We stated that the anomalous dimensions could be written in the form (1.6), and this means that using the form (4.1) and (4.2) for the evolution equations we may write,

$$\tilde{\gamma}(N,Q^2) = \sum_{n=0}^{\infty} \alpha_s^n(Q^2) \sum_{m=1-n}^{\infty} \tilde{\gamma}^{nm} \alpha_s^m(Q^2) N^{-m} \equiv \sum_{n=0}^{\infty} \alpha_s^n(Q^2) \tilde{\gamma}^n(\alpha_s(Q^2)/N). \tag{4.39}$$

So, the $\tilde{\gamma}^0$'s only differ from the γ^0's by the normalization factor of b_0, but at higher orders the relationship is more complicated. In particular, in the $\overline{\rm MS}$ renormalization and factorization scheme we may write

$$\tilde{\gamma}_{gg}(N,\alpha_s(Q^2)) = \sum_{n=0}^{\infty} \alpha_s^n(Q^2) \tilde{\gamma}_{gg}^n(\alpha_s(Q^2)/N), \tag{4.40}$$

where the series expansion for $\tilde{\gamma}^0_{gg}(\alpha_s/N)$ is known to all orders (all the coefficients being positive). This expression for $\tilde{\gamma}^0_{gg}(\alpha_s/N)$ is renormalization scheme independent since any change in renormalization scheme can only bring about a change in the coupling at $\mathcal{O}(\alpha_s^2)$, and this does not require a change in $\tilde{\gamma}^0_{gg}(\alpha_s/N)$ to keep physical quantities invariant at leading order in this expansion scheme, as we will soon see. $\tilde{\gamma}^1_{gg}(\alpha_s/N)$ is of course renormalization scheme dependent, since it must change to absorb part of the effect of the $\mathcal{O}(\alpha_s^2)$ change in the coupling on $\tilde{\gamma}^0_{gg}(\alpha_s/N)$. The renormalization scheme independence is also true for $\tilde{\gamma}^0_{gf}(\alpha_s/N)$, which obeys (1.7), and also for $\tilde{\gamma}^0_{ff}(\alpha_s/N)$ and $\tilde{\gamma}^0_{fg}(\alpha_s/N)$ which are both zero. There is also a renormalization–scheme–independent relationship between $\tilde{\gamma}^1_{ff}(\alpha_s/N)$ and $\tilde{\gamma}^1_{fg}(\alpha_s/N)$, which tells us that

$$\tilde{\gamma}^1_{ff}(\alpha_s/N) = \frac{4}{9}\left(\tilde{\gamma}^1_{fg}(\alpha_s/N) - \frac{2N_f}{6\pi b_0}\right), \tag{4.41}$$

where the second term in the brackets is the one–loop contribution to $\tilde{\gamma}^1_{fg}(\alpha_s/N)$. It is also known that the nonsinglet anomalous dimension has no poles at $N = 0$, and neither does the nonsinglet coefficient function. Hence, the nonsinglet sector makes very little contribution to the structure function at small x, and as such we will ignore it for the remainder of this section.

A general change in factorization scheme may be expressed by writing an element of the transformation matrix U as

$$U_{ab}(N, \alpha_s(Q^2)) = \sum_{n=-k}^{\infty} \alpha_s^n(Q^2) \sum_{m=1-n}^{\infty} U_{ab}^{nm} \alpha_s^m(Q^2) N^{-m} \equiv \sum_{n=-k}^{\infty} \alpha_s^n(Q^2) U_{ab}^n(\alpha_s(Q^2)/N), \tag{4.42}$$

with condition on the U_{ab}^{nm} such that U obeys $U_{ab} = \delta_{ab} + \mathcal{O}(\alpha_s)$. This flexibility in the factorization scheme means that all the above results on the low–order $\tilde{\gamma}$'s are in principle factorization scheme dependent. In particular, the expressions for the $\tilde{\gamma}$'s in (4.39) may be insufficient in general, and should be replaced by

$$\tilde{\gamma}(N, Q^2) = \sum_{n=-k}^{\infty} \alpha_s^n(Q^2) \sum_{m=1-n}^{\infty} \tilde{a}_{nm} \alpha_s^m(Q^2) N^{-m}. \tag{4.43}$$

However, this is only necessary if one makes a change of scheme away from a standard scheme, such as the $\overline{\text{MS}}$ scheme, using a matrix U containing terms such as $\alpha_s^n(Q^2)N^{-m}$ where $m > n$. For simplicity and because, of course, all physical results are ultimately factorization scheme independent by definition using any expansion, we will forbid such singular changes of scheme and demand that $k = 0$ in (4.42).

However, a change of factorization scheme with $k = 0$ can still introduce scheme dependence into the $\tilde{\gamma}^0_{ab}(\alpha_s/N)$'s. Using (2.12) we see that

$$\tilde{\tilde{\gamma}}^0_{ab}(\alpha_s/N) = \sum_c \sum_d U^0_{ac}(\alpha_s/N) \tilde{\gamma}^0_{cd}(\alpha_s/N) (U^{-1})^0_{db}(\alpha_s/N). \tag{4.44}$$

Insisting that $\tilde{\gamma}^0_{gg}(\alpha_s/N)$, $\tilde{\gamma}^0_{gf}(\alpha_s/N)$, $\tilde{\gamma}^0_{ff}(\alpha_s/N)$ and $\tilde{\gamma}^0_{fg}(\alpha_s/N)$ are unaltered by scheme changes leads to the requirement,

$$\begin{aligned} &U^0_{fg}(N,\alpha_s) = 0, \qquad\qquad U^0_{ff}(N,\alpha_s) = 1, \\ &U^0_{gg}(\alpha_s/N) = 1 + \sum_{n=1}^{\infty} U^{0,n}_{gg}(\alpha_s(Q^2)/N)^n, \qquad U^0_{gf}(\alpha_s/N) = \tfrac{4}{9}(U^0_{gg}(\alpha_s/N) - 1). \end{aligned} \tag{4.45}$$

This requirement also preserves the relationship (4.41) between $\tilde{\gamma}^1_{ff}(\alpha_s/N)$ and $\tilde{\gamma}^1_{fg}(\alpha_s/N)$. Again for simplicity, and due to factorization scheme invariance of physical quantities, we will only consider factorization scheme changes away from $\overline{\rm MS}$ scheme of the type (4.45). The $U^n_{ab}(\alpha_s/N)$ for $n > 0$ will have no restrictions.[14]

Restricting ourselves to these schemes we may now write

$$C^S_2(N,\alpha_s) = 1 + \sum_{n=1}^{\infty} \alpha_s^n(Q^2) \sum_{m=1-n}^{\infty} C^S_{2,n,m} \alpha_s^m(Q^2) N^{-m} \equiv 1 + \sum_{n=1}^{\infty} \alpha_s^n(Q^2) C^S_{2,n}(\alpha_s/N), \tag{4.46}$$

and all other coefficient functions as

$$C^a_i(N,\alpha_s) = \sum_{n=1}^{\infty} \alpha_s^n(Q^2) \sum_{m=1-n}^{\infty} C^a_{i,n,m} \alpha_s^m(Q^2) N^{-m} \equiv \sum_{n=1}^{\infty} \alpha_s^n(Q^2) C^a_{i,n}(\alpha_s/N). \tag{4.47}$$

All the $C^a_{i,n}(\alpha_s/N)$ are both renormalization–scheme– and factorization–scheme–dependent quantities. Indeed, all of the $C^a_{i,n,m}$ are renormalization scheme and factorization scheme dependent, except for the $C^a_{L,n,1-n}$, which come from the one–loop longitudinal coefficient functions which, as we saw in the previous subsection, are totally scheme independent. There are also two renormalization and factorization scheme (with our restrictions) independent relationships between coefficient functions:

$$C^S_{2,1}(\alpha_s/N) = \frac{4}{9}\Big(C^g_{2,1}(\alpha_s/N) - C^g_{2,1,0}\Big), \tag{4.48}$$

where the second term in brackets is the one–loop contribution to $C^g_{2,1}(\alpha_s/N)$, which is itself renormalization scheme and factorization scheme dependent (being equal to $(N_f/6\pi)$ in $\overline{\rm MS}$ scheme); and

$$\Big(C^S_{L,1}(\alpha_s/N) - \frac{2}{3\pi}\Big) = \frac{4}{9}\Big(C^g_{2,1}(\alpha_s/N) - \frac{2N_f}{6\pi}\Big), \tag{4.49}$$

where the second terms in the brackets are the one–loop contributions to $C^S_{L,1}(\alpha_s/N)$ and $C^g_{L,1}(\alpha_s/N)$, both of which are renormalization and factorization scheme independent.

[14] To our knowledge, no–one has yet bothered considering the effects of any transformations of the type we have forbidden (of course, there is no point, since physical quantities are factorization scheme invariant). A number of the type described above with $U^0_{gg}(N,\alpha_s) \neq 0$ have been considered (however, once again, physical quantities are invariant under such scheme changes).

Working in an arbitrary factorization scheme (up to the above restrictions) and using the general expressions for the $\tilde{\gamma}$'s and coefficient functions, we may find expressions for the structure functions. The first step towards this is solving the renormalization group equations for the parton distributions. The lowest–order part of the equation is,

$$\alpha_s^2(Q^2)\frac{d}{d\alpha_s(Q^2)}\begin{pmatrix} f_0^S(N,Q^2) \\ g_0(N,Q^2)\end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ \frac{4}{9}\tilde{\gamma}_{gg}^0(\alpha_s/N) & \tilde{\gamma}_{gg}^0(\alpha_s/N)\end{pmatrix}\begin{pmatrix} f_0^S(N,Q^2) \\ g_0(N,Q^2)\end{pmatrix}. \tag{4.50}$$

This may easily be solved to give

$$\begin{aligned} f_0^S(N,Q^2) &= f_0^S(N,Q_0^2), \\ g_0(N,Q^2) &= (g_0(N,Q_0^2)+\tfrac{4}{9}f_0^S(N,Q_0^2))\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] - \tfrac{4}{9}f_0^S(N,Q_0^2). \end{aligned} \tag{4.51}$$

This is analogous to the lowest–order solution within the loop expansion and contains two factors, one of which must appear in all the higher terms in the expansion: instead of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\tilde{\gamma}_{+,-}^{0,l}(N)}$, corresponding to the two eigenvalues of $\tilde{\gamma}^{0,l}(N)$ in the loop expansion, we have $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$ and 1, corresponding to the two eigenvalues of $\tilde{\gamma}^0(\alpha_s/N)$ in the leading–$\ln(1/x)$ expansion, i.e. $\tilde{\gamma}_{gg}^0(\alpha_s/N)$ and 0. In particular we notice that $f_0^S(N,Q^2)$ is Q^2–independent.

Let us briefly digress in order to discuss the form of the inputs for the parton distributions. In particular, we examine how the terms in our expression change under the change in input scale $Q_0^2 \to (1+\delta)Q_0^2$, leading to the change in coupling at the starting scale (4.7). Rather trivially, our expression for $f_0^S(N,Q^2) \equiv f_0^S(N,Q_0^2)$ is unchanged by the change in the coupling at the starting scale, and as such can be chosen to be independent of Q_0^2. Hence, as for the lowest order input in the loop expansion we may write $f_0^S(N,Q^2) \equiv f_0^S(N,Q_0^2) \equiv f_0^S(N)$. The expression for the gluon involves a little more work. The change in the evolution term under a change in starting scale is

$$\begin{aligned} &\Delta\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ &\qquad = \exp\left[\int_{\alpha_s(Q_0^2)}^{\alpha_s(((1+\delta)Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)) - 1\right]\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right], \end{aligned} \tag{4.52}$$

and using (4.7) we find that

$$\begin{aligned} &\Delta\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ &\qquad = (\exp[-\delta b_0\tilde{\gamma}_{gg}^0(\alpha_s(Q_0^2)/N)) - 1]\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ &\qquad + \text{higher order in } \alpha_s \text{ and/or } N. \end{aligned} \tag{4.53}$$

We choose $g_0(N,Q_0^2)$ so that the change in $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right]$ can be compensated for by a change in $g_0(N,Q_0^2)$ up to corrections of higher order. Hence, the gluon input may be written as

$$g_0(N,Q_0^2) = g_0(N) + (g_0(N) + \tfrac{4}{9}f_0^S(N))\sum_{m=1}^{\infty} g_{0,m}(Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{N}\right)^m \equiv g_0(N) + \tilde{g}_0(N,Q_0^2). \quad (4.54)$$

The change in $g_0(N,Q_0^2)$ under a change in starting scale necessary to make $g_0(N,Q^2)$ invariant under changes of Q_0^2 up to higher orders may easily be calculated from (4.53), and is equal to

$$\begin{aligned}\Delta g_0(N,Q_0^2) &= \delta b_0\tilde{\gamma}^0_{gg}(\alpha_s(Q_0^2)/N)(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2)) + \text{higher order}\\ &\equiv \delta\gamma^0_{gg}(\alpha_s(Q_0^2)/N)(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2)) + \text{higher order},\end{aligned} \quad (4.55)$$

in the limit of small δ. Thus, an appropriate expression for the input which satisfies this condition is given by

$$\begin{aligned}&(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2))\\ &\qquad = (g_0(N) + \tfrac{4}{9}f_0^S(N))\left(1 + \sum_{m=1}^{\infty}\tilde{g}_{0,m}\left(\frac{\alpha_s(Q_0^2)}{N}\right)^m\right)\exp[\ln(Q_0^2/A_{gg})\gamma^0_{gg}(\alpha_s(Q_0^2)/N)],\end{aligned} \quad (4.56)$$

where A_{gg} is an unknown scale. The series $\sum_{m=1}^{\infty}\tilde{g}_{0,m}(\alpha_s(Q_0^2)/N)^m$ is at yet undetermined, but is potentially renormalization and factorization scheme dependent. It will only be determined when we come to construct the structure functions themselves.

We also consider the form of the N–dependence of our inputs. If we take the point of view that any steep behaviour in the parton distributions only comes about due to perturbative effects, then we assume that $f_0^S(N)$ and $g_0(N)$ are both soft, i.e. either flat or even valence–like when the transform to x–space is performed (or at most going like a finite, small power of $\ln(1/x)$). This requires that they both be analytic for $N > 0$. We also note that the soft parts of the input are common for the whole of the gluon input, as shown in (4.54), i.e. $\tilde{g}_0(N,Q_0^2)$ is just the soft $(g_0(N) + \frac{4}{9}f_0^S(N))$ multiplied by a series of the form $\sum_{m=1}^{\infty}a_m(\alpha_s(Q_0^2)/N)^m$. Thus, as in the loop expansion, we may think of $g_0(N)$, and $f_0^S(N)$ as fundamentally soft, nonperturbative parts of the input. [15] The parts multiplying these are then really determined by perturbation theory. Since we are meant to be expanding our solution for the structure functions in powers of both α_s and N, it might be argued that we should expand $f_0^S(N)$ and $g_0(N)$ in powers of N. We feel this is not really appropriate since it is the perturbative part of the solution for which we are able to solve, and thus

[15] Of course, we should be trying to find fundamentally nonperturbative inputs for the structure functions rather than the partons, as discussed for the loop expansion. However, in this expansion scheme $f_0^S(N)$ and $g_0(N)$ are trivially related to $F_{2,0}(N)$ and $\hat{F}_{L,0}(N)$ as we will see in the next subsection.

which we are able to order correctly, and in the expressions for the structure functions the whole of the nonperturbative inputs should multiply the well–ordered perturbative parts of the solution.

Before using our solutions for the parton densities to construct expressions for the structure functions we will solve higher–order renormalization group equations in order to determine the general form of the solutions for the parton distributions. This is made easier by the fact that $\tilde{\gamma}^0_{ff}(\alpha_s/N)$ and $\tilde{\gamma}^0_{fg}(\alpha_s/N)$ are both zero, and hence,

$$\alpha_s(Q^2)\frac{d\,f_1^S(N,Q^2)}{d\alpha_s(Q^2)} = -(\tilde{\gamma}^1_{ff}(\alpha_s/N)f_0^S(N,Q^2) + \tilde{\gamma}^1_{fg}(\alpha_s/N)g_0(N,Q^2)). \tag{4.57}$$

All of the above quantities on the right–hand side are in principle already known. Thus, we have an expression for $(d\,f_1^S(N,Q^2)/d\alpha_s(Q^2))$ which can be written in the simple form

$$\alpha_s^2(Q^2)\frac{d\,f_1^S(N,Q^2)}{d\alpha_s(Q^2)} = -\,\alpha_s(Q^2)\tilde{\gamma}^1_{fg}(\alpha_s/N)(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2))\times \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] + \alpha_s(Q^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}f_0^S(N), \tag{4.58}$$

where (4.41) has been used. Thus, $\alpha_s^2(Q^2)d\,f_1^S(N,Q^2)/d\alpha_s(Q^2)$ is simply a sum of two power series of the form $\alpha_s\sum_{m=0}^{\infty}a_m(\alpha_s/N)^m$ multiplying input densities and our eigenvalue determined evolution factors (the second series rather trivially having $a_m = 0, m \geq 0$). Thus, all terms in the expression are of the same order in this expansion scheme.

Integrating (4.58) we get

$$f_1^S(N,Q^2) = -\,\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)f_0^S(N) + f_1^S(N,Q_0^2) + (g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2))\times \int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}\exp\left[\int_{\alpha_s(q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(r^2)/N)}{\alpha_s^2(r^2)}d\,\alpha_s(r^2)\right]d\alpha_s(q^2). \tag{4.59}$$

We may express the last term differently by integrating by parts. This gives

$$\begin{aligned} f_1^S(N,Q^2) =& \left(\alpha_s(Q^2)\frac{\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)}{\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]\right. \\ &\left. - \alpha_s(Q_0^2)\frac{\tilde{\gamma}^1_{fg}(\alpha_s(Q_0^2)/N)}{\tilde{\gamma}^0_{gg}(\alpha_s(Q_0^2)/N)}\right)(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2)) \\ &- \int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{d}{d\alpha_s(q^2)}\left(\alpha_s(q^2)\frac{\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}\right)\times \\ &\exp\left[\int_{\alpha_s(q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(r^2)/N)}{\alpha_s^2(r^2)}d\,\alpha_s(r^2)\right]d\alpha_s(q^2)(g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2)) \\ &- \frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)f_0^S(N) + f_1^S(N,Q_0^2). \end{aligned} \tag{4.60}$$

$\alpha_s(\tilde{\gamma}^1_{fg}(\alpha_s/N)/\tilde{\gamma}^0_{gg}(\alpha_s/N))$ is a power series of the form $N\sum_{m=0}^{\infty}a_m(\alpha_s/N)^m$. The integration by parts may be repeated indefinitely, producing power series which behave like $N\alpha_s^k(Q^2)\sum_{m=-k}^{\infty}a_m(\alpha_s(Q^2)/N)^m$, with k increasing by one at each integration. Hence, the solution to the next–to–leading–order renormalization group equation for $f^S(N,Q^2)$ is not of a single consistent order in this expansion scheme. We may write it as

$$f_1^S(N,Q^2) = (g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times$$
$$\left(\alpha_s(Q^2)\frac{\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)}{\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]-\alpha_s(Q_0^2)\frac{\tilde{\gamma}^1_{fg}(N,\alpha_s(Q_0^2))}{\tilde{\gamma}^0_{gg}(N,\alpha_s(Q_0^2))}\right)$$
$$+f_1^S(N,Q_0^2)-\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)f_0^S(N)+\text{higher order in }\alpha_s\text{ and/or }N. \tag{4.61}$$

Let us examine the form of our solution for $f_1^S(N,Q^2)$. No part of it is of the same form as the zeroth–order parton distributions, i.e. either just a constant soft distribution or such a constant soft distribution multiplied by powers of $\alpha_s(Q_0^2)/N$ and the evolution term $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$. Thus, the lowest–order corrections to this zeroth–order solution are given by the lowest–order parts of (4.61).

In the same manner as for the zeroth–order gluon distribution we can determine the general form of our input, in this case $f_1^S(N,Q_0^2)$, by considering a change of starting scale. Under the change of scale leading to the change of $\alpha_s(Q_0^2)$ in (4.7) each of the $\alpha_s(Q_0^2)$–dependent terms in (4.61) will change. The change of $\tilde{g}_0(N,Q_0^2)$ has already been chosen to cancel (at leading order) the change due to $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$, and hence the first term in (4.61) is stable to the change in starting scale. The last explicit term in (4.61) is also stable since the change induced in the $\ln(\alpha_s(Q_0^2))$ term is of higher order in $\alpha_s(Q_0^2)$. The second term does however vary; the change due to the variation of $\alpha_s(Q_0^2)\tilde{\gamma}^1_{fg}(\alpha_s(Q_0^2)/N)/\tilde{\gamma}^0_{gg}(\alpha_s(Q_0^2)/N)$ is of higher order in $\alpha_s(Q_0^2)$, but $\tilde{g}_0(N,Q_0^2)$ changes as prescribed in (4.55), and hence the second term changes by an amount of the same order as itself, i.e. by a series of the type $\alpha_s(Q_0^2)\sum_n a_n(\alpha_s(Q_0^2)/N)^n$. In order to keep the whole of (4.61) unchanged at this order, $f_1^S(N,Q_0^2)$ must change under a change in Q_0^2 in a manner which compensates this change in the second term. Hence, we choose

$$f_1^S(N,Q_0^2) = (g_0(N)+\tfrac{4}{9}f_0^S(N))N\sum_{m=0}^{\infty}f_{1,m}^S(Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{N}\right)^m. \tag{4.62}$$

The change of $f_{1,m}^S(Q_0^2)$ under a change in Q_0^2 may easily be calculated using the known change in $\tilde{g}_0(N,Q_0^2)$, i.e.

$$\Delta f_1^S(N,Q_0^2) = \delta\alpha_s(Q_0^2)\gamma^1_{fg}(\alpha_s(Q_0^2)/N)(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))+\text{higher order}, \tag{4.63}$$

in the limit of small δ. Rather obviously, the simplest choice which satisfies the requirement is

$$f_1^S(N,Q_0^2) = (g_0(N) + \tfrac{4}{9} f_0^S(N))\alpha_s(Q_0^2)\frac{\gamma_{fg}^1(\alpha_s(Q_0^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}\left(1 + \sum_{m=0}^{\infty} \tilde{g}_{0,n}\left(\frac{\alpha_s(Q_0^2)}{N}\right)^m\right) \times \exp[\ln(Q_0^2/A_{gg})\gamma_{gg}^0(\alpha_s(Q_0^2)/N)],$$

$$\equiv (g_0(N) + \tfrac{4}{9} f_0^S(N) + \tilde{g}_0(N,Q_0^2))\alpha_s(Q_0^2)\frac{\gamma_{fg}^1(\alpha_s(Q_0^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}. \tag{4.64}$$

However, as when deriving the $\mathcal{O}(\alpha_s(Q_0^2))$ input for the nonsinglet structure function within the loop expansion (4.22), the invariance of the parton distribution under changes in Q_0^2 is not the sole consideration. We also require that at lowest order the expression for the input for the structure function is renormalization scheme independent, and that at all orders it is factorization scheme independent. Also, we think of the scale A_{gg} (or A_{NS} *etc.*) as the value of Q_0^2 where the input for the structure function becomes just the nonperturbative input, so $f_1^S(N,Q_0^2)$ should be consistent with this. If we add a series of the form

$$\tilde{f}_1^S(N,Q_0^2) = N\sum_{n=0}^{\infty} \tilde{f}_{1,m}^S\left(\frac{\alpha_s(Q_0^2)}{N}\right)^m (g_0(N) + \tfrac{4}{9} f_0^S(N)) \tag{4.65}$$

to (4.64) then the resulting expression still satisfies (4.63), but gives us the flexibility to satisfy the other requirements above. Thus, similarly to (4.24) we write

$$f_1^S(N,Q_0^2) = (g_0(N) + \tfrac{4}{9} f_0^S(N) + \tilde{g}_0(N,Q_0^2))\alpha_s(Q_0^2)\frac{\gamma_{fg}^1(\alpha_s(Q_0^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)} + \tilde{f}_1^S(N,Q_0^2), \tag{4.66}$$

as the appropriate input, where $\tilde{f}_1^S(N,Q_0^2)$ is potentially renormalization and factorization scheme dependent, and is not yet determined.

Therefore, the leading correction to the zeroth–order input $f_0^S(N)$ is a series of the form $N\sum_{m=0}^{\infty}(\alpha_s(Q_0^2)/N)^m$ multiplying $(g_0(N) + \frac{4}{9} f_0^S(N))$. The lowest–order parts of the Q^2–dependent expression for $f^S(N,Q^2)$ are a series of the form $N\sum_{m=0}^{\infty}(\alpha_s(Q^2)/N)^m$ multiplying $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\gamma_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} d\alpha_s(q^2)\right]$ and a zeroth–order input, and also a logarithm of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ multiplying the other common factor of unity and a zeroth–order input. $f_1^S(N,Q^2)$ then contains other terms which are subleading, i.e. of higher order in α_s and/or N, to these.

Hence, this solution for the next–to–leading evolution equation for the singlet quark distribution clearly demonstrates that we will not obtain well–ordered, factorization–scheme–independent expressions in the leading–$\ln(1/x)$ expansion in as straightforward a manner as when using the loop expansion.

In order to investigate this further, we now consider the NLO renormalization group equation for $g_1(N,Q^2)$. This is a little more complicated that that for $f_1^S(N,Q^2)$:

$$\alpha_s^2(Q^2)\frac{d\,g_1(N,Q^2)}{d\alpha_s(Q^2)} = -\Big(\alpha_s\tilde{\gamma}_{gf}^1(\alpha_s/N)f_0^S(N,Q^2) + \alpha_s\tilde{\gamma}_{gg}^1(\alpha_s/N)g_0(N,Q^2) + \tilde{\gamma}_{gf}^0(\alpha_s/N)f_1^S(N,Q^2) + \tilde{\gamma}_{gg}^0(\alpha_s/N)g_1(N,Q^2)\Big). \tag{4.67}$$

This may be solved, and integrating some of the resulting terms by parts, we obtain

$$\begin{aligned} g_1(N,Q^2) =& \Bigg(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N) + \frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\times \\ & (g_0(N) + \tfrac{4}{9}f_0^S(N) + \tilde{g}_0(N,Q_0^2)) + g_1(N,Q_0^2)\Bigg)\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ & - \tfrac{4}{9}f_0^S(N)\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right) + \text{higher order in } \alpha_s \text{ and/or } N. \end{aligned} \tag{4.68}$$

The integral in the first term gives a series of the form $\sum_{m=1}^{\infty}a_m(\alpha_s/N)^m$ plus a term of the sort $\ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))$. So those terms explicitly shown above are the lowest–order terms in $g_1(N,Q^2)$. The input $g_1(N,Q_0^2)$ is present in order to compensate for the change in $\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)$ under a change in Q_0^2. However, this means it is a series of the form $\alpha_s(Q_0^2)\sum_n a_n(\alpha_s(Q_0^2)/N)^n$, and as such is higher order than the other terms present, and may be dropped. We see that the solution for $g_1(N,Q^2)$ contains terms of the same apparent order as the zeroth–order solutions for the parton distributions. Therefore $g_0(N,Q^2)$ seems to be only part of the lowest–order expression for the gluon distribution, and the terms shown above appear to be on the same footing. These are of a similar form to the leading parts of $f_1^S(N,Q^2)$, with one term being a logarithm of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ multiplying $f_0^S(N,Q_0^2)$, but the series multiplying $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$ is a power of N, or equivalently of α_s, lower than the corresponding series in the case of $f_1^S(N,Q^2)$.

So we have the rather undesirable situation that solutions to the NLO renormalization group equations for both the quark and gluon distribution functions result in solutions which are of mixed order, but fortunately, ones from which we can clearly extract the leading–$\ln(1/x)$ behaviour. However, the situation gets worse, as may be seen by solving the NNLO renormalization group equations. For the quark distribution function this is

$$\alpha_s(Q^2)\frac{d\,f_2^S(N,Q^2)}{d\alpha_s(Q^2)} = -\Big(\alpha_s(Q^2)\tilde{\gamma}_{ff}^2(\alpha_s/N)f_0^S(N,Q^2) + \alpha_s(Q^2)\tilde{\gamma}_{fg}^2(\alpha_s/N)g_0(N,Q^2) + \tilde{\gamma}_{ff}^1(\alpha_s/N)f_1^S(N,Q^2) + \tilde{\gamma}_{fg}^1(\alpha_s/N)g_1(N,Q^2)\Big). \tag{4.69}$$

As with the equation for $f_1^S(N,Q^2)$ all terms on the right are already known in principle (it is clear that this will be true for the equations for all the $f_n^S(N,Q^2)$). Moreover, part of the right–hand side of (4.69) is of the same order as the right–hand side of (4.57), as is seen by inserting the leading parts of $f_1^S(N,Q^2)$ and $g_1(N,Q^2)$, i.e.

$$\alpha_s^2(Q^2)\frac{d\,f_2^S(N,Q^2)}{d\alpha_s(Q^2)} = -\alpha_s(Q^2)\tilde{\gamma}_{fg}^1(\alpha_s/N)\Bigg(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\times$$
$$(g_0(N)+\tfrac{4}{9}f_0^S(N))+\tilde{g}_0(N,Q_0^2)\Bigg)\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$$
$$-\,f_0^S(N)\alpha_s(Q^2)\left(\frac{4}{9}\frac{2N_f}{6\pi b_0}\right)^2\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)+\text{higher order in }\alpha_s\text{ and/or }N. \tag{4.70}$$

Solving (4.69) we find that $\tilde{\gamma}_{ff}^2(\alpha_s/N)$ and $\tilde{\gamma}_{fg}^2(\alpha_s/N)$ play no role as far as the leading part of the solution is concerned, and

$$f_2^S(N,Q^2) = \alpha_s(Q^2)\frac{\tilde{\gamma}_{fg}^1(\alpha_s(Q^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q^2)/N)}\Bigg(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\times$$
$$(g_0(N)+\tfrac{4}{9}f_0^S(N))+\tilde{g}_0(N,Q_0^2)\Bigg)\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$$
$$+\frac{1}{2}\left(\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right)^2 f_0^S(N)+\text{higher order in }\alpha_s\text{ and/or }N. \tag{4.71}$$

Thus, the leading part of $f_2^S(N,Q^2)$ has terms of the same order as the leading part of $f_1^S(N,Q^2)$, i.e. a sum of the form $N\sum_{m=0}^{\infty}a_m(\alpha_s/N)^m$ multiplying $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$ and soft inputs. We also explicitly include above the parts of $f_2^S(N,Q^2)$ which have terms of the form $\ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ directly multiplying $f_0^S(N)$, even though these are not of the same form as in $f_1^S(N,Q^2)$. We will take these terms to be part of the lowest–order parton distribution since they are not a power of α_s or N higher than any terms already produced. However, we ignore terms of the form $\ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ multiplying series of the form $N\sum_{m=0}^{\infty}a_m(\alpha_s/N)^m$, such as $f_1^S(N,Q_0^2)$, because they are clearly of higher order in N than the term consisting of logarithms multiplying $f_0^S(N)$.

Similarly, for the gluon the presence of lowest–order terms is not limited to $g_1(N,Q^2)$, but is also seen in $g_2(N,Q^2)$:

$$g_2(N,Q^2) = \frac{1}{2}\left(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\right)^2\times$$
$$(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \tag{4.72}$$
$$+\tfrac{4}{9}f_0^S(N)\frac{1}{2}\left(\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right)^2+\text{higher order in }\alpha_s\text{ and/or }N.$$

This phenomenon persists to all orders, but is independent of the $\tilde{\gamma}^n$'s for $n \geq 1$. It can be proved by induction in a straightforward manner that the leading part of the solutions to the $n_{\rm th}$–order renormalization group equations are

$$\begin{aligned} f_n^S(N,Q^2) =& \alpha_s(Q^2)\frac{\tilde{\gamma}_{fg}^1(\alpha_s(Q^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q^2)/N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]\times \\ & \frac{1}{(n-1)!}\left(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\right)^{n-1}\times \\ & (g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ & +\frac{(-1)^n}{n!}\left(\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right)^n f_0^S(N)+\text{higher order in }\alpha_s\text{ and/or }N, \end{aligned} \tag{4.73}$$

for $n \geq 2$, and

$$\begin{aligned} g_n(N,Q^2) =& \frac{1}{n!}\left(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{(\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N))}{\alpha_s(q^2)}d\alpha_s(q^2)\right)^n\times \\ & (g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right] \\ & -\tfrac{4}{9}f_0^S(N)\frac{(-1)^n}{n!}\left(\frac{4}{9}\frac{2N_f}{6\pi b_0}\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right)^n+\text{higher order in }\alpha_s\text{ and/or }N, \end{aligned} \tag{4.74}$$

for $n \geq 0$. One can therefore find well–ordered parts of the full solutions for $f^S(N,Q^2)$ and $g(N,Q^2)$ by summing all such terms, leading to

$$\begin{aligned} f^S(N,Q^2) =& \alpha_s(Q^2)\frac{\tilde{\gamma}_{fg}^1(\alpha_s(Q^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q^2)/N)}(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times \\ & \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right] \\ & +f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}}+\tilde{f}_1^S(N,Q_0^2)+\text{higher order in }\alpha_s\text{ and/or }N, \end{aligned} \tag{4.75}$$

and,

$$\begin{aligned} g(N,Q^2) =& (g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times \\ & \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}_{gg}^1(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}_{fg}^1(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right] \\ & -\tfrac{4}{9}f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}}+\text{higher order in }\alpha_s\text{ and/or }N. \end{aligned} \tag{4.76}$$

We may also write the expression for the rate of change of the quark distribution

$$-\alpha_s^2(Q^2)\frac{d\,f^S(N,Q^2)}{d\alpha_s(Q^2)} = \alpha_s(Q^2)\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right]$$
$$-\tfrac{4}{9}\alpha_s(Q^2)f_0^S(N)\frac{2N_f}{6\pi b_0}\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N. \tag{4.77}$$

This last expression will be important since at leading order $\alpha_s^2(Q^2)(d\,F_2(N,Q^2)/d\alpha_s(Q^2))$ is directly related to $d\,F_2(N,Q^2)/d\ln Q^2$ which we will wish to study as well as $F_2(N,Q^2)$.

4.3. The Leading–$\ln(1/x)$ Expansion: Structure Functions.

It is now possible to examine the form of the solutions for the structure functions. We can construct the leading part of the full solutions by combining our leading solutions for the parton distributions with the zeroth– and first–order coefficient functions. This gives

$$F_L(N,Q^2) = \alpha_s(Q^2)C^g_{L,1}(\alpha_s(Q^2)/N)(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right]$$
$$+\alpha_s(Q^2)(C^S_{L,1,0}-\tfrac{4}{9}C^g_{L,1,0})f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N, \tag{4.78}$$

where (4.49) has been used. Also

$$F_2(N,Q^2) = \alpha_s(Q^2)\left(\frac{\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)}{\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N)}+C^g_{2,1}(\alpha_s(Q^2)/N)\right)(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right] \tag{4.79}$$
$$+f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \tilde{f}_1^S(N,Q_0^2) + \text{higher order in } \alpha_s \text{ and/or } N.$$

We can also write

$$-\alpha_s^2(Q^2)\frac{d\,F_2(N,Q^2)}{d\alpha_s(Q^2)} = (\alpha_s(Q^2)\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)+$$
$$\alpha_s(Q^2)C^g_{2,1}(\alpha_s(Q^2)/N)\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N))(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right]$$
$$-\alpha_s(Q^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N. \tag{4.80}$$

In each case, since we only consider the singlet structure functions in this expansion scheme, we drop the superscript S for the structure functions for the rest of this subsection.

It is clear that each of these expressions will be factorization scheme independent since each represents the expression for a physical quantity up to corrections of a form different from the terms explicitly appearing and which we have deemed to be higher order in our expansion scheme. $(\alpha_s^2(Q^2)(dF_2(N,Q^2)/d\alpha_s(Q^2))$ has as much right to be considered as a physical quantity as $F_2(N,Q^2)$, since at leading order it is proportional to $(dF_2(N,Q^2)/d\ln Q^2)$. However, we would like to find out if each of these expressions may itself be more rigorously ordered and, as a by–product, split into factorization–scheme invariant pieces. This would be desirable from a theoretical point of view, since the current expressions will soon be seen to be incompatible with renormalization scheme consistency. Also, from the practical point of view we do not know all of the terms appearing in the explicit parts of these expressions in any factorization scheme, i.e. $\tilde{\gamma}_{gg}^1(\alpha_s(Q^2)/N)$ is not at present known, and we would clearly like to have some sort of scheme–independent expression for the structure functions involving terms we already know.

In order to see if we can obtain scheme–independent and thus physically relevant subsets of the solutions (4.78)–(4.80), we shall examine in detail the form of these solutions. Since it is the least complicated, and because we will find it useful, we begin with the expression (4.78) for the longitudinal structure function. Since this expression contains two terms of rather different form, i.e. the first depending on the factor $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$ and the second having an evolution unenhanced by leading–$\ln(1/x)$ terms, it would be surprising if these terms were not separately factorization scheme invariant. It is simple to check that this is indeed the case.

Beginning with the second term, it is clear that this is factorization scheme independent. The power of $(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ comes from the one–loop contribution to $\tilde{\gamma}_{fg}^1(\alpha_s/N)$ which is both factorization scheme and renormalization scheme invariant and which we have already written in its actual numerical form. $\alpha_s(Q^2)$ and $f_0^S(N)$ are both clearly renormalization and factorization scheme dependent, and the one–loop contributions to the longitudinal coefficient functions are also, i.e. $C_{L,1,0}^S = (2/3\pi)$ and $C_{L,1,0}^g = (2N_f/6\pi)$ in all schemes. Hence the whole term is both factorization scheme and renormalization scheme independent, as it must be, depending only on leading–order quantities.

The first term is not as simple to deal with. It must be factorization scheme independent; however, many of the pieces appearing in the term are clearly not, e.g. $\tilde{\gamma}_{fg}^1(\alpha_s/N)$ and $C_{L,1}^g(\alpha_s/N)$. So, to a certain extent it must be the interplay between the $\tilde{\gamma}$'s, the coefficient function and the parton inputs which leads to a factorization–scheme–independent result. Remembering our discussion of the solutions for the structure function in the loop expansion, we see that there is at

least one way of writing this term as the product of two factorization–scheme–independent pieces, i.e.

$$\begin{aligned}
&\alpha_s(Q^2)C^g_{L,1}(\alpha_s(Q^2)/N)(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\times \\
&\qquad \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}+\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\,\alpha_s(q^2)\right] \\
&=\alpha_s(Q_0^2)C^g_{L,1,0}(g_0(N)+\tfrac{4}{9}f_0^S(N))\left(\frac{C^g_{L,1}(\alpha_s(Q_0^2)/N)}{C^g_{L,1,0}}\cdot\left(1+\frac{\tilde{g}_0(N,Q_0^2)}{g_0(N)+\frac{4}{9}f_0^S(N)}\right)\right)\times \\
&\qquad \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]\exp\left[-\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right]\times \\
&\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}-\frac{d}{d\alpha_s(q^2)}\left(\ln\left(\frac{C^g_{L,1}(\alpha_s(q^2)/N)}{C^g_{L,1,0}}\right)\right)d\,\alpha_s(q^2)\right].
\end{aligned} \tag{4.81}$$

The left–hand side is factored into its value at Q_0^2 and a term which gives the evolution from this value. Clearly these are physically distinguishable and as such must be individually factorization scheme invariant. Also, we already know that $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]$ is factorization scheme invariant on its own, and $\exp[-\ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$ clearly is. Hence, we make the definition

$$\Phi_1^+(Q^2,Q_0^2)=\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^1_{gg}(\alpha_s(q^2)/N)+\frac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(q^2)/N)}{\alpha_s(q^2)}-\frac{d}{d\alpha_s(q^2)}\left(\ln\left(\frac{C^g_{L,1}(\alpha_s(q^2)/N)}{C^g_{L,1,0}}\right)\right)d\,\alpha_s(q^2), \tag{4.82}$$

where $\Phi_1^+(Q^2,Q_0^2)$ must now be factorization–scheme independent (as can be checked using the rules (2.10)–(2.12)) and $\Phi_1^+(Q_0^2,Q_0^2)=1$.

Having isolated the factorization–scheme–independent parts, we may factorize (4.78) completely into factorization–scheme–invariant input and evolution parts, i.e.

$$\begin{aligned}
F_L(N,Q^2)&=\alpha_s(Q_0^2)\frac{2N_f}{6\pi}(g_0(N)+\tfrac{4}{9}f_0^S(N))\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\gamma}^0_{gg}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\,\alpha_s(q^2)\right]\times \\
&\left(\frac{C^g_{L,1}(\alpha_s(Q_0^2)/N)}{C^g_{L,1,0}}\cdot\left(1+\frac{\tilde{g}_0(N,Q_0^2)}{g_0(N)+\frac{4}{9}f_0^S(N)}\right)\right)\exp\left[\Phi_1^+(Q^2,Q_0^2)-\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right] \\
&+\alpha_s(Q^2)\left(\frac{18-4N_f}{27\pi}\right)f_0^S(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q_0^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}-1}+\text{higher order in }\alpha_s\text{ and/or }N.
\end{aligned} \tag{4.83}$$

It is now possible to attach direct physical significance to each of the factorization–scheme–independent pieces appearing in this expression.

We first consider the inputs. Going to (4.79) for the moment we see that $f_0^S(N)$ is the only term in $F_2(N,Q_0^2)$ which is zeroth–order in our expansion scheme. As such it is the zeroth order input for $F_2(N,Q^2)$, and we may write

$$f_0^S(N)=F_{2,0}(N), \tag{4.84}$$

and this is one of our two fundamentally nonperturbative inputs. Also, from (4.83) we see that the total $\alpha_s(Q_0^2)$–independent input for $F_L(N,Q_0^2)$ (once we have divided out a single power of $\alpha_s(Q_0^2)/(2\pi)$) is

$$\frac{2N_f}{3}(g_0(N)+\tfrac{4}{9}f_0^S(N))+\left(\frac{36-8N_f}{27}\right)f_0^S(N). \tag{4.85}$$

This whole expression may therefore be written as $\hat{F}_{L,0}(N)$, and we have

$$\hat{F}_{L,0}(N)=\frac{2N_f}{3}(g_0(N)+\frac{2}{N_f}f_0^S(N)). \tag{4.86}$$

$\hat{F}_{L,0}(N)$ is our other fundamentally nonperturbative physical input. We note that the expression for $F_{2,0}(N)$ in terms of the parton inputs is the same in this case, i.e. (4.84), as in the loop expansion (4.36). However, the definition of $F_{L,0}(N)$ in terms of the parton inputs, (4.86), is not the same as in the loop expansion (4.37). Thus, the definition of the nonperturbative gluon, $g_0(N)$, is different in the two expansion schemes.

We also make a similar definition for the part of the input for $\hat{F}_L(N,Q^2)$ which is of the form $\sum_{n=1}^{\infty}a_n(Q_0^2)(\alpha_s(Q_0)/N)^n$ multiplying $(g_0(N)+\frac{4}{9}f_0^S(N))$, or more correctly multiplying $(\hat{F}_{L,0}(N)-((36-8N_f)/27)F_{2,0}(N))$, to complete our definition of the input in (4.83) and write

$$\begin{aligned}\frac{2N_f}{3}(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\left(\frac{C_{L,1}^g(\alpha_s(Q_0^2)/N)}{C_{L,1,0}^g}\right)\\ =\hat{F}_{L,0}(N)-\left(\frac{36-8N_f}{27}\right)F_{2,0}(N)+\tilde{\hat{F}}_{L,0}(N,Q_0^2).\end{aligned} \tag{4.87}$$

This whole expression must be both factorization scheme and renormalization scheme independent, facts which reveal information about the form of the gluon input. $C_{L,1}^g(N,\alpha_s(Q_0^2))$ is both renormalization and factorization–scheme dependent, so the scheme dependence of $\tilde{g}_0(N,Q_0^2)$ must be precisely so as to cancel this out, i.e.

$$\begin{aligned}&(g_0(N)+\tfrac{4}{9}f_0^S(N)+\tilde{g}_0(N,Q_0^2))\\ &\quad\equiv(g_0(N)+\tfrac{4}{9}f_0^S(N))\left(1+\sum_{m=1}^{\infty}\tilde{g}_{0,m}(\alpha_s(Q_0^2)/N)^m\right)\exp[\ln(Q_0^2/A_{LL})\gamma_{gg}^0(\alpha_s(Q_0^2)/N)]\\ &\quad=(g_0(N)+\tfrac{4}{9}f_0^S(N))\mathrm{g}_0(N,Q_0^2)(C_{L,1,0}^g/C_{L,1}^g(\alpha_s(Q_0^2)/N))\exp[\ln(Q_0^2/A_{LL})\gamma_{gg}^0(\alpha_s(Q_0^2)/N)],\end{aligned} \tag{4.88}$$

where $\mathrm{g}_0(N,Q_0^2)$ is scheme independent. Hence, there is no reason for $\tilde{\hat{F}}_{L,0}(\alpha_s(Q_0^2)/N)$ to depend on the leading–order longitudinal gluon coefficient function at all, and indeed, a natural choice seems to be that $(1+\sum_{m=1}^{\infty}\tilde{g}_{0,m}(\alpha_s(Q_0^2)/N)^m)$ is chosen equal to $(C_{L,1,0}^g/C_{L,1}^g(N,\alpha_s(Q_0^2)))$ (it is difficult to see what else it could be chosen equal to), i.e. $\mathrm{g}_0(N,Q_0^2)=1$, and therefore that

$$\begin{aligned}\hat{F}_{L,0}(N)-\left(\frac{36-8N_f}{27}\right)F_{2,0}(N)+\tilde{\hat{F}}_{L,0}(N,Q_0^2)\\ =\left(\hat{F}_{L,0}(N)-\left(\frac{36-8N_f}{27}\right)F_{2,0}(N)\right)\exp[\ln(Q_0^2/A_{LL})\gamma_{gg}^0(\alpha_s(Q_0^2)/N)].\end{aligned} \tag{4.89}$$

Hence, we have a prediction for the input for the longitudinal structure function at small x in terms of the nonperturbative inputs and some scale A_{LL} (where $A_{LL} = A_{gg}$ from the previous subsection). As with A_{NS} earlier, A_{LL} is the scale at which the input is equal to the nonperturbative input alone, and hence we would expect it to be typical of the scale where perturbation theory starts to break down. As we have already stressed, Q_0^2 is a completely free parameter. We have constructed the solution to be insensitive to Q_0^2 at leading order, but there is clearly some residual Q_0^2–dependence. Hence, there will also be some optimum Q_0^2 to choose as the starting scale.

We may now examine the terms governing the evolution, continuing the convention started in (4.82). $\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} d\alpha_s(q^2)\right]$ is the factorization–scheme–independent factor coming from the eigenvalue of the zeroth–order anomalous dimension which will govern the small–x growth with Q^2. As such we make the definition

$$\Phi_0^+(Q^2, Q_0^2) = \int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\gamma}_{gg}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} d\alpha_s(q^2). \tag{4.90}$$

Of course, the other evolution factor resulting from the eigenvalues of the zeroth–order anomalous dimension was simply unity and as such $\Phi_0^-(Q^2, Q_0^2)$ does exist, but is implicitly zero. However, there is a correction to this factor of unity in (4.78)–(4.80), and we make the definition

$$\Phi_1^-(Q^2, Q_0^2) = -\frac{4}{9}\frac{2N_f}{6\pi b_0} \ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right). \tag{4.91}$$

Having made these factorization–scheme–invariant definitions for the inputs and evolution we may write the solution for $F_L(N, Q^2)$ as

$$\begin{aligned} F_L(N, Q^2) = {} & \frac{\alpha_s(Q_0^2)}{2\pi}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N, Q_0^2) - \left(\frac{36 - 8N_f}{27}\right) F_{2,0}(N)) \times \\ & \exp\left[\Phi_0^+(Q^2, Q_0^2) + \Phi_1^+(Q^2, Q_0^2) - \ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right] \\ & + \frac{\alpha_s(Q_0^2)}{2\pi}\left(\frac{36 - 8N_f}{27}\right) F_{2,0}(N) \exp\left[\Phi_1^-(Q^2, Q_0^2) - \left(\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right)\right] \\ & + \text{higher order in } \alpha_s \text{ and/or } N. \end{aligned} \tag{4.92}$$

It is now relatively obvious how we may separate out the "leading part" from this expression. $[\Phi_1^+(Q^2, Q_0^2) - \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$ contains the same type of terms as $\Phi_0^+(Q^2, Q_0^2)$, but each is a power of N higher. Thus $[\Phi_1^+(Q^2, Q_0^2) - \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$ is subleading to $\Phi_0^+(Q^2, Q_0^2)$, and indeed $\Phi_1^+(Q^2, Q_0^2)$ is a renormalization–scheme–dependent quantity, as it must be in order to absorb the change in $\Phi_0^+(Q^2, Q_0^2)$ resulting from a change in the definition of $\alpha_s(Q^2)$ under a change in renormalization scheme when working beyond leading order. Hence, we should factor $\exp[\Phi_1^+(Q^2, Q_0^2) - \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$ out of the first term. Since $[\Phi_1^-(Q^2, Q_0^2) - \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$

is of the same form as the zeroth–order–in–N part of $\Phi_1^+(Q^2,Q_0^2)$, then it should also be factored out of the leading–order expression for $F_L(N,Q^2)$. Thus, we are left with

$$F_L^0(N,Q^2) = \frac{\alpha_s(Q_0^2)}{2\pi}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\exp[\Phi_0^+(Q^2,Q_0^2)] + \frac{\alpha_s(Q_0^2)}{2\pi}\left(\frac{36-8N_f}{27\pi}\right)F_{2,0}(N). \tag{4.93}$$

Using the one–loop running coupling constant, as is appropriate for a leading–order expression, the whole of (4.93) is not only manifestly factorization scheme independent, but also renormalization scheme independent, as one would hope. Also, $\tilde{\hat{F}}_{L,0}(N,Q_0^2)$ is constructed precisely so as to make the expression unchanged, at this order, under a change in starting scale. Hence, within this expansion scheme (4.93) is genuinely the leading–order expression for $F_L(N,Q^2)$.

We now turn our attention to the more phenomenologically important case of the structure function $F_2(N,Q^2)$. Here we immediately have an ambiguity: should we consider the expression for the structure function itself in this expansion scheme, or that for its α_s–derivative, $\alpha_s^2(Q^2)(dF_2(N,Q^2)/d\alpha_s(Q^2))$ (which, using the definition of the running coupling (2.5), is directly related to $(dF_2(N,Q^2)/d\ln Q^2)$ at leading order)? It may be argued that in certain senses the latter is more natural because it is a "real" perturbative quantity, beginning at first order in α_s, as does $F_L(N,Q^2)$.

There is in fact a distinction between $F_2(N,Q^2)$ and $(d\,F_2/d\,\ln Q^2)$ in the usual loop expansion. Differentiating a fixed–order expression for $F_2(N,Q^2)$ and using the β–function evaluated to the appropriate order in α_s results in the fixed–order expression for $(d\,F_2/d\,\ln Q^2)$[16] plus terms of higher order in $\alpha_s(Q^2)$, which depend on the β–function beyond lowest order (and thus are absent when working to leading order only). Hence, the size of these extra terms is of the same order as the renormalization scheme uncertainty, and therefore the distinction between the fixed–order expressions for $F_2(N,Q^2)$ and $(d\,F_2/d\,\ln Q^2)$ is of similar magnitude to the distinction between renormalization schemes.

There is also a distinction when using the small–x expansion, and it appears more graphically, and is not only dependent on terms in the β-function beyond lowest order. We distinguished between $F_2(N,Q^2)$ and $\alpha_s^2(Q^2)(d\,F_2(N,Q^2)/d\alpha_s(Q^2))$ in (4.79) and (4.80) because the expression to a given order in α_s and N even for $\alpha_s^2(Q^2)(d\,F_2(N,Q^2)/d\alpha_s(Q^2))$ is no longer simply obtained by differentiating $F_2(N,Q^2)$ at given order, and conversely given order in $F_2(N,Q^2)$ is not obtained just by integrating $\alpha_s^2(Q^2)(d\,F_2(N,Q^2)/d\alpha_s(Q^2))$ with respect to α_s at fixed order. This was clearly illustrated when finding the leading part of $f_1^S(N,Q^2)$ by solving the equation for $\alpha_s^2(Q^2)(d\,f_1^S(N,Q^2)/d\alpha_s(Q^2))$ earlier, and it leads to a distinction between even the LO

[16] $(d\,F_2(N,Q^2)/d\,\ln Q^2)$ is analogous to the longitudinal structure function, i.e. it has an overall power of $\alpha_s(Q^2)$ and the lowest–order expression consists of first order–in–$\alpha_s(Q_0^2)$ inputs multiplying the eigenvalue–determined evolution terms $\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\gamma_{+,-}^{0,l}-1}$.

$(d\,F_2/d\ln Q^2)$ and the derivative of the LO $F_2(N,Q^2)$ in this expansion scheme. Thus, even at leading order we have to decide which of the two expressions to use, although we would hope that there is not too much difference between the choices in practice. We will postpone the decision for the present and will derive the form of the leading–order expressions for both $F_2(N,Q^2)$ and its derivative. We can then examine these expressions and use them in order to help us decide.

For technical simplicity we begin with the expression for $\alpha_s^2(Q^2)(d\,F_2(N,Q^2)/d\alpha_s(Q^2))$. Using the definitions introduced in our discussion for the longitudinal structure function we may write (4.80) as

$$\begin{aligned}-\alpha_s^2(Q^2)\frac{d\,F_2(N,Q^2)}{d\alpha_s(Q^2)} =&(\alpha_s(Q^2)\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)+\alpha_s(Q^2)C^g_{2,1}(\alpha_s(Q^2)/N)\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N))\times\\ &\frac{3}{2N_f}\left(\frac{C^g_{L,1,0}}{C^g_{L,1}(\alpha_s(Q^2)/N)}\right)(\hat{F}_{L,0}(N)+\tilde{\hat{F}}_{L,0}(N,Q_0^2)-\left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times\\ &\exp[\Phi_0^+(Q^2,Q_0^2)+\Phi_1^+(Q_0^2,Q^2)]-\alpha_s(Q^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2)]\\ &+\text{higher order in }\alpha_s\text{ and/or }N.\end{aligned} \tag{4.94}$$

The factorization scheme independence of this complete expression guarantees that the term $(\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)+C^g_{2,1}(\alpha_s(Q^2)/N)\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N))(C^g_{L,1,0}/C^g_{L,1}(\alpha_s(Q^2)/N))$ is a factorization–scheme–invariant quantity, of the form $\sum_{m=0}^{\infty}a_m(\alpha_s(Q^2)/N)^m$. It is straightforward to verify this. Indeed, it was shown by Catani and Hautmann that $(\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)+C^g_{2,1}(\alpha_s(Q^2)/N)\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N))$ and $C^g_{L,1}(\alpha_s(Q^2)/N)$ could each always be expressed in terms of the product of a factorization–scheme– and renormalization–scheme–independent factor (which they calculated) and a scheme–dependent factor, where the scheme–dependent part was the same for both [18]. Using their results for the scheme–independent parts it is a trivial matter to find that

$$\begin{aligned}(\gamma^1_{fg}(\alpha_s(Q^2)/N)+C^g_{2,1}(\alpha_s(Q^2)/N)\gamma^0_{gg}(\alpha_s(Q^2)/N)))&\frac{3}{2N_f}\left(\frac{C^g_{L,1,0}}{C^g_{L,1}(\alpha_s(Q^2)/N)}\right)\\ &=\frac{1}{2\pi}\left(\frac{3}{2}\gamma^0_{gg}(\alpha_s(Q^2)/N)+\sum_{n=0}^{\infty}(\gamma^0_{gg}(\alpha_s(Q^2)/N))^n\right),\end{aligned} \tag{4.95}$$

which is clearly both factorization scheme and renormalization scheme independent. Thus, making the definition

$$\alpha_s(Q^2)\tilde{\gamma}^1_{2L}(\alpha_s(Q^2)/N)=\frac{1}{2\pi b_0}\left(\frac{3}{2}+\sum_{n=0}^{\infty}(\alpha_s(Q^2)\gamma^0_{gg}(\alpha_s(Q^2)/N))^n\right), \tag{4.96}$$

our expression for the explicit part of (4.80) is entirely in terms of factorization–scheme–invariant, and hence physically meaningful quantities, i.e.

$$-\alpha_s^2(Q^2)\frac{dF_2(N,Q^2)}{d\alpha_s(Q^2)} = \alpha_s(Q^2)\tilde{\gamma}_{2L}^1(\alpha_s(Q^2)/N)(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times$$
$$\exp[\Phi_0^+(Q^2,Q_0^2) + \Phi_1^+(Q_0^2,Q^2)] - \alpha_s(Q^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2)]$$
$$+ \text{higher order in } \alpha_s \text{ and/or } N. \tag{4.97}$$

We now have an expression which is analogous to that for $\hat{F}_L(N,Q^2)$ in (4.92), except that it has still not been explicitly separated into inputs and evolution terms. In order to do this we must rewrite (4.97) as

$$-\alpha_s^2(Q^2)\frac{dF_2(N,Q^2)}{d\alpha_s(Q^2)} = \alpha_s(Q_0^2)\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times$$
$$\exp[\Phi_0^+(Q^2,Q_0^2) + \Phi_1^+(Q_0^2,Q^2) + \tilde{\Phi}_1^+(Q_0^2,Q^2)]$$
$$-\frac{4}{9}\frac{2N_f}{6\pi b_0}\alpha_s(Q_0^2)F_{2,0}(N)\exp\left[\Phi_1^-(Q^2,Q_0^2) - \ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right] + \text{higher order in } \alpha_s \text{ and/or } N, \tag{4.98}$$

where $\tilde{\Phi}_1^+(Q^2,Q_0^2)$ is a series of the same form as $\Phi_1^+(Q^2,Q_0^2)$, defined as

$$\tilde{\Phi}_1^+(Q^2,Q_0^2) = -\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{d}{d\alpha_s(q^2)}\ln(\alpha_s(q^2)\tilde{\gamma}_{2,L}^1(N,\alpha_s(q^2)))d\alpha_s(q^2). \tag{4.99}$$

It is now clear how we obtain the "leading part" of this expression. All of the inputs are of leading order, but we must factor out the subleading parts of the evolution. $[\Phi_1^+(Q^2,Q_0^2) + \tilde{\Phi}_1^+(Q^2,Q_0^2)]$ is subleading to $\Phi_0^+(Q^2,Q_0^2)$ (we note that $\tilde{\Phi}_1^+(Q^2,Q_0^2)$ is a renormalization–scheme–independent contribution to this subleading evolution), and must be factored out. Since it is of the same order, so must $[\Phi_1^-(Q^2,Q_0^2) - \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))]$. This leaves us with the leading–order expression

$$-\left(\alpha_s^2(Q^2)\frac{dF_2(N,Q^2)}{d\alpha_s(Q^2)}\right)_0 = \alpha_s(Q_0^2)\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)\times$$
$$(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\exp(\Phi_0^+(Q^2,Q_0^2))$$
$$-\alpha_s(Q_0^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}F_{2,0}(N). \tag{4.100}$$

Again, this expression is insensitive to changes in starting scale, up to higher order, and using the one–loop coupling constant, is renormalization scheme independent as well as factorization scheme independent.

Finally we consider the expression for $F_2(N,Q^2)$ itself. It is now a relatively simple matter to write this in terms of factorization–scheme–independent quantities and in terms of inputs and evolution terms. Using the definitions we have already made and defining $\hat{\Phi}_1^+(Q^2,Q_0^2)$ by

$$\hat{\Phi}_1^+(Q^2,Q_0^2) = -\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{d}{d\,\alpha_s(q^2)} \ln\left(\frac{\alpha_s(q^2)\gamma_{2,L}^1(\alpha_s(q^2)/N)}{\gamma_{gg}^0(N,\alpha_s(q^2))}\right) d\alpha_s(q^2). \tag{4.101}$$

we may write (4.79) as

$$\begin{aligned} F_2(N,Q^2) =& \alpha_s(Q_0^2)\frac{\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q_0^2)/N)}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times \\ & \exp[\Phi_0^+(Q^2,Q_0^2) + \Phi_1^+(Q_0^2,Q^2) + \hat{\Phi}_1^+(Q^2,Q_0^2)] + F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2)] \\ & + \tilde{f}_1^S(N,Q_0^2) + \text{higher order in } \alpha_s \text{ and/or } N. \end{aligned} \tag{4.102}$$

So the factorization–scheme–independent input $F_2(N,Q_0^2)$ is

$$F_{2,0}(N) + \alpha_s(Q_0^2)\frac{\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q_0^2)/N)}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N)) + \tilde{f}_1^S(N,Q_0^2). \tag{4.103}$$

By construction it is guaranteed that the change of (4.103) under a change in starting scale will cancel the change in the evolution in the first term in (4.102) under a change in starting scale up to higher orders. It is also clear that (4.103) is both renormalization scheme and factorization scheme independent, as we require, as long as $\tilde{f}_1^S(N,Q_0^2)$ is scheme independent. In fact, the requirement that if $Q_0^2 = A_{LL}$ the input reduces to the nonperturbative input $F_{2,0}(N)$ determines $\tilde{f}_1^S(N,Q_0^2)$ uniquely. It must be the scheme–independent quantity $-\alpha_s(Q_0^2)(\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)/\tilde{\gamma}_{gg}^0(\alpha_s(Q_0^2)/N))(\hat{F}_{L,0}(N) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))$. Hence, the input for $F_2(N,Q^2)$ is

$$F_2(N,Q_0^2) = F_{2,0}(N) + \alpha_s(Q_0^2)\frac{\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)}{\tilde{\gamma}_{gg}^0(\alpha_s(Q_0^2)/N)}\tilde{\hat{F}}_{L,0}(N,Q_0^2) + \text{higher order in } \alpha_s \text{ and/or } N, \tag{4.104}$$

and the expression for $F_2(N,Q^2)$ is

$$\begin{aligned} F_2(N,Q^2) =& \alpha_s(Q_0^2)\frac{\tilde{\gamma}_{2L}^1(N,\alpha_s(Q_0^2))}{\tilde{\gamma}_{gg}^0(N,\alpha_s(Q_0^2))}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times \\ & \exp[\Phi_0^+(Q^2,Q_0^2) + \Phi_1^+(Q_0^2,Q^2) + \hat{\Phi}_1^+(Q^2,Q_0^2)] + F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2)] \\ & - \alpha_s(Q_0^2)\frac{\tilde{\gamma}_{2L}^1(N,\alpha_s(Q_0^2))}{\tilde{\gamma}_{gg}^0(N,\alpha_s(Q_0^2))}(\hat{F}_{L,0}(N) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N)) \\ & + \text{higher order in } \alpha_s \text{ and/or } N. \end{aligned} \tag{4.105}$$

By comparing (4.104) with (4.93) we now see that there is a very direct relationship between the inputs for our two structure functions at small x, i.e. we have a definite prediction, up to

additive nonperturbative parts which are flat at small x, for one in terms of the other, as well as approximate predictions for the form of each. Also comparing with the form of (4.100), these two inputs for the structure functions are directly related to the slope of $(dF_2/d\ln Q^2)$ for small x at Q_0^2. As already mentioned, we do not yet know at what Q_0^2 it is most appropriate to choose the inputs, but it is a nontrivial requirement that the inputs for the three expressions are of the correct form and related in the above manner at any Q_0^2.

Our expression for $F_2(N,Q^2)$ cannot be split into leading and next–to–leading pieces in quite such a clear and symmetric way as our previous two examples, essentially because it begins at zeroth order, not at first order in α_s. Looking at (4.105) it is clear that there is only one LO (in our expansion scheme) input multiplied by a LO evolution, and that is $F_{2,0}(N)$ multiplying unity. Hence we take the leading–order expression to be given simply by

$$\tilde{F}_{2,0}(N,Q^2) = F_{2,0}(N). \tag{4.106}$$

This is obviously completely independent of α_s, and is rather trivial. At next–to–leading order, or equivalently, at leading–α_s–dependent order, we include the whole of (4.105) except that we factor $\exp[\Phi_1^+(q^2,Q_0^2) + \hat{\Phi}_1^+(Q^2,Q_0^2)]$, out of the first term, i.e. we have NLO inputs multiplying LO evolution, and vice versa, as well as $\tilde{F}_{2,0}(N,Q^2)$. Hence, at leading–α_s–dependent order we have

$$\begin{aligned}
F_{2,0}(N,Q^2) &= F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2)] \\
&+ \alpha_s(Q_0^2)\frac{\gamma_{2L}^1(\alpha_s(Q_0^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\exp[\Phi_0^+(Q^2,Q_0^2)] \\
&- \alpha_s(Q_0^2)\frac{\gamma_{2L}^1(\alpha_s(Q_0^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}(\hat{F}_{L,0}(N) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N)) \\
&+ \text{higher order in } \alpha_s \text{ and/or } N.
\end{aligned} \tag{4.107}$$

In this expression there are clearly no terms which mix if we were to make a change in definition of the coupling $\alpha_s \to \alpha_s + \epsilon\alpha_s^2$, and hence we can consider it as a leading–order expression. If the one–loop coupling is used, it is both factorization scheme and renormalization scheme independent.

We now have the full set of LO expressions in the leading–$\ln(1/x)$ expansion scheme. We could obtain the correct scheme–independent expressions for the structure functions at higher orders within this expansion scheme. We choose to finish at leading order, however. The labour required to obtain higher–order expressions becomes progressively greater and we would obtain expressions requiring unknown anomalous dimensions and coefficient functions. Working to NLO we would need to calculate all the NLO evolutions and all the NLO inputs. This would require all the NLO expressions for the anomalous dimensions and coefficient functions, i.e. the $\tilde{\gamma}_{g,a}^1$'s, $\tilde{\gamma}_{f,a}^2$'s and $C_{i,2}^a$'s. There is optimism that these NLO terms will soon be known [35], and once this is so the full NLO scheme–independent expressions should be calculated.

We should make some comments about our LO scheme–independent expressions for $F_L(N,Q^2)$, $F_2(N,Q^2)$ and $(dF_2(N,Q^2)/d\ln(Q^2))$. First we note that all depend on factorization–scheme–independent combinations of the $\tilde{\gamma}^0_{ga}$'s, $\tilde{\gamma}^1_{fa}$'s and $C^a_{i,1}$'s (along with the input parton distributions, where $g_0(N,Q_0^2)$ is also factorization scheme dependent). There is, however, no terribly simple prescription for how one uses the anomalous dimensions and coefficient functions in order to arrive at the expressions. One does certainly not take the known anomalous dimensions, solve the full renormalization group equations, and combine with the known coefficient functions. This introduces terms we do not even include in (4.78)–(4.80), let alone in (4.93), (4.100) and (4.107). The additional terms will depend on the factorization scheme used, and can be very large. The method of determining LO scheme–independent expressions is also more complicated than solving the renormalization group equations to a given order and combining with coefficient functions to a given combined order, as in the loop expansion. The only way to obtain the correct expressions using the parton model directly seems to be to calculate carefully and keep all terms of a given type, as explained.

We should also make some mention of why factorization scheme dependence can be very large in this expansion scheme. In order to do this let us consider (4.81) as an example. A representative example of the way in which factorization–scheme–dependent calculations are done is to consider this expression evaluated with $\tilde{\gamma}^0_{gg}(\alpha_s(Q^2)/N)$, $\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N)$ and $C^g_{L,1}(\alpha_s(Q^2)/N)$ known in some particular factorization scheme, but $\tilde{\gamma}^1_{gg}(\alpha_s(Q^2)/N)$, which is unknown, either set equal to zero, or guessed by imposing some ansatz such as momentum conservation (which may well give completely the wrong answer).

As a first comment we consider the input. The input in terms of parton distributions is multiplied by $C^g_{L,1}(\alpha_s(Q_0^2)/N)$, a series of the form $\sum_{m=0}^{\infty} a_m(\alpha_s(Q_0^2)/N)^m$ which, in general, becomes singular at $N=\lambda(Q_0^2)$. Under a transformation of the type (4.45) this series will be multiplied by $(U^0_{gg}(\alpha_s(Q_0^2)/N))^{-1}$ in order to compensate for the change in the input parton distributions. A number of scheme transformations that are considered have the series $U^0_{gg}(\alpha_s(Q_0^2)/N) = 1+\sum_{m=0}^{\infty} U^{0,m}_{gg}(\alpha_s(Q_0^2)/N)^m$ also becoming nonsingular at $N=\lambda(Q_0^2)$ (e.g. [24][25]). This nonsingular behaviour leads to powerlike growth of the form $x^{-1-\lambda(Q_0^2)}$ as $x\to 0$, but the magnitude of the powerlike behaviour and the manner in which in which it is approached depends on the strength of the singularity and/or on the precise behaviour of the coefficients in the series. Hence, changes in definition of the form in (4.45) with this type of singularity can lead to very marked differences in the form of the gluon at small x, or if the gluon is kept roughly constant (e.g. it is attempted to predict the form of the input structure function by assuming a form for the input gluon), to significant changes in the form of $F_L(x,Q_0^2)$. Very similar considerations also hold for $F_2(x,Q^2)$ because the input depends strongly on $C^g_{2,1,0}(\alpha_s(Q_0^2)/N)$, which transforms in the same way as $C^g_{L,1,0}(\alpha_s(Q_0^2)/N)$ under changes in factorization scheme; i.e. the parton input (or prediction for the structure function if the parton input is assumed, e.g. to be flat) depends

very strongly on scheme. We also note that some of the scheme changes away from the standard $\overline{\rm MS}$ scheme (e.g. the SDIS scheme) involve $U^0_{gg}(\alpha_s(Q_0^2)/N)$ with larger coefficients in the power series than in $C^g_{L,1}(\alpha_s(Q_0^2)/N)$ defined in the $\overline{\rm MS}$ scheme. This leads to $C^g_{L,1}(\alpha_s(Q_0^2)/N)$ defined in the new scheme to have negative coefficients and therefore to $F_L(x,Q_0^2)$ developing a powerlike behaviour with negative magnitude unless either the input gluon or quark (or both) has a powerlike growth of the form $x^{-1-\lambda(Q_0^2)}$ itself, with a large enough multiplicative factor, in order to counter this effect. This must be borne in mind when using such schemes.

Examining (4.81) we can also see how the evolution may be strongly factorization scheme dependent. As we have already mentioned, the whole of $\Phi_1^+(Q^2,Q_0^2)$ must be used in order to obtain a factorization–scheme–independent expression. If $\gamma^1_{gg}(\alpha_s(Q^2)/N)$ is omitted, or the wrong $\gamma^1_{gg}(\alpha_s(Q^2)/N)$ used, then this can give a completely misleading result for the evolution. The integrand in $\Phi_1^+(Q^2,Q_0^2)$ is a series which is a power of $\alpha_s(Q^2)$ down on the series $\gamma^0_{gg}(\alpha_s(Q^2)/N)$. However, in many popular factorization schemes the coefficients in the incomplete, or incorrect series for this integrand are much larger than those in $\gamma^0_{gg}(\alpha_s(Q^2)/N)$ (helped by the fact that many of the early coefficients in $\gamma^0_{gg}(\alpha_s(Q^2)/N)$ are zero), e.g. they commonly behave roughly like $(12\ln 2/\pi)^n n^{-3/4}$, whereas the coefficients in $\gamma^0_{gg}(\alpha_s(Q^2)/N)$ behave roughly like $(12\ln 2/\pi)^n n^{-3/2}$. Hence, the incorrect $\Phi_1^+(Q^2,Q_0^2)$ can have a dominant effect on the evolution. Under changes of factorization scheme the coefficients in the factorization–scheme–dependent series can change by amounts similar to their own magnitude, i.e. $\gamma^1_{gg}(\alpha_s(Q^2)/N)$ is very unstable under factorization scheme changes as can be seen from (2.12). Therefore, the evolution of the structure functions in terms of a given input can appear to have a very strong factorization scheme dependence. Once again, this is true for the evolution of $F_2(x,Q^2)$ as well as for $F_L(x,Q^2)$: the influence of the incorrect $\Phi_1^+(Q^2,Q_0^2)$ can be more important than that of $\Phi_0^+(Q^2,Q_0^2)$ and $\hat{\gamma}_{2L}(N,\alpha_s(Q_0^2))$ combined, where for the latter the coefficients in the series are again relatively small.

Simply using an incorrect calculational procedure, such as solving for the parton distribution using the renormalization group equations up to some order, and then combining with the coefficient functions to a given (combined) order leads to expressions which are not only similar to (4.78)–(4.80) with incorrect or missing $\gamma^1_{gg}(\alpha_s(Q^2)/N)$, but which have additional factorization–scheme–dependent terms. (Solving by using the full known anomalous dimensions and combining the resulting parton distributions with all known coefficient functions is even worse.) These will be formally of higher order than the terms in (4.78)–(4.80), but again can have very large coefficients in the series expansions. This can lead to even more dramatic effects than those outlined above. One clear example of such incorrect effects is found in [22].

Once $\gamma^1_{gg}(N,\alpha_s(Q^2))$ is known in a given scheme and (4.78)–(4.80) can be calculated correctly there is no guarantee that the correct $\Phi_1^+(Q^2,Q_0^2)$ is not larger than $\Phi_0^+(Q^2,Q_0^2)$. If this is the case, $\Phi_1^+(Q^2,Q_0^2)$ will then have a large, but at least definite, effect. However, because it is a formally NLO correction to the structure functions, if $\Phi_1^+(Q^2,Q_0^2)$ is introduced then the full set of NLO

expressions, both evolution and input factors, must be calculated at the same time. The correct calculational method respects renormalization scheme independence as well as factorization scheme dependence. As mentioned earlier, this requires many more terms than just $\Phi_1^+(Q^2,Q_0^2)$. Hopefully, the complete NLO expression, as well as being factorization scheme independent, will also cause only fairly small changes to the LO expressions.

Having obtained our full set of leading–order expressions, we can also now examine the difference between $(d\,F_2(N,Q^2)/d\ln(Q^2))_0$ and $\alpha_s(Q^2)d\,F_{2,0}(N,Q^2)/d\ln(Q^2)$, i.e. the LO derivative of $F_2(N,Q^2)$ and the derivative of the LO $F_2(N,Q^2)$. If we differentiate (4.107) with respect to $\alpha_s(Q^2)$ we obtain

$$\begin{aligned}-\alpha_s^2(Q^2)\frac{d\,F_{2,0}(N,Q^2)}{d\alpha_s(Q^2)} =&\alpha_s(Q_0^2)\tilde{\gamma}_{2L}^1(\alpha_s(Q_0^2)/N)\left(\frac{\gamma_{gg}^0(\alpha_s(Q^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}\right)\times \\ &(\hat{F}_{L,0}(N)+\tilde{\hat{F}}_{L,0}(N,Q_0^2)-\left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\exp[\Phi_0^+(Q^2,Q_0^2)] \\ &-\alpha_s(Q^2)\frac{4}{9}\frac{2N_f}{6\pi b_0}F_{2,0}(N)\exp[\Phi_1^-(Q^2,Q_0^2].\end{aligned} \tag{4.108}$$

This is clearly not exactly the same as (4.100), the difference being due to three additional terms in the above expression as compared to (4.100). These are the factor $(\gamma_{gg}^0(\alpha_s(Q^2)/N)/\gamma_{gg}^0(\alpha_s(Q_0^2)/N))$ in the first term, and the factors $(\alpha_s(Q^2)/\alpha_s(Q_0^2))$ and $\exp[\Phi_1^-(Q^2,Q_0^2)]$ in the second term. All of these factors are unity at the boundary of the evolution, and the two expressions are therefore identical in this limit, i.e the inputs are the same. Therefore, it is the evolution terms which are different when comparing (4.100) and (4.108). Writing

$$\left(\frac{\gamma_{gg}^0(\alpha_s(Q^2)/N)}{\gamma_{gg}^0(\alpha_s(Q_0^2)/N)}\right)=\exp\left[-\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{d}{d\,\alpha_s(q^2)}\ln(\gamma_{gg}^0(\alpha_s(q^2)/N))d\alpha_s(q^2)\right], \tag{4.109}$$

and

$$\alpha_s(Q^2)=\alpha_s(Q_0^2)\exp\left[-\ln\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)\right] \tag{4.110}$$

we see that the terms present in (4.108) but absent in (4.100) are NLO evolution terms. Thus, as in the loop expansion, the difference between the fixed–order expression for $(d\,F_2/d\ln Q^2)$ and the $\ln Q^2$ derivative of the fixed–order F_2 consists of terms of higher order. However, in the leading–$\ln(1/x)$ expansion this difference exists between even LO expressions. Close to the boundary the effect of the additional evolution terms in (4.108) is very small, but they clearly become more significant for $Q^2 >> Q_0^2$ or $Q^2 << Q_0^2$. So, as we might hope, despite the formal difference between the derivative of the LO structure function and the LO expression for the derivative of the structure function, within the region of expected applicability, i.e. x small and Q^2 relatively near Q_0^2, the two are very similar in practice. As one goes away from the boundary, especially at large x,

the terms begin to differ markedly.[17] This highlights the fact that the main region of applicability of the leading-$\ln(1/x)$ expansion is small x and $Q^2 \sim Q_0^2$.

Finally, we should also make some comment about longitudinal momentum conservation. The first moment of the parton distributions is interpreted as the fraction of momentum carried by that type of parton. As such it is usually required that $f^S(1,Q^2) + g(1,Q^2) = 1$, i.e. the total momentum carried by the partons is that of the proton. For this to be true for all Q^2 then, of course, $(d(f^S(1,Q^2) + g(1,Q^2))/d\ln Q^2) = 0$, and from the renormalization group equations this is true if

$$\gamma_{ff}(1,\alpha_s(Q^2)) + \gamma_{gf}(1,\alpha_s(Q^2)) = 0, \qquad \gamma_{fg}(1,\alpha_s(Q^2)) + \gamma_{gg}(1,\alpha_s(Q^2)) = 0. \tag{4.111}$$

This is assumed to be true for the all–orders anomalous dimensions, and thus momentum will be conserved for the all–orders parton distributions. When expanding the anomalous dimensions order by order in $\alpha_s(Q^2)$, i.e. as in (1.3), it is easy to specify that (4.111) be true for the anomalous dimensions at each order, and to define a wide variety of factorization schemes which maintain this. It is not difficult to see from §2.2 that this guarantees that the fraction of momentum carried by the $n_{\rm th}$–order parton distributions is conserved at each n. However, it does not necessarily tell us anything about the amount of momentum carried by the $n_{\rm th}$–order inputs, for any particular n. Sometimes, all the momentum is designated to be carried by the zeroth order part of the solution, but this need not be the case, and indeed, we see no good reason why it should be. Most often the input is implicitly assumed to be the all–orders input. In this case it is true that it must carry all the momentum, but this method destroys the strict ordering of the solution.

As always seems to be the case, the situation is not as simple for the leading-$\ln(1/x)$ expansion. As can be seen from the matrix in (4.50), the leading–order γ contains entirely positive entries for $N = 1$, and is clearly not consistent with momentum conservation: $f_0^S(Q^2)$ carries a constant amount of momentum while that carried by $g_0(Q^2)$ is constantly increasing with Q^2. In a general factorization scheme we assume that there is no reason that working to a finite higher order will restore the relationship (4.111).

Two general methods have been proposed to restore momentum conservation. The first multiplies the known γ^0's and γ^1's in some factorization scheme by some finite power series in N which vanishes at $N = 1$ [21], e.g. the simplest example is $(1 - N)$. The evolution equations are then solved using the whole of these anomalous dimensions and momentum conservation is guaranteed. However, this prescription destroys any sense of ordering the solution correctly and is extremely

[17] Similarly, in the loop expansion significant differences appear between the derivative of fixed order $F_2(N,Q^2)$ and $(d\,F_2(N,Q^2)/d\ln Q^2)$ at the same fixed order when higher powers of $\alpha_s(Q^2)$ become more important, i.e. at low Q^2. Hence the differences are small in the expected region of validity of the loop expansion, i.e. large Q^2.

scheme dependent, causing the sort of large factorization scheme variations described above. Also, since the full anomalous dimensions have singularities at $N = -1$ the power series expansion about $N = 0$ for given order in α_s does not even converge at $N = 1$. Thus, it seems inappropriate to demand that the first few terms in the expansion about $N = 0$ should approximate the correct value at $N = 1$. An examination of the first few terms in the expansion of the two–loop anomalous dimensions about $N = 0$ shows that a truncation of each after a few terms is not at all similar to the above prescription.

An alternative method [20] is to assume that the relationship

$$\begin{aligned}\alpha_s(Q^2)\gamma^1_{ff}(1,\alpha_s(Q^2)) + \gamma^0_{gf}(1,\alpha_s(Q^2)) + \alpha_s(Q^2)\gamma^1_{gf}(1,\alpha_s(Q^2)) = 0 \\ \alpha_s(Q^2)\gamma^1_{fg}(1,\alpha_s(Q^2)) + \gamma^0_{gg}(1,\alpha_s(Q^2)) + \alpha_s(Q^2)\gamma^1_{gg}(1,\alpha_s(Q^2)) = 0,\end{aligned} \tag{4.112}$$

is satisfied, and to determine the unknown $\gamma^1_{ga}(\alpha_s/N)$ this way. In fact it has been proved that it is always possible to choose a factorization scheme where this is true [36], and suggested that this might be some sort of preferred scheme. This will guarantee momentum conservation if one truncates the series for the γ's at the γ^1's and solves the whole renormalization group equation using this truncated γ. However, this does not lead to a well–ordered solution for the parton distributions in any sense. If one were to believe that solving the renormalization group equations order by order and combining with the coefficient functions up to a given combined order led to an ordered solution for the structure functions in this expansion scheme, as in the loop expansion (and as is used in some calculations), then the zeroth–order solutions for the partons do not conserve momentum, and even if (4.112) is true then adding (4.50), (4.57) and (4.67) still leaves us with

$$\sum_{i=0,1} -\alpha_s(Q^2)\frac{d(f_i^S(1,Q^2)+g_i(1,Q^2))}{d\alpha_s(Q^2)} = \tilde{\gamma}^0_{gf}(1,\alpha_s(Q^2))f_1^S(1,Q^2) + \tilde{\gamma}^0_{gg}(1,\alpha_s(Q^2))g_1(1,Q^2), \tag{4.113}$$

as well as terms coming from the difference between $\tilde{\gamma}^1_{gg}$ and γ^1_{gg}. Thus, momentum violation will not be zero in general, and may be quite large. Thus, it seems that in order to enforce momentum conservation strictly within this expansion scheme one must make a guess at the full anomalous dimensions (and coefficient functions) in terms of some truncated form of them, and hope that this is a good approximation to the full solution. This sacrifices any possibility of making a well–ordered expansion for the structure functions.

Of course, in this paper we advocate that the most sensible approach is to obtain a well–ordered solution for the structure functions, and thus definitely eliminate any questions of scheme dependence; i.e. regard a correct treatment of the physical quantities as of paramount importance. Doing this, there does not seem to be any way to ensure that what we choose to define as the parton distributions within our final expressions are such as to conserve momentum. Thus, we simply take the hint offered us by the zeroth–order anomalous dimension, and accept the fact that momentum is not conserved order by order in this method of expansion. How badly it appears to

be violated, however, will depend very much on which factorization scheme we claim to work in. It will always be possible to choose one where momentum violation is very small and, if one wishes, one may choose to think of this as a "physical scheme" for the partons. However, the structure functions themselves will be completely unaffected by this choice, and the real physical relevance is therefore rather questionable. Because momentum is not in general conserved order by order in this expansion scheme the amount carried by a certain order will vary with Q^2. Hence, using an arbitrary factorization scheme there seems little reason to demand that the momentum carried by the zeroth–order inputs for the partons should sum to one; it seems more sensible to share the momentum amongst the different orders in the inputs. It is clear that the momentum carried by the zeroth–order parton distributions, for example, will increase quickly with Q^2, and thus it makes sense when choosing their inputs to choose distributions which sum to less than unity. As higher order corrections to the evolution come in, acting to curb this growth in momentum of the parton distributions (hopefully countered by increased growth with Q^2 of the first moment of the structure functions coming from the effects of the coefficient functions, leading to the overall behaviour of the structure functions being largely unchanged) they can bring inputs carrying positive momentum with them. If we could work to all orders then the momentum carried by the inputs would eventually sum to unity, and stay at this value for all Q^2, but this is of course not possible.

We can also discuss the relationship between our scheme–independent solutions and the ones which would be obtained using Catani's physical anomalous dimensions. One can solve for the LO structure functions using these effective anomalous dimensions in exactly the same way as we solved for the parton distributions in the previous subsection. In the same way that we have the relationships between the LO anomalous dimensions, (1.7) and (4.41), we have relationships between the effective anomalous dimensions (3.13), i.e

$$\Gamma^0_{L2}(\alpha_s(Q^2)/N) = -\left(\frac{36-8N_f}{27}\right)\Gamma^0_{LL}(\alpha_s(Q^2)/N), \tag{4.114}$$

and

$$\Gamma^1_{22}(\alpha_s(Q^2)/N) = -\left(\frac{36-8N_f}{27}\right)\Gamma^1_{2L}(\alpha_s(Q^2)/N) - \tfrac{4}{9}\alpha_s(Q^2)\frac{2N_f}{6\pi}, \tag{4.115}$$

where the second term in (4.115) is the one–loop contribution to $\Gamma^1_{2L}(\alpha_s(Q^2)/N)$. Using these relationships it is straightforward to follow through the steps in (4.50)–(4.77) to obtain the analogous expressions to (4.75), (4.76) and (4.77):

$$\begin{aligned} F_2(N,Q^2) =& \alpha_s(Q^2)\frac{\Gamma^1_{2L}(\alpha_s(Q^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q^2)/N)}(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times \\ & \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} + \frac{\tilde{\Gamma}^1_{LL}(\alpha_s(q^2)/N) - \left(\frac{36-8N_f}{27}\right)\tilde{\Gamma}^1_{2L}(\alpha_s(q^2)/N)}{\alpha_s(q^2)} d\alpha_s(q^2)\right] \\ & - \tilde{F}_{2,1}(N,Q_0^2) + F_{2,0}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N, \end{aligned} \tag{4.116}$$

$$\hat{F}_L(N,Q^2) = (\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} + \frac{\tilde{\Gamma}^1_{LL}(\alpha_s(q^2)/N) - \left(\frac{36-8N_f}{27}\right)\tilde{\Gamma}^1_{2L}(\alpha_s(q^2)/N)}{\alpha_s(q^2)} d\alpha_s(q^2)\right]$$
$$+ \left(\frac{36-8N_f}{27}\right)F_{2,0}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N, \tag{4.117}$$

and

$$-\alpha_s^2(Q^2)\frac{dF_2(N,Q^2)}{d\alpha_s(Q^2)} = \alpha_s(Q^2)\tilde{\Gamma}^1_{2L}(\alpha_s(Q^2)/N)(\hat{F}_{L,0}(N) + \tilde{\hat{F}}_{L,0}(N,Q_0^2) - \left(\frac{36-8N_f}{27}\right)F_{2,0}(N))\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} + \frac{\tilde{\Gamma}^1_{LL}(\alpha_s(q^2)/N) - \left(\frac{36-8N_f}{27}\right)\tilde{\Gamma}^1_{2L}(\alpha_s(q^2)/N)}{\alpha_s(q^2)} d\alpha_s(q^2)\right]$$
$$- \tfrac{4}{9}\alpha_s(Q^2)F_{2,0}(N)\frac{2N_f}{6\pi b_0}\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{4}{9}\frac{2N_f}{6\pi b_0}} + \text{higher order in } \alpha_s \text{ and/or } N, \tag{4.118}$$

where $\tilde{\hat{F}}_{L,0}(N,Q_0^2)$ is a function of Γ^0_{LL} rather than γ^0_{gg}.

We see that, once we make the identifications

$$\gamma^0_{gg}(\alpha_s(Q^2)/N) = \Gamma^0_{LL}(\alpha_s(Q^2)/N), \tag{4.119}$$

$$\gamma^1_{2L}(\alpha_s(Q^2)/N) = \Gamma^1_{2L}(\alpha_s(Q^2)/N), \tag{4.120}$$

and

$$\tilde{\gamma}^1_{gg}(\alpha_s(Q^2)/N) + \tfrac{4}{9}\tilde{\gamma}^1_{fg}(\alpha_s(Q^2)/N) - \frac{d}{d\alpha_s(Q^2)}\left(\ln\left(\frac{C^g_{L,1}(\alpha_s(Q^2)/N)}{C^g_{L,1,0}}\right)\right)$$
$$= \tilde{\Gamma}^1_{LL}(\alpha_s(Q^2)/N) - \left(\frac{36-8N_f}{27}\right)\tilde{\Gamma}^1_{2L}(\alpha_s(Q^2)/N), \tag{4.121}$$

(4.117) is identical to (4.92), (4.118) is identical to (4.97) and (4.116) is identical to (4.105). Of course, the identifications (4.119)–(4.120) are exactly what we obtain from the definitions of the physical anomalous dimensions in §3.

Thus, we are able to reach these expressions for the structure functions somewhat more directly by using the physical anomalous dimensions, and do not have to worry about problems with factorization scheme dependence (though we do have to calculate the physical anomalous dimensions in terms of known coefficient functions and anomalous dimensions of course). Once we have obtained these expressions using the physical anomalous dimensions we may then separate each of the terms into input parts and evolution parts (where more of this has already been done automatically when using the physical anomalous dimensions) and keep the most leading parts, obtaining once again

the LO expressions (4.93), (4.100) and (4.107). So, using the physical anomalous dimensions leads us in a rather more direct manner to the correct leading–order expressions. If we were to work to higher orders, the amount of simplification obtained by using the physical anomalous dimensions rather than working in terms of parton densities would increase significantly. However, we stress that one will always automatically obtain factorization–scheme–independent answers by working to well–defined orders in physical quantities even when working in terms of partons. Also, even if one uses the physical anomalous dimensions, care is still needed to obtain expressions which are consistent with renormalization scheme dependence.

In this section, we have derived well–ordered, factorization–scheme–independent expressions for structure functions in the leading–$\ln(1/x)$ expansion (which should be useful at small x and moderate Q^2) up to the order which is useful at present. This expansion does, however, sacrifice any attempt to describe the structure functions at large x at any reasonable distance from Q_0^2 (we will discuss later what large x and small x turn out to be), in the same way that the loop expansion should show signs of failing at very low values of x. We would hope there is some expansion scheme which will be useful at all values of x. In the next section we will show that there is indeed an expansion scheme which satisfies this criterion, and argue that it is the only really correct expansion scheme.

4.4. The Renormalization–Scheme–Consistent Expansion.

In order to devise an expansion scheme which is useful at both large and small x we would *a priori* expect that we would need to use the known anomalous dimensions and coefficient functions at low orders in both α_s and in the leading–$\ln(1/x)$ expansion. There have already been various methods along these lines, and the phrase "double leading expansion" was coined in [20]. However, these methods have all suffered from scheme dependence. As when deriving our expressions for the structure functions when using the leading–$\ln(1/x)$ expansion we will ensure that we obtain results which are invariant under changes of factorization scheme and, as a stronger constraint, demand complete consistency of our expressions for physical quantities with renormalization scheme invariance (which in itself automatically guarantees factorization scheme independence). Consequently, our approach will be rather different from those used previously, and the results and conclusions will also be somewhat different.

To begin, let us consider what we have meant by "consistency with renormalization scheme dependence" so far in this paper. In both the loop expansion and the leading–$\ln(1/x)$ expansion we demanded that once we had chosen a particular renormalization scheme and chosen to work to a particular order in this renormalization scheme then we would include all terms in our expressions for the structure function which were of greater magnitude than the uncertainty due to the freedom of choice of renormalization scheme (i.e. the uncertainty in the definition of the coupling constant),

and no others. In both cases the leading–order term consisted of the lowest–order inputs multiplying the lowest–order evolution terms. If working with the n–loop coupling constant in a particular renormalization scheme the uncertainty in its definition is of order α_s^{n+1}. Thus, the uncertainty of the input or evolution when working to $n_{\rm th}$–order is the change in the leading–order input or evolution if the coupling changes by $\delta\alpha_s = \epsilon\alpha_s^{n+1}$, i.e $\alpha_s \to \alpha_s(1+\epsilon\alpha_s^n)$. Hence, the uncertainty in the whole structure function is of the order of the change of the leading–order part under such a change in the coupling. Therefore the $n_{\rm th}$–order renormalization–scheme–independent expression includes all complete terms smaller than this change.

This definition does give us a well–defined way of building up an ordered solution to the structure functions, but relies upon the definition of a given expansion scheme. It leaves an ambiguity about how we define the leading–order expressions and in how we define the order of terms compared to this leading–order term. Our two examples, i.e. the loop expansion, where the size of a term is determined simply by its order in α_s, and the leading–$\ln(1/x)$ expansion, where $\ln(1/x)$ is put on an equal footing to α_s, are just the two most commonly used examples of expansion schemes (even though the $\ln(1/x)$ expansion has not previously been presented in the quite same way as in this paper). Both have potential problems: in the former one does not worry about the fact that the large–$\ln(1/x)$ terms can cause enhancement at small x of terms which are higher order in α_s, and in the latter one does not worry about the fact that at large x, especially as Q^2 increases, it is the terms which are of lowest order in α_s that are most important. Hence, one would think that both have limited regions of validity.

The shortcomings of these two expansion schemes come about because, even though any given order contains no terms which are inconsistent with working to the same given order in a particular renormalization scheme, in neither case does it include every one of the terms which are consistent to working to a given order in the renormalization scheme. In each expansion scheme some of the terms appearing at what we call higher orders are not actually subleading in α_s to any terms which have already appeared. Thus, despite the fact that for a given expansion method these terms are formally of the same order as uncertainties due to the choice of renormalization scheme, they are not terms which can actually be generated by a change in renormalization scheme.[18]

In order to demonstrate this point more clearly we consider a simple toy model. Let us imagine some hypothetical physical quantity which can be expressed in the form

$$H(N,\alpha_s(Q^2)) = \sum_{m=1}^{\infty}\alpha_s(Q^2)\sum_{n=-m}^{\infty}a_{mn}N^n \equiv \sum_{i=0}^{\infty}\alpha_s^i(Q^2)\sum_{j=1-i}^{\infty}b_{ij}\left(\frac{\alpha_s(Q^2)}{N}\right)^j, \qquad (4.122)$$

where the expansion in powers of N about $N=0$ is convergent for all N. The first way of writing $H(N,\alpha_s(Q^2))$ as a power series corresponds to the loop expansion, where we work order by order

[18] Similarly, they cannot be generated by a change in renormalization scale.

in m, out to $m = k$, and use the k–loop coupling. The second corresponds to the leading–$\ln(1/x)$ expansion where we work order by order in i, out to $i = l$, and use the $(l+1)$–loop coupling. Let us, for a moment, consider the LO expression in the loop expansion, $\alpha_s(Q^2)\sum_{n=-1}^{\infty} a_{1n}N^n$. The coupling is uncertain by $\mathcal{O}(\alpha_s^2(Q^2))$ and hence the uncertainty of the leading–order expression (i.e. the change due to a change of the coupling) is $\sim \alpha_s^2(Q^2)\sum_{n=-1}^{\infty} b_{1n}N^n$. There is no change with powers of N less than -1, and hence any such term is not really subleading. Similarly, the uncertainty of the leading–order expression in the leading $\ln(1/x)$ expansion contains no terms at first order in α_s (or with positive powers of N), and such terms are not really subleading either. The full set of terms contained within the combination of both leading–order expressions is genuinely leading order, and is therefore renormalization scheme independent by definition.

Perhaps the best way in which to write our expression for $H(N,\alpha_s(Q^2))$ in order to appreciate these points is

$$H(N,\alpha_s(Q^2)) = \sum_{m=-1}^{\infty} N^m \sum_{n=1}^{\infty} c_{mn}\alpha_s^n(Q^2) + \sum_{m=2}^{\infty} N^{-m} \sum_{n=m}^{\infty} c_{mn}\alpha_s^n(Q^2), \tag{4.123}$$

i.e. as an infinite number of power series in $\alpha_s(Q^2)$, one for each power on N. Each of these series in $\alpha_s(Q^2)$ is independent of the others, and the lowest order in $\alpha_s(Q^2)$ of each is therefore renormalization scheme independent and part of the complete LO expression for $H(N,\alpha_s(Q^2))$. The full LO expression for $H(N,\alpha_s(Q^2))$ is therefore

$$\begin{aligned} H_0(N,\alpha_s(Q^2)) &= \sum_{m=-1}^{\infty} N^m c_{m1}\alpha_s(Q^2) + \sum_{m=2}^{\infty} c_{mm}N^{-m}\alpha_s^m(Q^2) \\ &\equiv \alpha_s(Q^2)\sum_{n=-1}^{\infty} a_{0n}N^n + \sum_{j=2}^{\infty} b_{0j}\left(\frac{\alpha_s(Q^2)}{N}\right)^j. \end{aligned} \tag{4.124}$$

Hence, the combined set of terms considered LO in both our expansion schemes comprise the full set of renormalization scheme invariant, and thus truly leading–order, terms. By considering $H(N,\alpha_s(Q^2))$ written in the form (4.123), and considering how the coefficients in the expression must change in order to make the whole expression invariant under a redefinition of the coupling constant, $\alpha_s(Q^2) \to \alpha_s(Q^2) + \mathcal{O}(\alpha_s^m(Q^2))$, we see that the $n_{\rm th}$–order expression for $H(N,\alpha_s(Q^2))$, which should be used with the n–loop coupling constant, consists of the sum of the first n terms in each of the power series in $\alpha_s(Q^2)$. Thus, the full $n_{\rm th}$–order expression always consists of the $n_{\rm th}$–order expression in the loop expansion plus additional terms with inverse powers of N greater than n.

Similar arguments have already been applied to the anomalous dimensions and coefficient functions, for example [18] and particularly [20]. The latter claims that one may, but need not, use expansions of the above form for the anomalous dimensions and coefficient functions, and moreover, in practice expresses the terms beyond fixed order in α_s as functions of x/x_0 for $x < x_0$ and sets them

to zero otherwise ($x_0 \leq 1$, and is in general $sim0.1$), thus reducing their effect (see also [28]). Here we take a strong, inflexible viewpoint and insist that the complete renormalization–scheme–consistent expressions, with no artificial suppression of leading–$\ln(1/x)$ terms, must be used. Furthermore, and very importantly, the expressions used must be those for the physical structure functions, not for the factorization–scheme– and renormalization–scheme–dependent coefficient functions and anomalous dimensions.

When considering the real structure functions the situation is technically a great deal more complicated than our toy model, but the principle is exactly the same. This can be seen by examining the the LO expressions for the structure functions in the two expansion schemes already considered. There is some overlap between the LO expressions for the structure functions when using the loop expansion and when using the leading–$\ln(1/x)$ expansion, but each contains an infinite number of terms not present in the other. However, we were previously happy to use the one loop coupling for both expressions. The uncertainty in the definition of this coupling is $\mathcal{O}(\alpha_s^2)$. Considering the change of each of our leading–order expressions under a change of coupling of $\mathcal{O}(\alpha_s^2)$, the changes in the expressions are rather complicated. However, it is not too difficult to see that, as with our toy model, the change in the LO structure functions in the loop expansion contains no terms in the LO expressions in the leading–$\ln(1/x)$ expansion, and *vice versa*. Thus, none of the terms contained within each of the LO expressions are generated by uncertainties at higher order in the opposing expansion scheme. Therefore, they should really all be regarded as genuinely LO, and be included in the full expressions for the structure functions which use the one–loop coupling constant.

Hence, as with the toy model, there should be some combined expansion–scheme–independent expressions for the structure functions which we can genuinely call the "leading order" expressions. Since these expressions will contain all the parts of the one–loop expressions, and also contain leading–$\ln(1/x)$ terms as well, they should be able to describe the data over the full range of parameter space (except very low Q^2, of course), as we would like from our correct LO expressions. We shall now demonstrate how we obtain these expressions.

There are two main complications when considering structure functions in comparison to our simple toy model. One is that the structure functions are combinations of perturbative evolution parts and input parts (which are viewed as partly perturbative with nonperturbative factors), rather than one simple power series in $\alpha_s(Q^2)$. The other is that in general the physical anomalous dimensions, out of which the perturbative parts are constructed, are nonanalytic functions which cannot be expressed as power series about a particular value N_0 for all N. The physical anomalous dimensions have singularities at $N = 0$ (in the case of the singlet structure function only), and also at negative powers of N (as well as possible $\alpha_s(Q^2)$–dependent nonanalyticities due to resummation effects, e.g. the branch point in $\Gamma^0_{LL}(N, \alpha_s(Q^2))$ at $N = \lambda(Q^2)$). We will deal with this second complication first.

Let us consider the perturbative parts of the expressions for the structure functions. The singularities at negative integer values of N mean that we cannot write any physically meaningful quantity as just a power series about $N = 0$ (or about $(N + 1)$ for the nonsinglet case). Any such power series expansion will have a radius of convergence of unity, whereas the physical moments of any x-space quantity will exist for all real N above some minimum N_{min}, which depends on the asymptotic form of the structure functions as $x \to 0$. A series expansion which applies over this whole range of N does not exist: the valid expression must include the nonanalytic functions explicitly. This seems to make it impossible to order the moment–space solution in such a simple way as in (4.122).

In order to overcome this problem let us consider making the inverse transformation of some physically relevant perturbative function $A(N, \alpha_s(Q^2))$ to x–space. The inverse of the Mellin transformation (2.1) is

$$\mathcal{A}(x, \alpha_s(Q^2)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-N} A(N, \alpha_s(Q^2)) dN, \tag{4.125}$$

where the line of integration is to the right of all nonanalyticities. Making the substitution $\xi = \ln(1/x)$ this becomes

$$\mathcal{A}(x, \alpha_s(Q^2)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\xi N] A(N, \alpha_s(Q^2)) dN. \tag{4.126}$$

Since $A(N, \alpha_s(Q^2))$ has, in general, singularities for all nonpositive integers, this whole integral may be evaluated by performing an infinite series of integrals, each with a contour centred on a given singularity, and not extending as far as unity from this singularity, i.e. not reaching any of the other singularities. Within each of these contours the function $A(N, \alpha_s(Q^2))$ may be expanded as a power series about the singularity, i.e. we may write

$$\mathcal{A}(x, \alpha_s(Q^2)) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} x^n \int_{c_n} \exp[\xi(N + n)] A_n((N + n), \alpha_s(Q^2)) d(N + n), \tag{4.127}$$

where $A_n((N + n), \alpha_s(Q^2))$ denotes $A(N, \alpha_s(Q^2))$ expanded as a power series about $N = -n$. The integrals will produce functions of ξ, which do not sum to integer powers of x, and hence each of the integrals in (4.127) will be independent and physically relevant in its own right.

So this is the solution to our problem of how to order the moment space expressions for physical expressions which are related to the real structure functions as powers series in α_s and N. We must consider the complete moment–space expression as an infinite number of expressions of the form (4.123), each one having power series expansions in terms of $(N + n)$, where $n = 0 \to \infty$. The expression for each n is then related to the part of the x–space expression behaving $\sim x^n$. Of course, in practice, unless we want to examine the details of the perturbative calculation of the structure function for x very close to 1, we can ignore all n greater than a finite, relatively small constant.

Thus, when we calculate the expressions for the perturbative part of the singlet structure functions, we will only be concerned about any LO terms beyond lowest order in α_s for the specific case of $n = 0$. For $n > 0$ we take the whole LO expression to be the one–loop expression. The terms we ignore by making this necessary decision are those which are LO in $\ln(1/x)$ at first order in x. Although these terms grow like $\alpha_s^m \ln^{2m-1}(1/x)$, as opposed to $\alpha_s^m \ln^m(1/x)$ at zeroth order in x, there is no evidence that their coefficients are any larger than those for the zeroth–order–in–x logarithms. Since the resummed terms at zeroth order in x only begin to make an impact as x falls to ~ 0.1 (as we will see), and only become dominant for x much smaller than this, the effect of terms like $x\alpha_s^m \ln^{2m-1}(1/x)$ should be extremely small in comparison. Indeed, the effect of those terms of the form $x\alpha_s^m \ln^{2m-1}(1/x)$ which are actually known, i.e. for $m = 2$, can indeed be shown to be negligible. In a similar manner, we will only consider the one–loop expressions for the nonsinglet structure functions in practice: the other LO parts of the expressions again lead to small–x enhancement of the form $x\alpha_s^m \ln^{2m-1}(1/x)$, which is very small compared to the leading singlet small–x enhancement, and there is only detailed data at very small x for the total structure function.[19]

Hence, we only really need to consider calculating a full LO renormalization–scheme–consistent (RSC) expression for the perturbative contributions to the structure functions expressed as power series in α_s and in N about $N = 0$, as for our toy model. However, we now have to return to our first problem, i.e. the fact that the structure functions are expressed in terms of both inputs and evolution parts. Using the results we have already obtained in the earlier subsections it is not too difficult to construct the full LORSC expressions for the inputs and for the evolution parts of structure functions. For the case of the nonsinglet structure functions the construction of the LORSC expression is then just the product of these two terms. Let us discuss this as an example first.

4.5. The Renormalization–Scheme–Consistent Nonsinglet Structure Functions.

We consider a nonsinglet longitudinal structure function. For the nonsinglet structure functions the physical anomalous dimensions contain no singularities at $N = 0$, so the leading–$\ln(1/x)$ behaviour comes from singularities at $N = -1$. Expanding about $N = -1$, the full LO physical anomalous dimension can be written in the form,

$$\Gamma_{L,0}^{NS}(N+1, \alpha_s(Q^2)) = \alpha_s(Q^2)\Bigg[\sum_{m=-1}^{\infty} a_m (N+1)^m + \sum_{m=1}^{\infty} b_m (N+1)^{-1}\left(\frac{\alpha_s(Q^2)}{(N+1)^2}\right)^m + \sum_{m=1}^{\infty} c_m \left(\frac{\alpha_s(Q^2)}{(N+1)^2}\right)^m\Bigg]. \tag{4.128}$$

[19] Also, the full leading–order–in–$\ln(1/x)$ physical anomalous dimension is not yet known for the nonsinglet structure functions, as will be discussed below.

The first sum is just $\Gamma^{NS}_{L,0,l}(N+1)$, the one–loop anomalous dimension expanded in powers of $(N+1)$. The second sum contains the leading singularities in $(N+1)$ for all other orders in $\alpha_s(Q^2)$. The final sum is included because, despite the obvious fact that it is a power of $(N+1)^{-1}$ down on the second sum, a series of this form cannot be created from the second sum by a change in the definition of the coupling of $\mathcal{O}(\alpha_s^2(Q^2))$. Therefore, the third sum is not subleading in $\alpha_s(Q^2)$ to the second sum, and must be renormalization scheme independent. Integrating (4.128) between Q_0^2 and Q^2, and including the overall power of $\alpha_s(Q^2)$ for the longitudinal structure function we obtain the leading–order evolution

$$\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{NS}_{L,0,l}(N+1)-1} \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \left(\sum_{m=1}^{\infty}\left(\frac{1}{N+1}b_m + c_m\right)\left(\frac{\alpha_s(q^2)}{(N+1)^2}\right)^m\right)\frac{d\alpha_s(q^2)}{b_0\alpha_s^2(q^2)}\right]. \quad (4.129)$$

In the loop expansion the lowest–order input for $F_L^{NS}(N,Q^2)$ was $\alpha_s(Q_0^2)C^{NS}_{L,1,l}(N)F^{NS}_{2,0}(N)$, where $F^{NS}_{2,0}(N)$ is a nonperturbative factor and $C^{NS}_{L,1,l}(N)$ has an expansion in powers of $(N+1)$ beginning at zeroth order. Factoring out the nonperturbative part, our lowest–order input is $\alpha(Q_0^2)$ multiplying a power series in $(N+1)$ which starts at zeroth order, i.e $C^{NS}_{L,1,l}(N+1)$. In order to construct the full LO input we must consider how the evolution term (4.129) changes under a change in starting scale $Q_0^2 \to (1+\delta)Q_0^2$, and therefore how the input must change in order to compensate for this. The change in the first term is $\propto \alpha_s(Q_0^2)(\tilde{\Gamma}^{NS}_{L,0,l}(N+1)-1)$. This can be absorbed into a change of the input at order $\alpha_s^2(Q_0^2)$, and hence the second–order–in–$\alpha_s(Q_0^2)$ input has a part $\propto (\tilde{\Gamma}^{NS}_{L,0,l}(N+1)-1)C^{NS}_{L,1,l}(N+1)$. The power expansion of this expression in terms of $(N+1)$ begins at order -1, and a term of this type cannot be generated by a change of the order–$\alpha_s(Q_0^2)$ input under renormalization scheme changes. Hence, this part of the $\alpha_s^2(Q_0^2)$ input, i.e $\propto \alpha_s^2(Q_0^2)a_{-1}C^{NS}_{L,1,l}(N=-1)(N+1)^{-1}$, belongs to the LORSC input. The rest of the $\mathcal{O}(\alpha_s^2(Q_0^2))$ input is genuinely subleading to the $\mathcal{O}(\alpha_s(Q_0^2))$ input and is renormalization scheme dependent. Extending this argument, and considering the form of the input required to compensate for the change of the whole of (4.129) under a change in starting scale (as with $g_0(N,Q_0^2)$ in subsection 4.2), we can see that the full LORSC input is

$$\alpha_s(Q_0^2)F^{NS}_{2,0}(N)\Bigg(C^{NS}_{L,1,l}(N+1) + C^{NS}_{L,1,l}(N=-1)\Bigg(\ln(Q_0^2/A_L^{NS})\alpha_s(Q_0^2)\times$$
$$\left(\frac{1}{N+1}\sum_{m=0}^{\infty} b_m\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m + \sum_{m=0}^{\infty}\left(c_m + \tfrac{1}{2}\sum_{n=0}^{m} b_n b_{m-n}\ln(Q_0^2/A_L^{NS})\right)\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m\right)\Bigg)\Bigg), \quad (4.130)$$

where $b_0 \equiv a_{-1}$ and $c_0 = 0$. The first term is just the lowest–order input in the loop expansion, while the second includes all the leading–$\ln(1/x)$ terms in a simple form which is compatible with making the full expression invariant under changes in starting scale Q_0^2. This second term only depends on the part of the one–loop coefficient function at zeroth order in $(N+1)$: the higher–order–in–$(N+1)$ parts multiply the part consisting of the leading–$\ln(1/x)$ terms to give the type of terms which can be generated from (4.130) by changes in the renormalization scheme.

Hence, the full LORSC expression for the nonsinglet longitudinal structure function, expanded about $N = -1$, is

$$F^{NS}_{L,RSC,0}(N,Q^2) = \alpha_s(Q_0^2)F^{NS}_{2,0}(N)\Bigg(C^{NS}_{L,1,l}(N+1) + C^{NS}_{L,1,l}(N=-1)\Bigg(\ln(Q_0^2/A_L^{NS})\alpha_s(Q_0^2)\times$$
$$\left(\frac{1}{N+1}\sum_{m=0}^{\infty} b_m\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m + \sum_{m=0}^{\infty}\left(c_m + \tfrac{1}{2}\sum_{n=0}^{m} b_n b_{m-n}\ln(Q_0^2/A_L^{NS})\right)\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m\right)\Bigg)\Bigg)\times$$
$$\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\mathrm{f}^{NS}_{L,0,l}(N+1)-1}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\left(\sum_{m=1}^{\infty}\left(\frac{1}{N+1}b_m + c_m\right)\left(\frac{\alpha_s(q^2)}{(N+1)^2}\right)^m\right)\frac{d\alpha_s(q^2)}{b_0\alpha_s^2(q^2)}\right]. \tag{4.131}$$

As we have already argued for the singlet case, as far as the expansion about the singularities at N to the left of the rightmost singularity are concerned, we may as well just take the one–loop expression. In practice, we will only use the one-loop expression for the nonsinglet structure functions for all the singularities in the anomalous dimensions. This is because of the phenomenological reasons given at the end of the last subsection, and also because of lack of knowledge of the full physical anomalous dimensions. In the $\overline{\mathrm{MS}}$ scheme the terms $\sim \alpha_s^m(N+1)^{2m-1}$ in the parton anomalous dimension are all known [37]. It does not appear as though the coefficient functions contribute to these sort of terms in the physical anomalous dimension (though there is no formal proof of this), and hence, it is believed all of these terms are known. However, there is little knowledge yet of the terms of the sort $\sim \alpha_s^m(N+1)^{2m-2}$. We have argued that these are an intrinsic part of the LORSC expression for the nonsinglet structure function, and they should be calculated and included in order to give a true indication of the effect of leading–$\ln(1/x)$ terms. Hence, we believe that calculations of the nonsinglet [23][38] (and polarized [39]) structure functions which claim to include leading–$\ln(1/x)$ corrections are incomplete, even at leading order, until the terms of the form $\alpha_s^m(N+1)^{2m-2}$ are known.

We now consider the nonsinglet structure function $F_2^{NS}(N,Q^2)$. This leads us back to our previous question of whether we should use the full RSC expression for $F_2(N,Q^2)$ or that for $(dF_2(N,Q^2)/d\ln Q^2)$. In order to illustrate the difference between the two, and help us make our choice, we consider the simpler nonsinglet case before the singlet case. We also pretend for the moment that the inverse powers of $(N+1)$ in the expressions do not increase as the power of α_s increases, i.e. there is no small–x enhancement at higher orders in α_s. This being the case, the LO term in the evolution is just

$$\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\mathrm{f}^{NS}_{2,0,l}(N+1)}. \tag{4.132}$$

Of course, the input may be written as a power series in $\alpha_s(Q_0^2)$, as we saw in §4.1, and is of the form

$$F^{NS}_{2,0}(N,Q_0^2) = F^{NS}_{2,0}(N)[1 + \alpha_s(Q_0^2)\ln(Q_0^2/A_2^{NS})\Gamma^{2,0,l}_{NS}(N+1) + \text{higher order in } \alpha_s(Q_0^2)]. \tag{4.133}$$

Therefore, we have the problem that as well as the lowest order α_s–, and hence Q_0^2–dependent part of the input there is also the "sub–lowest–order", Q_0^2–independent part. These two terms are clearly of different order, but under a change in renormalization scheme, and hence in the definition of the coupling, both remain unchanged and both should therefore appear in the LO definition of the structure function. This mixing of orders seems rather unsatisfactory, and comes about because for $F_2^{NS}(N,Q^2)$ the structure function still exists in the formal limit of $\alpha_s \to 0$, i.e. the parton model limit, being equal to the simple Q_0^2–independent function $F_{2,0}^{NS}(N)$. Hence, it is not a perturbative quantity in quite the sense way as $F_L^{NS}(N,Q^2)$ or $(d\,F_2^{NS}(N,Q^2)/d\ln Q^2)$, both of which vanish in this limit.

So in our simplified model the LORSC expression for $F_2^{NS}(N,Q^2)$, obtained by combining the LO input and evolution, is

$$F_{2,RSC,0}^{NS}(N,Q^2) = F_{2,0}^{NS}(N)[1+\alpha_s(Q_0^2)\ln(Q_0^2/A_2^{NS})\Gamma_{NS}^{2,0,l}(N+1)]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}_{2,0,l}^{NS}(N+1)}. \quad (4.134)$$

This consists of two parts which are clearly of different magnitude, i.e. one is a power of α_s down on the other with no small–x enhancement. This seems against the spirit of a well–ordered calculation. Indeed, the second part of this LO expression is of the same order of magnitude as part of the LO input multiplying the NLO evolution, i.e.

$$F_{2,0}^{NS}(N)(\alpha_s(Q_0^2)-\alpha_s(Q^2))\hat{\Gamma}_{NS}^{2,1,l}(N+1)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}_{2,0,l}^{NS}(N+1)}. \quad (4.135)$$

Even when we take into account the higher inverse powers of $(N+1)$ at higher powers of α_s, (4.134) will be part of the LO expression for $F_2^{NS}(N,Q^2)$, and this unsatisfactory behaviour remains. It is also clear that the same effect will be seen for the singlet structure function.

If we instead consider $(d\,F_2^{NS}(N,Q^2)/d\ln Q^2)$ (again ignoring small–x enhancement for the moment) the situation improves. In this case the full expression is

$$\frac{d\,F_2^{NS}(N,Q^2)}{d\ln Q^2} = \Gamma_{2,NS}(N+1,\alpha_s(Q_0^2))F_2^{NS}(N,Q_0^2)\times \exp\left[\int_{\ln Q_0^2}^{\ln Q^2}\left(\Gamma_{2,NS}(N+1,\alpha_s(q^2)) - \frac{d}{d\ln q^2}\ln(\Gamma_{2,NS}(N+1,\alpha_s(q^2)))\right)d\ln q^2\right]. \quad (4.136)$$

Hence, the input may be written as

$$\left(\frac{d\,F_{2,0}^{NS}(N,Q^2)}{d\ln Q^2}\right)_{Q_0^2} = F_{2,0}^{NS}(N)[\alpha_s(Q_0^2)\Gamma^{2,0,l}(N+1)+\alpha_s^2(Q_0^2)(\ln(Q_0^2/A_2^{NS})(\Gamma_{NS}^{2,0,l}(N+1))^2 + \Gamma^{2,1,l}(N+1)) + \mathcal{O}(\alpha_s^3(Q_0^2))]. \quad (4.137)$$

Of course, the $\mathcal{O}(\alpha_s(Q_0^2))$ piece is renormalization scheme independent by definition. The $\mathcal{O}(\alpha_s^2(Q_0^2))$ piece is renormalization scheme dependent ($\Gamma^{2,1,L}(N)$ is renormalization scheme dependent) in order to absorb changes in the $\mathcal{O}(\alpha_s(Q_0^2))$ piece under a change in the coupling of $\mathcal{O}(\alpha_s^2(Q_0^2))$. So this time we have a LORSC input which is of a given order in $\alpha_s(Q_0^2)$. The full LO expression for $(dF_2(N,Q^2)/d\ln Q^2)$ is then

$$\left(\frac{d\,F_{2,0}^{NS}(N,Q^2)}{d\ln Q^2}\right)_0 = F_{2,0}^{NS}(N)\alpha_s(Q_0^2)\Gamma_{2,0,l}^{NS}(N+1)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}_{2,0,l}^{NS}(N+1)-1}. \tag{4.138}$$

This is rather more satisfactory than the renormalization–scheme–independent expression for $F_2^{NS}(N,Q^2)$ itself, and hence we choose $(dF_2(N,Q^2)/d\ln Q^2)$ to be the perturbative quantity we calculate, in both the nonsinglet and singlet case.

If we wish to calculate the structure function $F_2(N,Q^2)$ itself to a given order we will do this by integrating the given order expression for $(dF_2(N,Q^2)/d\ln Q^2)$ between Q_0^2 and Q^2, and adding it to $F_2(N,Q_0^2)$ evaluated to the same order. For example, in our simplified nonsinglet model we would integrate (4.138) and add this to the explicitly written part of (4.133). This results in the effective LO expression

$$F_{2,RSC,0}^{NS}(N,Q^2) = F_{2,0}^{NS}(N)\left[\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}_{2,0,l}^{NS}(N+1)} + \alpha_s(Q_0^2)\ln(Q_0^2/A_2^{NS})\Gamma_{NS}^{2,0,l}(N+1)\right]. \tag{4.139}$$

This is perhaps more sensible than (4.134), since the two terms are now of a more comparable size. Moreover, this expression is more stable under changes in Q_0^2.

Of course, this whole discussion of $F_2^{NS}(N,Q^2)$ has been rather simplified by the assumption that the higher–order–in–α_s terms in the physical anomalous dimension do not contain higher singularities in $(N+1)$. Recognizing that they do, we obtain expressions which are more complicated, as in the case of $F_L^{NS}(N,Q^2)$. With a little work it is possible to see that the full leading–order expressions are

$$\begin{aligned}\left(\frac{d\,F_{2,RSC,0}^{NS}(N,Q^2)}{d\ln Q^2}\right)_0 &= \alpha_s(Q_0^2)F_{2,0}^{NS}(N)\Bigg(\Gamma_{2,0,l}^{NS}(N+1) + \frac{1}{N+1}\times \\ &\left(\sum_{m=1}^{\infty}(\tilde{b}_m + (N+1)\tilde{c}_m)\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m + \frac{\tilde{a}_{-1}\alpha_s(Q_0^2)}{(N+1)}\ln(Q_0^2/A_2^{NS})\sum_{m=0}^{\infty}\tilde{b}_m\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m\right)\times \\ &\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}_{2,0,l}^{NS}(N+1)-1}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\left(\sum_{m=1}^{\infty}\left(\frac{1}{N+1}\tilde{b}_m + \tilde{c}_m\right)\left(\frac{\alpha_s(q^2)}{(N+1)^2}\right)^m\right)\frac{d\alpha_s(q^2)}{b_0\alpha_s^2(q^2)}\right]\times \\ &\left(\frac{1+\sum_{m=1}^{\infty}\tilde{b}_m(\alpha_s^{m+1}(Q^2)/(N+1)^{2m})}{1+\sum_{m=1}^{\infty}\tilde{b}_m(\alpha_s^{m+1}(Q_0^2)/(N+1)^{2m})}\right)\Bigg)\end{aligned} \tag{4.140}$$

and

$$F^{NS}_{2,RSC,0}(N,Q_0^2) = F^{NS}_{2,0}(N)\Bigg(1+\ln(Q_0^2/A_2^{NS})\alpha_s(Q_0^2)\Bigg(\Gamma^{NS}_{2,0,l}(N+1)+ \frac{1}{N+1}\sum_{m=1}^{\infty}\tilde{b}_m\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m+\sum_{m=0}^{\infty}\Big(\tilde{c}_m+\tfrac{1}{2}\sum_{n=0}^{m}\tilde{b}_n\tilde{b}_{m-n}\ln(Q_0^2/A_2^{NS})\Big)\left(\frac{\alpha_s(Q_0^2)}{(N+1)^2}\right)^m\Bigg)\Bigg), \tag{4.141}$$

where for $F_2^{NS}(N,Q^2)$ the coefficients in the series in $(N+1)^{-1}$are not necessarily the same as for the longitudinal structure function, hence the slightly different notation. However, $a_{-1} = \tilde{a}_{-1}$ and, if the hypothesis that in the $\overline{\rm MS}$ renormalization and factorization scheme all the contribution to the $\alpha_s^m/(N+1)^{2m-1}$ singularities in the physical anomalous dimensions is due to the parton anomalous dimension, $b_m = \tilde{b}_m$ as well. The c_m are not equal to the $\tilde{c}_m$ in general though, so there is no guarantee that $F^{NS}_{2,RSC,0}(x,Q^2)$ and $F^{NS}_{L,RSC,0}(x,Q^2)$ will behave in the same way in the small–x limit.

When all the b_m and c_m are known they can be used to present an argument for the form of the small–x behaviour of nonsinglet structure functions, e.g. to explain any discrepancy between the input power at small x for these structure functions and predictions from Regge physics. This seems to us to be an interesting project, and we look forward to the calculation of these coefficients. Until this happens our discussion of the LORSC calculation of the nonsinglet structure functions is rather academic. However, it has enabled us to discuss many of the issues in a simpler framework than if we had gone directly to the singlet structure functions. We will discuss these singlet structure functions next.

4.6. The Renormalization–Scheme–Consistent Singlet Structure Functions.

When calculating the singlet structure functions we cannot just construct the complete LO evolution and input and combine these to obtain the LO expression because the evolution mixes the two different structure functions. Each of the component parts of the LO expressions for $F_2(N,Q^2)$ and $F_L(N,Q^2)$ (we omit the superscript S in this section) must consist of LO input parts and evolution parts, but it is not obvious what these are. In order to find the full LORSC expressions for the singlet structure functions we will have to work in steps. We will consider only the full LO expression with the perturbative factors expanded about the particular value of $N = 0$ (the non-perturbative inputs are the full nonanalytic expressions for $\hat{F}_L(N)$ and $F_2(N)$), and the simplest way to proceed is to work directly with the physical quantities, solving the evolution equations in terms of physical anomalous dimensions and structure functions. We have already proved that in the loop expansion the LO expressions only depend on the one–loop physical anomalous dimensions, and in the leading–$\ln(1/x)$ expansion the LO expressions depend only on $\Gamma^0_{LL}(\alpha_s(Q^2)/N)$, $\Gamma^0_{L2}(\alpha_s(Q^2)/N)$, $\Gamma^1_{2L}(\alpha_s(Q^2)/N)$ and $\Gamma^1_{22}(\alpha_s(Q^2)/N)$. Hence, it is only the combination of these anomalous dimensions which is considered in our solution.

We cannot simply write the physical anomalous dimension matrix

$$\begin{pmatrix} \alpha_s(Q^2)\Gamma^{0,l}_{LL}(N) + \Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) & \alpha_s(Q^2)\Gamma^{0,l}_{L2}(N) + \Gamma^{\delta}_{L2}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma^{0,l}_{2L}(N) + \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N) & \alpha_s(Q^2)\Gamma^{0,l}_{22}(N) + \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{22}(\alpha_s(Q^2)/N) \end{pmatrix} \quad (4.142)$$

(where $\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) = \Gamma^{0}_{LL}(\alpha_s(Q^2)/N)$ with the one–loop component subtracted out, etc.), and solve the renormalization group equations. With this anomalous dimension matrix there is no simple closed form for the solution of these equations, and the full solution contains terms which are not properly of leading order. We must choose some way of solving for the structure functions systematically which enables us to extract the true LO behaviour in as simple a manner as possible.

In order to do this we take account of the fact that the one–loop solutions for $F_L(N,Q^2)$ and $(dF_2(N,Q^2)/d\ln Q^2)$ must be part of the complete LORSC solutions. Hence, we split our anomalous dimension matrix up into the form

$$\alpha_s(Q^2)\begin{pmatrix} \Gamma^{0,l}_{LL}(N) + & \Gamma^{0,l}_{L2}(N) \\ \Gamma^{0,l}_{2L}(N) & \Gamma^{0,l}_{22}(N) \end{pmatrix} + \begin{pmatrix} \Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) & \Gamma^{\delta}_{L2}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N) & \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{22}(\alpha_s(Q^2)/N) \end{pmatrix}, \quad (4.143)$$

and solve by treating the second matrix as a perturbation to the first. Doing this we obtain the one–loop solutions as the lowest–order solutions and can systematically calculate corrections to this, extracting the parts of these "corrections" which are leading order.

So, first let us consider the solution to the renormalization group equation

$$\frac{d}{d\ln Q^2}\begin{pmatrix} \hat{F}^{0,l}_L(N,Q^2) \\ F^{0,1}_2(N,Q^2) \end{pmatrix} = \alpha_s(Q^2)\begin{pmatrix} \Gamma^{0,l}_{LL}(N) & \Gamma^{0,l}_{L2}(N) \\ \Gamma^{0,l}_{2L}(N) & \Gamma^{0,l}_{22}(N) \end{pmatrix}\begin{pmatrix} \hat{F}^{0,l}_L(N,Q^2) \\ F^{0,1}_2(N,Q^2) \end{pmatrix}, \quad (4.144)$$

with boundary conditions $\hat{F}^{0,l}_L(N,Q^2_0) = \hat{F}_L(N)$ and $F^{0,l}_2(N,Q^2_0) = F_2(N)$. We may write the solution for the longitudinal structure function as

$$\hat{F}^{0,l}_L(N,Q^2) = \hat{F}^{0,l,+}_L(N)\left(\frac{\alpha_s(Q^2_0)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)} + \hat{F}^{0,l,-}_L(N)\left(\frac{\alpha_s(Q^2_0)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N)}, \quad (4.145)$$

where $\tilde{\Gamma}^{0,l,+,-}(N)$ are the two eigenvalues of the zeroth–order physical anomalous dimension matrix (which are the same as the eigenvalues of the zeroth–order parton anomalous dimension matrix), and $\hat{F}^{0,l,+}_L(N) + \hat{F}^{0,l,-}_L(N) = \hat{F}_L(N)$. Having chosen to write the lowest–order solution for the longitudinal structure function in this way we may then write the lowest–order solution for $F_2(N,Q^2)$ as

$$\hat{F}^{0,l}_2(N,Q^2) = e^+(N)\hat{F}^{0,l,+}_L(N)\left(\frac{\alpha_s(Q^2_0)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)} + e^-(N)\hat{F}^{0,l,-}_L(N)\left(\frac{\alpha_s(Q^2_0)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N)}, \quad (4.146)$$

where

$$e^+(N) = \left(\frac{\Gamma^{0,l,+}(N) - \Gamma^{0,l}_{LL}(N)}{\Gamma^{0,l}_{L2}(N)}\right), \qquad e^-(N) = \left(\frac{\Gamma^{0,l,-}(N) - \Gamma^{0,l}_{LL}(N)}{\Gamma^{0,l}_{L2}(N)}\right), \quad (4.147)$$

and $e^+(N)\hat{F}_L^{0,l,+}(N) + e^-(N)\hat{F}_L^{0,l,-}(N) = F_2(N)$. In practice

$$e^+(N) = N/6 + \mathcal{O}(N^2), \qquad e^-(N) = \left(\frac{27}{36-8N_f}\right) + \mathcal{O}(N), \tag{4.148}$$

and

$$\hat{F}_L^{0,l,+}(N) = \hat{F}_L(N) - \frac{36-8N_f}{27}F_2(N) + \mathcal{O}(N), \quad \hat{F}_L^{0,l,-}(N) = \frac{36-8N_f}{27}F_2(N) + \mathcal{O}(N). \tag{4.149}$$

It is then simple to see that

$$\begin{aligned} \hat{F}_2^{0,l,+}(N) &\equiv e^+(N)F_L^{0,l,+}(N) = \frac{N}{6}\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)\right)F_2(N) + \mathcal{O}(N^2), \\ \hat{F}_2^{0,l,-}(N) &\equiv e^-(N)F_L^{0,l,-}(N) = F_2(N) + \mathcal{O}(N). \end{aligned} \tag{4.150}$$

The first correction to the one–loop solution may be obtained by solving the equation,

$$\begin{aligned} \frac{d}{d\ln Q^2}\begin{pmatrix} \hat{F}_L^{c1}(N,Q^2) \\ F_2^{c1}(N,Q^2) \end{pmatrix} &= \alpha_s(Q^2)\begin{pmatrix} \Gamma_{LL}^{0,l}(N) & \Gamma_{L2}^{0,l}(N) \\ \Gamma_{2L}^{0,l}(N) & \Gamma_{22}^{0,l}(N) \end{pmatrix}\begin{pmatrix} \hat{F}_L^{c1}(N,Q^2) \\ F_2^{c1}(N,Q^2) \end{pmatrix} \\ &+ \begin{pmatrix} \Gamma_{LL}^{\delta}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma_{LL}^{\delta}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma_{2L}^{\mathrm{I}}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma_{2L}^{\mathrm{I}}(\alpha_s(Q^2)/N) \end{pmatrix}\begin{pmatrix} \hat{F}_L^{0,l}(N,Q^2) \\ F_2^{0,l}(N,Q^2) \end{pmatrix}, \end{aligned} \tag{4.151}$$

where we have used the relationships in (4.114) and (4.115) in order to simplify the second matrix. We proceed as follows. First we define the vectors

$$\underline{e}^+(N) = \begin{pmatrix} 1 \\ e^+(N) \end{pmatrix}, \qquad \underline{e}^-(N) = \begin{pmatrix} 1 \\ e^-(N) \end{pmatrix}, \qquad \underline{F}^{c1}(N,Q^2) = \begin{pmatrix} F_L^{c1}(N,Q^2) \\ F_2^{c1}(N,Q^2) \end{pmatrix}, \tag{4.152}$$

and write

$$\underline{F}^{c1}(N,Q^2) = \underline{e}^+(N)F^{c1,+}(N,Q^2) + \underline{e}^-(N)F^{c1,-}(N,Q^2). \tag{4.153}$$

We also define projection operators $\underline{p}^+(N)$ and $\underline{p}^-(N)$ by

$$\begin{aligned} \underline{p}^+(N)\cdot\underline{e}^+(N) &= 1, & \underline{p}^+(N)\cdot\underline{e}^-(N) &= 0, \\ \underline{p}^-(N)\cdot\underline{e}^+(N) &= 0, & \underline{p}^-(N)\cdot\underline{e}^-(N) &= 1, \end{aligned} \tag{4.154}$$

which in practice gives

$$\underline{p}^+(N) = \begin{pmatrix} 1 \\ \frac{8N_f-36}{27} \end{pmatrix} + \mathcal{O}(N), \qquad \underline{p}^-(N) = \begin{pmatrix} 0 \\ \frac{27}{8N_f-36} \end{pmatrix} + \mathcal{O}(N). \tag{4.155}$$

Multiplying (4.151) by $\underline{p}^+(N)$ now leads to the straightforward first–order differential equation

$$\begin{aligned} \frac{dF^{c1,+}(N,Q^2)}{d\ln Q^2} &= \alpha_s(Q^2)\Gamma^{0,l,+}(N)F^{c1,+}(N,Q^2) + \\ &\underline{p}^+(N)\cdot\begin{pmatrix} \Gamma_{LL}^{\delta}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma_{LL}^{\delta}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma_{2L}^{\mathrm{I}}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma_{2L}^{\mathrm{I}}(\alpha_s(Q^2)/N) \end{pmatrix}\begin{pmatrix} \hat{F}_L^{0,l}(N,Q^2) \\ F_2^{0,1}(N,Q^2) \end{pmatrix}. \end{aligned} \tag{4.156}$$

We also write the zeroth–order solution $\underline{F}^{0,l}(N,Q^2)$, in the form

$$\underline{F}^{0,l}(N,Q^2) = \underline{e}^{+}(N)F^{0,l,+}(N,Q^2) + \underline{e}^{-}(N)F^{0,l,-}(N,Q^2), \tag{4.157}$$

where $F^{0,l,+}(N,Q^2) = F_L^{0,l,+}(N)(\alpha_s(Q_0^2)/\alpha_s(Q^2))^{\hat{\mathbf{f}}^{0,l,+}(N)}$ and similarly for $F^{0,l,-}(N,Q^2)$. Doing this (4.151) becomes

$$\begin{aligned}\frac{d\,F^{\mathrm{cl},+}(N,Q^2)}{d\ln Q^2} &= \alpha_s(Q^2)\Gamma^{0,l,+}(N)F^{\mathrm{cl},+}(N,Q^2)+ \\ &\underline{p}^{+}(N)\cdot\begin{pmatrix}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N)\\ \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\times \\ &\left(\underline{e}^{+}(N)F_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\mathbf{f}}^{0,l,+}(N)} + \underline{e}^{-}(N)F_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\mathbf{f}}^{0,l,-}(N)}\right).\end{aligned} \tag{4.158}$$

This equation can now be solved by using the power series expansions of $\underline{e}^{+(-)}(N)$ and $\underline{p}^{+(-)}(N)$ in terms of N. Only a small part of the overall solution contributes at LO. Considering the contraction of the matrices between the two vectors in (4.158) we obtain

$$\begin{aligned}\underline{p}^{+}(N)\cdot\begin{pmatrix}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N)\\ \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\underline{e}^{+}(N) \\ = \Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) + \text{higher order},\end{aligned} \tag{4.159}$$

and

$$\begin{aligned}\underline{p}^{+}(N)\cdot\begin{pmatrix}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N)\\ \alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma^{\mathrm{I}}_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\underline{e}^{-}(N) \\ = \mathcal{O}\left(\alpha_s(Q^2)\left(\frac{\alpha_s(Q^2)}{N}\right)^{m}\right).\end{aligned} \tag{4.160}$$

Therefore (4.158) becomes

$$\begin{aligned}\frac{d\,F^{\mathrm{cl},+}(N,Q^2)}{d\ln Q^2} &= \alpha_s(Q^2)\Gamma^{0,l,+}(N)F^{\mathrm{cl},+}(N,Q^2)+ \\ &\Gamma^{\delta}_{LL}(\alpha_s(Q^2)/N)\hat{F}_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\mathbf{f}}^{0,l,+}(N)} + \text{higher order}.\end{aligned} \tag{4.161}$$

Thus

$$\begin{aligned}F^{\mathrm{cl},+}(N,Q^2) = &\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^{\delta}_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\hat{F}_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\mathbf{f}}^{0,l,+}(N)} \\ &+ \text{higher order}.\end{aligned} \tag{4.162}$$

This method of solution demonstrates the particularly nice feature that the leading–order "corrections" to the one–loop solution which are proportional to $\underline{e}^{+}(N)$ are completely independent of

$F^{0,l,-}(N,Q^2)$. Since this is due to the form of the "correction" matrix contracted between $\underline{p}^+(N)$ and $\underline{e}^-(N)$, we can write the equation for the $n_{\rm th}$–order correction to $F^{0,l,+}(N,Q^2)$ as

$$\frac{d\,F^{cn,+}(N,Q^2)}{d\ln Q^2} = \alpha_s(Q^2)\Gamma^{0,l,+}(N)F^{cn,+}(N,Q^2)$$
$$+\underline{p}^+(N)\cdot\begin{pmatrix}\Gamma^0_{LL}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\Gamma^0_{LL}(\alpha_s(Q^2)/N)\\ \alpha_s(Q^2)\Gamma^1_{2L}(\alpha_s(Q^2)/N) & -\frac{(36-8N_f)}{27}\alpha_s(Q^2)\Gamma^1_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\begin{pmatrix}\hat{F}_L^{c(n-1)}(N,Q^2)\\ F_2^{c(n-1)}(N,Q^2)\end{pmatrix}, \tag{4.163}$$

and precisely the same argument goes through each time. Therefore,

$$\frac{d\,F^{cn,+}(N,Q^2)}{d\ln Q^2} = \alpha_s(Q^2)\Gamma^{0,l,+}(N)F^{cn,+}(N,Q^2)+\Gamma^0_{LL}(\alpha_s(Q^2)/N)F^{c(n-1),+}(N,Q^2)+\text{higher order}. \tag{4.164}$$

Thus, it is easy to prove by induction that

$$F^{cn,+}(N,Q^2) = \frac{1}{n!}\left(\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right)^n \hat{F}_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)} \tag{4.165}$$
$$+\text{higher order}.$$

This leads to the straightforward expression for the whole of the leading–order part of $F^+(N,Q^2)$,

$$F^+_{RSC,0}(N,Q^2) = F_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^0_{LL}(N,\alpha_s(q^2))}{\alpha_s(q^2)}d\alpha_s^2(q^2)\right]. \tag{4.166}$$

Of course, we have not yet considered the corrections to the one–loop input. We could have built up these corrections to the input at the same time as we built up the evolution. However, it is easier to simply examine the change of the whole of (4.166) under a change in starting scale, and choose the input necessary to make the expression insensitive to such changes. In order to do this it is simplest to rewrite (4.166) in the form,

$$F^+_{RSC,0}(N,Q^2) = \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)}\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times$$
$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]+\text{higher order in } N, \tag{4.167}$$

where the second term contains those parts of the one–loop input which are higher order in N. The factor $(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N))$ must be multiplied by $\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)]$ in order to absorb changes in the evolution term under a change in starting scale $Q_0^2\to(1+\delta)Q_0^2$. This does not absorb the whole of the change of the evolution term, but all other changes required in the input are subleading to this. Similarly the change in the input of the "higher–order–in–N" terms needed to absorb the change in the evolution is entirely subleading in $\alpha_s(Q_0^2)$ to the one–loop input,

or to $(\hat{F}_L(N) - \frac{(36-8N_f)}{27}F_2(N))\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)]$. Hence, the full LO expression for $F^+(N,Q^2)$ is

$$F^+_{RSC,0}(N,Q^2) = \left[\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right)(\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)] - 1) + \hat{F}_L^{0,l,+}(N)\right]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)} \exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)} \frac{\tilde{\Gamma}^0_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)} d\alpha_s(q^2)\right]. \tag{4.168}$$

We can solve for the corrections which are proportional to $\underline{e}^-(N)$ in exactly the same manner as above. Multiplying (4.151) by $\underline{p}^-(N)$, instead of $\underline{p}^+(N)$ leads to

$$\frac{dF^{c1,-}(N,Q^2)}{d\ln Q^2} = \alpha_s(Q^2)\Gamma^{0,l,-}(N)F^{c1,-}(N,Q^2) + \underline{p}^-(N)\cdot\begin{pmatrix} \Gamma^0_{LL}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\Gamma^0_{LL}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\begin{pmatrix}\hat{F}_L^{0,l}(N,Q^2) \\ F_2^{0,l}(N,Q^2)\end{pmatrix}. \tag{4.169}$$

Again considering the contraction of the matrix between the two vectors in the last term we obtain

$$\underline{p}^-(N)\cdot\begin{pmatrix} \Gamma^0_{LL}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\Gamma^0_{LL}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\underline{e}^+(N) = \left(\frac{36-8N_f}{27}\right)(\alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N) - \frac{N}{6}\Gamma^0_{LL}(\alpha_s(Q^2)/N)) + \text{higher order}, \tag{4.170}$$

and

$$\underline{p}^-(N)\cdot\begin{pmatrix} \Gamma^0_{LL}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\Gamma^0_{LL}(\alpha_s(Q^2)/N) \\ \alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N) & -\left(\frac{36-8N_f}{27}\right)\alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N)\end{pmatrix}\underline{e}^-(N) = \mathcal{O}\left(\alpha_s^2(Q^2)\left(\frac{\alpha_s(Q^2)}{N}\right)^m\right). \tag{4.171}$$

Therefore we can write (4.169) in the form

$$\frac{dF^{c1,-}(N,Q^2)}{d\ln Q^2} = \alpha_s(Q^2)\Gamma^{0,l,-}(N)F^{c1,-}(N,Q^2) + \left(\frac{36-8N_f}{27}\right)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)} \times (\alpha_s(Q^2)\Gamma^I_{2L}(\alpha_s(Q^2)/N) - \frac{N}{6}\Gamma^0_{LL}(\alpha_s(Q^2)/N))\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right) + \text{higher order}. \tag{4.172}$$

Rather than solving this equation, it is easier to solve for the whole of the correction to $F_L^{0,l,-}(N,Q^2)$ in one go. Using (4.170) and (4.171) we easily obtain

$$\sum_{n=1}^{\infty}\left[\left(\frac{dF^{cn,-}(N,Q^2)}{d\ln Q^2}\right)-\alpha_s(Q^2)\Gamma^{0,l,-}(N)F^{cn,-}(N,Q^2)\right]=\left(\frac{36-8N_f}{27}\right)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)}$$
$$(\alpha_s(Q^2)\Gamma^{\hat{I}}_{2L}(\alpha_s(Q^2)/N)-\frac{N}{6}\Gamma^{\hat{0}}_{LL}(\alpha_s(Q^2)/N))\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times$$
$$\exp[\ln(Q_0^2/A_{LL})\hat{\Gamma}^0_{LL}(\alpha_s(Q_0^2)/N)]\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^{\hat{0}}_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$$
$$+\text{ higher order.} \tag{4.173}$$

The solution to this is relatively simple if we wish to keep only the LO parts. Letting $F^{cfull,-}(N,Q^2)=\sum_{n=1}^{\infty}F^{cn,-}(N,Q^2)$ we obtain,

$$F^{cfull,-}(N,Q^2)=F^{cfull,-}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,-}(N)}$$
$$+\left(\frac{\alpha_s(Q_0^2)\Gamma^{\hat{I}}_{2L}(\alpha_s(Q_0^2)/N)-\frac{N}{6}\Gamma^{\hat{0}}_{LL}(\alpha_s(Q_0^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)}\right)\left(\frac{36-8N_f}{27}\right)\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times$$
$$\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^{\hat{0}}_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$$
$$-\left(\frac{\alpha_s(Q_0^2)\Gamma^{\hat{I}}_{2L}(\alpha_s(Q_0^2)/N)-\frac{N}{6}\Gamma^{\hat{0}}_{LL}(\alpha_s(Q_0^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)}\right)\left(\frac{36-8N_f}{27}\right)\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times$$
$$\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,-}(N)}+\text{ higher order.} \tag{4.174}$$

Adding to the solution at one–loop, the whole of the LO part of $F^-(N,Q^2)$ is

$$F^-_{\overline{RSC},0}(N,Q^2)=\left(F_L^{0,l,-}(N)-\left(\frac{36-8N_f}{27}\right)\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times\right.$$
$$\left.\left(\frac{\alpha_s(Q_0^2)\Gamma^{\hat{I}}_{2L}(\alpha_s(Q_0^2)/N)-\frac{N}{6}\Gamma^{\hat{0}}_{LL}(\alpha_s(Q_0^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)}\right)\right)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,-}(N)}+$$
$$\left(\frac{36-8N_f}{27}\right)\left(\frac{\alpha_s(Q_0^2)\Gamma^{\hat{I}}_{2L}(\alpha_s(Q_0^2)/N)-\frac{N}{6}\Gamma^{\hat{0}}_{LL}(\alpha_s(Q_0^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)}\right)\left(\hat{F}_L(N)-\left(\frac{36-8N_f}{27}\right)F_2(N)\right)\times$$
$$\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\hat{\Gamma}^{0,l,+}(N)}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}^{\hat{0}}_{LL}(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]$$
$$+\text{ higher order,} \tag{4.175}$$

where, in order to make the expression invariant under changes in starting scale, and also ensure

that for $Q_0^2 = A_{LL}$ we have $F^-(N,Q_0^2) = \hat{F}_L^{0,l,-}(N)$, we make the choice

$$F^{cfull,-}(N,Q_0^2) = \left(\frac{36-8N_f}{27}\right)\left(\frac{\alpha_s(Q_0^2)\Gamma_{2L}^1(\alpha_s(Q_0^2)/N) - \frac{N}{6}\tilde{\Gamma}_{LL}^0(\alpha_s(Q_0^2)/N)}{\Gamma_{LL}^0(\alpha_s(Q_0^2)/N)}\right)\times$$
$$\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right)(\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(\alpha_s(Q_0^2)/N)] - 1). \tag{4.176}$$

We have been able to obtain, without too much difficulty, the leading parts of $F^+(N,Q^2)$ and $F^-(N,Q^2)$. We must now use these in order to obtain LORSC expressions for the structure functions. The way in which we have set up the calculation makes this very straightforward for the longitudinal structure function: we multiply $F_{RSC,0}^+(N,Q^2)$ and $F_{RSC,0}^-(N,Q^2)$ by $(\alpha_s(Q^2)/2\pi)$. However, we notice that all the parts of $F_{RSC,0}^-(N,Q^2)$, except the one–loop expression, are subleading to $F_{RSC,0}^+(N,Q^2)$: all the terms in the inputs in the former are a power of $\alpha_s(Q_0^2)$ higher for the same power of N. Thus, the only part of $F_{RSC,0}^-(N,Q^2)$ which contributes to the LORSC expression for $F_L(N,Q^2)$ is the one–loop part. Adding this to (4.168), and multiplying by $(\alpha_s(Q^2)/2\pi)$, we obtain

$$F_{L,RSC,0}(N,Q^2) = \frac{\alpha_s(Q_0^2)}{2\pi}\left[\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)-1}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}_{LL}^0(\alpha_s(q^2)/N)}{\alpha_s(q^2)}d\alpha_s(q^2)\right]\times\right.$$
$$\left(F_L^{0,l,+}(N) + \left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right)(\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(\alpha_s(Q_0^2)/N)] - 1)\right)$$
$$\left. + F_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N)-1}\right]. \tag{4.177}$$

It is slightly more involved to find the LORSC expressions for $F_2(N,Q^2)$. In this case we wish to find the LORSC expressions for the derivative $(dF_2(N,Q^2)/d\ln Q^2)$ and for the input $F_2(N,Q_0^2)$. We consider the former first. Using the form of $e^+(N)$ and $e^-(N)$ in (4.148) it is clear that, besides for the one–loop contributions, the LORSC expression will come from $(27/(36-8N_f))\times(dF_2^-(N,Q^2)/d\ln Q^2)$ and from $N/6\times(dF_2^+(N,Q^2)/d\ln Q^2)$. Explicitly we obtain

$$\left(\frac{dF_2(N,Q^2)}{d\ln Q^2}\right)_{RSC,0} = \alpha_s(Q_0^2)\left[e^-(N)\Gamma^{0,l,-}(N)\hat{F}_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N)-1}\right.$$
$$+\left(e^+(N)\Gamma^{0,l,+}(N)\hat{F}_L^{0,l,+}(N) - \Gamma_{2,L}^{1,0}(N)\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right)\right.$$
$$\left.+\Gamma_{2L}^1(\alpha_s(Q_0^2)/N)\left(\hat{F}_L(N) - \left(\frac{36-8N_f}{27}\right)F_2(N)\right)\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(\alpha_s(Q_0^2)/N)]\right)\times$$
$$\left.\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\tilde{\Gamma}_{LL}^0(\alpha_s(q^2)/N)}{\alpha_s^2(q^2)}d\alpha_s(q^2)\right]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)-1}\right], \tag{4.178}$$

where we have used the relationship

$$\frac{N}{6}(\Gamma^0_{LL}(\alpha_s(Q^2)/N) - \Gamma^0_{LL}(\alpha_s(Q^2)/N)) = \alpha_s(Q^2)(\Gamma^1_{2L}(\alpha_s(Q^2)/N) - \Gamma^1_{2L}(\alpha_s(Q^2)/N)), \quad (4.179)$$

and $\Gamma^{1,0}_{2,L}(N)\Big(\hat{F}_L(N) - \Big(\frac{36-8N_f}{27}\Big)F_2(N)\Big)$ is the part of the input which is common to both the one–loop and the leading–$\ln(1/x)$ input, and is subtracted from the former in order to avoid double counting. Similarly the input for $F_2(N,Q^2)$ is given by the relatively simple form

$$\begin{aligned}
F_{2,RSC,0}(N,Q_0^2) &= F_2(N) \\
&+ \alpha_s(Q_0^2)\frac{\Gamma^1_{2L}(\alpha_s(Q_0^2)/N)}{\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)}\Big(\hat{F}_L(N) - \frac{(36-8N_f)}{27}F_2(N)\Big)(\exp[\ln(Q_0^2/A_{LL})\Gamma^0_{LL}(\alpha_s(Q_0^2)/N)] - 1) \\
&+ \ln(Q_0^2/A_{LL})\alpha_s(Q_0^2)\Big(e^+(N)\Gamma^{0,l,+}(N)\hat{F}_L^{0,l,+}(N,Q_0^2) + e^-(N)\Gamma^{0,l,-}(N)\hat{F}_L^{0,l,-}(N,Q_0^2) \\
&\qquad - \Gamma^{1,0}_{2,L}(N)\Big(\hat{F}_L(N) - \Big(\frac{36-8N_f}{27}\Big)F_2(N)\Big)\Big).
\end{aligned} \quad (4.180)$$

The third term is the renormalization–scheme–invariant order–$\alpha_s(Q_0^2)$ input, which must compensate for changes in the one–loop evolution under changes in Q_0^2. Again we explicitly extract a term $\propto \ln(Q_0^2/A)\Gamma^{1,0}_{2,L}(N)\Big(\hat{F}_L(N) - \Big(\frac{36-8N_f}{27}\Big)F_2(N)\Big)$ in order to avoid double counting. The fact that this term can be thought of as appearing from two different sources leads us to choose both our unknown scale constants in the input equal to the same value A_{LL}. We note that our choice of inputs not only ensures Q_0^2–invariance up to higher orders, but are also such that our expressions for $\hat{F}_{L,RSC,0}(N,Q_0^2)$ and $F_{2,RSC,0}(N,Q_0^2)$ reduce to the nonperturbative inputs if $Q_0^2 = A_{LL}$. Having obtained our LORSC expressions for $(dF_2(N,Q^2)/d\ln Q^2)$ and $F_2(N,Q_0^2)$, then as already argued, in order to obtain our expression for $F_2(N,Q^2)$ we integrate $(dF_2(N,Q^2)/d\ln Q^2)_{RSC,0}$ from Q_0^2 to Q^2 and add to the input $F_{2,RSC,0}(N,Q_0^2)$.

Thus, we have our complete leading–order, including leading–$\ln(1/x)$ terms, renormalization–scheme–consistent expressions for the structure functions. These are significantly different from both the one–loop expressions and the leading–$\ln(1/x)$ expressions, although they do reduce to them in the appropriate limits; i.e. to the one–loop expressions for very large x, and to the leading–$\ln(1/x)$ expressions in the limit of very small x and near the boundary of evolution, $Q^2 \approx Q_0^2$. Indeed, once we include the $\mathcal{O}(\alpha_s(Q_0^2))$ inputs for $F_2(N,Q^2)$ in the definition of the leading–order inputs in the loop–expansion (as we should), each of the terms in the inputs and evolution terms in (4.177), (4.178) and (4.180) contains a part which appears in both the LO expression in the loop expansion and the LO expression in the leading $\ln(1/x)$ expansion. Our full LORSC expressions are obtained rather more easily than above by simply letting each of the input and evolution terms become the combination of the terms in the two expansion schemes. In a sense this result is obvious, but it is necessary to verify this by deriving the expressions as above.

Let us comment on the form of our final LORSC expressions. We note that as for the leading–$\ln(1/x)$ expansion, the LORSC expansion still leads to predictions for all the small–x inputs: predictions of each in terms of the nonperturbative inputs (which we imagine should be quite flat) and the nonperturbative scale A_{LL}, and also stronger predictions for the relationships between the inputs (although the scale Q_0^2 at which they should be chosen is not determined).

Finally, we notice that each of the terms appearing in our expressions is manifestly renormalization scheme invariant, and it is clear that no terms are subleading in α_s to any other terms, either in the input or in the evolution. If we had simply solved the renormalization group equations using the whole of the anomalous dimension matrix (4.142) then we would have obtained many terms which do not appear in our full leading–order expressions (4.177), (4.178) and (4.180). These would still be renormalization scheme independent (and trivially factorization scheme independent of course), since our input anomalous dimensions are renormalization scheme independent. However, all of these extra terms would be of the same form as terms which must be renormalization scheme dependent in order to absorb the changes of the leading–order expressions under a change of the coupling coming from a renormalization scheme change, i.e. $\alpha_s(Q^2) \to \alpha_s(Q^2) + \mathcal{O}(\alpha_s^2(Q^2))$ (e.g. we saw in 4.2 that the subleading–in–$\ln(1/x)$ evolution $\Phi_1^+(Q^2, Q_0^2)$ has a manifestly renormalization–scheme–independent part depending on $\Gamma_{2L}^1(\alpha_s(Q^2)/N)$ as well as a renormalization–scheme–dependent part depending on $\Gamma_{LL}^1(\alpha_s(Q^2)/N)$). Thus, these terms should be dropped, and (4.177), (4.178) and (4.180) are the correct expressions for the structure functions to be used with the one–loop coupling constant. Finally, as already stated, when considering the expressions for the structure functions expanded about negative integer values of N we will simply use the one–loop expressions, due both to lack of knowledge of the explicit resummed anomalous dimensions which would occur, and also because it almost certainly makes practically no difference to do this as far as phenomenology is concerned.

Now that we have our desired expressions, (4.177), (4.178) and (4.180), we can see how they compare to the data. The first step towards this is to consider the solution in x space. After briefly doing this we will consider detailed fits to the data.

5. x–Space Solutions.

We shall now discuss how we use the expressions (4.177), (4.178) and (4.180) in order to obtain our expressions for the x–space structure functions and ultimately compare with data. The data on $\mathcal{F}_2(x, Q^2)$ exist over a range of Q^2 from $\sim 0.2\text{GeV}^2$ to 5000GeV^2, so it is clear that we will need expressions for the structure functions which cross quark thresholds. Thus, before presenting details of the x–space solutions we must consider how we treat these quark thresholds.

In practice we will impose a lower cut on the data of $Q^2 = 2\text{GeV}^2$, except for the HERA data where we choose $Q^2 = 1.5\text{GeV}^2$ simply in order not to lose some of the very low x data. The

threshold of heavy quark production is $W^2 = 4m_H^2$, where W is the invariant mass of the hadronic system created from the struck proton (or neutron) and in the limit of zero proton (and/or neutron) mass is given by $W^2 = Q^2(x^{-1} - 1)$. m_H is the mass of the heavy quark. Thus, we clearly work in the limit where the up, down and strange quarks are effectively massless. However, we cross the b–quark threshold of $W^2 \approx 20\text{GeV}^2$, and at the lower end of our range are in the region of the c–quark threshold of $W^2 \approx 2\text{GeV}^2$. The correct treatment of these heavy quark thresholds is less well established than the treatment of effectively massless quarks, and is certainly more complicated (especially when considering leading–$\ln(1/x)$ terms). Hence, we will delay a more sophisticated treatment to a future article, and in this paper use a relatively simple treatment.

We use the prescription for treating heavy quarks outlined in [40]. This involves treating all the quarks as massless, but only allowing the heavy quarks to become active above the simple threshold $Q^2 = m_H^2$. Hence, the value of N_f appearing explicitly in any expressions changes discontinuously at this threshold. The running coupling constant, which itself depends on N_f, is defined to be continuous at the thresholds. It is determined by the relationship

$$\alpha_{s,n}(Q^2) = \alpha_{s,n+1}(Q^2)\left(1 + \frac{\alpha_{s,n+1}(Q^2)}{6\pi}\ln(m_{n+1}^2/Q^2)\right), \tag{5.1}$$

where the central $\alpha_s(Q^2)$ with $N_f = 4$ is defined by

$$\alpha_{s,4}(Q^2) = \frac{12\pi}{(33 - 2\cdot(N_f = 4))\ln(Q^2/\Lambda_{QCD,4}^2)}. \tag{5.2}$$

This leads to a kink in $\alpha_s(Q^2)$ at the thresholds, with the coupling used below a threshold being larger than the continuation downwards of the coupling used above the threshold.

This complete prescription for treatment of the heavy quarks is consistent with the decoupling theorem, as it is guaranteed to provide the correct expressions far above or below any threshold, as discussed in [40]: the increase in the coupling below a threshold compensating for the absence of virtual heavy quarks in calculations below this threshold. It is clearly going to be rather unsatisfactory in the region of the threshold, with heavy quark structure functions having an abrupt threshold in Q^2 rather than the physically correct smooth threshold in W^2. Work to rectify this is in progress, and will certainly involve the use of the heavy quark coefficient functions at leading order in $\ln(1/x)$ already calculated [13][30]. We note, however, that the treatment of quark thresholds in this paper is no less rigorous than in most of the other calculations of structure functions currently performed, and is perhaps better than some.

The number of active quark flavours, N_f, appears in (4.177), (4.178) and (4.180), in a number of places other than the implicit dependence in $\alpha_s(Q^2)$. It appears in the one–loop physical anomalous dimension eigenvalues $\tilde{\Gamma}^{0,l,+(-)}(N)$, in the eigenvector factors $e^{+(-)}(N)$, and in the definition of

$\hat{F}_L(N)$; i.e. $C^q_{L,1,l}(N) \propto N_f$ and hence, from the definitions (4.36) and (4.37) which define the nonperturbative inputs for the structure functions, we have

$$\hat{F}_L(N) - \hat{C}^f_{L,1,l}(N)F_2(N) = \hat{F}_L(N) - \frac{8}{3(N+2)}F_2(N) \propto N_f. \tag{5.3}$$

The first two forms of dependence have very little impact in practice, the N_f–dependence being relatively weak. The last dependence, the proportionality of what is in practice almost all of $\hat{F}_L(N)$ to N_f, has a large impact. It means that the expressions for the longitudinal structure function and for the $\ln Q^2$ derivative of $F_2(N, Q^2)$ are discontinuous at $Q^2 = m_c^2$ and at $Q^2 = m_b^2$. Also the definition of the input $F_2(N, Q_0^2)$ is sensitive to where Q_0^2 is chosen with respect to the thresholds.

This dependence of $\hat{F}_L(N)$ on N_f is obviously rather unsatisfactory. When treating the heavy quark thresholds more rigorously we should subtract out of the full structure functions those contributions where there are particles with heavy quarks in the final state. Doing this, we could define intrinsic light quark structure functions where the final state particles do not contain heavy quarks, and which have a fixed number of active flavours. Our definition of the nonperturbative inputs could come from these light quark structure functions alone. There would then also be heavy quark structure functions which could be calculated separately from and entirely in terms of the light quark structure functions. However, using our somewhat simplistic prescription for turning on the heavy quark contributions, even if we were to define heavy quark structure functions completely separately from the light ones, the expressions for $F_{L,H}(N, Q^2)$ and $(d\,F_{2,H}(N, Q^2)/d\ln(Q^2))$ would be discontinuous at the thresholds. Thus, in this paper we feel that the treatment of quark thresholds makes it more appropriate to define the whole structure function in terms of a discontinuous number of active quark flavours. The contribution to this complete structure function from the production of heavy quarks can then be extracted straightforwardly. Hence, we treat the problem of quark thresholds as described above. We will discuss the effect of the discontinuities at the thresholds on fits to data in detail in the next section. We note, however, that the expressions for the whole structure functions in terms of the singlet and nonsinglet components in (3.1) depend on the average value of the squared charge of the active quarks, and this will also change discontinuously at the thresholds. This means that overall the effects of heavy quark thresholds on a best fit to $F_2(x, Q^2)$ data are in practice rather small. The effects of the b–quark contribute only like the charge squared, i.e. 1/9, and hence have little impact. The effects due to the c–quark threshold are proportionally much larger, but because they are at $Q^2 \approx 2\text{GeV}^2$ they only affect a very small proportion of the complete set of data.

Having defined our treatment of quark thresholds we can now discuss the form of the x–space solutions for the structure functions. As already discussed it is the perturbative part of the moment–space structure functions for which we can produce well–ordered, RSC expressions and the nonperturbative parts of the expressions for these structure functions, $F_2(N)$ and $\hat{F}_L(N)$, will

be nonanalytic, complicated functions of N. Our complete moment–space expression for a general structure function (or derivative of a structure function) will be

$$F_i(N,Q^2) = P_{i,2}(N,\alpha_s)F_2(N) + P_{i,L}(N,\alpha_s)\hat{F}_L(N), \tag{5.4}$$

where the $P_{ij}(N,\alpha_s)$'s are the calculable perturbative components of the complete expressions which can be expanded as power series in both α_s and N. Hence, the $P_{ij}(N,\alpha_s)$'s are an example of the physically relevant perturbative functions we discussed in §4.4. Taking the inverse Mellin transformation of (5.4) back to x–space we obtain

$$x\mathcal{F}_i(x,Q^2) = \mathcal{P}_{i,2}(x,\alpha_s) \otimes \mathcal{F}_2(x) + \mathcal{P}_{i,L}(x,\alpha_s) \otimes \mathcal{F}_L(x), \tag{5.5}$$

where $\otimes$ denotes the convolution of two quantities, i.e.

$$\mathcal{A}(x) \otimes \mathcal{B}(x) = x\int_x^1 \frac{dz}{z}\mathcal{A}(x/z)\mathcal{B}(z). \tag{5.6}$$

Making an ordered calculation of the structure function is then equivalent to making an ordered calculation of the $\mathcal{P}_{ij}(x,\alpha_s)$'s. As proved in §4.4, this is equivalent to calculating these perturbative parts of the expressions for structure functions as RSC expansions about each of their nonanalyticities in N, and then taking the inverse Mellin transformation of each of these expansions by integrating around curves enclosing each nonanalyticity.

Our moment–space expressions (4.177), (4.178) and (4.180), are part of the full expressions for the structure functions in the form in (5.4). They contain the complete nonanalytic expressions for $F_2(N)$ and $\hat{F}_L(N)$, but the factors multiplying these are LORSC expansions of the perturbative parts of the expressions about $N = 0$. Thus, in a sense the expressions (4.177), (4.178) and (4.180) are misleading. They need to be interpreted correctly. We cannot simply take an inverse Mellin transformation of these expressions because the result would depend on all the singularities in the nonperturbative functions, whereas the perturbative factors in these expressions are defined as power series expansions in N about $N = 0$ with radii of convergence of only unity. The expressions do not have a direct meaning for general values of N. Instead, we must interpret them as providing us with the perturbative factors multiplying $F_2(N)$ and $\hat{F}_L(N)$ which are LORSC when expanded about $N = 0$.

Hence, we use (4.177), (4.178) and (4.180) to obtain the x–space structure functions as follows. We can factor out the perturbative part of each of these expressions which is proportional to either $\hat{F}_L(N)$ or $F_2(N)$, and take the inverse Mellin transformation of this perturbative function by integrating around a contour encircling $N = 0$. This provides us with the leading–in–ln$(1/x)$, at lowest–order–in–x parts of the perturbative $\mathcal{P}(x,\alpha_s)$'s as well as all the lowest–order–in–α_s parts at lowest–order–in–x. In order to obtain the leading–order–in–ln$(1/x)$–and–α_s, at lowest–order–in–x, part of the perturbatively calculated structure functions we then convolute with the whole of the

nonperturbative $\hat{F}_L(x)$ and $F_2(x)$ which are obtained by a complete inverse Mellin transformation of $\hat{F}_L(N)$ and $F_2(N)$, including contributions from singularities at many values of N. In principle, the full LORSC x–space structure functions are calculated by repeating this procedure for the LORSC moment–space structure functions where the perturbative parts are expanded about N equal to all negative integers. This would systematically include all leading–order–in–$\ln(1/x)$–and–α_s terms at progressively higher orders in x. As already explained, in practice one only really needs include the one–loop, or leading–in–α_s part of the full LORSC perturbative parts for all negative integers, and also only need to work to a relatively small negative integer unless one is concerned with values of x very close to one. Thus, it is this type of calculation we aim to make in order to compare our expressions with data.

In practice the calculation of the structure functions is performed in a rather different manner. The structure functions are calculated using a modification of the computer program that is used by Martin, Roberts and Stirling in their global fits to structure function data. This works in terms of parton densities and also directly in x–space. Input parton densities are specified at some scale $\tilde{Q}_0^2$ and the Q^2 evolution is calculated on a grid in x and Q^2. This Q^2 evolution is obtained by integrating up the complete renormalization group equations involving the complete parton distributions and full specified splitting functions. The structure functions are calculated by numerically performing the convolutions of the resulting Q^2–dependent parton distributions with coefficient functions. Thus, in fact we obtain the factorization–scheme–independent LORSC structure functions by working in terms of parton densities obtained from the full solution of the renormalization group equations and choosing coefficient functions and splitting functions (i.e. an effective factorization scheme) which will reproduce the correct expressions as closely as possible.

In order to do this we choose to work in a DIS–type scheme for simplicity, and take the longitudinal coefficient functions to be just the one–loop expressions. By breaking the expressions for the LORSC structure functions in terms of the structure functions themselves into expressions involving partons with the above choice of coefficient functions, and also using the power series expansions of the one–loop anomalous dimensions and eigenvalues in terms of N it is possible to choose effective splitting functions which when used in the computer program will reproduce the correct expressions for $(d\mathcal{F}_2(x,Q^2)/d\ln(Q^2))$, $\mathcal{F}_2(x,Q_0^2)$ and $\mathcal{F}_L(x,Q^2)$ to very good accuracy. We note that this choice has to be different when calculating $\mathcal{F}_2(x,Q^2)$ from when calculating $\mathcal{F}_L(x,Q^2)$. Due to the method of calculation used by the MRS program, it is extremely difficult to obtain the exact behaviour one desires of the structure functions, i.e. a given evolution away from a particular input at some fixed Q_0^2, with any choice of splitting functions. However, we aim to get as close as possible to this exact behaviour with relatively simple choices.

We will briefly describe our choice of effective splitting functions as follows. At first order in α_s we choose the normal parton splitting functions. We then add corrections to these in order to reproduce our desired results. Denoting these corrections by $\Delta\gamma_{ab}(N,\alpha_s(Q^2))$, for a given parton

to parton splitting function, and expressing these in moment space and in terms of previously discussed quantities for simplicity we get for $\mathcal{F}_L(x,Q^2)$

$$\begin{aligned}
\Delta\gamma_{fg}(N,\alpha_s(Q^2)) &= 0 \\
\Delta\gamma_{ff}(N,\alpha_s(Q^2)) &= 0 \\
\Delta\gamma_{gg}(N,\alpha_s(Q^2)) &= \Gamma^0_{LL}(\alpha_s(Q^2)/N) \\
\Delta\gamma_{gf}(N,\alpha_s(Q^2)) &= \tfrac{4}{9}\Gamma^0_{LL}(\alpha_s(Q^2)/N).
\end{aligned} \tag{5.7}$$

This form is not too difficult to understand by looking at e.g. (4.177): the only leading–$\ln(1/x)$ enhancement of the one-loop expression comes from the evolution in the longitudinal sector which is directly related to the enhanced gluon evolution and hence corrections to the gluon anomalous dimensions. For $\mathcal{F}_2(x,Q^2)$ the corrections are a little more involved:

$$\begin{aligned}
\Delta\gamma_{fg}(N,\alpha_s(Q^2)) &= \alpha_s(Q^2)\hat{C}^g_{2,1,l}(N)\Gamma^1_{2L}(\alpha_s(Q_0^2)/N) \\
\Delta\gamma_{ff}(N,\alpha_s(Q^2)) &= \alpha_s(Q^2)\left[\left(\frac{8N_f}{27}-\frac{4}{3}\right)+\frac{2}{N_f}\hat{C}^g_{2,1,l}(N)\right]\Gamma^1_{2L}(\alpha_s(Q_0^2)/N) \\
\Delta\gamma_{gg}(N,\alpha_s(Q^2)) &= \Gamma^0_{LL}(\alpha_s(Q^2)/N) - \tfrac{4}{9}\Delta\gamma_{fg}(N,\alpha_s(Q^2)) \\
\Delta\gamma_{gf}(N,\alpha_s(Q^2)) &= \tfrac{4}{9}\Gamma^0_{LL}(\alpha_s(Q^2)/N) - \tfrac{4}{9}\Delta\gamma_{ff}(N,\alpha_s(Q^2)).
\end{aligned} \tag{5.8}$$

The corrections to the quark anomalous dimensions are functions of Q_0^2 rather than Q^2, except for the single power of $\alpha_s(Q^2)$, because $\Gamma^1_{2L}(\alpha_s/N)$ appears only in the input terms in (4.178). In the small–x, or small–N limit we have the simplifications that $\Delta\gamma_{fg}(N,\alpha_s(Q^2)) \to \alpha_s(Q^2)(2N_f/3)\Gamma^1_{2L}(\alpha_s(Q_0^2)/N)$ and $\Delta\gamma_{ff}(N,\alpha_s(Q^2)) \to \frac{4}{9}\Delta\gamma_{fg}(N,\alpha_s(Q^2))$. Thus, in this limit these corrections to the one–loop quark anomalous dimensions take on the standard form of $\alpha_s(Q^2)\gamma^1_{fg}(\alpha_s(Q_0^2)/N)$ and $\alpha_s(Q^2)\gamma^1_{ff}(\alpha_s(Q_0^2)/N)$ minus their one–loop components. The prefactors multiplying these terms, which depend on the one–loop longitudinal coefficient functions, ensure that the leading–$\ln(1/x)$ enhancement of the rate of growth of $\mathcal{F}_2(x,Q^2)$ is directly coupled to the longitudinal structure, not to the gluon. This delays the enhancement to slightly smaller values of x, and reduces it a little. The contributions to the gluon anomalous dimensions beyond LO in $\ln(1/x)$ are present to counter the exponentiation of the corrections to the quark anomalous dimensions, i.e. this is similar to choosing $\gamma^1_{gg}(\alpha_s/N)$ equal to $-\frac{4}{9}\gamma^1_{fg}(\alpha_s/N)$ in (4.78)–(4.80) in order to ensure that the exponentiation of $\gamma^1_{fg}(\alpha_s/N)$ has no effect in these expressions. The effect of these terms is in practice rather small.

We may now perform checks to see if the particular choice above does indeed lead to an accurate representation of the correct expressions. In order to do this we obtain solutions for the structure functions which are the same as those obtained by the MRS program for our given choice of splitting functions, by analytically performing the same calculation as performed by this program, but in moment space, where the calculation is possible. The resulting expressions can

be compared with the correct LORSC moment–space expressions in the region of relatively small x ($x \lesssim 0.1$) by expanding both expressions in powers of N, and an accurate approximation of the differences between the two found.[20] These differences between the expressions can then be estimated by transforming back to x–space. With the choice of the splitting functions above the differences between the expressions are very small.

One may also take a more direct approach. For a simple, but physically reasonable, choice of input parton densities, and hence input structure functions, one can use the types of techniques outlined in [22] to begin with the exact expressions (4.177), (4.178) and (4.180) in moment space and then transform to x–space obtaining analytic solutions for the structure functions. These expressions are very complicated, but are almost exact for very low values of x. They begin to become approximate for larger values of x, particularly at high Q^2, due to the effects of truncating power series in N. The range of x for which the expressions are reliable depends on the severity of the truncation but it is not too difficult to have accurate expressions for $x \lesssim 0.1$. These enable one to compare the results obtained from the inverse Mellin transformation of our correct moment–space expressions with the calculations performed using the computer program in x-space. For $x \lesssim 0.1$ our choice of the anomalous dimensions leads to agreement with the analytic expressions to an accuracy much better than the errors on the data in any appropriate range of parameter space. There is no reason to believe that input distributions which are very similar in form, but which need to be described by more complicated expressions, should lead to deviations much different from our

[20] The ease of obtaining an accurate expression at small x is aided immensely by the fact that the expansions about $N = 0$ of the one–loop parton anomalous dimensions, and subsequently their eigenvalues, eigenvectors and projection operators have coefficients which stay roughly constant as the power of N increases, or even where the lowest powers have the largest coefficients. For example, the expansion of $\tilde{\gamma}^{0,l,+}(N)$ about $N = 0$ goes like $(6/N - 5.64 + 0.24N - 2.37N^2 + 3.38N^3 + \cdots)$. This enables one to truncate these series after a very small number of terms when working at small x, or conversely work to quite large x without keeping too many terms in the series. It is also the reason why the "double asymptotic scaling" formula, which effectively truncates the above series after the second term, is such a good approximation to the full one–loop solution (with flat parton inputs) up to quite high values of x. In the $\overline{\rm MS}$ renormalization and factorization scheme, or the DIS factorization scheme, the two–loop anomalous dimensions, and in the former case coefficient functions, do not exhibit nearly such a nice behaviour. e.g. the $\overline{\rm MS}$ scheme $\tilde{\gamma}_{qg}^{1,l}(N)$ expanded about $N = 0$ goes like $(26.7/N - 57.7 + 104N - 176N^2 + \cdots)$, and the expansion of the coefficient function is even worse, $C_{2,1,l}^{g}(N) \propto (1.33 - 6.16N + 9.2N^2 - 12N^3 + \cdots)$. Hence, truncations of the two–loop anomalous dimensions and coefficient functions expanded as series about $N = 0$ give a far worse approximation to the effects of the complete anomalous dimensions and coefficient functions at small x than in the one–loop case. This has been explicitly checked, and indeed, evolution and convolution of a given set of parton distributions using the series expansions for the two–loop anomalous dimensions and coefficient functions truncated at lowest order in N lead to structure functions up to a factor of two larger than those obtained from calculations performed using the full anomalous dimensions and coefficient functions.

particular choice. Hence, using this direct test and the more general, if slightly more approximate check above, we are confident that our choices do lead to the correct x–space expressions for $x \lesssim 0.1$ up to very small corrections.

At higher x the check is simpler, in fact almost trivial. Simply setting to zero all terms in the MRS program other than those coming from one loop, we observe that the expressions for the structure functions for $x > 0.1$, particularly at high Q^2, are very close to the one–loop expressions alone (becoming essentially identical for $x > 0.3$). This result agrees with calculations taking the inverse Mellin transformation of the LORSC moment–space expressions. Hence, the direct x–space calculation must be the same as our desired expression in this relatively high x range. Therefore, the correct choice of effective splitting functions leads to the MRS program producing a very accurate approximation to our correct LORSC expressions for the structure functions over the complete range of parameter space.[21]

We note that with our choice of splitting functions and definitions of parton densities then momentum is not conserved by the evolution: in the best fit to $\mathcal{F}_2(x, Q^2)$ discussed in the next section the total momentum carried by the partons at $Q^2 = 2\text{GeV}^2$ is 87% and at $Q^2 = 5000\text{GeV}^2$ it is about 94%. Hence, the amount of momentum violation is at the level of a few percent. We have already defended this violation of momentum conservation in subsection 4.3. We now also point out that starting with our definition of the partons we could define new parton densities and splitting functions by defining non–zero $C_{2,1}^{f(g)}(\alpha_s/N)$ and $C_{L,1}^{f(g)}(\alpha_s/N)$ beyond one–loop, and use the transformation rules in §2 to keep the structure functions unchanged. If these coefficient functions had negative coefficients then, compared to our prescription above, the low Q^2 parton distributions would need to be larger, and would hence carry more momentum. As Q^2 increased the effect of these new coefficient functions would decrease, and the extra amount of momentum carried by the new parton distributions compared to the original ones would decrease. Thus, the effect of such a redefinition of parton distributions would be to increase the amount of momentum carried by the partons at low Q^2 and also to slow the growth of momentum with Q^2 (or even to turn it into a fall). With a judicious redefinition of coefficient functions of this sort (with perhaps some dependence on $\alpha_s(Q_0^2)$ as well as $\alpha_s(Q^2)$) it should clearly be possible to find an effective factorization scheme where the structure functions are identical to those given by our prescription, but where the momentum violation will be extremely small. As already mentioned, one may think of this as a "physical scheme". We have not seriously investigated this redefinition of parton densities in any quantitative manner since it will not affect any physical quantities.

Now that it has been determined that our choice of splitting functions is correct, one can vary the input parameters in the standard way to obtain the best fit. In practice the starting

[21] In fact the agreement between the result obtained from the computer program and the "correct" result is certainly better than the agreement between NLO calculations performed using different programs and different prescriptions for truncating the solution at NLO [34].

scale for the numerical evolution is chosen to be $\tilde{Q}_0^2 = m_c^2$ so that the charm contribution to the structure functions is guaranteed to turn on at this scale. Evolving upwards, the bottom quark contribution turns on at $Q^2 = m_b^2$. Thus, the parton distributions and, via convolutions with coefficient functions, the structure functions are input at $\tilde{Q}_0^2$. However, this does not mean that the true starting scale, Q_0^2, has to be the same as $\tilde{Q}_0^2$. This true starting scale is the scale at which some of the couplings in the anomalous dimensions in our full LORSC expressions have to be fixed. When performing the numerical integration upwards form a lower scale we simply interpret the values of $\mathcal{F}_L(x,Q^2)$, $(d\mathcal{F}_2(x,Q^2)/d\ln Q^2)$ and $\mathcal{F}_2(x,Q^2)$ at the fixed scale Q_0^2 as our input values for these structure functions. We then demand that at Q_0^2 the relationship between these quantities is the same (up to an accuracy much better than the errors on the data) as that demanded by the expressions (4.177), (4.178) and (4.180). We also demand that the value of each of these quantities at Q_0^2 is compatible with them being written as the convolution of our completely determined, up to the scale A_{LL}, perturbative parts and our nonperturbative soft inputs. In practice we demand that these nonperturbative inputs must be flat at small x, i.e. they must be described well by a function of the form

$$\mathcal{F}_i(x) = F_i(1-x)^{\eta_i}(1+\epsilon_i x^{0.5}+\gamma_i x). \tag{5.9}$$

It is by satisfying these requirements on the inputs, while simultaneously choosing Q_0^2 as the fixed scale appearing in the anomalous dimension or splitting function dependent input terms in our expressions, which enables us to identify Q_0^2 as the true starting scale. The evolution which takes place upwards from $\tilde{Q}_0^2(=m_c^2)$ to Q_0^2 would be identical if it were to take place backwards from Q_0^2 to $\tilde{Q}_0^2$. The former manner of performing the evolution is simply more convenient in practice.[22]

We should perhaps comment more on this starting scale Q_0^2, and on its relationship to A_{LL}. A_{LL} has been defined as the particular choice of Q_0^2 for which the inputs for the structure functions reduce to the purely nonperturbative inputs. Hence, we imagine it is a scale typical of nonperturbative physics, i.e. $A_{LL} \lesssim 1\text{GeV}^2$. If we choose Q_0^2 different from (in practice higher than) A_{LL}, then A_{LL} is not precisely the scale at which the full expressions for the structure functions reduce to the nonperturbative values, although it will remain fairly close to this. Hence, A_{LL} is simply some phenomenological parameter which we will fine tune in order to get the best fit, though it does maintain its interpretation as a scale typical of the transition between perturbative and nonperturbative physics, and it would be surprizing if it were much greater than say 1GeV^2. As already mentioned, the choice of Q_0^2 is undetermined. However, it should be such that it does make our full

[22] As already mentioned, a more sophisticated treatment of the heavy quarks would involve calculating heavy quark structure functions separately from the light quark structure functions. More sophisticated treatments within the usual loop expansion, e.g [41], have the thresholds for heavy quark production built into the coefficient functions, and it is not important to begin numerical evolution at, or below any particular value of Q^2 in this case.

expressions relatively insensitive to its value. (We also reduce this sensitivity by letting A_{LL} be a free parameter for each choice of Q_0^2, though it should not vary a great deal.) This insensitivity to Q_0^2 imposes two restrictions on its value. Firstly, Q_0^2 should not be too small, otherwise the value of $\alpha_s(Q_0^2)$ will be very sensitive to our choice of Q_0^2, making it difficult to make our expressions Q_0^2 insensitive. However, we cannot simply choose very high Q_0^2. This is because terms beyond LO in the full RSC expressions will contain terms of higher order in $\ln(Q_0^2/A_{LL})$, albeit along with higher orders in $\alpha_s(Q_0^2)$. Hence, we do not wish to choose Q_0^2 to be a great deal larger than the scale A_{LL}. Thus, Q_0^2 is probably some intermediate value. It will, of course be determined in practice by the quality of the fits to data using the expressions (4.177), (4.180) and (4.178) evaluated at particular values of Q_0^2.

Now that we know precisely how we will perform our calculations of structure functions we can make some comment on the general form the structure functions have to take. The expressions for $\mathcal{F}_L(x,Q^2)$ and $\mathcal{F}_2(x,Q^2)$ depend on $\Gamma^0_{LL}(\alpha_s/N)$ and $\Gamma^1_{2L}(\alpha_s/N)$. Both of these series have coefficients which behave like $n^{-3/2}(12\ln 2)/\pi$ for very large n. This leads to a cut in the N–plane at $N = \bar{\alpha}_s 4\ln 2$ in both cases, and to the structure functions having asymptotic behaviour $\mathcal{F}(x,Q^2) \sim (\ln x)^{-3/2} x^{-4\ln 2\bar{\alpha}_s}$ as $x \to 0$, where $\alpha_s = \alpha_s(Q_0^2)$ and $Q^2 \geq Q_0^2$.

However, this is only for the strict asymptotic limit $x \to 0$. It was convincingly demonstrated in [20] (and discussed from a different point of view in [22]) that one only need keep a finite number of terms in the leading-$\ln(1/x)$ series if working at finite $\ln(1/x)$. In practice, if one works down to $x \approx 10^{-5}$ and the nonperturbative inputs behave roughly like $(1-x)^5$, then keeping terms up to 10_{th} order in the series for $\Gamma^0_{LL}(\alpha_s/N)$ and $\Gamma^1_{2L}(\alpha_s/N)$ is more than sufficient (truncating at 8_{th} order leads to only tiny errors). The series up to this order have the explicit form

$$\Gamma^0_{LL}(\alpha_s/N) = \left(\frac{\bar{\alpha}_s}{N}\right) + 2.40\left(\frac{\bar{\alpha}_s}{N}\right)^4 + 2.07\left(\frac{\bar{\alpha}_s}{N}\right)^6 + 17.3\left(\frac{\bar{\alpha}_s}{N}\right)^7 + 2.01\left(\frac{\bar{\alpha}_s}{N}\right)^8 + 39.8\left(\frac{\bar{\alpha}_s}{N}\right)^9 + 168.5\left(\frac{\bar{\alpha}_s}{N}\right)^{10} + \cdots, \tag{5.10}$$

and

$$2\pi\Gamma^1_{2L}(\alpha_s/N) = 1 + 2.5\left(\frac{\bar{\alpha}_s}{N}\right) + \left(\frac{\bar{\alpha}_s}{N}\right)^2 + \left(\frac{\bar{\alpha}_s}{N}\right)^3 + 7.01\left(\frac{\bar{\alpha}_s}{N}\right)^4 + 5.81\left(\frac{\bar{\alpha}_s}{N}\right)^5 + 13.4\left(\frac{\bar{\alpha}_s}{N}\right)^6 + 58.1\left(\frac{\bar{\alpha}_s}{N}\right)^7 + 64.7\left(\frac{\bar{\alpha}_s}{N}\right)^8 + 196.8\left(\frac{\bar{\alpha}_s}{N}\right)^9 + 650\left(\frac{\bar{\alpha}_s}{N}\right)^{10} + \cdots. \tag{5.11}$$

It is clear that the low–order terms in both these series have coefficients which generally grow far less quickly than the asymptotic relationship $a_{n+1} = 4\ln 2 a_n$; some fall, and in the case of $\Gamma^0_{LL}(\alpha_s/N)$ are even zero. Thus, in the range of parameter space we are considering we will have

rather less steep behaviour than the asymptotic limit of $\mathcal{F}(x,Q^2) \sim (\ln x)^{-3/2} x^{-4\ln 2\bar{\alpha}_s}$. Using the techniques in [22] to derive analytic expressions for the structure functions in the small–x limit, it becomes clear that the extreme smallness of the first few coefficients in the series for $\Gamma^0_{LL}(\alpha_s/N)$ leads to the small–x form of $\mathcal{F}_L(x,Q^2)$ being driven largely by the first term, with the $4_{\rm th}$–order term having a reasonable effect at the smallest x values. Similarly, the small–x form of $\mathcal{F}_2(x,Q^2)$ is determined largely by the series $\Gamma^1_{2L}(\alpha_s/N)$ and the first term in $\Gamma^0_{LL}(\alpha_s/N)$, again with the $4_{\rm th}$–order term in $\Gamma^0_{LL}(\alpha_s/N)$ playing some role at the smallest x values.

One can be a little more quantitative. Choosing $Q_0^2 = 25\text{GeV}^2$ and making a guess that $A_{LL} = 0.8\text{GeV}^2$ we obtain $\ln(Q_0^2/A_{LL}) \approx 3.4$. We also choose nonperturbative inputs which behave like $F_i(1-x)^5$ for $i = L, 2$. As argued in [22], for values of $x \lesssim 0.01$ it is a good approximation to replace the $(1-x)^5$ behaviour with the behaviour $\Theta(0.1-x)$.[23] Doing this we can derive simple expressions for the form of the inputs for $x \lesssim 0.01$ which will be accurate up to a few percent error. Using the series (5.10) and (5.11) and the expressions (4.177) and (4.180) we can take the inverse Mellin transformations (using the techniques of [22]), obtaining

$$\mathcal{F}_L(x,Q_0^2) \approx \alpha_s(Q_0^2)/2\pi\Bigg[\left(F_L + \frac{4}{27}F_2\right)\left(1 + 3.4\bar{\alpha}_s(Q_0^2)\xi + 5.8\frac{(\bar{\alpha}_s(Q_0^2)\xi)^2}{2!} + 6.6\frac{(\bar{\alpha}_s(Q_0^2)\xi)^3}{3!} + 13.7\frac{(\bar{\alpha}_s(Q_0^2)\xi)^4}{4!} + 31.5\frac{(\bar{\alpha}_s(Q_0^2)\xi)^5}{5!} + 56\frac{(\bar{\alpha}_s(Q_0^2)\xi)^6}{6!} + 136\frac{(\bar{\alpha}_s(Q_0^2)\xi)^7}{7!}\right) - \frac{4}{27}F_2\Bigg], \tag{5.12}$$

and

$$\mathcal{F}_2(x,Q_0^2) \approx 3.4\alpha_s(Q_0^2)/2\pi\left(F_L + \frac{4}{27}F_2\right)\left(1 + 4.2\bar{\alpha}_s(Q_0^2)\xi + 7.1\frac{(\bar{\alpha}_s(Q_0^2)\xi)^2}{2!} + 9.1\frac{(\bar{\alpha}_s(Q_0^2)\xi)^3}{3!} + 19.7\frac{(\bar{\alpha}_s(Q_0^2)\xi)^4}{4!} + 44.1\frac{(\bar{\alpha}_s(Q_0^2)\xi)^5}{5!} + 84\frac{(\bar{\alpha}_s(Q_0^2)\xi)^6}{6!} + 206\frac{(\bar{\alpha}_s(Q_0^2)\xi)^7}{7!}\right) + F_2, \tag{5.13}$$

where $\xi = \ln(0.1/x)$, and $\alpha_s(Q_0^2) \approx 0.2$ if $\Lambda_{QCD,4} \approx 100\text{MeV}$. Putting in values of $F_L = 2.5$ and $F_2 = 1$, choices which, as with the $(1-x)^5$ behaviour, are roughly compatible with the high x data, we have a rough estimate of the form of the input structure functions at $Q_0^2 = 25\text{GeV}^2$. We see that the coefficients in the series in $\alpha_s(Q_0^2)\xi$ multiplying $(F_L + 4/27F_2)$ grow a little more quickly for $\mathcal{F}_2(x,Q_0^2)$ than for $\mathcal{F}_L(x,Q_0^2)$. However, in the former case this contribution which rises with falling x is accompanied by the flat F_2, whereas in the latter it is accompanied only by the nearly insignificant $-4\alpha_s(Q_0^2)/(54\pi)F_2$. Thus, we find that for $x \approx 10^{-4}$ $\mathcal{F}_L(x,25)$ behaves approximately like $0.1x^{-0.3}$ but that $\mathcal{F}_2(x,25)$ is slightly flatter, behaving approximately like $0.6x^{-0.28}$. These powers of x are clearly somewhat less steep than the asymptotic $x^{-0.5}$. Comparison of the estimate of $\mathcal{F}_2(x,25)$ with the data in [1] and [2] shows a very reasonable qualitative agreement. Of course,

[23] The argument of the Θ–function depends on the power of $(1-x)$. Higher powers would require the step to occur at lower x and *vice versa*.

there is no data for $\mathcal{F}_L(x,25)$ for values of x anything like this low. Being rather more general, we find that for any Q_0^2 between 10GeV^2 and 100GeV^2, and with any sensible choice of A_{LL} (e.g $0.2\text{GeV}^2 \geq A_{LL} \geq 2\text{GeV}^2$), then both $F_2(x,Q_0^2)$ and $F_L(x,Q_0^2)$ behave roughly like $x^{-0.3}$ for $0.01 \geq x \geq 0.00001$. Hence, this type of behaviour can be taken as a prediction of the theory.

One can also use the same techniques to make an estimate of the structure functions at values of Q^2 away from 25GeV^2. The expressions involved are relatively straightforward to derive but become rather less concise than (5.12) and (5.13), and we will not write them explicitly here. With the particular inputs above they do lead to good qualitative agreement with the data below $x = 0.01$. Therefore we have every reason to feel encouraged by our results and believe that we really are proceeding correctly. However, the real test of our approach will be a complete global fit to the available data for $\mathcal{F}_2(x,Q^2)$ using the rather more accurate calculations, especially at large x, of the MRS program. We will therefore discuss these detailed fits next.

6. Fits to the Data and Predictions.

As explained in detail in the previous section we indirectly use the expressions (4.177)–(4.180) to calculate the lowest–order–in–x, RSC part of the perturbative part of the solution for the x–space singlet structure functions, and convolute this with the nonperturbative parts. The–higher–order–in–x parts of the perturbative solution for the singlet structure function, along with the nonsinglet structure functions are calculated using the normal one–loop prescription. Phenomenologically this procedure is practically identical to using the full LORSC solution for the structure functions. Let us now explain how we use the procedure to obtain fits to structure function data.

Once we have the general LORSC expression then by combining the singlet and nonsinglet components and varying all the free parameters (A_{LL}, the soft inputs for $\mathcal{F}_L^S(x,Q^2)$ and $\mathcal{F}_2^S(x,Q^2)$ and the soft nonsinglet inputs), we obtain the best fit for the available $\mathcal{F}_2$ structure function data using a particular starting scale Q_0^2. We note that at this starting scale the input $\mathcal{F}_2^S(x,Q_0^2)$ and the evolution $(d\mathcal{F}_2(x,Q^2)/d\ln Q^2)_{Q_0^2}$ are forced by (4.178) and (4.180) to be trivially related at small x. This is not the case when working at fixed order in α_s, and indeed, is a new feature of the approach to the small–x structure functions in this paper, and in particular of the manner of determining the inputs that is used. Choosing the renormalization scale to be Q^2, the one–loop value for $\Lambda_{N_f=4}$ is fixed at 100MeV, thus giving $\alpha_s(M_Z^2) = 0.115$.[24] This precise value is not determined by a best fit, but a value near to this is certainly favoured.

[24] Of course, since we are using a genuinely leading–order expression, any change in renormalization scale $\mu_R^2 \to kQ^2$ is exactly countered by a change in $\Lambda_{N_f=4}$ of the form $\Lambda_{N_f=4} \to k^{-1}\Lambda_{N_f=4}$. However, it is encouraging that making the simple choice $\mu_R^2 = Q^2$ leads to a value of $\alpha_s(M_Z^2)$ which is nicely compatible with the usual value.

There are some further details we should mention. Firstly, when obtaining a fit using the above approach, the values of $\mathcal{F}_2(x,Q^2)$ used are not precisely those published in [1] and [2]. This is because it is not $\mathcal{F}_2(x,Q^2)$ that is measured directly at HERA, but the differential cross–section. This is related to the structure functions as follows

$$\frac{d^2\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{Q^4 x}\left(2 - 2y + \frac{y^2}{1+R}\right)\mathcal{F}_2(x,Q^2), \tag{6.1}$$

where $y = Q^2/xs \approx Q^2/90000x$ for the HERA experiment and $R = \mathcal{F}_L(x,Q^2)/(\mathcal{F}_2(x,Q^2) - \mathcal{F}_L(x,Q^2))$. Over most of the range of parameter space y is very small and $\mathcal{F}_L(x,Q^2)$ is likely to be small. Hence, the measurement is essentially directly that of $\mathcal{F}_2(x,Q^2)$. However, at the lowest x values, especially as Q^2 increases, the value of $\mathcal{F}_2(x,Q^2)$ must be extracted using some prescription for the value of $\mathcal{F}_L(x,Q^2)$. Both the H1 and ZEUS collaborations, roughly speaking, obtain their values of $\mathcal{F}_L(x,Q^2)$ from predictions coming from NLO calculations of $\mathcal{F}_2(x,Q^2)$; H1 use the GRV structure function parameterization while ZEUS perform their own fit to the data on $\mathcal{F}_2(x,Q^2)$.[25] Using these values of $\mathcal{F}_L(x,Q^2)$ can lead to an increase in $\mathcal{F}_2(x,Q^2)$ of about 12% for the highest values of y compared to the values of $\mathcal{F}_2(x,Q^2)$ obtained using the assumption that $\mathcal{F}_L(x,Q^2) = 0$. Since the approach in this paper leads in practice to a somewhat lower prediction of $\mathcal{F}_L(x,Q^2)$ than the standard NLO–in–α_s approach, the values of $\mathcal{F}_2(x,Q^2)$ used in the fit must be altered to take account of the fact that our predictions for $\mathcal{F}_L(x,Q^2)$ are not the same as those used by H1 and ZEUS. Thus, the $\mathcal{F}_2(x,Q^2)$ values are a little (at most about 6%) lower for the largest values of Q^2/x than in those in [1] and [2]. In practice, the best fit is first obtained using the published values of $\mathcal{F}_2(x,Q^2)$, a prediction obtained for $\mathcal{F}_L(x,Q^2)$, the values of $\mathcal{F}_2(x,Q^2)$ altered accordingly and the best fit obtained once again. The values of $\mathcal{F}_2(x,Q^2)$ are not altered a second time using the corrected prediction for $\mathcal{F}_L(x,Q^2)$ since this changes by only a small amount, leading to further changes in $\mathcal{F}_2(x,Q^2)$ of much less than 1%.

We should also comment on our choice of m_c^2 for the best fit. This is chosen to be equal to 4GeV^2 in order to obtain a reasonable description of the available data on the charm structure function coming from EMC [42] and also from preliminary measurements at HERA [43]. The quality of the fit is shown in fig. 1. As one can see the fit is of a fair quality, with the predicted $\mathcal{F}_2^c(x,Q^2)$ perhaps being a little large in general at large x, and a little small at small x. This result is qualitatively consistent with the fact that we have used a threshold at $Q^2 = m_c^2$, rather than at

[25] We note that both NLO calculations of $\mathcal{F}_2(x,Q^2)$ use a smooth treatment of the charm threshold while the predictions for $\mathcal{F}_L(x,Q^2)$ use a LO formula which uses NLO parton distributions and assumes 4 massless quarks in the expression for the longitudinal coefficient functions. Thus, there seems to be an internal inconsistency in this method of determining $\mathcal{F}_2(x,Q^2)$. Fortunately the errors due to using the LO formula in general cancel with those from using 4 massless quarks. The net effect is an overestimate of at most about $2-3\%$ on the extracted value of $\mathcal{F}_2(x,Q^2)$ at the largest values of y. For most of the parameter space the effect is negligible.

$W^2 = Q^2(x^{-1} - 1) = 4m_c^2$, since in the latter case at very low x the threshold will effectively occur at lower values of Q^2, and hence $\mathcal{F}_2^c(x, Q^2)$ should increase relatively, while at large x the threshold will be at higher values of Q^2, and $\mathcal{F}_2^c(x, Q^2)$ will decrease relatively. Hopefully, a correct treatment of the charm quark threshold will support these qualitative conclusions. Since our treatment of the charm threshold is so primitive we do not regard our value of m_c^2 as a determination of the charm mass. Rather, m_c is used simply as a phenomenological parameter, ensuring that the charm contribution to the full structure function is qualitatively correct. Of course, $m_c^2 = 4\text{GeV}^2$ is somewhat high compared with the values obtained from reliable determinations. We will comment on this value later.

The strange quark is treated as being massless in our calculations, which is presumably a good approximation for the values of Q^2 considered. However, we insist that the strange contribution to the structure function is 0.2 of the singlet structure function minus the valence contributions (i.e. the "sea structure function") at $Q^2 = m_c^2$. This ensures compatibility with the data on neutrino–induced deep inelastic di–muon production obtained by the CCFR collaboration [44].

Finally we consider the form of the gluon at large x. Within our effective factorization scheme we would expect the gluon distribution to be quite similar to that in the $\overline{\text{MS}}$ scheme at NLO–in–α_s for very large x. Thus, we demand that our gluon is qualitatively similar to that obtained from the WA70 prompt photon data [45] at $x \geq 0.3$. This means that the gluon distribution must be roughly of the form $2.5(1-x)^6$ at $Q^2 = 20\text{GeV}^2$ for values of x this large. Encouragingly, this is the type of large x behaviour that the best fit chooses for the gluon, and no strong constraint is needed.

The fit is performed for a wide variety of data: the H1 [1] and Zeus [2] data on $\mathcal{F}_2^{ep}(x, Q^2)$ with $0.000032 \geq x \geq 0.32$ and $1.5\text{GeV}^2 \geq Q^2 \geq 5000\text{GeV}^2$; The BCDMS data [46] on $\mathcal{F}_2^{\mu p}(x, Q^2)$ with $0.07 \geq x \geq 0.75$ and $7.5\text{GeV}^2 \geq Q^2 \geq 230\text{GeV}^2$; the new NMC data [47] on $\mathcal{F}_2^{\mu p}(x, Q^2)$ and $\mathcal{F}_2^{\mu d}(x, Q^2)$ with $0.008 \geq x \geq 0.5$ and $2.5\text{GeV}^2 \geq Q^2 \geq 65\text{GeV}^2$; NMC data on the ratio of $\mathcal{F}_2^{\mu n}(x, Q^2)$ to $\mathcal{F}_2^{\mu p}(x, Q^2)$ [48] with $0.015 \geq x \geq 0.7$ and $5.5\text{GeV}^2 \geq Q^2 \geq 160\text{GeV}^2$, CCFR data [49] on $\mathcal{F}_2^{\nu N}(x, Q^2)$ and $\mathcal{F}_3^{\nu N}(x, Q^2)$ with $0.125 \geq x \geq 0.65$ and $5\text{GeV}^2 \geq Q^2 \geq 501.2\text{GeV}^2$; and the E665 [50] data on $\mathcal{F}_2^{\mu p}(x, Q^2)$ with $0.0037 \geq x \geq 0.387$ and $2.05\text{GeV}^2 \geq 64.3\text{GeV}^2$.[26] For each data set the lowest x bins cover a range of Q^2 from the minimum up to somewhat less than the maximum, and similarly the highest x bins cover a range of Q^2 from somewhat higher than

[26] It has recently been realized that the MRS fit program has been treating the errors for the E665 data incorrectly; the errors as a fraction of the value of $\mathcal{F}_2(x, Q^2)$ have been interpreted as the absolute errors, and the errors used have therefore been on average about 3 times too big. This has resulted in the very low χ^2 in, for example, [26] and [31]. In this paper the correct errors are used, though the MRSR fits are not performed again. As will be seen, including the correct errors raises the χ^2 for the fit to the E665 data by a factor of about 8, as we would expect. However, it does not really affect any comparison of the best fits since all the fits we perform give descriptions of this data which are almost identical in quality.

the minimum up to the maximum. We use the NMC data on the ratio $\mathcal{F}_2^{\mu n}(x,Q^2)/\mathcal{F}_2^{\mu p}(x,Q^2)$ as well as on $\mathcal{F}_2^{\mu p}(x,Q^2)$ and $\mathcal{F}_2^{\mu d}(x,Q^2)$ since the parameter space covered by the two types of data sets is by no means identical. One can see that the full range of data used in the fit covers an extremely wide range of both x and Q^2 and thus provides a very stringent test of any approach used to describe it.[27]

The result of the best fit to this data using the leading–order, including leading–$\ln(1/x)$ terms, RSC expressions (henceforth referred to as the LO(x) fit) with Q_0^2 chosen to be equal to 40GeV^2, $m_c^2 = 4\text{GeV}^2$ and $m_b^2 = 20\text{GeV}^2$ is shown in table 1. Also the result of the fit to the small–x data is shown in fig. 2. As one can see there is a very good quality fit to the whole selection of data, and thus over the whole range of x and Q^2. Only the NMC $\mathcal{F}_2^{\mu n}(x,Q^2)/\mathcal{F}_2^{\mu p}(x,Q^2)$ data give a χ^2 of much more than one per point, but this is due to the scatter of the data points and is true for any global fit. Overall, the fit gives a χ^2 of 1105 for 1099 data points.[28]

The fit shown is for the particular starting scale $Q_0^2 = 40\text{GeV}^2$, but the quality of the fit is extremely insensitive to changes in this scale, as we expect from the method of construction of the solutions. The fit is essentially unchanged over the range $20-80\text{GeV}^2$, and we choose 40GeV^2 as the geometric mean. When Q_0^2 drops below 20GeV^2 the fit immediately gets markedly worse. This effect is due to the bottom quark threshold, and we can explain it briefly. Both $d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2$ and $\mathcal{F}_2^S(x,Q_0^2)$ are nearly proportional to $F_L^S(x,Q_{(0)}^2)$ and hence nearly proportional to N_f at small x, as can be seen from (4.180) and (4.178) and from (4.36) and (4.37). Thus, when starting at Q_0^2 above $Q^2 = 20\text{GeV}^2$ they are both $\propto N_f = 5$. As we go below the bottom quark threshold, $Q^2 = 20\text{GeV}^2$, $d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2$ becomes smaller, and is $\propto 4/5\mathcal{F}_2^S(x,Q_0^2)$. However, if $Q_0^2 < 20\text{GeV}^2$, then $d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2$ is proportional to $\mathcal{F}_2^S(x,Q_0^2)$, with the same constant of proportionality as above, when below $Q^2 = 20\text{GeV}^2$, and is $\propto 5/4\mathcal{F}_2^S(x,Q_0^2)$ above $Q^2 = 20\text{GeV}^2$. In the latter case, choosing the correct value of $\mathcal{F}_2^S(x,Q_0^2)$ to fit the data leads to $d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2$ being too large at very small x. The quality of the fit then gets continuously worse as Q_0^2 lowers further,

[27] We have not included ZEUS data for $Q^2 \geq 2000\text{GeV}^2$ since this was not included in the MRSR fits [26] against which we compare our results. The fit to this high x, high Q^2 data is practically identical for all the fits discussed in this paper, giving a χ^2 of about 17 for the 7 data points. This fairly large value of χ^2 is due to the large scatter of these points.

[28] The LO(x) fit is a little different from that in [31] since $m_c^2 = 4\text{GeV}^2$ is chosen here in order to give a rather better description of the charm structure function data. In fact this value of m_c^2 leads to the best overall fit for the data also: lower m_c^2 leads to a slightly worse fit to the HERA data with the fit to the other data remaining more or less unchanged, while higher m_c^2 leads to a slightly worse fit to the large x data. We also point out that there was a problem in the computer program when at $Q^2 < m_c^2$ in [31]; the slope of $d\mathcal{F}_2(x,Q^2)/d\ln Q^2$ was a little too small in this region. Thus, in fig. 1 in [31] this slope should be a little larger below $Q^2 = 3\text{GeV}^2$, and hence the kink at this value of Q^2 becomes less obvious. As far as the fit is concerned, the effect of this error is minute since the error is small, and because there are so few data points below $Q^2 = 3\text{GeV}^2$. In practice the overall χ^2 should be ~ 2 better than quoted in [31].

becoming completely uncompetitive long before reaching m_c^2. We expect that a correct treatment of quark thresholds would lead to similar results to above for $Q_0^2 \approx 40\text{GeV}^2$ ($d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2 \propto \mathcal{F}_2^S(x,Q^2)$ at the input here, but $d\mathcal{F}_2^S(x,Q^2)/d\ln Q^2$ falls off towards 3/4 of this as Q^2 falls to $\sim 4\text{GeV}^2$ and the charm quark effects die away) and then a smooth falling off of the quality of the fit as Q_0^2 was lowered, with it again beginning to deteriorate somewhere in the region of 20GeV^2, due to $\alpha_s(Q_0^2)$ becoming too large below this value.

There are 18 free parameters used in the fit: $\Lambda_{QCD,4}$ (which we choose to describe as a parameter since, although it is fixed at 100MeV, and small variations would no doubt improve the fit slightly, it can certainly not vary much); four parameters for each of the valence quark contributions, which are described by functions of the form (4.38) where the normalization is set by the number of valence quarks; four parameters for the nonperturbative inputs for the two singlet structure functions, which are of the form (5.9); and the unknown scale A_{LL}, where we allow A_{LL} to be a free parameter for each Q_0^2. We do not consider Q_0^2 as a free parameter, since it can take a very wide range of values. The parameter A_{LL}, which we have argued should be a scale typical of soft physics, turns out to be 0.55GeV^2 for the fit starting at $Q_0^2 = 40\text{GeV}^2$. This decreases a little as Q_0^2 increases and *vice versa*. For $Q_0^2 = 40\text{GeV}^2$ the soft inputs for the fit are roughly

$$\hat{F}_L^S(x) \approx 3(1-x)^5(1+0.1x^{0.5}-0.2x), \qquad F_2^S(x) \approx (1-x)^{3.4}(1-0.65x^{0.5}+4.5x), \quad (6.2)$$

where they have been forced to be flat as $x \to 0$. These nonperturbative inputs lead to complete inputs of $\hat{F}_L^S(x,Q_0^2) \approx 3x^{-0.33}$ and $F_2^S(x,Q_0^2) \approx 0.65x^{-0.3}$ for $0.01 \geq x \geq 0.0001$, with the effective λ increasing in both cases for even smaller x. Instead of forcing the nonperturbative inputs to be flat as $x \to 0$ we could allow an asymptotic behaviour $x^{-\lambda}$, where $\lambda \lesssim 0.08$. This leads to an equally good fit as for the flat nonperturbative inputs.

Thus, the LO(x) fit seems to be a success. However, in order to gauge its true quality it is helpful to have some points of comparison. Hence, we will discuss some alternative fits. As in [31], the most recent MRS fits R_1 and R_2 are shown. These are obtained using the standard two–loop method, where R_1 allows $\Lambda_{\overline{\text{MS}}}^{N_f=4}$ to be free (giving $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 241\text{MeV}$) and R_2 fixes $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 344\text{MeV}$ to force a better fit to the HERA data. The new NMC data for $\mathcal{F}_2^{\mu p}(x,Q^2)$ and $\mathcal{F}_2^{\mu d}(x,Q^2)$ are used rather than the previous sets which were used in the MRS fits. This new data is not used directly in the MRS fits, i.e they are not performed again using these new data, the χ^2 values for the new NMC data sets are simply calculated using the same input parameters as in [26]. However, there is little indication that these would be changed much by the new data. The MRS fits are useful for purposes of comparison for a number of reasons. Firstly the treatment of the errors in this paper is identical to that in the MRS fits, so consistency in this respect is guaranteed. Also, any systematic differences due to differences in computer programs is guaranteed to be absent. Finally, the cuts in Q^2 for each data set are chosen to be the same in this paper as for the MRS fits.

The number of free parameters in the NLO–in–α_s fit is the same as in the LO(x) fit. There is one more parameter in the NLO–in–α_s fit due to the powerlike forms of the input gluon and singlet quark at small x being independent[29] whereas the small–x shape of the inputs for both structure functions is completely determined by the one parameter A_{LL} in the LO(x) fit. However, there is one less parameter in the NLO fit due to the normalization of the gluon being determined by momentum conservation. Essentially, the LO(x) fit is a little less constrained at large x than the NLO fit, but rather more predetermined at small x than the NLO fit. Indeed, the LO(x) fit has a certain amount of predictive power in the low x region, as seen at the end of the last section, whereas the NLO has none.

Comparing the LO(x) fit and the MRS fits, it is clear that the LO(x) scheme–independent fit is much better for the HERA data (even when compared to R_2), much better for the BCDMS data (even when compared to R_1) and similar in standard for the rest of the data. The overall fit is ~ 200 better for the whole data set; clearly a lot better. However, this direct comparison with the MRS fits is rather unfair, since there are a number of ways in which it differs from the LO(x) fit. First, and not terribly importantly, the normalizations of the data sets are allowed to vary more in the LO(x) fits than in the MRS fits, as described in the caption to table 1. (There is a small systematic difference in the normalization required by the H1 data and ZEUS data.) Rather more importantly the MRS fits use the SLAC data [51] on $\mathcal{F}_2^{ep}(x,Q^2)$ in the fits as well as the data sets mentioned in table 1. This covers the range $0.07 \geq x \geq 0.65$ and $2\text{GeV}^2 \geq Q^2 \geq 22.2\text{GeV}^2$, i.e. the same sort of high x range as the BCDMS data, but rather lower Q^2 in general. Hence the SLAC data is not included in the LO(x) fit due to much greater sensitivity of this data to potential higher twist effects than any of the other data sets. If it is included in a fit, then in certain regions of parameter space it is incompatible with the BCDMS data, as can be seen in fig. 4 of [26]. Hence a best fit to both the BCDMS data and the SLAC data must, to a certain extent, be a compromise between the two. Therefore the fit to the BCDMS data in the MRS fits is worsened somewhat by the need to fit the SLAC data. If the SLAC data were ignored the fits to the BCDMS data would improve by about 40. The LO(x) fit also differs from the MRS fits in the starting scale of the evolution and in the treatment of the quark thresholds. The charm contribution to the structure function in the MRS fits turns on at $Q_0^2 = 1\text{GeV}^2$ but is suppressed by a phenomenological factor of a function of $(3.5\text{GeV}^2/Q^2)$ in order to give a reasonable description of the charm data. The

[29] We claim that the allowed powerlike form of the inputs at small x is against the spirit of a well–ordered perturbative expansion, and that the small–x form of the inputs should be determined in terms of the nonperturbative inputs and the requirement that the expressions are insensitive to the choice of Q_0^2, as already discussed in §4.2. However, enforcing this rule would mean that the quality of the fit to the small–x data was very poor. Thus, we allow the more usual, unjustified choice for the small–x form of the inputs. In fact, at $Q_0^2 = 1\text{GeV}^2$ the gluon is strongly valence–like, while the sea–quark distribution $\sim x^{-1-0.15}$ at small x.

bottom contribution to the structure function, which is very small, is not included at all. Finally, the fact that the quark and gluon distributions are input at 1GeV^2 in the MRS fits is not helpful to the quality of the fit. If the best fit is obtained for the HERA data using the NLO calculation with massless quarks with inputs at $Q_0^2 \approx 4\text{GeV}^2$, as in [27], and evolution performed downwards in Q^2 then the gluon becomes negative at very small x by the time $Q^2 = 1\text{GeV}^2$ is reached. Hence, the valence–like input gluon distributions in MRSR_1 and MRSR_2 are not the ideal distributions for fitting the HERA data. A parameterization which would allow the gluon distribution to turn over and become negative at very small x would be better, as would simply starting at larger Q^2 where the gluon distribution is happy to be positive everywhere.[30]

Thus, in order to make a more meaningful comparison of the LO(x) fit with a NLO–in–α_s fit we have ourselves performed a NLO–in–α_s fit, called NLO_1, allowing the normalizations to vary in the same way as in the LO(x) fit, with exactly the same treatment of quark thresholds as the LO(x) fit, with the input parton distributions chosen at $Q_0^2 = m_c^2$ and with the fit to exactly the same data, i.e. to those sets shown in table 1, and with the constraint on the gluon from the WA70 prompt photon data.[31] As in the MRS fits the values of $\mathcal{F}_2(x, Q^2)$ at small x are those quoted in [2] and [1]. Strictly speaking they should be altered to take account of fact that the predicted values of $\mathcal{F}_L(x, Q^2)$ from these fits are not precisely the same as those used in [1] and [2] to extract $\mathcal{F}_2(x, Q^2)$ from the measurement of the differential cross-section, i.e. an iterative procedure should be used as with the LO(x) fit. However, the differences between the values of $\mathcal{F}_L(x, Q^2)$ in [1] and [2] and from these NLO–in–α_s fits are far smaller than the differences between those in [1] and [2] and the LO(x) fit. Hence, in practice the errors introduced by not altering the values of $\mathcal{F}_L(x, Q^2)$ are very small (the χ^2 for the fit to the HERA data might improve by a couple of points if the formally correct procedure was used). The value of m_c^2 is chosen to provide a good description of the data on the charm structure function. The value needed is $m_c^2 = 2.75\text{GeV}^2$, somewhat lower than in the LO(x) fit, and the fit to the charm structure function data is shown in fig. 3. [32] The value of $\Lambda_{\overline{\text{MS}}}^{N_f=4}$ determined by the fit is intermediate to those chosen by the MRS fits, being equal to 299MeV. The quality of the fit is shown in table 1.

[30] Starting the evolution at $Q_0^2 < 1\text{GeV}^2$ quickly leads to much worse fits to the data. Also starting at fairly high Q_0^2, i.e. $Q_0^2 \gtrsim 20\text{GeV}^2$, leads to a worsening of the fit due to the given form of parameterization not being adequate to describe the parton distributions at these scales, see e.g. [28]. Thus, Q_0^2 is no less a parameter in a NLO fit as in the LO(x) fit.

[31] Once again we allow the small–x form of the inputs to be unjustified powerlike behaviours. At $Q_0^2 = 2.75\text{GeV}^2$ the gluon is quite flat while the sea–quark distribution $\sim x^{-1-0.22}$.

[32] As in the LO(x) fit the choice of m_c^2 which gives a good description of the charm data is also the choice which gives the best global fit. Raising the value of m_c^2 leads to the value of $\Lambda_{\overline{\text{MS}}}^{N_f=4}$ chosen going up, which improves the fit to the HERA data, but the fit to the rest of the data deteriorates by more than this improvement. Lowering the value of m_c^2 leads to the value of $\Lambda_{\overline{\text{MS}}}^{N_f=4}$ chosen going down, improving the fit to the large x data, but again leading to overall deterioration.

As one can see, the different treatment of the quark thresholds and the higher starting scale for evolution has produced a better fit to the HERA data than the MRS fits, even though the value of $\alpha_s(Q^2)$ is lower than in the R_2 fit. However, the best fit comes from allowing the normalization of the H1 data to be at the lower limit allowed by the error on the normalization[33], and is still not as good as the LO(x) fit to the HERA data. Not including the SLAC data, and in the case of the R_2 fit the lowering of $\Lambda_{\overline{MS}}^{N_f=4}$, leads to a much better fit to the BCDMS data than the MRS fits, but again this is clearly not as good as the LO(x) fit. The fit to the NMC data is much the same for the LO(x) fit as for the NLO_1 fit, and the NLO_1 fit is a little better for the CCFR data. The overall quality of the NLO_1 fit is 1169 for the 1099 points, and thus is 64 worse than the LO(x) fit. Therefore, the LO(x) fit is still clearly better than the NLO_1 fit, but not nearly as convincingly as appeared to be the case when compared to the MRS fits. Nevertheless, it is encouraging that, while the overall fits to the relatively high x data, i.e. the BCDMS, NMC, E665 and CCFR data are similar in the NLO_1 fit and the LO(x) fit, it is the fit to the small–x HERA data that is definitely better for the LO(x) fit, as we would expect. This can be seen even more clearly if we simply examine the quality of the fit by separating the data into two sets: one where $x < 0.1$ and one where $x \geq 0.1$. This is shown in table 2, and demonstrates that the LO(x) fit is superior at small x, while not quite as good as the NLO_1 fit at large x.

This qualitative result above is exactly what we would expect. The importance of the leading–$\ln(1/x)$ terms in the LORSC calculation can be quantitatively judged by how they affect the fit. If, after obtaining the best LO(x) fit, all terms other than those in the one–loop expressions are set to zero, the quality of the fit is unchanged above $x = 0.3$, begins to alter slightly below this, and is clearly worse by the time we reach $x = 0.1$. Thus, the leading–$\ln(1/x)$ terms are important by this value of x. However, much of this effect is due to the terms at $\mathcal{O}(\alpha_s^2)$, so the NLO expression at fixed order in α_s should be insensitive to higher–order–in–α_s leading–$\ln(1/x)$ terms down to x somewhat lower than 0.1. Therefore, above $x = 0.1$ the NLO fit should in principle be better than the LO(x) fit since it contains terms at NLO in α_s which are important at large x. However, the NLO fit should be considerably worse at small x since it does not contain many important leading–$\ln(1/x)$ terms. This is qualitatively in agreement with the comparison of the NLO_1 fit and the LO(x) fit.

However, the above comparison is somewhat incorrect because in the process of obtaining the best fit for all the data the NLO_1 fit may choose some parameters, particularly $\Lambda_{\overline{MS}}^{N_f=4}$, such that they mimic the effects of the leading–$\ln(1/x)$ terms, and a decent fit for the small x data is obtained to the detriment of the fit to the large x data. In order to check this hypothesis we have also performed a NLO–in–α_s fit with $\Lambda_{\overline{MS}}^{N_f=4}$ fixed at 250MeV, which we will denote by NLO_2. The

[33] The fit to the H1 data continues to improve slightly for a normalization going down to 0.96 if this is allowed.

results of this fit in terms of the different data sets is shown in table 1. The fit clearly improves compared to the NLO_1 fit for the BCDMS and CCFR data, and gives the best overall fit for the high x data sets. It worsens somewhat for the NMC data, and also for the HERA data, and overall is slightly worse than the NLO_1 fit, having a χ^2 of 1184 for the 1099 data points. Nevertheless, it is perhaps a truer representation of a real NLO–in–α_s fit than NLO_1 since, as can be seen in table 2, it gives a better fit to high x data, but not such a good fit to the small–x data which presumably require the leading–$\ln(1/x)$ terms. Whether one believes this argument or not, it is certainly clear that the LO(x) fit does provide a better description of the data than any standard NLO–in–α_s fit.

We can be even more general in our NLO–in–α_s type fit. We have already argued that, when working at fixed order in α_s, despite the fact that the amount of momentum carried by the partons can be constant, it does not have to sum to unity, i.e. higher–order inputs can carry some fraction of the momentum. Thus, we have also performed a fit where the total momentum of the partons is not constrained. When doing this we have again allowed the small–x form of the parton inputs to be $\propto x^{-1-\lambda}$ which, as we have already stressed, we do not believe is correct procedure for a NLO–in–α_s calculation. However, we will allow as much freedom in this fit as possible. Doing so we obtain a fit which is nearly, but still not as good as the LO(x) fit. The fit to the data for $x \geq 0.1$ is about 25 better than the LO(x) fit, i.e. similar to the NLO_2 fit, but still about 45 worse for $x < 0.1$. The value of $\Lambda_{\overline{\rm MS}}^{N_f=4}$ chosen by the fit is 230MeV. The amounts of momentum carried by the quarks are similar to the other NLO–in–α_s fits; these are constrained by fitting the data near to the input scale. The momentum carried by the gluon is 57%, far higher than in the other NLO–in–α_s fits. This large amount of gluon momentum is necessary in order make the gluon large at small x and hence provide a good fit to the small–x data for $\mathcal{F}_2(x,Q^2)$ even with the small value of $\Lambda_{\overline{\rm MS}}^{N_f=4}$. The large amount of gluon momentum leads to the total momentum carried by the NLO partons being 114% of the proton's momentum. We do not believe the validity of this fit, having already argued that the LORSC expressions are the theoretically correct expressions, and that leading–$\ln(1/x)$ terms are necessary for the correct description of small–x data. Indeed, this fit leads to very different predictions for other quantities from those obtained using the LO(x) fit, as we will soon see. However, we do believe that the lack of the momentum constraint on the inputs allows this NLO–in–α_s fit to be perhaps the most realistic at high x: the fact that the NLO–in–α_s fit is not designed to work at small x due to the lack of leading–$\ln(1/x)$ terms can be largely compensated by an overly large input gluon distribution at small x, and the high x part of the fit can be as good as possible without having to compromise itself to match the low x data. In fact, at $x \geq 0.1$ the details of this fit are very similar to the NLO_2 fit, but the gluon is much larger at smaller x in this fit.

Let us comment briefly on other possible NLO–in–α_s fits. As already mentioned it is difficult to compare directly with global fits from groups other than MRS since there may be differences in treatment of errors or systematic differences in the outputs of the computer program. However,

we can make some comments. The latest CTEQ fits [52], e.g. CTEQ4M, are performed in a very similar manner to the NLO_1 fit, though with different cuts on data, and the results also appear to be very similar.[34] Hence, there seems little more to say than that the NLO_1 fit is similar to the CTEQ results.

We also feel we must comment on the fit performed by H1 in [1]. This takes account of the charm quark threshold rather more correctly than in the present paper, i.e. there is a smooth threshold at $W^2 = 4m_c^2$. However, it is a fit to only the H1, NMC and BCDMS data. It is difficult to make a direct comparison with this fit due to the completely different way of performing the calculation. It is however possible to perform a NLO–in–α_s fit to the same data as the H! fit using the same prescription as for NLO_1 and NLO_2. This has been done, and many similarities to features of the H1 fit noticed. It is possible to obtain a NLO–in–α_s fit for the H1, NMC and BCDMS data of a comparable quality to that in the LO(x) fit. However, if one simply inserts the values of the ZEUS and CCFR data after finding the best fit one finds that the χ^2 for the ZEUS fit is over 300, i.e. there is a certain degree of incompatibility between finding the best fit to the H1 data and the ZEUS data, and the fit to the CCFR data is very poor indeed. Perhaps more importantly, since the fit does not put any direct constraint on the gluon, then as in [1] one obtains a gluon which at $Q^2 = 5\text{GeV}^2$ behaves like $2x^{-0.2}$ at small x, which helps the fit to the H1 data, but behaves like $2(1-x)^8$ at large x. This leads to a gluon like $2(1-x)^{8.5}$ at large x at $Q^2 = 20\text{GeV}^2$ rather than the $2.5(1-x)^6$ required by the WA70 prompt photon data. For values of $x \approx 0.4$, $2(1-x)^{8.5}$ is clearly a great deal smaller than $2.5(1-x)^6$, and so the gluon obtained from this type of approach is hopelessly incompatible with the prompt photon data.

We feel that similar, if less dramatic, considerations are probably true of the type of approach adopted in [20], [27] and [28], where fits are performed to small–x data alone. Despite the fact that the input parton distributions are constrained to be very similar to standard parton distributions at large x, e.g. the MRSD0 distributions, it must be remembered that these parton distributions provide good descriptions of the large x data when evolved at NLO in α_s only when using a particular value of $\Lambda_{\overline{\text{MS}}}^{N_f=4}$. Evolving the MRSD0 parton distributions, which should have $\Lambda_{\overline{\text{MS}}}^{N_f=4} =$ 230MeV, using instead $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 360\text{MeV}$ will produce very different results. Indeed, as some sort of comparison we are able to obtain a fit to the H1 data and NMC data (thus constraining the high x parton distributions) alone to a quality rather better than in the full LO(x) fit by using a NLO–in–α_s fit. However, the fit does indeed choose $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 360\text{MeV}$. The resulting fit to virtually all the other data is very poor.

[34] There is a clear improvement to the fit to the BCDMS data compared to the NLO_1 fit, and clear worsening of the fit to the CCFR $\mathcal{F}_2(x, Q^2)$ data. These seem to be typical systematic differences between all CTEQ fits and MRS type fits and are presumably due to systematic differences in the treatment of errors or in the numerical calculation. The comparison of the codes in [34] certainly supports the latter conclusion.

Thus, we have a number of clear demonstrations of the dangers of performing fits to only a restricted set of data. In order to avoid very obvious inconsistencies with some particular data it is necessary to perform proper global fits. One may investigate the possibility that the data in some region of parameter space is not really expected to be fit well using a particular expression. Indeed, this is what we have done in the NLO_2 fit, and believe that we have sound theoretical arguments for doing so, and moreover that the results of the fits back these up.

This leads us to the question of the determination of $\alpha_s(M_Z^2)$ using global fits to structure function data. The complete RSC expression for structure functions only exists at leading order, and, as already mentioned, this leaves the renormalization scale completely undetermined. Hence, a determination of $\alpha_s(M_Z^2)$ does not really take place. (Note, however, the previous comments on this question.) It is not yet possible to extend the RSC calculation beyond the leading order due to lack of knowledge of NLO–in–$\ln(1/x)$ terms, but hopefully these will shortly become available [35], and when they do the NLO versions of (4.177)–(4.180) can be derived and put to use. Only then should the NLO coupling constant really be used in any global fit. Until the full renormalization–scheme–consistent NLO expressions become available, we believe that it is incorrect to use NLO–in–α_s fits to small–x structure function data in order to determine the NLO coupling constant (unless, of course, direct measurements of $\mathcal{F}_L(x,Q^2)$ and other less inclusive quantities at very small x turn out to verify standard two–loop predictions). However, as already argued, the fixed–order–in–α_s expressions should be accurate for CCFR, BCDMS and NMC data (except perhaps at the lowest x values), which after all are still much more precise than HERA data, and fits to these data alone will provide the best determination of the NLO $\alpha_s(Q^2)$. However, let us be wary here. One might think that the above comments mean that we believe that the value of $\Lambda_{\overline{\mathrm{MS}}}^{N_f=4}$ at NLO is about 250MeV, since our fits to large x data at NLO–in–α_s support this. To a certain extent this is true, but we also believe that our naive treatment of the heavy quark thresholds can introduce an error in the determination of $\alpha_s(M_Z^2)$ which could easily be as large as 0.005. Hence, until this naive treatment is improved, we have nothing concrete to say about the value of $\alpha_s(Q^2)$.

We have given what we hope are convincing arguments for our advocated approach for calculating structure functions. Not only do we claim theoretical correctness, and limited predictive power, but we also have good quality, very comprehensive fits to data on $\mathcal{F}_2(x,Q^2)$. However, we are well aware that only further experimental tests can prove us right (or wrong). Hence, we now discuss our predictions for $\mathcal{F}_L(x,Q^2)$.

So far we have only probed $\mathcal{F}_L(x,Q^2)$ indirectly, i.e. it is simply related to the $\ln Q^2$ derivative of $\mathcal{F}_2(x,Q^2)$ (as well as to the input $\mathcal{F}_2(x,Q_0^2)$). Having tied down the nonperturbative inputs and A_{LL} and Q_0^2 from our fit to $\mathcal{F}_2(x,Q^2)$, we have a prediction for $\mathcal{F}_L(x,Q^2)$. The result of this prediction for the fit with $Q_0^2 = 40\mathrm{GeV}^2$ is shown in fig. 4, where it is compared to the prediction using the NLO–in–α_s approach, in particular the NLO_1 fit. As one can see, it is smaller than the

NLO_1 $\mathcal{F}_L(x,Q^2)$, but becomes steeper at very small x. The NLO_2 fit gives a very similar form of $\mathcal{F}_L(x,Q^2)$ to the NLO_1 fit, while the NLO–in–α_s fit where momentum is not conserved leads to a similar form of $\mathcal{F}_L(x,Q^2)$ at high x but an even larger value than the NLO_1 fit at $x < 0.1$. The LORSC prediction for $\mathcal{F}_L(x,Q^2)$ is weakly dependent on the value of Q_0^2 chosen: the value at $Q^2 = 5\text{GeV}^2$ and $x = 10^{-4}$ varies by $\pm 10\%$ within our range of Q_0^2 (increasing with Q_0^2), and by less than this for higher x and Q^2.[35] The very recent results on $\mathcal{F}_L(x,Q^2)$ for $0.01 \gtrsim x \gtrsim 0.1$ from NMC [47] are matched far better by the LO(x) $\mathcal{F}_L(x,Q^2)$ than the NLO_1 $\mathcal{F}_L(x,Q^2)$ (the latter being rather large). However, it is fair to say that any problems with the NLO_1 $\mathcal{F}_L(x,Q^2)$ can almost certainly be attributed to the naive treatment of the charm quark threshold, i.e. the predicted $\mathcal{F}_L(x,Q^2)$ from the NLO–in–α_s calculation in the last of [9] matches the data well. Actually, it is rather uncertain whether in this region of parameter space $\mathcal{F}_L(x,Q^2)$ would be better described by the LORSC calculation, which ignores some $\mathcal{O}(\alpha_s^2)$ effects, or the NLO–in–α_s calculation, which ignores the leading–$\ln(1/x)$ effects beyond second order in α_s.

Measurements of $\mathcal{F}_L(x,Q^2)$ at $x < 10^{-2}$ would be a better discriminant between fixed–order–in–α_s calculations and those involving leading–$\ln(1/x)$ terms. However, the sort of "determination" of $\mathcal{F}_L(x,Q^2)$ already performed by H1 [54] is really only a consistency check for a particular NLO–in–α_s fit. It is by no means a true measurement of $\mathcal{F}_L(x,Q^2)$. In essence, all it proves is that the measurements of the cross–section are consistent with a particular NLO–in–α_s fit to $\mathcal{F}_2(x,Q^2)$ when the relationship between the cross–section and the values of $\mathcal{F}_2(x,Q^2)$ is determined assuming the correctness of the NLO–in–α_s expressions for both structure functions. This is a perfectly correct procedure, and should be adopted for any fit to $\mathcal{F}_2(x,Q^2)$ data, as it has been for the LO(x) fit. It would cause concern about the validity of the particular fit being used if it were to fail, but in itself says nothing whatever about the validity of a different approach, or about the actual value of $\mathcal{F}_L(x,Q^2)$. Hence, real measurements of $\mathcal{F}_L(x,Q^2)$ are needed in order to find the real values of $\mathcal{F}_L(x,Q^2)$. From fig. 4 it is clear that such measurements at HERA would be an important (and probably essential) way of determining the validity of the approach in this paper, and the genuine importance of leading–$\ln(1/x)$ terms in structure functions.

Another potentially important discriminant between different methods of calculating structure functions is the measurement of the charm structure function. Both the LO(x) fit and the NLO_1 fit can provide good fits to the currently available charm data, as already seen, and the value of m_c^2

[35] The author has submitted two conference proceedings on the topic of the current paper [53] and should point out that these are both incomplete. In the former the expression (4.180) was not used for the input for $\mathcal{F}_2(x,Q^2)$. Thus, the fit imposed far less constraint on Q_0^2 than the full procedure, and a value of 5GeV^2 was used, which resulted in a prediction of $\mathcal{F}_L(x,Q^2)$ that is much too small. The latter claimed that an input for $\mathcal{F}_L(x,Q^2)$ a little smaller than that consistent with the full set of expressions was needed for the best fit, even at the optimum Q_0^2. This was due to there being no account whatever taken of the b-quark threshold. The correction makes very little difference to the phenomenological results in this latter case.

in the NLO_1 fit is rather more satisfactory than that in the LO(x) fit. However, these calculations involve incorrect treatments of the charm quark threshold. A correct treatment of this threshold at NLO in α_s [41] shows that the value of m_c^2 must be at least halved in order to produce the same sort of value of the charm structure function as the approach used in this paper. Indeed, as seen in [55], correct NLO–in–α_s calculations with $m_c^2 = 2.25\text{GeV}^2$ undershoot the small–x data (other than the H1, calculation which has a form of the gluon already criticized and which would certainly badly undershoot the larger x EMC data). In fact, standard NLO–in–α_s calculations using the coefficient functions for charm production in [41] only produce a sufficiently large charm structure function if $m_c^2 \approx 1\text{GeV}^2$ [56], i.e. somewhat lower than expected.

A correct treatment of the charm quark threshold within the framework advocated in this paper has not been completely worked out (although, as already mentioned, work is in progress), and has certainly not been tested. However, using the naive treatment in this paper, the quark mass required in the LORSC approach is somewhat higher than for the NLO–in–α_s approach. If the same sort of conclusion is true when using a correct treatment of quark thresholds, as we might expect (but cannot guarantee), then this will offer further support for the LORSC calculation. More generally, a comparison of the theoretical calculations with the ever–improving data on the charm structure function seems potentially to be a very useful way of discriminating between different methods of calculation.[36]

In principle there are many other quantities which could be calculated within the LORSC framework and compared to those calculated using the NLO–in–α_s approach (or any other method) and to experimental data. Particularly obvious examples are the distribution of the transverse energy flow in the final state in lepton–hadron scattering and the the cross–section for forward jet production, for both of which there exists some experimental data which does not seem to be terribly well described by the order–by–order–in–α_s approach, particularly in the latter case, see e.g. [57] and [58]. We have not performed a LORSC calculation of any such quantities, and it is very difficult to estimate the results, other than guess that they will probably lie somewhere between the fixed–order–in–α_s predictions and those obtained using BFKL physics naively. Such calculations are obviously a priority, and work will begin soon. Only by comparing our theoretical predictions with a wide variety of experimental data can we determine unambiguously which theoretical approach is correct. For the moment we leave the theoretical arguments and the results of tables 1 and 2 as the evidence supporting our particular approach.

[36] We also note that as seen in [41] the longitudinal structure function is more sensitive to the treatment of the charm quark threshold than $\mathcal{F}_2(x,Q^2)$, with there being a sizeable discontinuity in $\mathcal{F}_L(x,Q^2)$ at $Q^2 = m_c^2$ when using the approach in this paper. Thus, for accurate predictions of $\mathcal{F}_L(x,Q^2)$ at values of Q^2 in the vicinity of m_c^2 a smooth treatment of the charm quark threshold is really needed in practice. The problem is not as acute at the bottom quark threshold because the total squared charge of the quarks changes proportionally much less at this threshold than at the charm quark threshold.

7. Conclusion and Summary.

In this paper we have derived expressions for the structure functions $\mathcal{F}_2(x,Q^2)$ and $\mathcal{F}_L(x,Q^2)$ in a theoretically correct, well–ordered manner. We have first done this for the particular expansions schemes which order the expressions strictly in orders of α_s, i.e. the standard loop expansion, or in terms of the leading powers of $\ln(1/x)$ for given power of α_s, i.e. the leading–$\ln(1/x)$ expansion. In both cases we have demonstrated that a correct calculation automatically leads to factorization–scheme–invariant results which may be expressed in terms of Catani's physical anomalous dimensions [30]. Thus, we refute any suggestion that these physical anomalous dimensions are simply an example of anomalous dimensions in a convenient, physically motivated factorization scheme, but insist that they are fundamental pieces in the correct expressions for the structure functions. Likewise, we insist that no particular factorization scheme is fundamentally any more "physical" than any other. However, we also demonstrate that in the case of the leading–$\ln(1/x)$ expansion the correct expressions are more difficult to obtain than in the loop expansion, and a correct calculation requires more than just the knowledge of these physical anomalous dimensions.

We have then argued that both the above expansion schemes are restrictive, and lead to only part of the correct solution at any given order. We have shown that the only calculational method which is truly consistent with working to a given order in α_s within a given renormalization scheme is the renormalization–scheme–consistent expansion, in which one works to a given order in both α_s and in $\ln(1/x)$ for given power of α_s. After presenting our argument that this is indeed the only correct expansion scheme, we have made use of Catani's physical anomalous dimensions (though this is not necessary, merely very convenient) to derive the leading–order, renormalization–scheme–consistent expressions for the structure functions $\mathcal{F}_2(x,Q^2)$ and $\mathcal{F}_L(x,Q^2)$.

As part of our overall approach we have also taken a different view of the starting scale Q_0^2 from that normally taken. Rather than trying to guess some form for the input at some particular Q_0^2, or simply allowing the inputs at some arbitrary Q_0^2 to take any form they wish within a given parameterization, we have demanded firstly that our expressions should be as insensitive as possible to the choice of the input scale, and secondly that any deviation from the flat Regge–type behaviour of the structure functions must come from perturbative effects. Thus, our inputs take the form of nonperturbative functions, which are flat at small x, convoluted with functions of the physical anomalous dimensions which are evaluated at Q_0^2 and determined by the requirement of insensitivity to the value of Q_0^2. This leads to our inputs being determined entirely in terms of the flat nonperturbative inputs and one arbitrary scale A_{LL} which roughly indicates the scale where perturbative physics should break down, i.e. $A_{LL} \lesssim 1\text{GeV}^2$. This gives us a great deal more predictive power than more usual approaches. We have some idea of the form of individual structure functions at small x: for a full range of sensible choices of Q_0^2 ($10\text{GeV}^2 - 100\text{GeV}^2$) and A_{LL} we obtain $\mathcal{F}_2(x,Q_0^2)$ roughly $\propto x^{-(0.25-0.33)}$ for $0.01 \geq x \geq 0.00001$, which is clearly in good qualitative agreement with the data. However, we also have a much stronger prediction for the relationship

between the small–x inputs for $\mathcal{F}_2(x,Q^2)$, $\mathcal{F}_L(x,Q^2)$ and for $d\mathcal{F}_2(x,Q^2)/d\ln Q^2$. Therefore, as well as some predictive (or at the very least, explanatory) power for individual structure functions, we have a consistency condition between the form of the different inputs, i.e. once we choose the input for $\mathcal{F}_2(x,Q^2)$ we have determined, up to a small amount of freedom, the small–x inputs for $d\mathcal{F}_2(x,Q^2)/d\ln Q^2$ and $\mathcal{F}_L(x,Q^2)$. This is a feature unique to the approach in this paper.

Not only are the features of the LORSC calculation unique and compelling, but they also work rather well in practice. The LORSC expressions, including this constraint on the small–x inputs, lead to very good fits to the data. To qualify this statement, the χ^2 for the LORSC fit to 1099 data on $\mathcal{F}_2(x,Q^2)$ ranging from from $0.75 \geq x \geq 0.000032$ and $1.5\text{GeV}^2 \leq Q^2 \leq 5000\text{GeV}^2$ is better by at least 60 than any NLO–in–α_s fit, even though the NLO–in–α_s fit is allowed arbitrary, unjustified powerlike behaviour at small x (i.e. the sea–quark distribution $\propto x^{-1-0.2}$), and the small–x inputs for $\mathcal{F}_2(x,Q^2)$ and $d\mathcal{F}_2(x,Q^2)/d\ln Q^2$ are allowed to vary with respect to each other a great deal more than in the LORSC fit. In fact, as we would hope, all of this superiority comes from the fit to the data with $x < 0.1$. Moreover, much the same quality of fit is obtained for a range of Q_0^2 from $20\text{GeV}^2 - 80\text{GeV}^2$. Let us also put the difference in the quality of the LO(x) and the NLO_1 fits, i.e. a difference of χ^2 of 64, into context: if one obtains fits to the data using the NLO–in–α_s approach and allows the fit to become up to 64 worse than the (best) NLO_1 fit, then values of $\Lambda_{\overline{\text{MS}}}$ from 190MeV to 380MeV are allowed, i.e. $\alpha_s(M_Z^2) = 0.109 \rightarrow 0.122$. As a caveat to the above, however, it is certainly true that the calculations in this paper must be improved to take account of massive quark thresholds in a better manner (as is the case with most NLO–in–α_s fits), and work towards this end is in progress. Nevertheless, with the present treatment we feel that the quality of the fit and the degree of explanatory (if not predictive) power, not to mention the theoretical correctness, give strong justification for the LORSC expressions.

We do, however, recognize that the quality of the fit alone, although impressive, is not such a substantial improvement on more standard approaches that it necessarily convinces one that this approach has to be correct. In order to obtain verification we must compare with more and different experimental data. Hence, we have presented a LORSC prediction for $\mathcal{F}_L(x,Q^2)$, comparing it to that obtained using the NLO–in–α_s approach. Hopefully there will be true measurements of $\mathcal{F}_L(x,Q^2)$ at HERA some time in the future with which we can compare these predictions. We stress the importance of such measurements to the understanding of the physics which really underlies hadron interactions. In the near future we will also have predictions of the charm contribution to the structure function within the framework of a correct treatment of the massive quark, and comparison with the ever–improving data on the charm structure function should also be a good discriminant between different theoretical approaches. Calculations and measurements of other, less inclusive quantities, such as forward jets, are another clear goal.

Finally, as we have noted, our calculation is at present only at leading order due to the lack of knowledge of next–to–leading–order–in–$\ln(1/x)$ coefficient functions and anomalous dimensions, or

equivalently of physical anomalous dimensions. This means that the NLO–in–α_s approach is still in principle superior to our approach at high x, where the leading–ln$(1/x)$ terms at third order in α_s and beyond are not important but the $\mathcal{O}(\alpha_s^2)$ terms are. Indeed, in practice the NLO–in–α_s fits, particularly the NLO_2 fit, are slightly better than the LO(x) fit for data at $x \geq 0.1$. We might therefore expect any predictions coming from the NLO–in–α_s approach to be more accurate than those coming from the LORSC approach down to values of x somewhere in the region of 0.05. The lack of the NLORSC expressions also means that a true determination of $\alpha_s(M_Z^2)$ from a global fit to structure function data is not yet possible, but that the best determination from fits to structure function data should at present come from using the NLO–in–α_s approach, with a correct treatment of heavy quark thresholds, using only large x data. For a really fair comparison between the renormalization–scheme–consistent method and the conventional order–by–order–in–α_s approach we really need the full next–to–leading–order, renormalization–scheme–consistent calculation. Hopefully the required NLO in ln$(1/x)$ quantities will soon become available [35], and with a little work we will be able to make such a comparison. We would expect that when using the NLORSC expressions the fit to the large x data would become at least as good as for the NLO–in–α_s approach, and that the fit to the small–x data would be of at least the same quality as for the LORSC fit. We wait expectantly to discover if this is indeed the case.

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Table 1

Comparison of quality of fits using the full leading–order (including leading–ln($1/x$) terms) renormalization–scheme–consistent expression, LO(x), and the two–loop fits $MRSR_1$, $MRSR_2$, NLO_1 and NLO_2. For the LO(x) fit the H1 data chooses a normalization of 1.00, the ZEUS data of 1.015, and the BCDMS data of 0.975. The CCFR data is fixed at a normalization of 0.95, and the rest is fixed at 1.00. Similarly, for the NLO_1 fit the H1 data is fixed at a normalization of 0.985, the ZEUS chooses a normalization of 0.99, and the BCDMS data of 0.975. Again the CCFR data is fixed at a normalization of 0.95, and the rest fixed at 1.00. Also, for the NLO_2 fit the H1 data is fixed at a normalization of 0.985, the ZEUS chooses a normalization of 0.985, and the BCDMS data of 0.97. Again the CCFR data is fixed at a normalization of 0.95, and the rest fixed at 1.00. In the R_1 and R_2 fits the BCDMS data has a fixed normalization of 0.98, the CCFR data of 0.95 and the rest of 1.00.

Experiment	data points	χ^2 LO(x)	NLO_1	NLO_2	R_1	R_2
H1 $\mathcal{F}_2^{ep}$	193	123	145	145	158	149
ZEUS $\mathcal{F}_2^{ep}$	204	253	281	296	326	308
BCDMS $\mathcal{F}_2^{\mu p}$	174	181	218	192	265	320
NMC $\mathcal{F}_2^{\mu p}$	129	122	131	148	163	135
NMC $\mathcal{F}_2^{\mu d}$	129	114	107	125	134	99
NMC $\mathcal{F}_2^{\mu n}/\mathcal{F}_2^{\mu p}$	85	142	137	138	136	132
E665 $\mathcal{F}_2^{\mu p}$	53	63	63	63	62	63
CCFR $\mathcal{F}_2^{\nu N}$	66	59	48	40	41	56
CCFR $\mathcal{F}_2^{\nu N}$	66	48	39	36	51	47

Table 2

Comparison of quality of fits using the full leading–order (including leading–ln($1/x$) terms) renormalization–scheme–consistent expression, LO(x), and the two–loop fits NLO_1 and NLO_2. The fits are identical to above, but the data are presented in terms of whether x is less than 0.1 or not.

	data points	χ^2 LO(x)	NLO_1	NLO_2
$x \geq 0.1$	551	622	615	595
$x < 0.1$	548	483	554	589
total	1099	1105	1169	1184

References

[1] H1 collaboration, *Nucl. Phys.* B470 (1996) 3.

[2] ZEUS collaboration: M. Derrick *et al*, *Zeit. Phys.* C69 (1996) 607;
preprint DESY 96–076 (1996), *Zeit. Phys.* C, in print.

[3] G. Altarelli and G. Parisi, *Nucl. Phys.* B126 (1977) 298;
Yu.L. Dokshitzer, *Sov. Jour. JETP* 46 (1977) 641;
L.N.Lipatov, *Sov. Jour. Nucl. Phys.* 20 (1975) 95;
V.N. Gribov and L.N. Lipatov, *Sov. Jour. Nucl. Phys.* 15 (1972) 438.

[4] A.D. Martin, R.G. Roberts and W.J. Stirling, *Phys. Rev.* D47 (1993) 867; *Phys. Lett.* B306 (1993) 145.

[5] A. Donnachie and P.V. Landshoff, *Nucl. Phys.* B244 (1984) 322; *Nucl. Phys.* B267 (1986) 690.

[6] A. de Rújula *et al*, *Phys. Rev.* D10 (1974) 1649.

[7] L.N. Lipatov, *Sov. Jour. Nucl. Phys.* 23 (1976) 338;
E.A. Kuraev, L.N. Lipatov and V.S. Fadin *Sov. Jour. JETP* 45 (1977) 199;
Ya. Balitskii and L.N. Lipatov, *Sov. Jour. Nucl. Phys.* 28 (1978) 6.

[8] A.D Martin, R.G. Roberts and W.J. Stirling, *Phys. Lett.* B354 (1995) 155;
H.L. Lai *et al*, *Phys. Rev.* D51 (1995) 4763.

[9] M. Glück, E. Reya and A. Vogt, *Zeit. Phys.* C48 (1990) 471; *Zeit. Phys.* C53 (1992) 127;
Phys. Lett. B306 (1993) 391; *Zeit. Phys.* C67 (1995) 433.

[10] R.D. Ball and S. Forte, *Phys. Lett.* B335 (1994) 77.

[11] R.D. Ball and S. Forte, *Phys. Lett.* B336 (1994) 77.

[12] J.C. Collins, D.E. Soper and G. Sterman, in: Perturbative Quantum Chromodynamics, ed. A.H. Mueller (World Scientific, Singapore, 1989), and references therein.

[13] S. Catani, M. Ciafaloni and F. Hautmann, *Phys. Lett.* B242 (1990) 97; *Nucl. Phys.* B366 (1991) 135; *Phys. Lett.* B307 (1993) 147.

[14] J.C. Collins and R.K. Ellis, *Nucl. Phys.* B360 (1991) 3.

[15] A.J. Askew, J. Kwieciński, A.D. Martin and P.J. Sutton, *Phys. Rev.* D49 (1994) 4402;
A.J. Askew *et al*, *Phys. Lett.* B325 (1994) 212.

[16] J. Kwieciński, A.D. Martin and P.J. Sutton, *Phys. Rev.* D53 (6094) 1996; *Zeit. Phys.* C71 (1996) 585.

[17] M. Ciafaloni, *Nucl. Phys.* B296 (1988) 49;
S. Catani, F. Fiorani and G. Marchesini, *Phys. Lett.* B234 (1990) 339; *Nucl. Phys.* B336 (1990) 18.

[18] S. Catani and F. Hautmann, *Phys. Lett.* B315 (1993) 157; *Nucl. Phys.* B427 (1994) 475.

[19] T. Jaroszewicz, *Phys. Lett.* B116 (1982) 291.

[20] R.D. Ball and S. Forte, *Phys. Lett.* B351 (1995) 313; *Phys. Lett.* B358 (1995) 365.

[21] R.K. Ellis, F. Hautmann and B.R. Webber, *Phys. Lett.* B348 (1995) 582.

[22] J.R. Forshaw, R.G. Roberts and R.S. Thorne, *Phys. Lett.* B356 (1995) 79.

[23] J. Blümlein, S. Riemersma and A. Vogt, Proc. of the International Workshop *QCD and QED in higher orders*, Rheinsburg, Germany, April, 1996, eds. J. Blümlein, F. Jegerlehner and T.

Riemann, *Nucl. Phys.* B (Proc. Suppl.), 51C (1996) p. 30; hep-ph/9607329, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print.
[24] S. Catani, *Zeit. Phys.* C70 (1996) 263.
[25] M. Ciafaloni, *Phys. Lett.* B356 (1995) 74.
[26] A.D. Martin, W.J. Stirling and R.G. Roberts, *Phys. Lett.* B387 (1996) 419.
[27] R.D. Ball and S. Forte, hep-ph/9607289, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print.
[28] S. Forte and R.D. Ball, hep-ph/9607291, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print.
[29] I. Bojak and M. Ernst, hep-ph/9609378, preprint DO–TH 96/18, September 1996.
[30] S. Catani, talk at UK workshop on HERA physics, September 1995, unpublished; hep-ph/9609263, preprint DDF 248/4/96, April 1996; hep-ph/9608310, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print.
[31] R.S. Thorne, hep-ph/9610334, *Phys. Lett.* B, in print.
[32] A.D. Martin, R.G. Roberts and W.J. Stirling, *Phys. Lett.* B266 (1991) 173.
[33] G. Altarelli, R.K. Ellis and G. Martinelli, *Nucl. Phys.* B157 (1979) 461.
[34] J. Blümlein *et al*, Proc. of the 1996 HERA Physics Workshop, eds. G. Ingelman, R. Klanner, and A. De Roeck, DESY, Hamburg, 1996, Vol. 1, p. 23.
[35] L.N. Lipatov and V.S. Fadin, *Sov. Jour. Nucl. Phys.* 50 (1989) 712;
V.S. Fadin, R. Fiore and A. Quartarolo, *Phys. Rev.* D53 (1996) 2729;
V. Del Duca, *Phys. Rev.* D54 (1996) 989;
G. Camici and M. Ciafaloni, *Phys. Lett.* B386 (1996) 341.
[36] R.D. Ball and S. Forte, *Phys. Lett.* B359 (1995) 362.
[37] R. Kirschner and L.N. Lipatov, *Nucl. Phys.* B213 (1983) 122.
[38] B.I. Ermolaev, S.I. Manaenkov and M.G. Ryskin, *Zeit. Phys.* C69 (1996) 259 ;
J. Blümlein and A. Vogt, *Acta Phys. Polon.* B27 (1996) 1309; *Phys. Lett.* B370 (1996) 149.
[39] J. Bartels, B.I. Ermolaev and M.G. Ryskin, hep-ph/9603204, preprint DESY–96–025, March 1996; *Zeit. Phys.* C70 (1996) 273;
J. Blümlein and A. Vogt, *Phys. Lett.* B386 (1996) 350; hep-ph/9610203, Proc. of the 1996 HERA Physics Workshop, eds. G. Ingelman, R. Klanner, and A. De Roeck, DESY, Hamburg, 1996, Vol. 2, p. 799; Proc. of the International Conference SPIN'96, Amsterdam, September 1996, eds. K. de Jager, P. Mulders *et al*, (World Scientific, Singapore, 1996), in print.
[40] J.C. Collins and W.K. Tung, *Nucl. Phys.* B278 (1986) 934.
[41] M. Glück, E. Reya and M. Stratmann, *Nucl. Phys.* B422 (1994) 37.
[42] EMC collaboration: J.J. Aubert *et al*, *Nucl. Phys.* B213 (1983) 31.
[43] H1 collaboration, preprint DESY 96–138 (1996), to be published in *Zeit. Phys.* C.
[44] CCFR collaboration: A.O. Bazarko *et al*, *Zeit. Phys.* C65 (1995) 189.
[45] WA70 collaboration: M. Bonesini *et al*, *Zeit. Phys.* C38 (1988) 371.
[46] BCDMS collaboration: A.C. Benvenuti *et al*, *Phys. Lett.* B223 (1989) 485.
[47] NMC collaboration, hep-ph/9610231, *Nucl. Phys.* B, in print.
[48] NMC collaboration: M. Arneodo *et al*, *Phys. Lett.* B364 (1995) 107.
[49] CCFR collaboration: P.Z. Quintas *et al*, *Phys. Rev. Lett.* 71 (1993) 1307.

[50] E665 collaboration: M.R. Adams *et al*, *Phys. Rev.* D54 (1996) 3006.
[51] L.W. Whitlow *et al*, *Phys. Lett.* B282 (1992) 475;
L.W. Whitlow *et al*, preprint SLAC–357 (1990).
[52] H.L. Lai *et al*, hep-ph/9606399, preprint MSUHEP–60426 or CTEQ–604, June 1996.
[53] R.S. Thorne, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print; Proc. of the 1996 HERA Physics Workshop, eds. G. Ingelman, R. Klanner, and A. De Roeck, DESY, Hamburg, 1996, Vol. 1, p. 107.
[54] H1 collaboration, hep-ex/9611017, preprint DESY 96–236.
[55] R.D. Ball and A. De Roeck, hep-ph/9609309, Proc. of the International Workshop on Deep Inelastic Scattering, Rome, April, 1996, in print.
[56] R.G. Roberts, private communication.
[57] H1 collaboration, paper pa02-073, submitted to ICHEP 1996, (Warsaw, July, 1996).
[58] H1 collaboration, paper pa03-049, submitted to ICHEP 1996, (Warsaw, July, 1996).

Figure Captions

Fig. 1. The description of the EMC and preliminary H1 data for $\mathcal{F}_2^c(x,Q^2)$ using the LO(x) fit.

Fig. 2. The curves correspond to the value of the proton structure function $\mathcal{F}_2(x,Q^2)$ obtained from the leading–order, renormalization–scheme–consistent (LO(x)) calculation at 12 values of x appropriate for the most recent HERA data. For clarity of display we add $0.5(12-i)$ to the value of $\mathcal{F}_2(x,Q^2)$ each time the value of x is decreased, where $i = 1 \rightarrow 12$. The data are assigned to the x value which is closest to the experimental x bin (for more details see the similar figure displaying the two–loop fits in [26]). E665 data are also shown on the curves with the five largest x values. The H1 and ZEUS data are normalized by 1.00 and 1.015 respectively in order to produce the best fit.

Fig. 3. The description of the EMC and preliminary H1 data for $\mathcal{F}_2^c(x,Q^2)$ using the NLO_1 fit.

Fig. 4. Comparison of predictions for $\mathcal{F}_L(x,Q^2)$ using the full leading–order, renormalization–scheme–consistent (LO(x)) fit and the two–loop NLO_1 fit. For both sets of curves $\mathcal{F}_L(x,Q^2)$ increases with increasing Q^2 at the lowest x values.

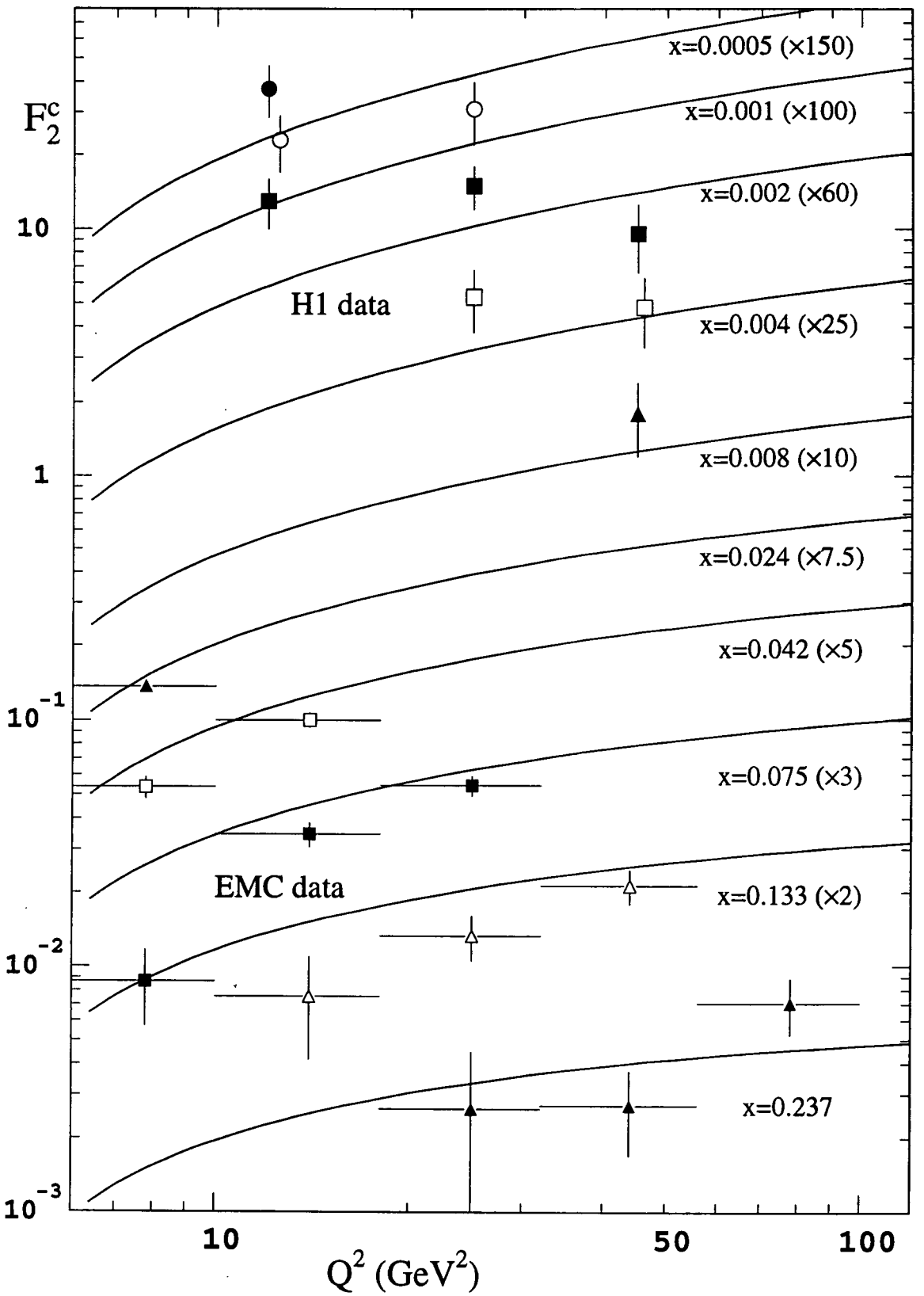
F_2^c
10
1
10^{-1}
10^{-2}
10^{-3}
x=0.0005 (×150)
x=0.001 (×100)
x=0.002 (×60)
H1 data
x=0.004 (×25)
x=0.008 (×10)
x=0.024 (×7.5)
x=0.042 (×5)
x=0.075 (×3)
EMC data
x=0.133 (×2)
x=0.237
10
50
100
Q^2 (GeV^2)

Fig. 1

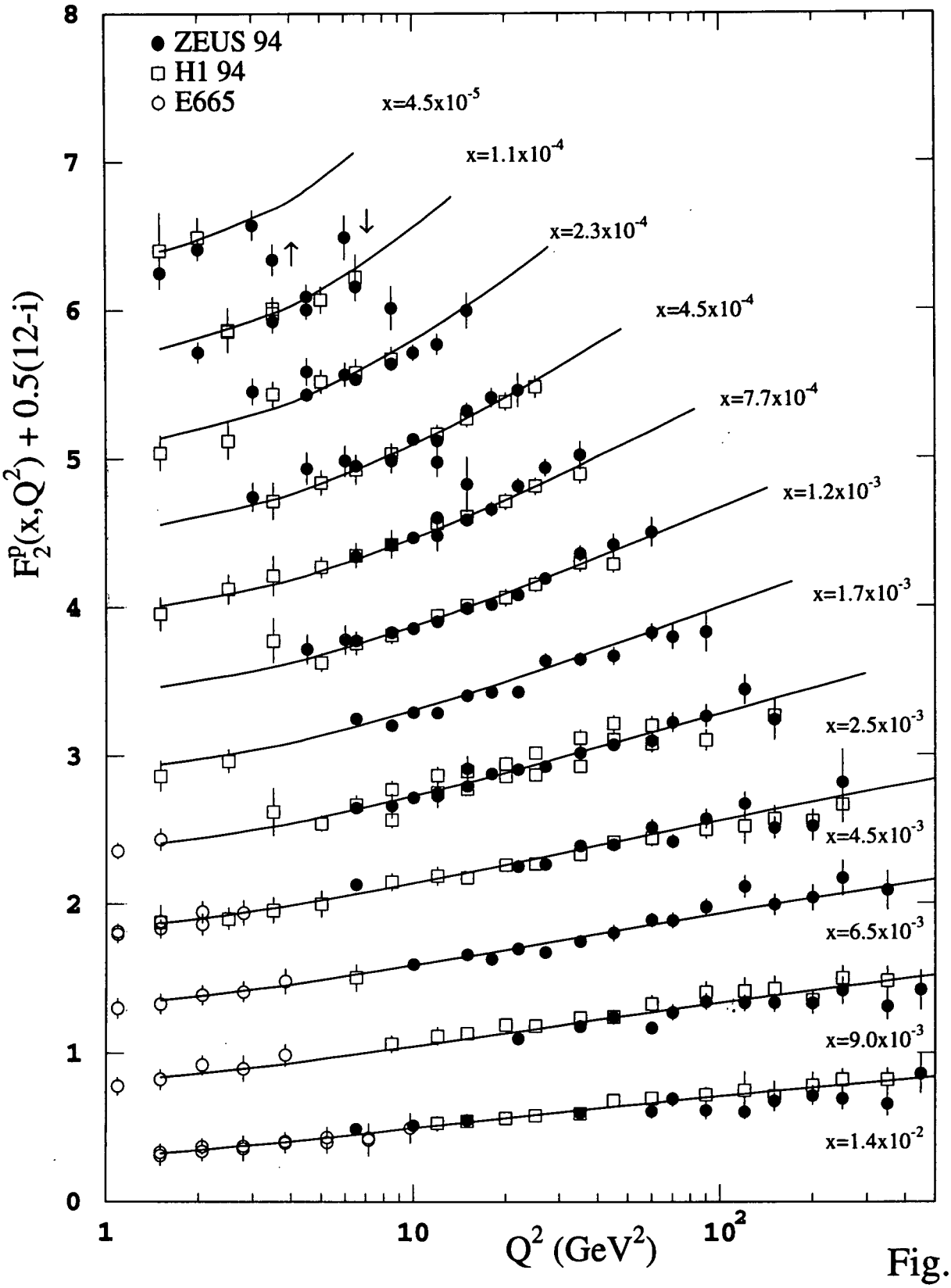
ZEUS 94
H1 94
E665
x=4.5x10^-5
x=1.1x10^-4
x=2.3x10^-4
x=4.5x10^-4
x=7.7x10^-4
x=1.2x10^-3
x=1.7x10^-3
x=2.5x10^-3
x=4.5x10^-3
x=6.5x10^-3
x=9.0x10^-3
x=1.4x10^-2
F2p(x,Q2) + 0.5(12-i)
Q2 (GeV2)
0
1
2
3
4
5
6
7
8
1
10
10^2

Fig. 2

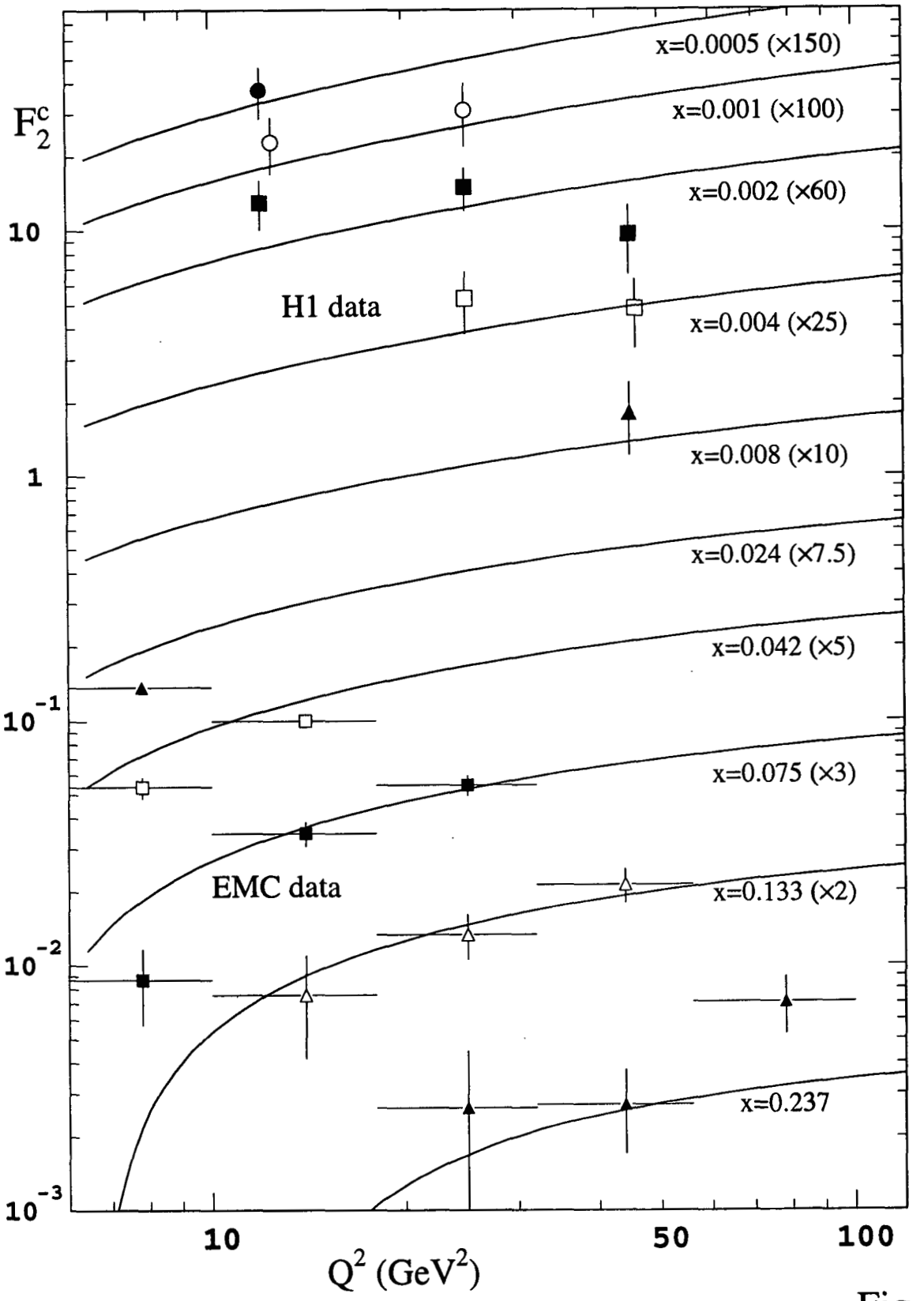
F_2^c
10
1
10^{-1}
10^{-2}
10^{-3}
x=0.0005 (×150)
x=0.001 (×100)
x=0.002 (×60)
H1 data
x=0.004 (×25)
x=0.008 (×10)
x=0.024 (×7.5)
x=0.042 (×5)
x=0.075 (×3)
EMC data
x=0.133 (×2)
x=0.237
10
50
100
Q^2 (GeV^2)

Fig. 3

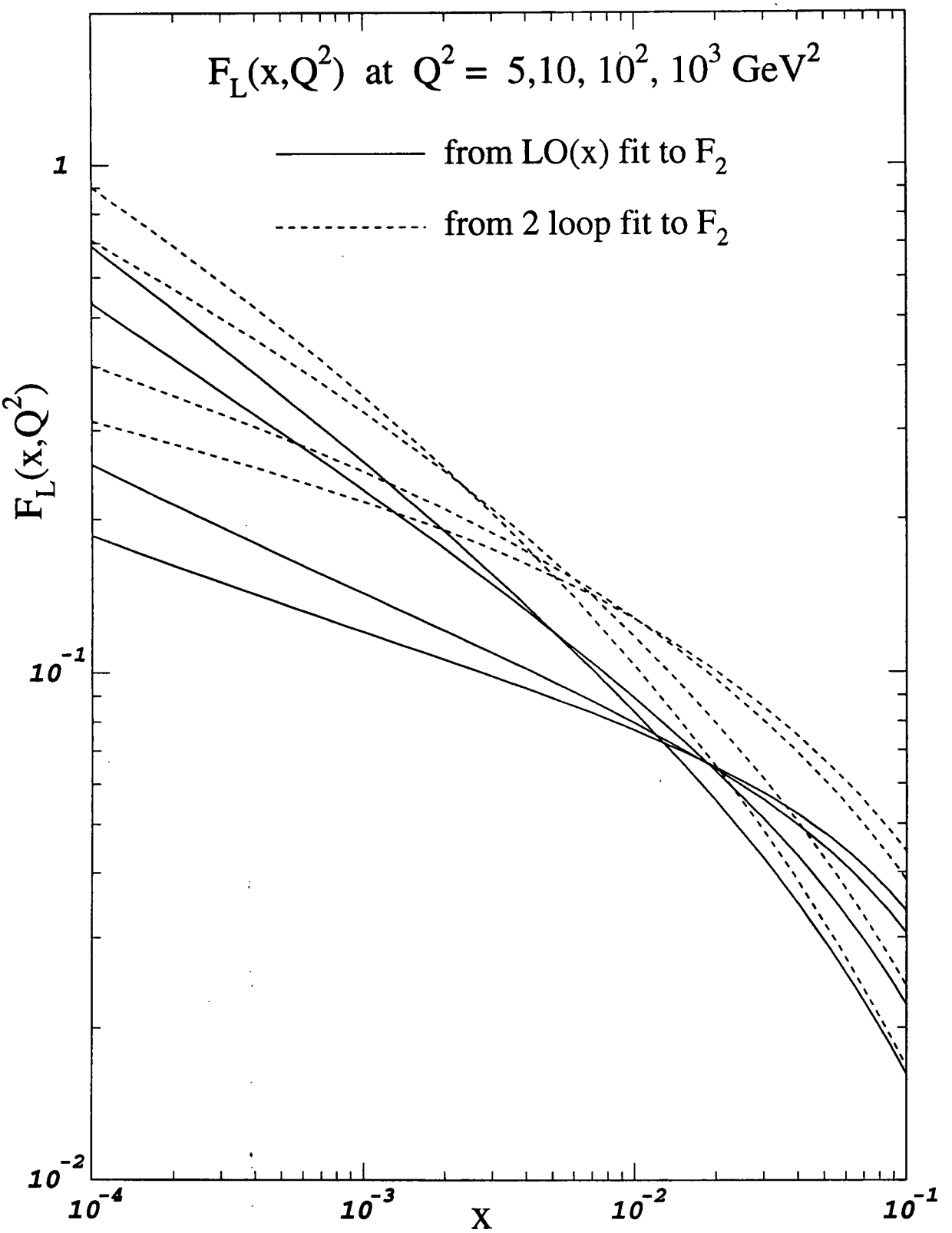
$F_L(x,Q^2)$ at Q^2 = 5,10, 10^2, 10^3 GeV^2
from LO(x) fit to F_2
from 2 loop fit to F_2
1
10^{-1}
10^{-2}
$F_L(x,Q^2)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
x

Fig. 4