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't Hooft's Order-Disorder Parameters and the Dual Potential

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Abstract

It is shown that the operator $B(C) = \mathrm{Tr}[P \exp i\tilde{g} \oint \tilde{A}_i(x) dx^i]$ constructed with the recently derived dual potential $\tilde{A}(x)$ and a coupling $\tilde{g}$ related to g by the Dirac quantization condition satisfies the correct commutation relation with the Wilson operator $\mathrm{Tr}[P \exp ig \oint A_i(x) dx^i]$ as required by 't Hooft for his order-disorder parameters.

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In his study of the confinement problem in nonabelian gauge theories, 't Hooft introduced 2 loop-dependent operators $A(C)$ and $B(C)$ with the following commutation relations: [1]

$$A(C)B(C') = B(C')A(C)\exp(2\pi in/N) \tag{1}$$

for $su(N)$ gauge symmetry, and any two spatial loops C and C' with linking number n between them. $A(C)$ was explicitly given as:

$$A(C) = \mathrm{Tr}\left[P\exp ig\oint_C A_i(x)dx^i\right], \tag{2}$$

and in the words of 't Hooft measures the magnetic flux through C and creates electric flux along C. On the other hand, $B(C)$ measures the electric flux through C and creates magnetic flux along C, and plays thus an exactly dual role to $A(C)$. For lack of a dual potential, however, $B(C)$ was not given a similar explicit expression.

In a recent paper [2], it was shown that a dual potential $\tilde{A}_\mu(x)$ does exist in nonabelian gauge theories, and the explicit though complicated transform between dual variables is given. That being so, one ought to have:

$$B(C) = \mathrm{Tr}\left[P\exp i\tilde{g}\oint_C \tilde{A}_i(x)dx^i\right] \tag{3}$$

as the explicit expression for $B(C)$, with the dual coupling $\tilde{g}$ related to g by a Dirac quantization condition. The purpose of the present note is to show that this is indeed the case.

Recall first the proof of (1) for abelian fields. In that case, one can ignore the trace and the ordering in (2) and (3) so that $A(C)$ and $B(C)$ are genuine exponentials of line integrals. Using Stokes' theorem, one of the exponents, say of $B(C')$, can be written as a surface integral, thus:

$$B(C') = \exp -i\tilde{e}\int\int_{\Sigma_{C'}} {}^*F_{ij}d\sigma^{ij}, \tag{4}$$

or, by the definition of the Hodge star (dual transform)

$${}^*F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{5}$$

in terms of the electric field strength $\mathcal{E}_i = F_{0i}$ as:

$$B(C') = \exp i\tilde{e} \int\int_{\Sigma_{C'}} \mathcal{E}_i d\sigma^i, \tag{6}$$

where $\Sigma_{C'}$ is some surface both spanning over and bounded by C'.

Consider first the simple case for linking number 1 between C and C'. The loop C in that case will intersect the surface $\Sigma_{C'}$ at some point x_0. (C may of course intersect $\Sigma_{C'}$ more than once, but the extra intersections occurring pairwise with opposite orientations, their contributions to the commutator will all cancel, leaving in effect just one intersection.) Except at this point x_0, all points on C are spatially separated from points on $\Sigma_{C'}$ so that, using the canonical commutation relation between $A_i(x)$ and $\mathcal{E}_i(x)$:

$$[\mathcal{E}_i(x), A_j(x')] = i\delta_{ij}\delta(x - x') \tag{7}$$

we have:

$$\left[ie \oint_C A_i(x)dx^i, i\tilde{e} \int\int_{\Sigma_{C'}} \mathcal{E}_j d\sigma^j\right] = ie\tilde{e}, \tag{8}$$

a c-number. Hence we conclude that

$$A(C)B(C') = B(C')A(C) \exp ie\tilde{e} \tag{9}$$

which by the Dirac quantization condition:

$$e\tilde{e} = 2\pi \tag{10}$$

gives the answer (1) for $n = 1$ as required. In case C' winds around C more than once, say n times, then C will intersect $\Sigma_{C'}$ at effectively n points for each of which the above applies, so that (1) still remains valid.

What happens when we generalize to the nonabelian case? Then $A(C)$ and $B(C)$ are each a trace of an ordered product of noncommuting factors for which no Stokes' Theorem applies. Nevertheless, one finds that one may still associate with each a surface in an analogous fashion. Take $B(C')$, for example. The phase factor:

$$\tilde{\Phi}(C') = P \exp i\tilde{g} \oint_{C'} \tilde{A}_i dx^i \tag{11}$$

of which $B(C')$ is the trace, can be written, according to ref. [2], as:

$$\tilde{\Phi}(C') = \prod_{t=0\to 2\pi} (1 - i\tilde{g}\tilde{W}[\eta|t]) \sim \prod_{t=0\to 2\pi} \exp -i\tilde{g}\tilde{W}[\eta|t], \tag{12}$$

where η, for $t = 0 \to 2\pi$, is a parametrization of C', and:

$$\tilde{W}[\eta|t] = \int_{\eta_0}^{\eta(t)} \delta\eta'^{\nu}(t)\tilde{E}_{\nu}[\eta'|t], \tag{13}$$

$\tilde{E}_{\nu}[\eta|t]$ and $\tilde{W}[\eta|t]$ being both 'segmental' quantities depending on a segment of C' around the point $\eta(t)$ on it. These 'segmental' quantities were used to establish dual symmetry for nonabelian theories in ref. [2], to which the reader is referred for detailed explanation of their significance. We note here only that the integral in (13) denotes a 'segmental' integral along some path from a reference point η_0 to the point $\eta(t)$, so that in ordinary space, this path appears as a ribbon. Piecing such ribbons together as $\eta(t)$ moves along C' in (12), one obtains a surface $\Sigma_{C'}$ spanning over and bounded by C' as suggested. In as much as the reference point η_0 and the path joining it to $\eta(t)$ are both arbitrary for (13) to hold, one can choose $\Sigma_{C'}$ to be completely space-like. This surface will again intersect the loop C of $A(C)$ at some point x_0. (Previous remarks in the abelian case about multiple intersections and higher linking numbers between C and C' will still apply and will not be repeated.)

To proceed further, write for convenience:

$$A(C) = \mathrm{Tr}\left[\prod_{s=0\to 2\pi} \phi(s)\right], \tag{14}$$

$$B(C') = \mathrm{Tr}\left[\prod_{t=0\to 2\pi} \tilde{\phi}(t)\right], \tag{15}$$

with

$$\phi(s) = \exp igA_i(\xi(s))\dot{\xi}^i(s)ds, \tag{16}$$

$$\tilde{\phi}(t) = \exp i\tilde{g}\int_{\eta_0}^{\eta(t)} \delta\eta'^{i}(t)\tilde{E}_i[\eta'|t], \tag{17}$$

where all products are meant to be properly ordered. Hence,

$$A(C) = \mathrm{Tr}[V\phi(s_0)], \tag{18}$$

$$B(C') = \mathrm{Tr}[\tilde{\phi}(t_0)\tilde{V}], \tag{19}$$

for

$$V = \prod_{s=0\to s_0} \phi(s) \prod_{s=s_0\to 2\pi} \phi(s), \tag{20}$$

$$\tilde{V} = \prod_{t=0\to t_0} \tilde{\phi}(t) \prod_{t=t_0\to 2\pi} \tilde{\phi}(t), \tag{21}$$

where s_0 and t_0 refer to the intersection of C with $\Sigma_{C'}$, namely such that $\xi(s) = x_0$, and that the ribbon for $t = t_0$, as defined by (17), passes through x_0. To avoid being entangled with the somewhat extraneous noncommutativity of these quantities which is due to their being elements of the gauge algebra, we rewrite them in terms of their internal symmetry components, thus:

$$A(C) = \sum_{a,b} V_{ab}\phi_{ba}, \;\; a, b = 1, ...N, \tag{22}$$

$$B(C') = \sum_{c,d} \tilde{\phi}_{cd}\tilde{V}_{dc}, \;\; c, d = 1, ..., N, \tag{23}$$

where the indexed quantities are now just c-numbers in internal symmetry space, though still operators in the quantum mechanical Hilbert space. We note, however, that except for the pair $\phi_{ab} = \phi_{ab}(s_0)$ and $\tilde{\phi}_{cd} = \tilde{\phi}_{cd}(t_0)$, all other factors are spatially separated and would thus mutually commute. That being the case, then supposing we assume that:

$$\phi_{ba}\tilde{\phi}_{cd} = \tilde{\phi}_{cd}\phi_{ba} \exp 2\pi i/N, \tag{24}$$

we would obtain (1) for $n = 1$ as desired.

Let us examine then the relation (24). It would be valid if the exponents of ϕ and $\tilde{\phi}$ in (16) and (17) satisfy the following commutation relation:

$$\left[igA_i(\xi(s_0))\dot{\xi}^i(s_0)ds_0, i\tilde{g}\int_{\eta_0}^{\eta(t_0)} \delta\eta'^j(t_0)\tilde{E}_j[\eta'|t_0] \right] = 2\pi i/N. \tag{25}$$

We note that in (25), although A_i and $\tilde{E}_j$ are both matrices in internal symmetry indices, they are to be regarded as elements of different algebras, since in (24) these indices are not summed. In other words, if we write:

$$A_i(x) = A_i^\alpha X_\alpha, \tag{26}$$

$$\tilde{E}_i[\eta|t] = \tilde{E}_i^\alpha[\eta|t]\tilde{X}_\alpha, \tag{27}$$

where although X_α and $\tilde{X}_\alpha$ are matrices representing the generators of the same gauge Lie algebra, they are to be regarded as corresponding to different degrees of freedom, like isospins of different particles. They therefore commute as far as (25) is concerned, their product there being the tensor product, not the ordinary matrix product.

To obtain (24), we write, recalling the dual transform defined in ref. [2]:

$$\tilde{E}_i[\eta|t] = -\frac{2}{\bar{N}}\epsilon_{ij\rho\sigma}\dot{\eta}^j(t)\int \delta\xi ds\, \omega(\xi(s))E^\rho[\xi|s]\omega^{-1}(\xi(s))\dot{\xi}^\sigma(s)\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)), \tag{28}$$

where using arguments similar to those given there for showing the abelian reduction of (28) to the Hodge star we can rewrite the right-hand side as:

$$-\frac{1}{\bar{N}}\epsilon_{ijk0}\dot{\eta}^j(t)\int \delta\xi ds\, \omega(\xi(s))U_\xi(s)F_\alpha^{k0}(\xi(s))\tilde{X}^\alpha U_\xi^{-1}(s)\omega^{-1}(\xi(s))\dot{\xi}^{-2}(s)\delta(\xi(s)-\eta(t)), \tag{29}$$

with $U_\xi(s)$ being an element of the gauge group. Then, using the canonical commutation relation for $\mathcal{E}_i^\alpha = F_{0i}^\alpha$:

$$[\mathcal{E}_i^\alpha(x), A_j^\beta(x')] = i\delta^{\alpha\beta}\delta_{ij}\delta(x-x'), \tag{30}$$

valid in the temporal gauge $A_0^\alpha = 0$, we obtain for the commutator in (25):

$$i\frac{g\tilde{g}}{\bar{N}}\omega(x_0)\left[\int \delta\xi U_\xi(s_0)X^\alpha\tilde{X}_\alpha U_\xi^{-1}(s_0)\dot{\xi}^{-2}(s_0)\delta(\xi(s_0)-x_0)\right]\omega^{-1}(x_0). \tag{31}$$

The quantity $X^\alpha\tilde{X}_\alpha$, however, is just a number [3]:

$$\sum_\alpha X_{ab}^\alpha\tilde{X}_{cd}^\alpha = \frac{1}{2}\left[\delta_{ad}\delta_{cb} - \frac{1}{N}\delta_{ab}\delta_{cd}\right]. \tag{32}$$

Hence, the integral in (31) can be done and cancels the normalisation factor $\tilde{N}$ in the denominator as defined in ref. [2] giving for the commutator in (25) just:

$$i g \tilde{g}\left(\pm\frac{1}{2}-\frac{1}{2N}\right) \tag{33}$$

for respectively the states symmetric or antisymmetric under the interchange of tilde with no tilde. We would then obtain the desired result in (25) if g and $\tilde{g}$ satisfy the Dirac quantization condition:

$$g\tilde{g} = 4\pi. \tag{34}$$

That this condition holds in the standard normalization convention adopted here[3], where the action is written as:

$$\mathcal{A} = -\frac{1}{4}\int d^4x \mathrm{Tr}[\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}] + \int d^4x \bar{\tilde{\psi}}(i\partial_\mu\gamma^\mu - m)\tilde{\psi} \tag{35}$$

can be seen by writing

$$\tilde{D}^\mu \tilde{F}_{\mu\nu} = -\tilde{g}(\bar{\tilde{\psi}}\gamma_\nu \tilde{X}^\alpha \tilde{\psi})\tilde{X}_\alpha \tag{36}$$

as a dual current in the direct (i.e. no tilde) description in terms of the loop space curvature $G_{\mu\nu}[\xi|s]$. For consistency with obtaining the correctly quantized monopole charge, we need:

$$\exp\left[-ig\tilde{g}\epsilon_{\mu\nu\rho\sigma}(\bar{\tilde{\psi}}\gamma^\rho \tilde{X}^\alpha \tilde{\psi})\dot{\xi}^\sigma \tilde{X}_\alpha \delta\xi^\mu \delta\xi^\nu\right] = \exp 2\pi i/N. \tag{37}$$

This condition being invariant, that it is satisfied by (34) can easily be checked by giving $\tilde{\psi}$ a specially simple orientation, say $(1,0,...,0)$, and using the standard representations for the $su(N)$ matrices $\tilde{X}_\alpha$.

With the Dirac condition (34) now established, the 'proof' of the validity of 't Hooft's commutation relation (1) for the operators $A(C)$ and $B(C)$ in (2) and (3) is then complete, although for lack of a general calculus for handling loop operations which is keenly felt throughout the scheme of ref. [2], the 'proof' is of necessity not as rigorous as one could desire.

[3]Notice that this is a different convention from that adopted in our earlier publications on the subject, e.g. [2]; hence the different form of the Dirac condition.

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[2] Chan Hong-Mo, J. Faridani, and Tsou Sheung Tsun, Phys. Rev. **D53**, 7293, (1996).

[3] See e.g. Graham G. Ross *Grand Unified Theories* (Benjamin/Cummings Pub. Co. Inc., California, 1985) Appendix A.