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R S Thorne

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Robert S. Thorne

Rutherford Appleton Laboratory,
Chilton, Didcot, Oxon., OX11 0QX, U.K.

Abstract

We present calculations of structure functions using a renormalization scheme consistent expansion which is leading order in both $\ln(1/x)$ and $\alpha_s(Q^2)$. There is no factorization scheme dependence, and the "physical anomalous dimensions" of Catani naturally appear. A relationship between the small x forms of the inputs $F_2(x, Q_0^2)$ and $F_L(x, Q_0^2)$ is predicted. Analysis of a very wide range of data for $F_2(x, Q^2)$ is performed, and a very good global fit obtained. The prediction for $F_L(x, Q^2)$ produced using this method is smaller than the usual NLO in $\alpha_s(Q^2)$ predictions for $F_L(x, Q^2)$, and different in shape.

1. Introduction

The recent measurements of $F_2(x, Q^2)$ at HERA have provided data on a structure function at far lower values of x than any previous experiments, and show that there is a marked rise in $F_2(x, Q^2)$ at very small x down to rather low values of Q^2 [1][2]. Indeed, the most recent measurements demonstrate that the rise persists for values of Q^2 as low as 1.5 GeV^2. These measurements have led to a great deal of interest in how one should best calculate structure functions.

The particular interest in the small x region comes about because it has long been known that there is potential small x enhancement of the structure functions at high orders in the strong coupling constant [3] which is known as BFKL physics: i.e. the effective splitting function governing the growth of the gluon Green's function at small x is of the form $P(x) = \sum_{m=1}^{\infty} a_m x^{-1} \alpha_s^m \ln^{m-1}(1/x)$ [4], where the a_m are such that an asymptotic growth $x^{-1-\bar{\alpha}_s 4 \ln 2}$ was predicted ($\bar{\alpha}_s = 3\alpha_s/\pi$). This implies that one needs more than the normal fixed order in α_s expansion in order to describe physics at small x. Qualitative studies incorporating these ideas to obtain the structure functions (rather than just the gluon Green's function) were in reasonable qualitative agreement with early HERA data [5].

However, it was shown to be possible to obtain very good fits to the same data whilst ignoring these leading $\ln(1/x)$ terms. Using the Altarelli–Parisi evolution equations at next to leading (NLO) order, or even leading order (LO) in α_s and starting with flat [6], or even valence–like inputs [7], predicted a steep (though not powerlike) rise in the small x structure functions at scales reasonably far above the starting scale Q_0^2, and gave a good fit as long as Q_0^2 was chosen to be low. This countered the BFKL approach which, after all, was derived using a less well-defined theoretical framework than the renormalization group approach based on the factorization of collinear singularities and ignored everything but the leading $\ln(1/x)$ terms. An approach somewhat intermediate between these extremes is also used, i.e. the fixed order evolution beginning from inputs for the parton distributions of the form $x^{-1-\lambda}$ at small x, with $\lambda \sim 0.2$, partially justifying the relatively steep input (significantly steeper than that expected from non-perturbative physics, if somewhat less than $\bar{\alpha}_s 4 \ln 2$) from BFKL physics, e.g. [8].

Recently, due to the work of Catani and Hautmann [9], it is possible to use the leading in $\ln(1/x)$ expressions for the anomalous dimensions and coefficient functions within the well–defined renormalization group approach. Using the k_T-factorization theorem [10][11][9] they verified the form of the gluon anomalous dimensions $\gamma_{gg}^0(N, \alpha_s)$ and $\gamma_{fg}^0(N, \alpha_s)$ (where $\gamma(N) = \int_0^1 x^N P(x) dx$) and derived expressions for $\gamma_{ff}^1(N, \alpha_s(Q^2))$ and $\gamma_{fg}^1(N, \alpha_s(Q^2))$ in certain factorization schemes (since $\gamma_{ff(g)}^0(N, \alpha_s(Q^2)) = 0$ the quark anomalous dimensions are a power of N^{-1} down on those of the gluon). They also derived expressions for the coefficient functions $C_{L,1}^g(N, Q^2)$, $C_{L,1}^f(N, Q^2)$, $C_{2,1}^g(N, Q^2)$ and $C_{2,1}^f(N, Q^2)$ (all zeroth order quantities being zero except $C_{2,0}^f$, which is unity). This facilitated calculations of structure functions within the normal renormalization group framework whilst including the leading $\ln(1/x)$ terms. Calculations were performed using these terms

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[12]–[14] and comparisons with data made. The calculations used different methods of solution, made rather different assumptions and used (different) ansätze for unknown terms. Consequently, differing results were obtained. The conclusions which could be drawn regarding the inclusion of the leading $\ln(1/x)$ terms depended very much on which approach was taken. However, including these terms did not seem to improve the best fits for the small x data using one– or two–loop evolution [12][14]. Indeed, many ways of inclusion significantly worsened the fits, particularly if they were more global, i.e. constrained by large x data [12]. Also, there seemed to be a very strong dependence on the factorization scheme used in the calculations when including the leading $\ln(1/x)$ terms [12][14][15].

The high precision of the most recent HERA data constrains theory far more than previously. The best recent global fits seem to come from those intermediate approaches which use NLO perturbation theory with a quite steep (unexplained) input for the singlet quark with $\lambda \sim 0.2$ and a similar form of small x input for the gluon [16] (unless Q_0^2 is less than $\sim 4\text{GeV}^2$, in which case it must be flatter or even valence–like). Fixed order perturbation theory with flat or valence–like inputs and low Q_0^2 fails at the lowest x values, and for fits to the small x data alone relatively steep inputs for the singlet quark, i.e. $\lambda \gtrsim 0.2$, seem to be required [17]. Approaches including the resummed terms now seem to fail [18] in practically all factorization schemes.

In this paper we will demonstrate that the apparent failure of approaches using the leading $\ln(1/x)$ terms, and certainly the factorization scheme dependence, is due to incorrect methods of incorporating these terms. The correct leading order renormalization scheme consistent (RSC) calculation naturally includes leading $\ln(1/x)$ terms in a form which has already been derived by Catani and called "physical anomalous dimensions" [19]. It also provides some limited predictive power at small x. We will discuss this method of calculation, then make detailed comparisons to data, and with the aid of the new HERA data demonstrate that this calculation leads to a very good global fit to all $F_2(x, Q^2)$ data. Indeed, the complete RSC calculation, including leading $\ln(1/x)$ terms, is clearly preferred by the latest data.

2. The Renormalization Scheme Consistent Expansion.

For simplicity we work in moment space in this paper, i.e. define the moment space structure functions by the Mellin transformation,

$$F(N, Q^2) = \int_0^1 x^{N-1} \mathcal{F}(x, Q^2) dx. \tag{2.1}$$

and similarly for the coefficient function. As with the anomalous dimension we define the moment space parton distribution as the Mellin transformation of a rescaled parton density i.e

$$f(N, Q^2) = \int_0^1 x^N f(x, Q^2) dx. \tag{2.2}$$

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The moment space expression for a structure function is then

$$F(N,Q^2) = \sum_a C^a(N, \alpha_s(Q^2)) f_a(N, Q^2), \tag{2.3}$$

and the parton distributions evolve according to the perturbative renormalization group equation

$$\frac{d f_a(Q^2)}{d \ln Q^2} = \sum_b \gamma_{ab}(\alpha_s(Q^2)) f_b(Q^2), \tag{2.4}$$

where we choose both the factorization and renormalization scale to be equal to Q^2. The coefficient functions and anomalous dimensions are factorization scheme dependent, of course.

There are two independent structure functions $F_2(N, Q^2)$ and $F_L(N, Q^2)$ which have both singlet and nonsinglet contributions. In general we may write

$$F_i(N, Q^2) = \frac{1}{N_f} \left(\sum_{j=1}^{N_f} e_j^2 \right) F_i^S(N, Q^2) + F_i^{NS}(N, Q^2) \qquad (i = 2, L), \tag{2.5}$$

where in terms of coefficient functions and parton densities

$$F_i^S(N, Q^2) = C_i^f(N, \alpha_s) f^S(N, Q^2) + C_i^g(N, \alpha_s) g(N, Q^2),$$

$$F_i^{NS}(N, Q^2) = C_i^{NS}(N, \alpha_s) \sum_{j=1}^{N_f} e_j^2 f_{q_j}^{NS}(N, Q^2). \tag{2.6}$$

N_f is the number of active quark flavours (we only consider massless quarks), and $f^S(N, Q^2)$ and $f_{q_j}^{NS}(N, Q^2)$ are the singlet and nonsinglet quark distribution functions respectively.

In order to devise an expansion scheme for the calculation of these structure functions which is useful at both large and small x, we would *a priori* expect that we would need to use the known anomalous dimensions and coefficient functions at low orders in both α_s and in the leading $\ln(1/x)$ expansion. There have already been some methods along these lines; however, these have suffered from scheme dependence. This is clearly incorrect since we do not expect factorization scheme dependence in a well ordered calculation of a physical quantity.

As already mentioned, Catani has shown how one may obtain factorization scheme independent results, even at small x, by the use of factorization scheme independent, or physical, anomalous dimensions. We refer to his papers [19], or for a slightly different presentation [20], for details. Very briefly, using (2.3) one writes parton distributions in terms of structure functions and coefficient functions, and substituting into (2.4) leads to evolution equations for the structure functions themselves in terms of physical anomalous dimensions $\Gamma_{22}(N, \alpha_s)$, $\Gamma_{2L}(N, \alpha_s)$, etc. These can be expressed in terms of coefficient functions and anomalous dimensions in any factorization scheme, but in scheme independent combinations. In order to perform our calculation of structure functions we do not work with these physical anomalous dimensions from the outset. Rather we will simply

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demand a complete leading order, renormalization scheme consistent (RSC) calculation. This leads to expressions which are unique, up to nonperturbative inputs, and which naturally contain the physical anomalous dimensions.

To begin, let us consider what we normally mean by "consistency with renormalization scheme dependence". In the loop expansion the order of a term is determined simply by its order in α_s, and in the leading $\ln(1/x)$ expansion $(\ln(1/x))^{-1}$ is put on an equal footing to α_s. In both forms of expansion one demands that, once we choose a particular renormalization scheme and work to a particular order in this renormalization scheme, we include all terms in our expressions for the structure function which are of greater order than the uncertainty due to the freedom of choice of renormalization scheme (i.e. the uncertainty in the definition of the coupling constant), and no others.[1] If working with the n-loop coupling constant, the ambiguity in its defintion due to renormalization scheme uncertainty is of order α_s^{n+1}. Thus, the uncertainty when working to n-th order is the change in the leading order expression under the change of coupling $\alpha_s \rightarrow \alpha_s(1 + \epsilon\alpha_s^n)$. Hence, the uncertainty in the whole structure function is of the order of the change of the leading order part under such a change in the coupling, and the n-th order renormalization scheme independent expression includes all complete terms of lower order than this change.

This definition gives a well–defined way of building up a solution to the structure functions, but relies upon the definition of a given expansion scheme. It leaves an ambiguity about how we define leading order expressions and about how we define the order of terms compared to this leading order. The loop expansion and leading $\ln(1/x)$ expansion are just the two most commonly used examples. Both have potential problems: in the former one does not worry about the fact that the large $\ln(1/x)$ terms can cause enhancement of terms which are higher order in α_s at small x, and in the latter one does not worry about the fact that at large x, especially as Q^2 increases, it is the terms that are of lowest order in α_s which are most important. Hence, one would think that both have limited regions of validity.

The shortcomings of these two expansion methods come about because even though a given order contains no terms which are inconsistent with working to this order in a particular renormalization scheme, in neither does it include every one of the terms which are consistent with working to a given order in the renormalization scheme. In each expansion scheme some of the terms appearing at what we call higher orders are not actually subleading in α_s to any terms which have already appeared. Thus, although (for a given expansion method) these terms are formally of the same order as uncertainties due to the choice of renormalization scheme, they are not terms which are actually generated by changes in renormalization scheme.[2]

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[1] Following this prescription one automatically obtains factorization scheme independent expressions in both the above expansion schemes. This is well known in the loop expansion, see [20] for a discussion of this and the more complicated case of the leading $\ln(1/x)$ expansion.

[2] Similarly, they are not generated by changes in renormalization scale.

In order to demonstrate this point more clearly we consider a simple toy model. Let us imagine some hypothetical physical quantity which can be expressed in the form

$$H(N, \alpha_s(Q^2)) = \sum_{m=1}^{\infty} \alpha_s(Q^2) \sum_{n=-m}^{\infty} a_{mn} N^n \equiv \sum_{i=0}^{\infty} \alpha_s^i(Q^2) \sum_{j=1-i}^{\infty} b_{ij} \left(\frac{\alpha_s(Q^2)}{N} \right)^j. \qquad (2.7)$$

The first way of writing $H(N, \alpha_s(Q^2))$ as a power series corresponds to the loop expansion, where we work order by order in m, out to $m = k$, and use the k-loop coupling. The second corresponds to the leading $\ln(1/x)$ expansion, where we work order by order in i, out to $i = l$, and use the $(l+1)$-loop coupling. Let us, for a moment, consider the leading order expression in the loop expansion, $\alpha_s(Q^2) \sum_{n=-1}^{\infty} a_{1n} N^n$. The coupling is uncertain by $\mathcal{O}(\alpha_s^2(Q^2))$ and hence the uncertainty of the leading order expression (i.e. the change due to a change of the coupling) is $\sim \alpha_s^2(Q^2) \sum_{n=-1}^{\infty} b_{1n} N^n$. We see that there is no change of any sort with a power of N less than -1, and hence any such term is not really subleading. Similarly, the uncertainty of the leading order expression in the leading $\ln(1/x)$ expansion contains no terms at first order in α_s (or with positive powers of N), and such terms are not really subleading either. The full set of terms in the combination of both leading order expressions is genuinely leading order, and renormalization scheme independent by definition.

Perhaps the best way in which to write our expression for $H(N, \alpha_s(Q^2))$ in order to appreciate these points is

$$H(N, \alpha_s(Q^2)) = \sum_{m=-1}^{\infty} N^m \sum_{n=1}^{\infty} c_{mn} \alpha_s^n(Q^2) + \sum_{m=2}^{\infty} N^{-m} \sum_{n=m}^{\infty} c_{mn} \alpha_s^n(Q^2), \qquad (2.8)$$

i.e. as an infinite number of power series in $\alpha_s(Q^2)$, one for each power of N. Each of these series in $\alpha_s(Q^2)$ is independent of the others, and the lowest order in $\alpha_s(Q^2)$ of each is therefore renormalization scheme independent and part of the complete leading order expression for $H(N, \alpha_s(Q^2))$. The full leading order expression for $H(N, \alpha_s(Q^2))$ is therefore

$$H_0(N, \alpha_s(Q^2)) = \sum_{m=-1}^{\infty} N^m c_{m0} \alpha_s(Q^2) + \sum_{m=2}^{\infty} c_{mm} N^{-m} \alpha_s^m(Q^2)$$
$$\equiv \alpha_s(Q^2) \sum_{n=-1}^{\infty} a_{0m} N^n + \sum_{j=2}^{\infty} b_{0j} \left(\frac{\alpha_s(Q^2)}{N} \right)^j. \qquad (2.9)$$

Hence, the combined set of terms considered to be leading order in both the previous expansion schemes comprise the full set of renormalization scheme invariant, and thus truely leading order, terms. By considering $H(N, \alpha_s(Q^2))$ written as (2.8), and considering a redefinition of the coupling constant, $\alpha_s(Q^2) \to \alpha_s(Q^2) + \mathcal{O}(\alpha_s^m(Q^2))$, we see that the n-th order expression for $H(N, \alpha_s(Q^2))$, which should be used with the n-loop coupling constant, is the sum of the first n terms in each of the power series in $\alpha_s(Q^2)$.

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Similar arguments have already been applied to the anomalous dimensions and coefficient functions ([10] and particularly [12]). Here we take a strong viewpoint and insist that the complete renormalization scheme consistent expressions, with no artificial supression of leading $\ln(1/x)$ terms, must be used. Futhermore, the expressions must be those for the physical structure functions, not for the factorization and renormalization scheme dependent coefficient functions and anomalous dimensions.

When considering the real structure functions the situation is a great deal more complicated technically than the toy model, but the principle is the same. One complication is that the structure functions are combinations of perturbative evolution parts and input parts (which are perturbative with nonperturbative factors), rather than one simple power series in $\alpha_s(Q^2)$. However, the physical consequence of the factorization theorem is that there is some fixed nonperturbative factor for the structure functions which we cannot calculate in perturbation theory, but we can predict the perturbative evolution of the structure functions in terms of this factor. Thus, we choose two independent variables for each structure function, the input at some starting scale Q_0^2, and the evolution away from this starting scale. We calculate the lowest order RSC expression for each of these, and combine to give the full LO expression. In this paper we only do this for the singlet structure function, since this is hugely dominant at small x. The procedure is also complicated here because the evolution of F_2^S and F_L^S is coupled, but the expression for each is a sum of terms consisting of input and evolution parts, and for each term we take the LO expression for the input and for the evolution.

Using the above prescription it is relatively straightforward, but rather involved to calculate the full leading order RSC expressions. The full details appear in [20], here we just present the results. In order to do this most succinctly we express the result in terms of physical quantities. Hence, in order to explain notation (which is similar to that in [19]), and also slightly elucidate the form of the final expressions we consider the solution to the 1-loop renormalization group equation first (this being part of the full solutions). Using boundary conditions $\hat{F}_L^{0,l}(N, Q_0^2) = \hat{F}_L(N)$ and $F_2^{0,l}(N, Q_0^2) = F_2(N)$, where $F_i(N)$ are nonperturbative inputs, and the superscript $0, l$ denotes one-loop quantities, we may write the solution for the longitudinal structure function as

$$\hat{F}_L^{0,l}(N, Q^2) = \hat{F}_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)} + \hat{F}_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N)}, \qquad (2.10)$$

where $\hat{F}_L(N, Q^2) = F_L(N, Q^2)/(\alpha_s(Q^2)/2\pi)$, $\tilde{\Gamma}^{0,l,+(-)}(N)$ are the two eigenvalues of the $\mathcal{O}(\alpha_s)$ physical anomalous dimension matrix, which are the same as those of the $\mathcal{O}(\alpha_s)$ parton anomalous dimension matrix, divided by $b_0\alpha_s(Q^2)$. (The superscript S is dropped for the rest of this section.) Having chosen to write the solution for $F_L^{0,l}(N, Q^2)$ in this way we may then write $F_2^{0,l}(N, Q^2)$ as

$$F_2^{0,l}(N, Q^2) = e^+(N)\hat{F}_L^{0,l,+}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,+}(N)} + e^-(N)\hat{F}_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\tilde{\Gamma}^{0,l,-}(N),} \qquad (2.11)$$

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where $e^+(N)\hat{F}_L^{0,l,+}(N) + e^-(N)\hat{F}_L^{0,l,-}(N) = F_2(N)$. The $e^{+(-)}(N)$ come from the eigenvectors of the $\mathcal{O}(\alpha_s)$ parton anomalous dimension and the $\mathcal{O}(\alpha_s)$ longitudinal coefficient functions. In practice

$$\hat{F}_L^{0,l,+}(N) = \hat{F}_L(N) - \frac{36 - 8N_f}{27}F_2(N) + \mathcal{O}(N), \quad \hat{F}_L^{0,l,-}(N) = \frac{36 - 8N_f}{27}F_2(N) + \mathcal{O}(N),$$

$$F_2^{0,l,+}(N) = \frac{N}{6}\left(\hat{F}_L(N) - \left(\frac{36 - 8N_f}{27}\right)\right)F_2(N) + \mathcal{O}(N^2), \quad F_2^{0,l,-}(N) = F_2(N) + \mathcal{O}(N),$$

$$(2.12)$$

where $F_2^{0,l,+(-)}(N) = e^{+(-)}(N)\hat{F}_L^{0,l,+(-)}(N)$.

Taking into account the leading $\ln(1/x)$ terms as well, the expressions acquire additional terms. Explicitly we obtain

$$F_{L,RSC,0}(N,Q^2) = \frac{\alpha_s(Q_0^2)}{2\pi}\left[\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{f^{0,l,+}(N)-1}\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\Gamma_{LL}^0(N,\alpha_s(q^2))}{b_0\alpha_s^2(q^2)}d\alpha_s(q^2)\right]\times\right.$$

$$\left(\hat{F}_L^{0,l,+}(N) + \left(\hat{F}_L(N) - \left(\frac{36 - 8N_f}{27}\right)F_2(N)\right)(\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(N,\alpha_s(Q_0^2))] - 1)\right)$$

$$\left. + \hat{F}_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{f^{0,l,-}(N)-1}\right],$$

$$(2.13)$$

for the longitudinal structure function, and

$$\left(\frac{dF_2(N,Q^2)}{d\ln Q^2}\right)_{RSC,0} = \alpha_s(Q_0^2)\left[e^-(N)\Gamma^{0,l,-}(N)\hat{F}_L^{0,l,-}(N)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{f^{0,l,-}(N)-1}\right.$$

$$+ \left(e^+(N)\Gamma^{0,l,+}(N)\hat{F}_L^{0,l,+}(N) - \Gamma_{2,L}^{1,0}(N)\left(\hat{F}_L(N) - \left(\frac{36 - 8N_f}{27}\right)F_2(N)\right)\right)$$

$$+ \Gamma_{2L}^1(N,\alpha_s(Q_0^2))\left(\hat{F}_L(N) - \left(\frac{36 - 8N_f}{27}\right)F_2(N)\right)\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(N,\alpha_s(Q_0^2))]\right)\times$$

$$\exp\left[\int_{\alpha_s(Q^2)}^{\alpha_s(Q_0^2)}\frac{\Gamma_{LL}^0(N,\alpha_s(q^2))}{b_0\alpha_s^2(q^2)}d\alpha_s(q^2)\right]\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{f^{0,l,+}(N)-1}\right],$$

$$(2.14)$$

and

$$F_{2,RSC,0}(N,Q_0^2) = F_2(N)$$

$$+ \alpha_s(Q_0^2)\frac{\Gamma_{2L}^1(N,\alpha_s(Q_0^2))}{\Gamma_{LL}^0(N,\alpha_s(Q_0^2))}\left(\hat{F}_L(N) - \frac{(36 - 8N_f)}{27}F_2(N)\right)(\exp[\ln(Q_0^2/A_{LL})\Gamma_{LL}^0(N,\alpha_s(Q_0^2))] - 1)$$

$$+ \ln(Q_0^2/A_{LL})\alpha_s(Q_0^2)\left(e^+(N)\Gamma^{0,l,+}(N)\hat{F}_L^{0,l,+}(N,Q_0^2) + e^-(N)\Gamma^{0,l,-}(N)\hat{F}_L^{0,l,-}(N,Q_0^2)\right.$$

$$\left. - \Gamma_{2,L}^{1,0}(N)\left(\hat{F}_L(N) - \left(\frac{36 - 8N_f}{27}\right)F_2(N)\right)\right).$$

$$(2.15)$$

for $F_2(N,Q^2)$.

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We should explain the terms in these expressions. $\Gamma^0_{LL}(N, \alpha_s)$ is the gluon anomalous dimension at leading order in $\ln(1/x)$, which it turns out governs the small x evolution of $F_L(N, Q^2)$, as seen in (2.13). It is also identical to the physical anomalous dimension, hence the notation. $\Gamma^{\mathbf{\emptyset}}_{LL}(N, \alpha_s)$ is $\Gamma^0_{LL}(N, \alpha_s)$ minus its one-loop component, which appears in $\Gamma^{0,l,+}(N)$. $\alpha_s \Gamma^1_{2L}(N, \alpha_s)$ is the leading order in $\ln(1/x)$ term governing the evolution of $F_2(N, Q^2)$ in terms of $F_L(N, Q^2)$. It is a power of α_s up on $\Gamma^0_{LL}(N, \alpha_s)$, but is sufficiently leading (i.e. is not subleading to any other contributions) to make an appearance in the input terms in (2.14) and (2.15). It is given by

$$\Gamma^1_{2L}(N, \alpha_s) = \frac{\gamma^1_{fg}(N, \alpha_s) + \gamma^0_{gg} C^1_{2,g}(N, \alpha_s)}{2\pi C^1_{L,g}(N, \alpha_s)}, \qquad (2.16)$$

and thus is equal again to one of the physical anomalous dimensions in [19]. ($\Gamma^{1,0}_{2L}(N)$ is the one-loop contribution to $\Gamma^1_{2L}(N, \alpha_s)$, and must be subtracted in some places in (2.14) and (2.15) in order to avoid double counting.) However, once again we stress that these anomalous dimensions are not needed to derive these expressions, but that they naturally appear in the end results.

The term $\exp[\ln(Q^2_0/A_{LL})\Gamma^0_{LL}(N, \alpha_s(Q^2_0))]$ appears in some of the input parts in our expressions. This leading $\ln(1/x)$ contribution to the inputs is derived by demanding that the form of our expressions is invariant under changes in the arbitrary starting scale Q^2_0 at the order at which they are calculated, i.e. the expression for the structure functions as a whole is genuinely of leading order. It is easy to see that the variation of this input term under a change in Q^2_0 cancels the leading order change in the leading $\ln(1/x)$ evolution. This procedure leaves us with an unknown scale A_{LL}, but at this scale the inputs become the nonperturbative inputs, and we would therefore expect A_{LL} to be the sort of scale where perturbation theory starts to break down. The other terms in (2.15) proportional to $\ln(Q^2_0/A_{LL})$ are likewise required to make the full expression invariant under changes in Q^2_0 at leading order.

Having obtained the full leading order RSC expressions for $(d F_2(N, Q^2)/d \ln Q^2)$ and $F_2(N, Q^2_0)$ we integrate $(d F_2(N, Q^2)/d \ln Q^2)_{RSC,0}$ from Q^2_0 to Q^2 and add to the input $F_{2,RSC,0}(N, Q^2_0)$ in order to get our expression for $F_2(N, Q^2)$. This is essentially because it is the derivative of $F_2(N, Q^2)$ that begins at first order in α_s and thus is a truly perturbative quantity. The difference between using this prescription and using the leading order $F_2(N, Q^2)$ directly is small.

It is easy to check that under a change in the coupling, $\alpha_s \rightarrow \alpha_s + \delta\alpha^2_s$ the change of each of our expressions is of higher order in α_s than any terms appearing. Thus, we have full leading order, including leading $\ln(1/x)$ terms, renormalization scheme consistent expressions for the structure functions. These are significantly different from both the one-loop expressions and the leading $\ln(1/x)$ expressions, although they clearly reduce to them in the appropriate limits. All terms in the expressions (2.13)-(2.15) are renormalization scheme as well as factorization scheme independent,

as we would hope, and they are the appropriate full expressions to use with the one–loop coupling constant.

Finally, we notice that this method of solution leads to a certain amount of predictive power. We know the precise form of the structure function inputs in terms of the nonperturbative inputs (which we imagine should be quite flat at small x). Thus, up to the absolute normalization and the scale A_{LL}, we have predictions for the small x form of the inputs for $F_L(N, Q^2)$ and $F_2(N, Q^2)$ (as well as for $(d F_2(N, Q^2)/d \ln Q^2)$). The normalization is fairly well set by the large x data, and we would expect $A_{LL} \lesssim 1\mathrm{GeV}^2$, so there is an estimate for the small x form of each input. Moreover, the unknown elements are the same for each input, so there is a strong prediction for the relationships between the small x inputs. However, the scale Q_0^2 which should be chosen is not determined. Nevertheless, it is a considerable consistency requirement that the relationships should be true for any choice of Q_0^2, and hopefully they can be well satisfied over a wide range of Q_0^2. In order for this to be true, $\alpha_s(Q_0^2)$ cannot be too sensitive to Q_0^2, so we would not expect Q_0^2 to be particularly low. Also, since we can largely choose our structure functions at Q_0^2, but then have no freedom in how we evolve up and down in Q^2, we would imagine that when performing a fit it is advantageous if Q_0^2 is near the centre of the range of our data.

3. Fits to The Data.

We use the expressions (2.13)-(2.15) to calculate the x-space singlet structure functions. The nonsinglet structure functions are calculated using the normal one–loop prescription. By combining the singlet and nonsinglet components and varying the free parameters (Q_0^2, A_{LL}, and the soft inputs for $F_L(x, Q^2)$ and $F_2(x, Q^2)$), we obtain the best fit for the available F_2 structure function data.[3] We note that the input $F_2^S(x, Q_0^2)$ and the evolution $d F_2(x, Q^2)/d \ln Q^2$ are forced by (2.14) and (2.15) to be trivially related at small x, which is not the case when working at fixed order in α_s. The one–loop value for $\Lambda^{N_f=4}$ is chosen to be 100MeV. This precise value is not determined by a best fit, but a value near this is certainly favoured. The published values of $F_2(x, Q^2)$ are altered to take account of the fact that our predictions for $F_L(x, Q^2)$ are not the same as (i.e. are somewhat lower than) those used by H1 and ZEUS in their determination of $F_2(x, Q^2)$. Thus, the $F_2(x, Q^2)$ values are a little lower for the largest values of Q^2/x than in [1] and [2].

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[3] In practice a variant of the program used by MRS is used for the fit, and inputs for the gluon and quarks of the standard form are specified. To calculate structure functions we use an effective factorization scheme which is of DIS type for $F_2(x, Q^2)$, where the longitudinal coefficient functions are the one–loop expressions, and the resummed anomalous dimensions and small x inputs are chosen to produce expressions for the structure functions matching (2.13)-(2.15) as closely as possible. For simple inputs where exact analytic expressions for the x-space forms of (2.13)-(2.15) can be found checks are made with the results of the evolution program, and any discrepancies are always much smaller than the errors in the data.

We treat the heavy quark thresholds rather naively. Taking $m_c^2 = 3\,\text{GeV}^2$ and $m_b^2 = 20\,\text{GeV}^2$, we simply change the number of active quark flavours discontinuously at these values of Q^2: since $F_L^S(N)$ has a large component proportional to N_f, $F_L(x, Q^2)$ is discontinuous at these values of Q^2, as is $d\,F_2(x, Q^2)/d\ln(Q^2)$. $\alpha_s(Q^2)$ is continuous at the thresholds, being defined by

$$\alpha_{s,n}(Q^2) = \alpha_{s,n+1}(Q^2)\left(1 + \frac{\alpha_{s,n+1}(Q^2)}{6\pi}\ln(m_{n+1}^2/Q^2)\right). \qquad (3.1)$$

This treatment of quark thresholds is consistent with the decoupling theorem, in so much that it guarantees the correct expressions far above or below thresholds [21]. It is clearly unsatisfactory near the thresholds and must be improved. However, the prescription has little effect on $F_2(x, Q^2)$ in the region of the fit: in the curves for $F_2(x, Q^2)$ the b-quark threshold is barely noticeable, while the kink at the c-quark threshold only really affects a handful of data points at very small x, tending to hinder the fit (see fig. 1).

The result of the best fit using the leading order, including leading $\ln(1/x)$ terms, RSC expressions (henceforth refered to as LO(x)) with $Q_0^2 = 40\,\text{GeV}^2$ is compared with fits obtained using the standard two–loop method, where R_1 allows $\Lambda_{\overline{\text{MS}}}^{N_f=4}$ to be free (giving $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 241\,\text{MeV}$) and R_2 fixes $\Lambda_{\overline{\text{MS}}}^{N_f=4} = 344\,\text{MeV}$ to force a better fit to the HERA data. The new NMC data [22] for $F_2^{\mu p}$ and $F_2^{\mu d}$ is used with a lower Q^2 cut of $2\,\text{GeV}^2$.[4] The results are shown in table 1 (full references for the experimental data can be found in [16]). As one can see, the LO(x) scheme independent fit is much better for the HERA data (even when compared to R_2), much better for the BCDMS data (even when compared to R_1) and similar in standard for the rest of the data. The overall fit is ~ 200 better for the whole data set. The results of the fit to the small x data is shown in fig. 1.

The leading order renormalization scheme consistent expressions clearly provide a very good fit to the $F_2(x, Q^2)$ data. The fit shown is for the particular starting scale $Q_0^2 = 40\,\text{GeV}^2$, but the quality of the fit is extremely insensitive to changes in this scale (where we allow A_{LL} to be a free parameter for each Q_0^2), as we expect from the method of construction of the solutions. The fit is essentially unchanged over the range $20 - 80\,\text{GeV}^2$, and we choose $40\,\text{GeV}^2$ as the (logarithmically) central value. When Q_0^2 drops below $20\,\text{GeV}^2$ the fit immediately gets markedly worse because of the discontinuity in $d\,F_2^S(x, Q^2)/d\ln Q^2$ at the threshold: $d\,F_2^S(x, Q^2)/d\ln Q^2$ suddenly becomes too large at small x if $F_2^S(x, Q_0^2)$ is the correct size to fit the data. The quality of the fit gets continuously worse as Q_0^2 lowers further, becoming completely uncompetitive long before reaching m_c^2. We expect that a correct treatment of quark thresholds would lead to a smooth falling off of the quality of the fit, but that it would begin to deteriorate somewhere in the region of $20\,\text{GeV}^2$, due to $\alpha_s(Q_0^2)$ becoming too large below this value.

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[4] The MRS fits are not performed again: the χ^2 for the new data is calculated using the same input parameters in [16] (there is little indication that these would be changed much by the new data).

The parameter A_{LL}, which should be a scale typical of soft physics, turns out to be 0.4GeV^2 for the fit starting at $Q_0^2 = 40\text{GeV}^2$. This decreases a little as Q_0^2 increases and *vice versa*. For $Q_0^2 = 40\text{GeV}^2$ the soft inputs for the fit are roughly

$$\hat{F}_L^S(x) \approx 2.65(1-x)^5, \qquad\qquad F_2^S(x) \approx (1-x)^4(1-0.6x^{0.5}+7x), \qquad (3.2)$$

where they have been forced to be flat as $x \to 0$. Allowing instead an asymptotic behaviour x^λ, where $\lambda \lesssim 0.08$, leads to an equally good fit. The importance of the leading $\ln(1/x)$ terms can be judged by how they affect the fit. If, after obtaining the best fit, all terms other than those in the one–loop expressions are set to zero, the quality of the fit is unchanged above $x = 0.3$, begins to alter slightly below this, and is clearly much worse by the time we reach $x = 0.1$. Thus, the leading $\ln(1/x)$ terms are important by this value of x.

It is not yet possible to extend the RSC calculation beyond the leading order due to lack of knowledge of NLO in $\ln(1/x)$ terms. There is hope that these will shortly become available [23], and when they do the NLO versions of (2.13)-(2.15) can be derived and put to use. Only then should the NLO coupling constant be used in any fit. As shown at leading order the $\ln(1/x)$ terms not present in the one–loop expressions become important above $x = 0.1$. However, much of this effect is due to the terms at $\mathcal{O}(\alpha_s^2)$, so the NLO expression at fixed order in α_s should be a good approximation to the full NLO RSC expression for x somewhat lower than 0.1, perhaps as low as $\sim 0.05 - 0.01$. However, until the full renormalization scheme consistent NLO expressions become available, we believe that it is premature to use fits to small x structure function data to determine the NLO coupling constant (unless, of course, direct measurements of $F_L(x, Q^2)$ and other less inclusive quantities at very small x turn out to verify standard two-loop predictions). The fixed order in α_s expressions should be accurate for CCFR, BCDMS and NMC data, which after all are still much more precise than HERA data, and fits to these data alone will provide the best determination of the NLO $\alpha_s(Q^2)$.

4. Conclusion

In this paper we have demonstrated that it is possible to derive expressions for the structure functions which incorporate the leading $\ln(1/x)$ terms in a way which is renormalization scheme consistent, and as a consequence avoids any factorization scheme dependence. We have also shown that these full leading order RSC expressions lead to very good fits to the data and that, futhermore, they are able to do so using as inputs only soft distributions for the singlet quark and gluon, i.e. all powerlike behaviour is generated perturbatively, and determined in terms of the nonperturbative flat inputs and a soft scale A_{LL}. Hence, this approach provides an explanation for the form of the small x structure function rather than just a way of fitting it. Futher details of both the theory and fits, as well as other related issues, are presented in [20]. It is certainly true that the

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calculations must be improved to take account of massive quark thresholds in a better manner, and work towards this end is in progress. Nevertheless, with the present treatment we feel that the quality of the fit and the degree of explanatory (if not predictive) power, not to mention the scheme independence, give strong justification for using this approach.

However, the quality of the fit alone is certainly not such a substantial improvement on more standard approaches that it necessarily convinces one that this approach must be correct. In order to obtain some degree of verification we must obtain more experimental data. So far we have only probed $F_L(x, Q^2)$ indirectly, i.e. it is simply related to the derivative of $F_2(x, Q^2)$ (as well as to the input $F_2(x, Q_0^2)$ using this method). Having tied down the nonperturbative inputs and A_{LL} and Q_0^2 from our fit to $F_2(x, Q^2)$, we have a prediction for $F_L(x, Q^2)$. The result of this prediction for the fit with $Q_0^2 = 40 \text{GeV}^2$ is shown in fig. 2, where it is compared to the prediction using the NLO in α_s approach and the MRSR$_1$ fit. As one can see, it is smaller than the MRSR$_1$ $F_L(x, Q^2)$, but becomes steeper at very small x. The prediction for $F_L(x, Q^2)$ is weakly dependent on the value of Q_0^2 chosen: the value at $Q^2 = 5 \text{GeV}^2$ and $x = 10^{-4}$ varies by $\pm 10\%$ within our range of Q_0^2 (increasing with Q_0^2), and by less than this for higher x and Q^2.[5] The very recent results on $F_L(x, Q^2)$ for $0.01 \gtrsim x \gtrsim 0.1$ from NMC [22] are matched far better by the LO(x) F_L than the MRSR$_1$ F_L (the latter being rather large). However, it is fair to say that any problems with the MRSR$_1$ F_L can very probably be assigned to the treatment of the charm quark threshold, i.e. the predicted $F_L(x, Q^2)$ in the last of [7] matches the data well. Measurements of $F_L(x, Q^2)$ at $x < 10^{-2}$ would be a better discriminant between fixed order in α_s calculations and those involving leading $\ln(1/x)$ terms. However, the sort of "determination" of $F_L(x, Q^2)$ already performed by H1 [25] is really only a consistency check for a particular fit, and is by no means a true measurement of $F_L(x, Q^2)$. Real, direct measurements of $F_L(x, Q^2)$ at HERA would be an important (and probably essential) way of determining the validity of the approach in this paper, and the genuine importance of leading $\ln(1/x)$ terms in structure functions.

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[5] The author has submitted two conference proceedings on the topic of the current paper [24] and should point out that these are both incomplete. In the former the expression (2.15) was not used for the input for $F_2(x, Q^2)$. Thus, the fit imposed far less constraint on Q_0^2 than the full procedure, and a value of 5GeV^2 was used, which resulted in a prediction of $F_L(x, Q^2)$ that is much too small. The latter claimed that an input for $F_L(x, Q^2)$ a little smaller than that consistent with the full set of expressions was needed for the best fit, even at the optimum Q_0^2. This was due to there being no account whatever taken of the b-quark threshold. The correction makes very little difference to phenomenological results.

Table 1

Comparison of quality of fits using full leading order (including $\ln(1/x)$ terms) renormalization scheme consistent expression, LO(x), and two-loop fits $MRSR_1$ and $MRSR_2$. For the LO(x) fit the H1 data is normalized by a factor of 0.995, the ZEUS data by 1.01, the BCDMS data by 0.98, the CCFR data by 0.95, and the rest by 1.00.

Experiment	data points	χ^2		
		LO(x)	R_1	R_2
H1 F_2^{ep}	193	128	158	149
ZEUS F_2^{ep}	204	256	326	308
BCDMS $F_2^{\mu p}$	174	190	265	320
NMC $F_2^{\mu p}$	129	124	163	135
NMC $F_2^{\mu d}$	129	109	134	99
NMC $F_2^{\mu n}/F_2^{\mu p}$	85	142	136	132
E665 $F_2^{\mu p}$	53	8	8	8
CCFR $F_2^{\nu N}$	66	51	41	56
CCFR $F_3^{\nu N}$	66	49	51	47

References

[1] H1 collaboration, *Nucl. Phys.* B **470** (1996) 3.

[2] ZEUS collaboration: M. Derrick *et al*, *Zeit. Phys.* C **69** (1996) 607;
Preprint DESY 96-076 (1996), to be published in *Zeit. Phys.* C.

[3] L.N. Lipatov, *Sov. Jour. Nucl. Phys.* **23** (1976) 338;
E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Sov. Jour. JETP* **45** (1977) 199;
Ya. Balitskii and L.N. Lipatov, *Sov. Jour. Nucl. Phys.* **28** (1978) 6.

[4] T. Jaroszewicz, *Phys. Lett.* B **116** (1982) 291.

[5] A.J. Askew, J. Kwiecinski, A.D. Martin and P.J. Sutton, *Phys. Rev.* D **49** (1994) 4402;
A.J. Askew *et al*, *Phys. Lett.* B **325** (1994) 212

[6] R.D. Ball and S. Forte, *Phys. Lett.* B **335** (1994) 77; *Phys. Lett.* B **336** (1994) 77.

[7] M. Glück, E. Reya and A. Vogt, *Zeit. Phys.* C **48** (1990) 471; *Zeit. Phys.* C **53** (1992) 127;
Phys. Lett. B **306** (1993) 391; *Zeit. Phys.* C **67** (1995) 433.

[8] A.D. Martin, W.J. Stirling and R.G. Roberts, *Phys. Lett.* B **354** (1995) 155;
H.L. Lai *et al*, *Phys. Rev.* D **51** (1995) 4763.

[9] S. Catani and F. Hautmann, *Phys. Lett.* B **315** (1993) 157; *Nucl. Phys.* B **427** (1994) 475.

[10] S. Catani, M. Ciafaloni and F. Hautmann, *Phys. Lett.* B **242** (1990) 97; *Nucl. Phys.* B **366** (1991) 135; *Phys. Lett.* B **307** (1993) 147.

[11] J.C. Collins and R.K. Ellis, *Nucl. Phys.* B **360** (1991) 3.

[12] R.D. Ball and S. Forte, *Phys. Lett.* B **351** (1995) 313; *Phys. Lett.* B **358** (1995) 365.

[13] R.K. Ellis, F. Hautmann and B.R. Webber, *Phys. Lett.* B **348** (1995) 582.

[14] J.R. Forshaw, R.G. Roberts and R.S. Thorne, *Phys. Lett.* B **356** (1995) 79.

[15] S. Catani, *Zeit. Phys.* C **70** (1996) 263.

[16] A.D. Martin, W.J. Stirling and R.G. Roberts, preprint DTP/96/44 or RAL-TR-96-037, May 1996, to be published in *Phys. Lett.* B.

[17] R.D. Ball and S.Forte, preprint Edinburgh 96/10 or DFTT 36/96, June 1996, to appear in proceedings of DIS 96, (Rome, April 1996).

[18] S. Forte and R.D. Ball, preprint Edinburgh 96/14 or DFTT 35/96, July 1996, to appear in proceedings of DIS 96, (Rome, April 1996);
I. Bojak and M. Ernst, DO-TH 96/18, September 1996.

[19] S. Catani, talk at UK workshop on HERA physics, September 1995, unpublished; preprint DDF 248/4/96, April 1996; preprint DDF 254-7-96, July 1996, to appear in proceedings of DIS 96, (Rome, April 1996).

[20] R.S. Thorne, preprint RAL-TR-96-065, to appear.

[21] J.C. Collins and W.K. Tung, *Nucl. Phys.* B **278** (1986) 934.

[22] NMC Collaboration, hep-ph/9610231, to be published in *Nucl. Phys.* B.

[23] L.N. Lipatov, and V.S. Fadin, *Sov. Jour. Nucl. Phys.* **50** (1989) 712;
V.S. Fadin, R. Fiore and A. Quartarolo, *Phys. Rev.* D **53** (1996) 2729;
V. Del Duca, *Phys. Rev.* D **54** (1996) 989;
G. Camici and M. Ciafaloni, Preprint DFF-250-6-96.

[24] R.S. Thorne, proceedings of DIS 96, (Rome, April 1996); proceedings of workshop "Future physics at HERA", (DESY, Hamburg, 1996).

[25] H1 collaboration, paper pa02-069, submitted to ICHEP 1996, (Warsaw, July, 1996).

Figure Captions

Fig. 1. The curves correspond to the value of the proton structure function $F_2(x, Q^2)$ obtained from the leading order renormalization scheme consistent (LO(x)) calculation at 12 values of x appropriate for the most recent HERA data. For clarity of display we add $0.5(12 - i)$ to the value of $F_2(x, Q^2)$ each time the value of x is decreased, where $i = 1 \rightarrow 12$. The data are assigned to the x value which is closest to the experimental x bin (for more details see the similar figure displaying the two–loop fits in [16]). E665 data is also shown on the curves with the five largest x values. The H1 and ZEUS data are normalized by 0.995 and 1.01 respectively in order to produce the best fit.

Fig. 2. Comparison of predictions for $F_L(x, Q^2)$ using the full renormalization scheme consistent (LO(x)) fit and the two–loop MRSR$_1$ fit. For both sets of curves $F_L(x, Q^2)$ increases with increasing Q^2 at the lowest x values.
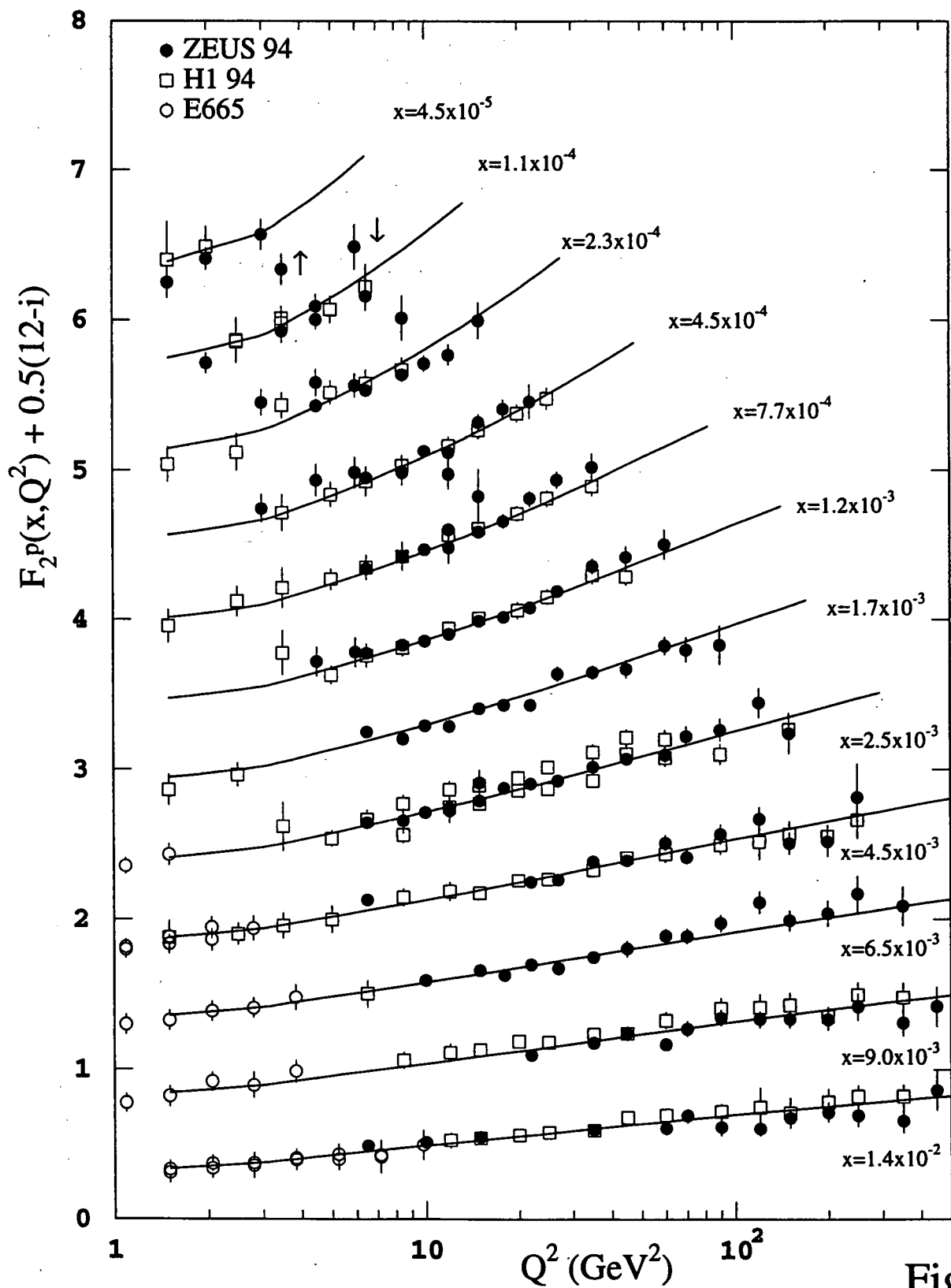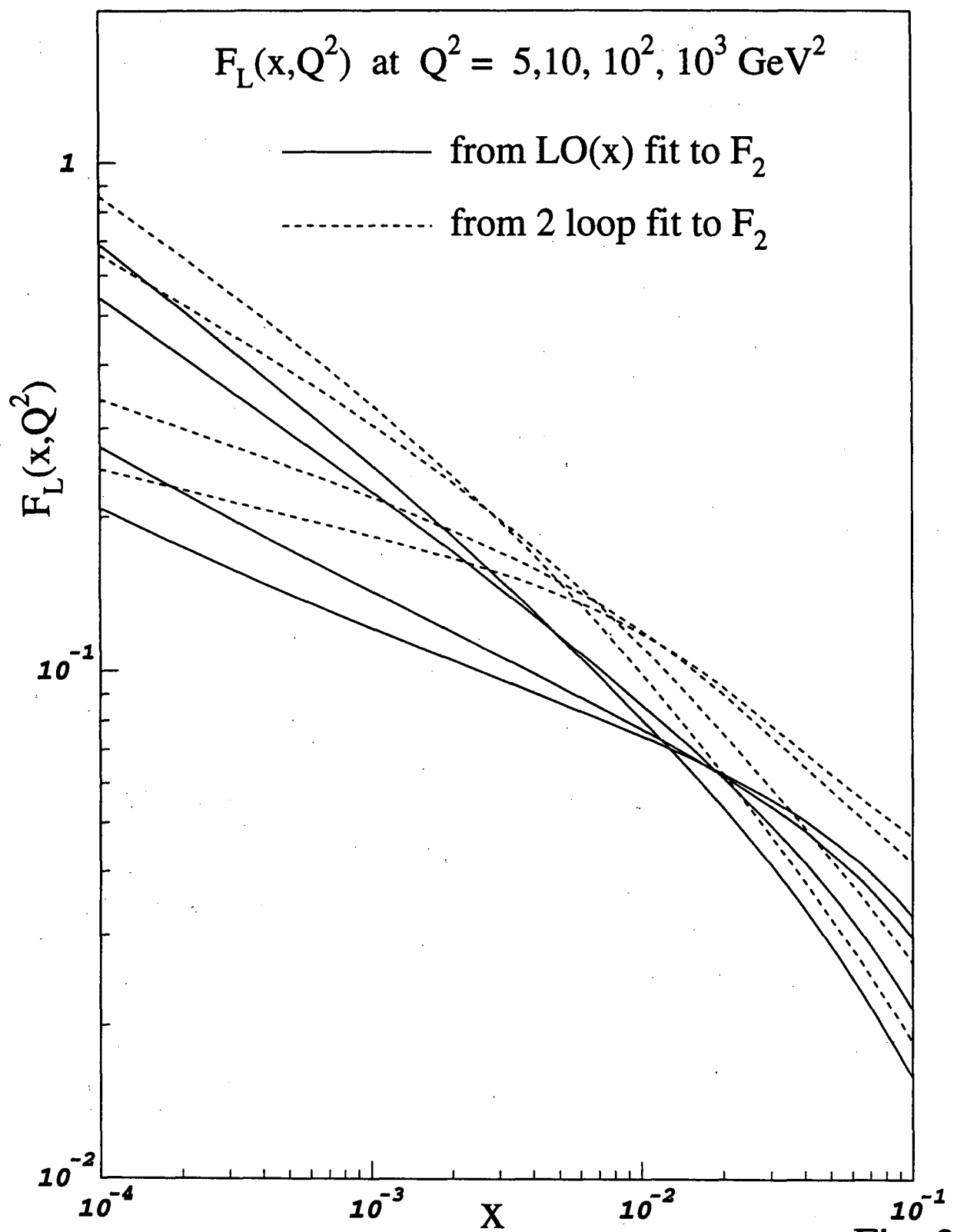
Fig. 1

$F_L(x,Q^2)$ at $Q^2 = 5, 10, 10^2, 10^3$ GeV^2

——————— from LO(x) fit to F_2

------------ from 2 loop fit to F_2

Fig. 2