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NULL-SPACE PRECONDITIONERS FOR SADDLE POINT SYSTEMS

JENNIFER PESTANA*, TYRONE REES†

Abstract. The null-space method is a technique that has been used for many years to reduce a saddle point system to a smaller, easier to solve, symmetric positive-definite system. This method can be understood as a block factorization of the system. Here we explore the use of preconditioners based on incomplete versions of a particular null-space factorization, and compare their performance with the equivalent Schur-complement based preconditioners. We also describe how to apply the non-symmetric preconditioners proposed using the conjugate gradient method (CG) with a non-standard inner product. This requires an exact solve with the (1,1) block, and the resulting algorithm is applicable in other cases where Bramble-Pasciak CG is used. We verify the efficiency of the newly proposed preconditioners on a number of test cases from a range of applications.

AMS subject classifications. 65F08, 65F10, 65F50

Key words. Preconditioning, saddle point systems, null-space method

1. Introduction. We consider the fast solution of saddle point systems of the form

$$\mathcal{A}\boldsymbol{w} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix} = \boldsymbol{b}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $m \leq n$. A kernel that solves systems with this structure is a vital component in numerous scientific computing algorithms; for example, such systems arise naturally in constrained optimization problems [7, 12, 15, 24], Stokes and Navier-Stokes equations in fluid mechanics [5, 9, 16], time-harmonic Maxwell equations [25, 28], and the application of Kirchhoff's laws in circuit simulation [41, 45]. For an overview of solution methods, see the survey article of Benzi, Golub and Liesen [4], and the references therein.

Here we focus on the case where A, and consequently $\mathcal{A}$, are large, sparse matrices, such that solving (1.1) by direct methods is infeasible. Furthermore, we suppose that B is of full rank, and that A is symmetric positive semi-definite, and positive definite on the kernel of B.

The *null-space method* is a technique for solving systems of the form (1.1). This method requires a particular solution $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$ such that $B\widehat{\boldsymbol{x}} = \boldsymbol{g}$, and a matrix Z whose columns span the null-space of B. Then, since $\boldsymbol{x} = Z\boldsymbol{x}_{\mathrm{n}} + \widehat{\boldsymbol{x}}$, we can solve (1.1) by first finding $\boldsymbol{x}_{\mathrm{n}}$ from

$$Z^T A Z \boldsymbol{x}_{\mathrm{n}} = Z^T(\boldsymbol{f} - A\widehat{\boldsymbol{x}}), \tag{1.2}$$

and then recovering $\boldsymbol{y}$ from the overdetermined system $B^T\boldsymbol{y} = \boldsymbol{f} - A\boldsymbol{x}$; see, e.g., [4, Chapter 6] for more details. There exist many methods for obtaining a null-space matrix Z but a common choice is the fundamental basis,

$$Z_f := \begin{bmatrix} -B_1^{-1}B_2 \\ I \end{bmatrix}, \tag{1.3}$$

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where $B = [B_1\ B_2]$ and, without loss of generality, $B_1 \in \mathbb{R}^{m\times m}$ is nonsingular. Note that because the rank of B is m, we can always permute B to obtain an invertible B_1.

Even for sparse $\mathcal{A}$ the symmetric positive definite matrix $Z^T AZ$ is often dense, and forming it could be costly. For larger problems, therefore, it may be better to solve (1.1) rather than (1.2). It is possible to write down a number of matrix factorizations that are equivalent to the null-space method—recently Rees and Scott gave an overview [39]. We can obtain one particular factorization of $\mathcal{A}$ which falls into this category by taking a 3×3 blocking of $\mathcal{A}$ as

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} & B_1^T \\ A_{21} & A_{22} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix},$$

where $B_1 \in \mathbb{R}^{m\times m}$ is, as in (1.3), assumed to be nonsingular. Consider a permuted version of $\mathcal{A}$,

$$\widehat{\mathcal{A}} := \Pi\mathcal{A}\Pi^T = \left[\begin{array}{cc|c} A_{11} & B_1^T & A_{12} \\ B_1 & 0 & B_2 \\ \hline A_{21} & B_2^T & A_{22} \end{array}\right] =: \begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} \end{bmatrix}, \quad \Pi = \begin{bmatrix} I_m & & \\ & & I_m \\ & I_{n-m} & \end{bmatrix}, \tag{1.4}$$

where I_m is the identity matrix of dimension m. The matrix $\widehat{A}$ is invertible with inverse

$$\widehat{A}^{-1} = \begin{bmatrix} 0 & B_1^{-1} \\ B_1^{-T} & -B_1^{-T} A_{11} B_1^{-1} \end{bmatrix}.$$

Therefore we can apply the standard block-LDL^T factorization for saddle point systems [4, Equation (3.1)], obtaining

$$\widehat{\mathcal{A}} = \begin{bmatrix} I & 0 \\ \widehat{B}\widehat{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \widehat{A} & 0 \\ 0 & A_{22} - \widehat{B}\widehat{A}^{-1}\widehat{B}^T \end{bmatrix} \begin{bmatrix} I & \widehat{A}^{-1}\widehat{B}^T \\ 0 & I \end{bmatrix}. \tag{1.5}$$

This factorization has a connection with standard Schur complement preconditioners and the range-space method. However, because of the permutation, it is also closely related to the null-space method with the fundamental basis since, using (1.3), we can re-write (1.5) as

$$\widehat{\mathcal{A}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ B_2^T B_1^{-T} & X B_1^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & B_1^T & 0 \\ B_1 & 0 & 0 \\ 0 & 0 & N \end{bmatrix} \begin{bmatrix} I & 0 & B_1^{-1}B_2 \\ 0 & I & B_1^{-T}X^T \\ 0 & 0 & I \end{bmatrix}, \tag{1.6}$$

where

$$X := Z_f^T \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \text{ and } N := Z_f^T A Z_f = A_{22} - \widehat{B}\widehat{A}^{-1}\widehat{B}^T. \tag{1.7}$$

Our assumption that A is positive definite on the null-space of B means that the null-space matrix N is symmetric positive definite. The approach in Section 2 of Rees and Scott [39] applied here shows that solving $\widehat{\mathcal{A}}(\Pi\boldsymbol{w}) = \Pi\boldsymbol{b}$ is equivalent to the null-space method with the fundamental null-space (1.3) and particular solution

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} B_1^{-1}\boldsymbol{g} \\ \mathbf{0} \end{bmatrix}. \tag{1.8}$$

Applying the inverse permutations Π and Π^T in (1.4) to $\widehat{\mathcal{A}}$ in (1.6) we can easily recover $\mathcal{A}$.

One interpretation of the null-space method is therefore that the null-space matrix $Z_f^T A Z_f$ is equivalent to the Schur complement of the sub-block $\widehat{A}$ in $\widehat{\mathcal{A}}$. Note that when the block A is invertible, the null-space matrix is equivalent to the Schur complement of the dual saddle point system of (1.1) [4, page 32].

In this paper we present four preconditioners based on incomplete or approximate versions of the factorization (1.6). Since these preconditioners are intimately related to the Schur-complement decomposition we can apply them using conjugate gradients in a non-standard inner product [10]. However, the usual application of such preconditioners requires certain quantities to be symmetric positive definite, and this is typically attained by scaling the blocks appropriately. In our case this is not possible without destroying the structure that we exploit, since we assume that we can solve with a certain submatrix of $\mathcal{A}$ exactly. Accordingly, we extend the current theory of conjugate-gradients in a non-standard inner product to allow for the case where one of the sub-block solves is exact.

The rest of this paper is as follows. In Section 2, we give an eigen-analysis of the preconditioned systems and show that the eigenvalues of the preconditioned matrices are clustered when a good approximation to the null-space matrix (1.7) is known. We describe the non-standard inner product CG method in Section 3, and describe how our constraint preconditioner can be used within a projected Krylov subspace method. In Section 4, we compare our preconditioners to standard Schur-complement based methods based on the factorization

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} = \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}, \tag{1.9}$$

where $S = C + BA^{-1}B^T$ is the Schur complement of A. Note that this factorization can only be applied if A is invertible, whereas the null-space method may be applied even when A is singular. The problems we consider range from academic to practical, and illustrate the merits and limitations of our preconditioners. We give some conclusions in Section 5.

2. Null-space preconditioners. Taking incomplete versions of the block LDL^T factorization (1.9) has proved to be an effective way of constructing preconditioners for saddle point problems—see, e.g., [7, 16, 27, 38, 46]. The key component of such methods is an approximation of the matrix S, which is often dense. In the following we present, and give theory for, preconditioners based on the alternative decomposition (1.5). In particular, we use the null-space decomposition

$$\mathcal{A} = \underbrace{\begin{bmatrix} I & 0 & 0 \\ B_2^T B_1^{-T} & I & XB_1^{-1} \\ 0 & 0 & I \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} A_{11} & 0 & B_1^T \\ 0 & N & 0 \\ B_1 & 0 & 0 \end{bmatrix}}_{\mathcal{D}} \underbrace{\begin{bmatrix} I & B_1^{-1}B_2 & 0 \\ 0 & I & 0 \\ 0 & B_1^{-T}X^T & I \end{bmatrix}}_{\mathcal{L}^T}, \tag{2.1}$$

as the basis of our preconditioners; see [4, Equation 10.35], [39]. We replace N by a symmetric positive definite approximation $\widetilde{N}$, and possibly drop one or both of $\mathcal{L}$ and $\mathcal{L}^T$.

2.1. The central-null preconditioner. First, we consider the preconditioner formed by dropping both the $\mathcal{L}$ and $\mathcal{L}^T$ terms from the factorization (2.1),

$$\mathcal{P}_{cn} = \begin{bmatrix} A_{11} & 0 & B_1^T \\ 0 & \tilde{N} & 0 \\ B_1 & 0 & 0 \end{bmatrix},$$

where $\tilde{N} \approx N = Z_f^T A Z_f$. This corresponds to the block diagonal preconditioner in the decomposition (1.9), because $\Pi\mathcal{P}_{cn}\Pi^T$ is block diagonal. First, we give an eigen-analysis of the preconditioned system.

THEOREM 2.1. *Let $\tilde{N}$ be a symmetric positive definite approximation to N. Then the generalized eigenvalues λ of*

$$\begin{bmatrix} A_{11} & A_{12} & B_1^T \\ A_{21} & A_{22} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} A_{11} & 0 & B_1^T \\ 0 & \tilde{N} & 0 \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix}, \tag{2.2}$$

satisfy $\lambda = 1$, or

$$\lambda = \frac{\mu + \sigma \pm \sqrt{(\sigma+\mu)^2 - 4\mu}}{2\mu}, \tag{2.3}$$

where $\sigma = \boldsymbol{y}^T A_{22}\boldsymbol{y}/\boldsymbol{y}^T N\boldsymbol{y}$ and $\mu = \boldsymbol{y}^T \widetilde{N}\boldsymbol{y}/\boldsymbol{y}^T N\boldsymbol{y}$.

Proof. Setting

$$\mathcal{P}_{cn} = \begin{bmatrix} A_{11} & 0 & B_1^T \\ 0 & \widetilde{N} & 0 \\ B_1 & 0 & 0 \end{bmatrix}$$

it follows that

$$\Pi\mathcal{P}_{cn}\Pi^T = \left[\begin{array}{cc|c} A_{11} & B_1^T & 0 \\ B_1 & 0 & 0 \\ \hline 0 & 0 & \widetilde{N} \end{array}\right] = \begin{bmatrix} \widehat{A} & \\ & \widetilde{N} \end{bmatrix}.$$

Since Π from (1.4) is orthogonal, the eigenvalues of $\mathcal{P}_{cn}^{-1}\mathcal{A}$ are the same as those of $\Pi\mathcal{P}_{cn}^{-1}\mathcal{A}\Pi^T = (\Pi\mathcal{P}_{cn}\Pi^T)^{-1}(\Pi\mathcal{A}\Pi^T)$, and therefore the eigenvalues needed are those of the generalized eigenvalue problem

$$\begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} \widehat{A} & \\ & \widetilde{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}. \tag{2.4}$$

The first block row of (2.4) gives $\widehat{B}^T\boldsymbol{y} = (\lambda - 1)\widehat{A}\boldsymbol{x}$, which implies that $\lambda = 1$ or

$$\boldsymbol{x} = \frac{1}{\lambda - 1}\widehat{A}^{-1}\widehat{B}^T\boldsymbol{y}.$$

The second block row of (2.4) gives $\widehat{B}\boldsymbol{x} + A_{22}\boldsymbol{y} = \lambda\widetilde{N}\boldsymbol{y}$. If we assume that $\lambda \neq 1$ then substituting for $\boldsymbol{x}$ gives that

$$\widehat{B}\widehat{A}^{-1}\widehat{B}^T\boldsymbol{y} + (\lambda - 1)A_{22}\boldsymbol{y} = \lambda(\lambda - 1)\widetilde{N}\boldsymbol{y}.$$

Using (1.7) we have that

$$(\lambda A_{22} - N)\boldsymbol{y} = \lambda(\lambda - 1)\widetilde{N}\boldsymbol{y}.$$

Premultiplying by $\boldsymbol{y}^T$ and setting $\mu := \boldsymbol{y}^T\widetilde{N}\boldsymbol{y}/\boldsymbol{y}^T N\boldsymbol{y}$ we get

$$\boldsymbol{y}^T N\boldsymbol{y} - \lambda\boldsymbol{y}^T A_{22}\boldsymbol{y} = \lambda(1-\lambda)\mu\boldsymbol{y}^T N\boldsymbol{y},$$

and hence

$$\lambda^2 - (1 + \sigma/\mu)\lambda + 1/\mu = 0,$$

where $\sigma = \boldsymbol{y}^T A_{22}\boldsymbol{y}/\boldsymbol{y}^T N\boldsymbol{y}$.

Thus,

$$\lambda = \frac{1 + \sigma/\mu \pm \sqrt{(\sigma/\mu)^2 + (2\sigma - 4)/\mu + 1}}{2} = \frac{\mu + \sigma \pm \sqrt{(\sigma+\mu)^2 - 4\mu}}{2\mu},$$

as required. □

We now present a few results that give a better understanding of the behaviour of these eigenvalues.

COROLLARY 2.2. *Suppose that λ is a real eigenvalue of the generalized eigenvalue problem* (2.2)*. Then*

$$\frac{1}{(\mu+\sigma)_{\max}} \le \lambda < 1 + \lambda_{max}(\widetilde{N}^{-1}A_{22})$$

where $0 \le \lambda_{max}(\widetilde{N}^{-1}A_{22})$ is the largest eigenvalue of $\widetilde{N}^{-1}A_{22}$ and $(\mu+\sigma)_{\max}$ is the largest value of $\mu + \sigma$.

Proof. For λ to be real we must have that $(\mu+\sigma)^2 \ge 4\mu$. The larger of the eigenvalues in (2.3), λ_+, satisfies

$$\lambda_+ = \frac{1}{2\mu}\left(\mu + \sigma + \sqrt{(\mu+\sigma)^2 - 4\mu}\right) < \frac{1}{2\mu}\left(\mu + \sigma + \sqrt{(\mu+\sigma)^2}\right) = 1 + \frac{\sigma}{\mu}.$$

Now, since

$$\frac{\sigma}{\mu} = \frac{\boldsymbol{y}^T A_{22}\boldsymbol{y}}{\boldsymbol{y}^T\widetilde{N}\boldsymbol{y}} \le \lambda_{max}(\widetilde{N}^{-1}A_{22})$$

we obtain the upper bound.

Now consider the smaller eigenvalue, λ_-. Note that $\lambda_- = (\gamma - \sqrt{\gamma^2 - \delta})/2$, where $\gamma = 1 + \sigma/\mu$, $\delta = 4/\mu$ and $\delta \le \gamma^2$. For any $x \in [0,1]$, $1 - \sqrt{1-x} \ge x/2$, and so

$$\lambda_- \ge \frac{1}{4}\frac{\delta}{\gamma} = \frac{1}{\mu+\sigma} \ge \frac{1}{(\mu+\sigma)_{\max}}.$$

□

It is clear from the lower bound that the real eigenvalues are positive.

REMARK 2.3. *In the ideal case where $\mu_{\min} = \mu_{\max} = 1$, which corresponds to $\widetilde{N} = N$, the lower bound becomes $\lambda_- > \frac{1}{1+\sigma_{\max}}$. Furthermore, in this case the eigenvalues are all real if $\sigma_{\min} \ge 1$.*

COROLLARY 2.4. *The complex eigenvalues of* (2.2) *with nonzero imaginary part satisfy*

$$\frac{1}{2} \leq Re(\lambda) < \frac{1}{\sqrt{\mu_{\min}}}, \qquad |Im(\lambda)| \leq \frac{1}{\sqrt{\mu_{\min}}}.$$

Proof. Since the eigenvalues of $N^{-1}A_{22}$ are nonnegative, the real part of any complex eigenvalue λ must be bounded below by $\frac{1}{2}$. Furthermore,

$$Re(\lambda) \leq \frac{1}{2}\left(1 + \frac{\sigma_{\max}}{\mu_{\min}}\right) < \frac{1}{\sqrt{\mu_{\min}}},$$

with the last step holding since $\mu + \sigma < 2\sqrt{\mu} \Rightarrow (\mu + \sigma)/2\mu < 1/\sqrt{\mu_{\min}}$. Now, the imaginary part satisfies

$$|Im(\lambda)|^2 = \frac{(\sigma+\mu)^2 - 4\mu}{4\mu^2} \leq \frac{(\sigma+\mu)^2}{4\mu^2},$$

and so we get

$$|Im(\lambda)| \leq \frac{1}{\sqrt{\mu_{\min}}}.$$

□

Note that the bound on the real part only holds for eigenvalues with a non-trivial imaginary part. For real eigenvalues there may be some $\lambda < 0.5$ if any eigenvalue of $N^{-1}A_{22}$ is larger than 1.5. It is also worth remarking that complex eigenvalues can be bounded independently of σ.

One final special case we wish to highlight is when A_{22} is zero and $\mu_{\min} = \mu_{\max} = 1$. Here the central-null preconditioned matrix has only three distinct eigenvalues, $\{1, \frac{1\pm\sqrt{3}i}{2}\}$, and this guarantees fast convergence of certain Krylov subspace methods. (We could also obtain these same three eigenvalues in this special case by Schur-complement based arguments, following approaches found in, e.g., [18, 34].) An example of where this structure arises naturally is in the interior point method for linear programs [49], where A_{22} approaches zero at convergence. Note that when $\mu_{\min} \approx 1$ and $\mu_{\max} \approx 1$, that is, when we have a good approximation $\widetilde{N}$ to N, the above conclusions for the ideal case are approximately satisfied.

2.2. The lower-null and upper-null preconditioners. Next, we drop the $\mathcal{L}^T$-term in (2.1) to form the preconditioner

$$\mathcal{P}_{ln} := \begin{bmatrix} A_{11} & 0 & B_1^T \\ A_{21} & \tilde{N} & B_2^T \\ B_1 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Since $\Pi\mathcal{P}_{ln}\Pi^T$ corresponds to a block lower-triangular preconditioner we refer to this as the lower-null preconditioner. Indeed, this preconditioner is 'psychologically block-lower triangular', in that we can easily identify blocks with which we can solve this system using a substitution method.

The following result holds for the eigenvalues.

THEOREM 2.5. *Let $\tilde{N}$ be an invertible approximation to N. Consider the generalized eigenvalue problem*

$$\begin{bmatrix} A_{11} & A_{12} & B_1^T \\ A_{21} & A_{22} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} A_{11} & 0 & B_1^T \\ A_{21} & \tilde{N} & B_2^T \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix}.$$

Then either $\lambda = 1$, or λ satisfies $N\boldsymbol{x} = \lambda\tilde{N}\boldsymbol{x}$.

Proof. As in the proof of Theorem 2.1, we use the fact that the required eigenvalues are the same as those of $(\Pi\mathcal{P}_{ln}\Pi^T)^{-1}\Pi\mathcal{A}\Pi^T$. Recalling (1.7) we have that

$$(\Pi\mathcal{P}_{ln}\Pi^T)^{-1}\Pi\mathcal{A}\Pi = \begin{bmatrix} I & \widehat{A}^{-1}\widehat{B} \\ 0 & \widetilde{N}^{-1}(A_{22} - \widehat{B}\widehat{A}^{-1}\widehat{B}^T) \end{bmatrix} = \begin{bmatrix} I & \widehat{A}^{-1}\widehat{B} \\ 0 & \widetilde{N}^{-1}N \end{bmatrix}.$$

The result follows. ∎

An apparent drawback of this preconditioner is that $\mathcal{P}_{ln}$ is a non-symmetric preconditioner for a symmetric problem, and hence we have to use a non-symmetric iterative method, such as GMRES [43] or BiCGStab [48]. However, Theorem 2.5 shows that if $N = \widetilde{N}$, GMRES applied to (1.1) with preconditioner $\mathcal{P}_{ln}$ converges in two steps. When $\widetilde{N} \neq N$ the eigenvalues may not tell us everything about convergence [23] although it is commonly observed that tightly clustered eigenvalues do predict convergence of GMRES in non-pathological cases—see, e.g., Pestana and Wathen [36] for a discussion. An additional benefit of using $\tilde{N}$ within a preconditioner for the whole matrix $\mathcal{A}$, and not explicitly for the null-space matrix, is that we never have to perform the (often costly) procedure of forming N.

If we instead drop the $\mathcal{L}$ term from (2.1) we get

$$\mathcal{P}_{un} := \begin{bmatrix} A_{11} & A_{12} & B_1^T \\ 0 & \tilde{N} & 0 \\ B_1 & B_2 & 0 \end{bmatrix}.$$

For reasons analogous to those given above we refer to this as the upper-null preconditioner, and we have the following result.

THEOREM 2.6. *Let $\tilde{N}$ be an invertible approximation to N. Consider the generalized eigenvalue problem*

$$\begin{bmatrix} A_{11} & A_{12} & B_1^T \\ A_{21} & A_{22} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} A_{11} & A_{12} & B_1^T \\ 0 & \tilde{N} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix}.$$

Then either $\lambda = 1$, or λ satisfies $N\boldsymbol{x} = \lambda\tilde{N}\boldsymbol{x}$.

Proof. Since $(\Pi\mathcal{P}_{un}\Pi^T)^{-1}(\Pi\mathcal{A}\Pi^T)$ is similar to $(\Pi\mathcal{A}\Pi^T)(\Pi\mathcal{P}_{un}\Pi^T)^{-1}$, a similar argument to that in the proof of Theorem 2.5 gives the result. ∎

The preconditioner $\mathcal{P}_{un}$ therefore has the same eigenvalues, with the same multiplicity, as $\mathcal{P}_{ln}$. In spite of possible effects of non-normality, in practice upper and lower block triangular preconditioners often exhibit similar behaviour (see [35] and the references therein), and this was our experience in the tests reported in Section 4.

2.3. A constraint preconditioner. Now consider the preconditioner obtained by taking the entire factorization (2.1) with the null space matrix replaced by $\tilde{N}$,

namely

$$\mathcal{P}_{con} = \begin{bmatrix} A_{11} & 0 & B_1^T \\ A_{21} & \widetilde{N} & B_2^T \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & B_1^{-1}B_2 & 0 \\ 0 & I & 0 \\ 0 & B_1^{-T}X^T & I \end{bmatrix}.$$

THEOREM 2.7. *The preconditioner $\mathcal{P}_{con}$ is a constraint preconditioner.*

Proof. Direct computation shows that

$$\Pi\mathcal{P}_{con}\Pi^T = \begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} - N + \widetilde{N} \end{bmatrix} \tag{2.6}$$

or that

$$\mathcal{P}_{con} = \begin{bmatrix} A_{11} & A_{12} & B_1^T \\ A_{21} & A_{22} - N + \widetilde{N} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix}.$$

□

We can show the following result about the eigenvalues for the constraint preconditioner here:

THEOREM 2.8. *Let $\widetilde{N}$ be an invertible approximation to N. Consider the generalized eigenvalue problem $\mathcal{A}\boldsymbol{z} = \lambda\mathcal{P}_{con}\boldsymbol{z}$. Then either $\lambda = 1$, or λ satisfies $N\boldsymbol{x} = \lambda\widetilde{N}\boldsymbol{x}$.*

Proof. As in previous sections, we can compute the eigenvalues of $\mathcal{P}_{con}^{-1}\mathcal{A}$ by solving a generalized eigenvalue problem to find the eigenvalues of $(\Pi\mathcal{P}_{con}\Pi^T)^{-1}(\Pi\mathcal{A}\Pi^T)$. In particular, using (2.6), we have that

$$\begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} - N + \widetilde{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}.$$

The first equation shows that $\lambda = 1$ or that $\widehat{A}\boldsymbol{x} + \widehat{B}^T\boldsymbol{y} = 0$. Since $\widehat{A}$ is invertible, in the latter case we have that $\boldsymbol{x} = -\widehat{A}^{-1}\widehat{B}^T\boldsymbol{y}$.

The second equation gives that $(1-\lambda)\widehat{B}\boldsymbol{x} + (1-\lambda)A_{22}\boldsymbol{y} = \lambda(\widetilde{N} - N)\boldsymbol{y}$. If $\lambda \neq 1$ then after substituting for $\boldsymbol{x}$ we find that

$$-(1-\lambda)\widehat{B}\widehat{A}^{-1}\widehat{B}^T\boldsymbol{y} + (1-\lambda)A_{22}\boldsymbol{y} = \lambda(\widetilde{N} - N)\boldsymbol{y}.$$

Using (1.7) and simplifying shows that $N\boldsymbol{y} = \lambda\widetilde{N}\boldsymbol{y}$ as required. □

Comparison with Theorems 2.5 and 2.6 shows that all three preconditioned matrices $\mathcal{P}_{ln}^{-1}\mathcal{A}, \mathcal{P}_{un}^{-1}\mathcal{A}$ and $\mathcal{P}_{con}^{-1}\mathcal{A}$ have the same eigenvalues, with the same multiplicities. The constraint preconditioner is more expensive to apply than the lower and upper null preconditioners, but it benefits from the advantages of constraint preconditioners; we expand on this point in Sections 2.4 and 3.4 below.

Note that this preconditioner is numerically the same as the preconditioner defined by the GALAHAD [20] subroutine `SBLS`, namely

$$\mathcal{P}_{\texttt{SBLS}} = \begin{bmatrix} A_{11} & 0 & I \\ A_{21} & I & B_2^T B_1^{-T} \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \widetilde{N} & 0 \\ I & 0 & -A_{11} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & B_1^T \\ 0 & I & 0 \\ I & B_1^{-1}B_2 & 0 \end{bmatrix}.$$

The authors are not aware of an eigen-analysis of this preconditioner in the literature.

2.4. Discussion. The preconditioners described above are the result of thinking about the null-space method in terms of a matrix factorization. The preconditioners $\mathcal{P}_{ln}$ and $\mathcal{P}_{un}$ are particularly promising. They have the drawback that applying them requires solves with B_1 and B_1^T, as well as the solve with $\tilde{N}$ and a number of matrix-vector multiplications. It is also somewhat jarring that we are proposing non-symmetric preconditioners for a symmetric problem (although a short-term recurrence method in a non-standard inner product can be applied as discussed in Section 3.3). Balancing these issues is the fact that the eigenvalue clustering is as good as possible.

Constraint preconditioners possess favourable properties, such as the iterates staying on a certain manifold [42], which can be useful in certain applications. If such properties are desired then $\mathcal{P}_{con}$ provides them, even with an approximate $\tilde{N}$, at the expense of extra solves with B_1 and B_1^T. This is in contrast to the equivalent Schur-complement formulation [7], which gives an *inexact* constraint preconditioner. As such, null-space preconditioners could be useful in optimization, where solution methods that remain on the constraint manifold are often required; see, e.g., [1]. Additionally, it is possible to use a projected CG or MINRES method in this case, as described in Section 3.4. We envisage that this preconditioner will be particularly useful in fields where many systems have to be solved with the same B block, possibly with A changing; an important example that generates linear systems with this structure is the interior point method in optimization [49].

Null-space preconditioners require us to find an invertible subset of the constraint matrix B, which is an additional computational cost that is not present in, for instance, Schur-complement based approaches. However, there are a number of applications we are aware of where this is not problematic; we discuss a few of these in more detail in Section 4.

We note that for problems with maximally rank-deficient A, i.e., problems for which $\text{rank}(A) = n - m$, alternative block diagonal preconditioners were recently proposed by Estrin and Greif [17] that rely on a matrix C whose columns span the null-space of A. Estrin and Greif show that under certain conditions, the preconditioned systems can be solved by (standard) conjugate gradients; otherwise a standard non-symmetric Krylov method can be used.

3. Using conjugate gradients and MINRES with the null-space preconditioners. As discussed in Section 2.4, although $\mathcal{P}_{ln}^{-1}\mathcal{A}$, $\mathcal{P}_{un}^{-1}\mathcal{A}$ and $\mathcal{P}_{con}^{-1}\mathcal{A}$ have nice spectra when $\widetilde{N}$ is a good approximation of N, the preconditioners are not symmetric positive definite. Accordingly, they cannot be used with standard MINRES or CG. However, since $\mathcal{P}_{con}$ is a constraint preconditioner, it can be used with the projected conjugate gradient [21] or projected MINRES [19] methods. On the other hand, although $\mathcal{P}_{ln}$ is non-symmetric, it can be used in conjunction with the conjugate gradient method in a non-standard inner product. We discuss both these approaches in this section.

3.1. Non-standard inner products. In this section, we show that it is possible to use the conjugate gradient method in a non-standard inner product to solve (1.1) with the preconditioner (2.5). We consider general equations of the form

$$\underbrace{\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} \tag{3.1}$$

where we assume that $\mathcal{A}$ is invertible, $A \in \mathbb{R}^{n\times n}$ is symmetric and invertible, $B \in \mathbb{R}^{m\times n}$, $m \leq n$, and $C \in \mathbb{R}^{m\times m}$ is symmetric. We additionally assume that the Schur complement $C - BA^{-1}B^T$ is positive definite, although our results extend trivially to the case where $C - BA^{-1}B^T$ is negative definite. Note that the proceeding results hold for $A = \widehat{A}$, $B = \widehat{B}$, $C = A_{22}$ in (1.4), as we show in Section 3.3, but are more generally applicable.

Although the saddle point matrix $\mathcal{A}$ is indefinite, so that the standard conjugate gradient method cannot be reliably applied to solve (1.1), a judicious choice of preconditioner can make $\mathcal{A}$ self-adjoint and positive definite with respect to a non-standard inner product. A number of preconditioners that achieve this goal have been proposed [6, 10, 14, 15, 29, 46]. Many, although not all, fall into the class of preconditioners and inner products discussed by Krzyżanowski [27], who showed the following:

PROPOSITION 3.1 (Krzyżanowski [27], Proposition 2.1). *Suppose we wish to solve the system* (3.1). *Consider the preconditioner given by*

$$\mathcal{P}^{-1} = \begin{bmatrix} I & -dA_0^{-1}B^T \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0^{-1} & 0 \\ 0 & S_0^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -cBA_0^{-1} & I \end{bmatrix}, \tag{3.2}$$

for fixed scalars c, d, and where A_0 and S_0 are symmetric and nonsingular. Let $\delta \in \{-1, +1\}$ and $\mathcal{H}$ be the block diagonal matrix

$$\mathcal{H} = \delta \begin{bmatrix} A_0 - cA & 0 \\ 0 & S_0 + cdBA_0^{-1}B^T - dC \end{bmatrix}. \tag{3.3}$$

Then $\mathcal{H}\mathcal{P}^{-1}\mathcal{A}$ is symmetric.

This means that, even though $\mathcal{P}^{-1}\mathcal{A}$ is non-symmetric, it is self-adjoint with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathcal{H}} = \boldsymbol{y}^T\mathcal{H}\boldsymbol{x}$. If A_0, S_0 and δ are chosen so that $\mathcal{H}$ is symmetric positive definite, and inner products are understood to be $\mathcal{H}$-inner products, then we can apply CG to solve (1.1). Algorithm 1, adapted from Algorithm 3.2 of Dollar, Gould, Stoll and Wathen [15], is one such method which does this.

Given $\boldsymbol{x}_0$, set $\boldsymbol{r}_0 = \boldsymbol{b} - \mathcal{A}\boldsymbol{x}_0$, $\boldsymbol{z}_0 = \mathcal{P}^{-1}\boldsymbol{r}_0$ and $\boldsymbol{p}_0 = \boldsymbol{z}_0$;
for $k = 0, 1, \ldots$ **do**
  $\alpha = \dfrac{\boldsymbol{z}_k^T\mathcal{H}\boldsymbol{z}_k}{\boldsymbol{p}_k^T\mathcal{H}\mathcal{P}^{-1}\mathcal{A}\boldsymbol{p}_k}$;
  $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha\boldsymbol{p}_k$;
  $\boldsymbol{r}_{k+1} = \boldsymbol{r}_k - \alpha\mathcal{A}\boldsymbol{p}_k$;
  $\boldsymbol{z}_{k+1} = \mathcal{P}^{-1}\boldsymbol{r}_{k+1}$;
  $\beta = \dfrac{\boldsymbol{z}_{k+1}^T\mathcal{H}\boldsymbol{z}_{k+1}}{\boldsymbol{z}_k^T\mathcal{H}\boldsymbol{z}_k}$;
  $\boldsymbol{p}_{k+1} = \boldsymbol{z}_{k+1} + \beta\boldsymbol{p}_k$;
end

Algorithm 1: CG in the scalar product defined by $\mathcal{H}$.

If $c = 0$, then $\mathcal{H}$ can only be symmetric positive definite if the approximation A_0 is as well. If c is non-zero we may need to scale the approximation A_0 so that $A_0 - cA$ is symmetric positive definite. In particular, this means that if we require a positive definite $\mathcal{H}$ we cannot use the exact $A_0 = A$, even if it is readily available, despite scaling adversely affecting eigenvalue clustering [34].

3.2. Semidefinite $\mathcal{H}$. Let us consider the use of a non-standard CG method for (1.1) with the lower-null preconditioner $\mathcal{P}_{ln}$ in (2.5). Although $\mathcal{P}_{ln}$ is not block lower triangular, $\Pi\mathcal{P}_{ln}\Pi^T$ is, while $\widehat{\mathcal{A}} = \Pi\mathcal{A}\Pi^T$ is a generalized saddle point matrix of the form in (3.1). Accordingly, we can apply the results of Krzyżanowski to this permuted matrix and preconditioner. Since the (1,1) block of $\Pi\mathcal{P}_{ln}\Pi^T$ is identical to that of $\widehat{\mathcal{A}}$, we are in precisely the situation that $c = 1$, $d = 0$ and $A_0 = A$ in (3.2). This may indicate that $\mathcal{P}_{ln}$ is a good preconditioner but it also means that $\mathcal{H}$ in (3.3) is singular. Despite this, a more careful examination of Algorithm 1 will reveal that this singularity causes no difficulties for computing the solution of (3.1).

Accordingly, we will consider the case $A_0 = A$, $c = 1$ and $d = 0$ in (3.2) and (3.3) in more detail, which arises in our application, but also more widely when solving, e.g., problems in PDE constrained optimization [37, 40]. We assume that S_0 is symmetric positive definite, giving

$$\mathcal{H}_{1,0} = \begin{bmatrix} 0 & 0 \\ 0 & S_0 \end{bmatrix}, \quad \mathcal{P}_{1,0} = \begin{bmatrix} A & 0 \\ B & S_0 \end{bmatrix}. \tag{3.4}$$

Thus, the matrix $\mathcal{H}_{1,0}$ is semidefinite with rank m.

We now show that we can apply Algorithm 1 with $\mathcal{H}_{1,0}$, $\mathcal{P}_{1,0}$ and $\mathcal{A}$ to solve (3.1). Let us first consider the initialization phase. With $\boldsymbol{x}_0 = \left[(\boldsymbol{x}_0^{(1)})^T \ (\boldsymbol{x}_0^{(2)})^T\right]$, $\boldsymbol{x}_0^{(1)} \in \mathbb{R}^n$, it follows from Algorithm 1 that

$$\boldsymbol{r}_0 = \begin{bmatrix} \boldsymbol{c} - A\boldsymbol{x}_0^{(1)} - B^T\boldsymbol{x}_0^{(2)} \\ \boldsymbol{d} - B\boldsymbol{x}_0^{(1)} - C\boldsymbol{x}_0^{(2)} \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}_0 = \boldsymbol{z}_0 = \begin{bmatrix} A^{-1}(\boldsymbol{c} - B^T\boldsymbol{x}_0^{(2)}) - \boldsymbol{x}_0^{(1)} \\ S_0^{-1}\left(\boldsymbol{d} - BA^{-1}\boldsymbol{c} - (C - BA^{-1}B^T)\boldsymbol{x}_0^{(2)}\right) \end{bmatrix}.$$

Let us move to the *for* loop. Given

$$\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{x}_k^{(1)} \\ \boldsymbol{x}_k^{(2)} \end{bmatrix}, \quad \boldsymbol{r}_k = \begin{bmatrix} \boldsymbol{r}_k^{(1)} \\ \boldsymbol{r}_k^{(2)} \end{bmatrix}, \quad \boldsymbol{p}_k = \begin{bmatrix} \boldsymbol{p}_k^{(1)} \\ \boldsymbol{p}_k^{(2)} \end{bmatrix}$$

and $\boldsymbol{z}_k = \mathcal{P}_{1,0}^{-1}\boldsymbol{r}_k$, where $\boldsymbol{x}_k^{(1)}, \boldsymbol{r}_k^{(1)}, \boldsymbol{p}_k^{(1)} \in \mathbb{R}^n$, the choice of inner product means that

$$\alpha = \frac{(\boldsymbol{z}_k^{(2)})^T S_0(\boldsymbol{z}_k^{(2)})}{(\boldsymbol{p}_k^{(2)})^T(C - BA^{-1}B^T)\boldsymbol{p}_k^{(2)}} \quad \text{and} \quad \beta = \frac{(\boldsymbol{z}_{k+1}^{(2)})^T S_0(\boldsymbol{z}_{k+1}^{(2)})}{(\boldsymbol{z}_k^{(2)})^T S_0(\boldsymbol{z}_k^{(2)})}.$$

Additionally, since $\boldsymbol{z}_{k+1} = \boldsymbol{z}_k - \alpha\mathcal{P}_{1,0}^{-1}\mathcal{A}\boldsymbol{p}_k$,

$$\boldsymbol{z}_{k+1} = \begin{bmatrix} \boldsymbol{z}_k^{(1)} - \alpha(\boldsymbol{p}_k^{(1)} + A^{-1}B^T\boldsymbol{p}_k^{(2)}) \\ \boldsymbol{z}_k^{(2)} - \alpha S_0^{-1}(C - BA^{-1}B^T)\boldsymbol{p}_k^{(2)} \end{bmatrix}.$$

Putting this together gives the equivalent Algorithm 2.

It is not yet clear that the iterates generated by Algorithms 1 and 2 converge to the solution of (3.1). To show that Algorithm 2 does indeed solve (3.1) we compare Algorithm 2 to the preconditioned conjugate gradient method applied to the system obtained from the range-space method, i.e, the method based on a factorization like (1.9).

The range-space method—which is equivalent to solving (3.1)—proceeds in two stages; first we seek a solution to

$$(C - BA^{-1}B^T)\boldsymbol{v} = \boldsymbol{d} - BA^{-1}\boldsymbol{c}, \tag{3.5}$$

Given $\boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{x}_0^{(1)} \\ \boldsymbol{x}_0^{(2)} \end{bmatrix}$, set $\boldsymbol{z}_0 =$
$\begin{bmatrix} A^{-1}(\boldsymbol{c} - B^T \boldsymbol{x}_0^{(2)}) - \boldsymbol{x}_0^{(1)} \\ S_0^{-1}(\boldsymbol{d} - BA^{-1}\boldsymbol{c} - (C - BA^{-1}B^T)\boldsymbol{x}_0^{(2)}) \end{bmatrix}$ and $\boldsymbol{p}_0 = \boldsymbol{z}_0$;
for $k = 0, 1, \ldots$ **do**
$\alpha = \dfrac{(\boldsymbol{z}_k^{(2)})^T S_0(\boldsymbol{z}_k^{(2)})}{(\boldsymbol{p}_k^{(2)})^T (C - BA^{-1}B^T)\boldsymbol{p}_k^{(2)}}$;
$\begin{bmatrix} \boldsymbol{x}_{k+1}^{(1)} \\ \boldsymbol{x}_{k+1}^{(1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_k^{(1)} + \alpha \boldsymbol{p}_k^{(1)} \\ \boldsymbol{x}_k^{(2)} + \alpha \boldsymbol{p}_k^{(2)} \end{bmatrix}$;
$\begin{bmatrix} \boldsymbol{z}_{k+1}^{(1)} \\ \boldsymbol{z}_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_k^{(1)} - \alpha(\boldsymbol{p}_k^{(1)} + A^{-1}B^T \boldsymbol{p}_k^{(2)}) \\ \boldsymbol{z}_k^{(2)} - \alpha S_0^{-1}(C - BA^{-1}B^T)\boldsymbol{p}_k^{(2)} \end{bmatrix}$;
$\beta = \dfrac{(\boldsymbol{z}_{k+1}^{(2)})^T S_0(\boldsymbol{z}_{k+1}^{(2)})}{(\boldsymbol{z}_k^{(2)})^T S_0(\boldsymbol{z}_k^{(2)})}$;
$\begin{bmatrix} \boldsymbol{p}_{k+1}^{(1)} \\ \boldsymbol{p}_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_{k+1}^{(1)} + \beta \boldsymbol{p}_k^{(1)} \\ \boldsymbol{z}_{k+1}^{(2)} + \beta \boldsymbol{p}_k^{(2)} \end{bmatrix}$;
end

Algorithm 2: Simplified CG in the scalar product defined by $\mathcal{H}_{1,0}$.

before recovering $\boldsymbol{u}$ by solving

$$\boldsymbol{u} = A^{-1}(\boldsymbol{c} - B^T \boldsymbol{v}). \tag{3.6}$$

More details can be found in e.g., [4, Chapter 5].

Since $C - BA^{-1}B^T$ is positive definite we can solve (3.5) by a preconditioned conjugate gradient method with a symmetric positive definite preconditioner S_0 as in Algorithm 3. Note that when applied to (1.2), our Schur complement approach is actually a null-space method, as we show in Section 3.3 (cf. Algorithm 2.1 in Gould, Hribar and Nocedal [21]).

Given $\boldsymbol{v}_0$, set $\boldsymbol{z}_0 = S_0^{-1}(\boldsymbol{d} - BA^{-1}\boldsymbol{c} - (C - BA^{-1}B^T)\boldsymbol{v}_0)$ and $\boldsymbol{p}_0 = \boldsymbol{z}_0$;
for $k = 0, 1, \ldots$ **do**
$\alpha = \dfrac{\boldsymbol{z}_k^T S_0 \boldsymbol{z}_k}{\boldsymbol{p}_k^T (C - BA^{-1}B^T)\boldsymbol{p}_k}$;
$\boldsymbol{v}_{k+1} = \boldsymbol{v}_k + \alpha \boldsymbol{p}_k$;
$\boldsymbol{z}_{k+1} = \boldsymbol{z}_k - \alpha S_0^{-1}(C - BA^{-1}B^T)\boldsymbol{p}_k$;
$\beta = \dfrac{\boldsymbol{z}_{k+1}^T S_0 \boldsymbol{z}_{k+1}}{\boldsymbol{z}_k^T S_0 \boldsymbol{z}_k}$;
$\boldsymbol{p}_{k+1} = \boldsymbol{z}_{k+1} + \beta \boldsymbol{p}_k$;
end

Algorithm 3: CG for the reduced system (3.5).

Comparison of Algorithms 2 and 3 show that whenever $\boldsymbol{x}_0^{(2)} = \boldsymbol{v}_0$, the vectors $\boldsymbol{z}_0^{(2)}$ and $\boldsymbol{p}_0^{(2)}$ in Algorithm 2 are the same as $\boldsymbol{z}_0$ and $\boldsymbol{p}_0$ in Algorithm 3. Moreover, since the scalars α and β in the two algorithms are equivalent, $\boldsymbol{x}_k^{(2)}$, $\boldsymbol{z}_k^{(2)}$ and $\boldsymbol{p}_k^{(2)}$ in

Algorithm 2 are the same as $\boldsymbol{v}_k$, $\boldsymbol{z}_k$ and $\boldsymbol{p}_k$ in Algorithm 3 for all iterations $k \geq 0$. It follows from the convergence theory for Algorithm 3 that α, β, $\boldsymbol{x}_k^{(2)}$, $\boldsymbol{z}_k^{(2)}$ and $\boldsymbol{p}_k^{(2)}$ are all well defined and that the iterates $\boldsymbol{x}_k^{(2)}$ in Algorithm 2 are approximations of $\boldsymbol{v}$. However, Algorithm 2 also yields approximations of $\boldsymbol{u}$ as the next result shows.

LEMMA 3.2. *Let $\boldsymbol{x}_k^{(1)}$, $\boldsymbol{x}_k^{(2)}$ and $\boldsymbol{z}_k^{(1)}$ be as in Algorithm 2, $k \geq 0$. Additionally, let*

$$\boldsymbol{u}_k = A^{-1}(\boldsymbol{c} - B^T \boldsymbol{x}_k^{(2)}). \tag{3.7}$$

Then $\boldsymbol{u}_k = \boldsymbol{x}_k^{(1)} + \boldsymbol{z}_k^{(1)}$.

Proof. We show that $\boldsymbol{z}_k^{(1)} = \boldsymbol{u}_k - \boldsymbol{x}_k^{(1)}$ by induction. First, since $\boldsymbol{z}_0^{(1)} = A^{-1}(\boldsymbol{c} - B^T \boldsymbol{x}_0^{(2)}) - \boldsymbol{x}_0^{(1)}$, we see from (3.7) that $\boldsymbol{z}_0^{(1)} = \boldsymbol{u}_0 - \boldsymbol{x}_0^{(1)}$.

Now assume that for some $j \geq 0$,

$$\boldsymbol{z}_j^{(1)} = \boldsymbol{u}_j - \boldsymbol{x}_j^{(1)}. \tag{3.8}$$

From the updates for $\boldsymbol{x}_{j+1}^{(1)}$ and $\boldsymbol{x}_{j+1}^{(2)}$ in Algorithm 2 we see that $\alpha \boldsymbol{p}_j^{(1)} = \boldsymbol{x}_{j+1}^{(1)} - \boldsymbol{x}_j^{(1)}$ and $\alpha \boldsymbol{p}_j^{(2)} = \boldsymbol{x}_{j+1}^{(2)} - \boldsymbol{x}_j^{(2)}$. Substituting these formulae into the equation for $\boldsymbol{z}_{j+1}^{(1)}$ and using (3.7) gives that

$$\boldsymbol{z}_{j+1}^{(1)} = \boldsymbol{z}_j^{(1)} + (\boldsymbol{x}_j^{(1)} - \boldsymbol{x}_{j+1}^{(1)}) + (\boldsymbol{u}_{j+1} - \boldsymbol{u}_j).$$

This shows that whenever (3.8) holds, $\boldsymbol{z}_{j+1}^{(1)} = \boldsymbol{u}_{j+1} - \boldsymbol{x}_{j+1}^{(1)}$. Since $\boldsymbol{z}_0 = \boldsymbol{u}_0 + \boldsymbol{x}_0^{(1)}$, the stated result is proved. □

Now, because $\boldsymbol{x}_k^{(2)}$ approximates $\boldsymbol{v}$, the vector $\boldsymbol{x}_k^{(1)} + \boldsymbol{z}_k^{(1)}$ approximates $\boldsymbol{u}$ and so we obtain approximations of both $\boldsymbol{u}$ and $\boldsymbol{v}$ from Algorithm 2. Thus, Algorithm 2 is well defined and can be used to solve (3.1).

As a final point, since to solve (3.1) we only need $\boldsymbol{u}_k = \boldsymbol{x}_k^{(1)} + \boldsymbol{z}_k^{(1)}$ and $\boldsymbol{p}_k$, it is straightforward to show that nothing is lost in Algorithm 2 by setting $\boldsymbol{p}_k^{(1)} = \boldsymbol{0}$, $k > 0$, and that some computational savings are made by avoiding the vector update. Additionally, in our experience such a step can be useful for reducing the effect of rounding errors.

3.3. A non-standard conjugate gradient method for $\mathcal{P}_{ln}$. Now let us apply the results of this section to our system (1.1) with preconditioner (2.5). Recall from (1.4) and (2.5) that

$$\widehat{\mathcal{A}} := \Pi \mathcal{A} \Pi^T = \left[\begin{array}{cc|c} A_{11} & B_1^T & A_{12} \\ B_1 & 0 & B_2 \\ \hline A_{21} & B_2^T & A_{22} \end{array}\right] =: \begin{bmatrix} \widehat{A} & \widehat{B}^T \\ \widehat{B} & A_{22} \end{bmatrix}, \quad \Pi \mathcal{P}_{ln} \Pi^T = \begin{bmatrix} \widehat{A} & \\ \widehat{B} & A_{22} \end{bmatrix},$$

where $N = A_{22} - \widehat{B}\widehat{A}^{-1}\widehat{B}^T$ is the null-space matrix with the fundamental basis Z_f (see (1.3) and (1.7)). Letting

$$\Pi \mathcal{H}_{\ln} \Pi^T = \begin{bmatrix} 0 & \\ & \widetilde{N} \end{bmatrix},$$

we see that $\Pi \mathcal{H}_{\ln} \Pi^T$, $\Pi \mathcal{P}_{ln} \Pi^T$ and $\Pi \mathcal{A} \Pi^T$ are in the form of $\mathcal{H}_{1,0}$ and $\mathcal{P}_{1,0}$ in (3.4) and $\mathcal{A}$ in (3.1). Accordingly, we can apply the non-standard conjugate gradient method in Algorithm 2 to

$$\Pi(\mathcal{H}_{\ln} \mathcal{P}_{ln}^T \mathcal{A}) \Pi^T (\Pi \boldsymbol{w}) = \Pi \mathcal{H}_{\ln} \mathcal{P}_{ln} \boldsymbol{b}, \tag{3.9}$$

where

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{g} \end{bmatrix},$$

with $\boldsymbol{x} = [\boldsymbol{x}_1^T,\ \boldsymbol{x}_2^T]^T$ and $\boldsymbol{f} = [\boldsymbol{f}_1^T,\ \boldsymbol{f}_2^T]^T$ and $\boldsymbol{x}_1, \boldsymbol{f}_1 \in \mathbb{R}^m$. As shown in the previous section, this is equivalent to applying preconditioned CG to the reduced system (3.5), given by

$$N\boldsymbol{v} = (A_{22} - \widehat{B}\widehat{A}^{-1}\widehat{B}^T)\boldsymbol{v} = \boldsymbol{d} - \widehat{B}\widehat{A}^{-1}\boldsymbol{c}$$

where, because of the permutation matrix Π, $\boldsymbol{v} = \boldsymbol{x}_2$, $\boldsymbol{d} = \boldsymbol{f}_2$ and $\boldsymbol{c} = [\boldsymbol{f}_1^T,\ \boldsymbol{g}^T]^T$. The vector $\boldsymbol{u}$ in (3.6) is $\boldsymbol{u} = [\boldsymbol{x}_1^T,\ \boldsymbol{y}^T]^T$.

Comparison with the null-space method shows that solving this reduced system is the same as using the null-space method with the fundamental basis Z_f and the particular solution (1.8). It follows that applying non-standard CG to (3.9) is equivalent to applying the null-space method using the fundamental basis (1.3).

3.4. The constraint preconditioner, $\mathcal{P}_{cn}$. Finally for this section, we mention the preconditioner $\mathcal{P}_{cn}$. As this is a constraint preconditioner, we can apply it with projected conjugate gradients [21], provided that the (1,1) block is symmetric positive definite on the nullspace of B. If A is indefinite, or if we require a method that minimizes the residual, then we can still utilize a short-term recurrence method by using projected MINRES [19]. Therefore standard methods work *out-of-the-box*, and no extra theory is needed in this case.

4. Numerical Results. In this section we apply null-space preconditioners to matrices arising in a number of applications, each chosen to highlight a specific feature of the proposed preconditioners. We compare their behaviour with Schur-complement based preconditioners

$$\mathcal{P}_{\text{us}} := \begin{bmatrix} A & B^T \\ 0 & -S_0 \end{bmatrix},\ \mathcal{P}_{\text{ls}} := \begin{bmatrix} A & 0 \\ B & -S_0 \end{bmatrix},\ \mathcal{P}_{\text{cs}} := \begin{bmatrix} A & 0 \\ 0 & S_0 \end{bmatrix},\ \mathcal{P}_{\text{cons}} := \begin{bmatrix} A & B^T \\ B & B^TA^{-1}B - S_0 \end{bmatrix}, \tag{4.1}$$

and use a similar approximation to the Schur complement and the null-space matrix in each case.

With Schur complement preconditioners it is often the case that an ideal approximation can be derived by considering the analytic setting of the problem; see, e.g., [30]. It is likely that for certain applications a similar derivation of an ideal preconditioner will be available for the null-space matrix. However, it is beyond the scope of this work to develop corresponding approximations here, as these will be necessarily problem-specific.

Unless otherwise stated we apply right-preconditioned GMRES or non-standard CG, which we terminate when the relative residual satisfies $\|\boldsymbol{r}_k\|_2/\|\boldsymbol{r}_0\|_2 < 10^{-8}$, or when $\min\{n+m, maxit\}$ iterations are reached, where $maxit$ is specified for each example below.

For some problems we approximate N and S by incomplete Cholesky factorizations. These we compute by the MATLAB routine `ichol` with a default drop tolerance of 10^{-2}. For both N and S, if `ichol` fails we reduce the drop tolerance by a factor of 10 until a factorization is computed. The smallest drop tolerance used is 10^{-8}.

4.1. Random saddle point matrices. EXAMPLE 4.1. *Consider the pseudo-random sparse matrix generated by the* MATLAB *code*

```
A = sprandsym(n,0.1,1e-2,1);
B = sprand(m,n,0.5);
K = [A B'; B sparse(m,m)];
```

We take `n = 100` *and* `m = 10, 50, 90`. For these problems $maxit = 200$.

In this example we test values of $n - m$ that are small, moderate, and large in comparison with n. We compare the null-space preconditioners $\mathcal{P}_{cn}$, $\mathcal{P}_{ln}$, $\mathcal{P}_{un}$, $\mathcal{P}_{con}$ and the Schur complement preconditioners in (4.1). For our approximations to the Schur complement and the null-space matrix we take the identity matrix of the appropriate dimension. We apply these preconditioners using MATLAB's inbuilt `GMRES` routine, and report the results in Figure 4.1.

Here, by choosing a weak approximation of the Schur complement and the null-space matrix (namely, the identity matrix), convergence depends entirely on how important this component piece is to the overall approximation. Therefore the null space based preconditioners do well for small $n - m$, the Schur complement based preconditioners do well for $n - m$ close to n, and there is no clear winner in the intermediate case when $n = m/2$. This suggests that null-space preconditioners may be a better choice over Schur complement preconditioners if we have an application where $n - m$ is small, particularly if we do not have a good Schur complement approximation.

It is also useful to compare the different null-space preconditioners with each other, and the different Schur complement preconditioners with each other. In all cases, the constraint, upper, and lower preconditioners take about the same number of iterations, and so which one of these we use will depend on the specific requirements of the application. The central approximations take roughly twice the number of iterations, but are cheaper to apply.

EXAMPLE 4.2. *Consider now the sparse matrix generated by the* MATLAB *code*

```
A = 10*speye(n,n);
B = sprand(m,n,0.5);
K = [A B'; B sparse(m,m)];
```

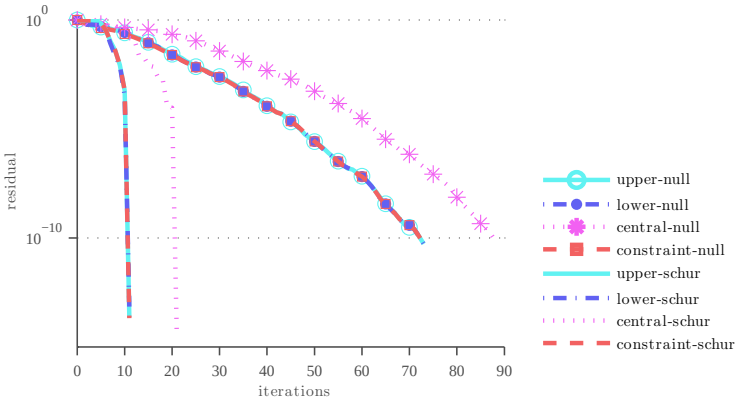
We take `n = 100` *and* `m = 10, 50, 90`. In Example 4.2 we have the same B as in Example 4.1, but instead take a scaled identity matrix for A. We test the same preconditioners that we used in Example 4.1, namely using an identity matrix as the Schur complement/null space approximation. We give the results in Figure 4.2. Again, $maxit = 200$.

Here, in contrast to Example 4.1, the null-space preconditioners perform well for both large and small values of $n - m$. This is because the (1,1) block in $\mathcal{A}$ is a scaled identity matrix here and, as a consequence of Theorem 2.1, the eigenvalues of $\mathcal{P}_{cn}^{-1}\mathcal{A}$ are well clustered. Again, the pattern that the central-based preconditioners take about twice the iterations of the others is in evidence.

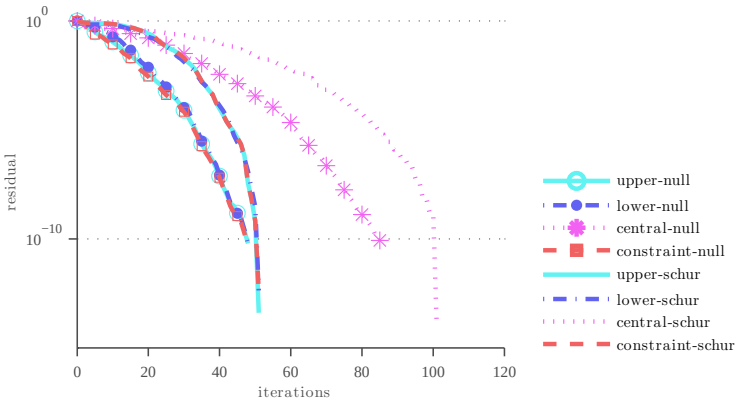
4.2. Optimization and interior point methods. Here we consider quadratic programming problems of the form

$$\begin{aligned} \min \ & \frac{1}{2}\boldsymbol{x}^T H\boldsymbol{x} + \boldsymbol{f}^T\boldsymbol{x} \\ \text{s.t. } & B\boldsymbol{x} = \boldsymbol{g}, \\ & \boldsymbol{x} \geq \underline{0}. \end{aligned} \tag{4.2}$$

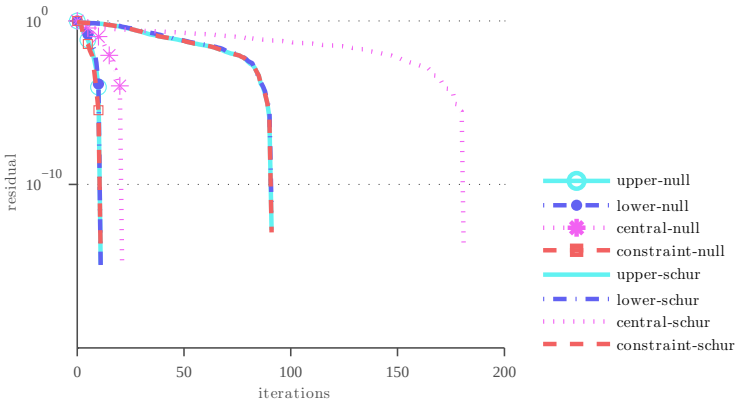
If we solve such a problem using a primal-dual interior point method [49] then at iteration k of the optimization algorithm we must solve a system of the form (1.1),



(a) $m = 10$

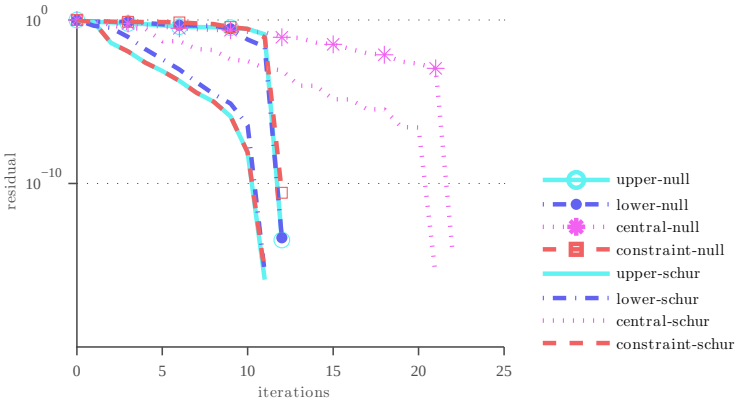


(b) $m = 50$

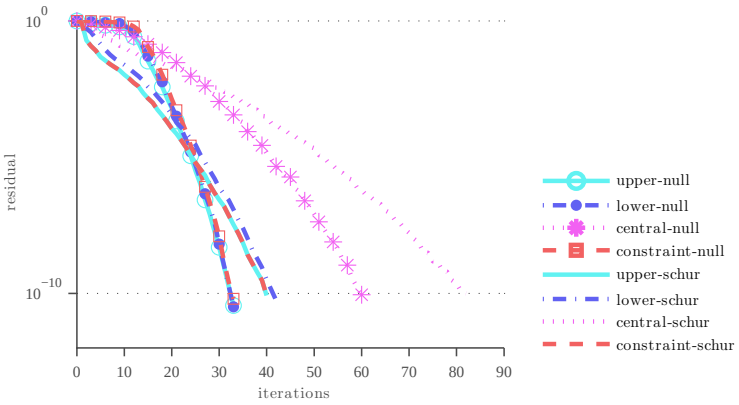


(c) $m = 90$

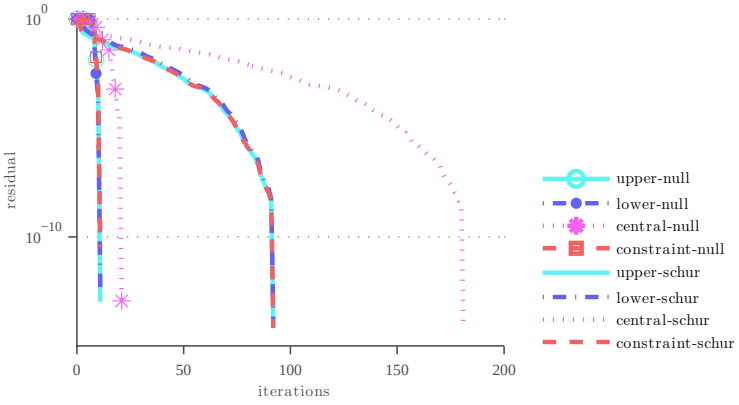
Fig. 4.1: Comparison: Schur complement and null space preconditioners, pseudo-random example from Example 4.1



(a) $m = 10$



(b) $m = 50$



(c) $m = 90$

Fig. 4.2: Comparison: Schur complement and null space preconditioners, pseudo-random example from Example 4.2

Matrix	n	m	Matrix	n	m
AUG3DC	3873	1000	GOULDQP3	699	349
AUG3DCQP	3873	1000	HUES-MOD	10000	2
CONT-050	2597	2401	HUESTIS	10000	2
CONT-100	10197	9801	LASER	1002	1000
CONT-101	10197	10098	LISWET1	10002	10000
CONT-200	40397	39601	LOTSCHD	12	7
CVXQP3_S	100	75	MOSARQP1	2500	700
DPKLO1	133	77	MOSARQP2	900	600
DTOC3	14999	9998	PRIMAL1	325	85
DUAL1	85	1	QPCSTAIR	467	356
DUAL2	96	1	STCQP2	4097	2052
DUAL3	111	1	YAO	2002	2000
DUAL4	75	1			

Table 4.1: Problem sizes, CUTEst set matrices from Example 4.2. Matrices for which A is singular are denoted by an asterisk.

where $A = H + X_k^{-1}Z_k$ for diagonal matrices X_k, Z_k.

In this context it is common to solve the linear system (1.1) by reducing it to the null-space matrix N, which the optimization community refer to as the reduced Hessian [8], [33, Section 16.2]. Since forming the matrix N is expensive, it is common in optimization to approximate this, e.g., by missing out cross terms [11, 32].

In this setting we need to solve a sequence of linear systems as the interior point method converges, but the 'constraint' blocks B do not change. Therefore we may justify the cost of using a direct method such as LUSOL [44], say, to find a basis of B, since we can reuse this splitting over all interior point iterations. Although we do not explore the possibility here, it is also possible to use the interior point method itself to predict an invertible sub-block B_1—see, e.g., Al-Jeiroudi, Gondzio, and Hall [2].

A thorough analysis of these methods, and subsequent development of a domain-specific approximation $\tilde{N}$, is beyond the scope of this work. However, to give a flavour of how we can expect such methods to work we run through some problems from the CUTEst test set [22], comparing standard Schur-complement preconditioners and null-space preconditioners. These problems, and their dimensions, are listed in Table 4.1. We highlight that, as described in a previous section, in optimization it is often important that the inexact solution of this subproblem remains on the constraint manifold. This is a property afforded by constraint preconditioners, and the only true constraint preconditioner tested here is $\mathcal{P}_{con}$.

In our experiments we choose X_k and Z_k so that $X_k^{-1}Z_k = I$, the identity matrix of dimension n; a system of this form may be used in practice to find an initial guess for the interior point method. Additionally, we set $maxit = 1000$.

Our first tests are for the ideal case where we take the exact matrices S or N; these are not practical, but give an idea of the best we can expect the respective method to work in practice. We give the results of these tests in Table 4.2. The fast convergence rates for both the Schur-complement and null-space preconditioners, with the exception of the central-null preconditioner, are to be expected from theoretical spectral results (see Theorems 2.5, 2.6 and 2.8 for the null space preconditioners and [26, 31] for the Schur complement preconditioners). Note that the performance of the central-null preconditioner, in contrast to the other null-space and Schur-complement preconditioners, depends on the eigenvalues of $N^{-1}A_{22}$ (see Theorem 2.1), which are not necessarily clustered. We find that iteration counts are

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
AUG3DC	2	2	3	1	1	2	2	27	1	1
AUG3DCQP	2	2	3	1	1	2	2	27	1	1
CONT-050	2	2	3	1	1	2	2	20	1	1
CONT-100	2	2	3	1	1	2	2	23	1	1
CONT-101	2	4	5	1	1	4	4	30	2	*
CONT-200	2	4	5	1	1	2	8	25	1	1
CVXQP3_S	1	2	2	1	1	2	2	34	1	1
DPKLO1	1	2	2	1	1	2	2	27	1	1
DTOC3	1	2	2	1	1	2	2	8	1	1
DUAL1	2	2	3	1	1	2	2	5	1	1
DUAL2	2	2	3	1	1	2	2	5	1	1
DUAL3	2	2	3	1	1	2	2	5	1	1
DUAL4	2	2	3	1	1	2	2	5	1	1
GOULDQP3	2	2	3	1	1	2	2	27	1	1
HUES-MOD	1	2	2	1	1	2	2	4	1	1
HUESTIS	1	2	2	1	1	2	3	4	2	2
LASER	1	2	2	1	1	2	2	3	1	1
LISWET1	2	4	3	1	2	2	2	4	1	1
LOTSCHD	1	2	2	1	1	2	2	11	1	1
MOSARQP1	2	2	3	1	1	2	2	21	1	1
MOSARQP2	2	2	3	1	1	2	2	19	1	1
PRIMAL1	1	2	2	1	1	2	2	22	1	1
QPCSTAIR	1	2	2	1	1	2	2	31	1	1
STCQP2	1	2	2	1	1	2	2	3	1	1
YAO	2	3	3	1	2	2	2	5	1	1

Table 4.2: Iteration counts for the Schur complement preconditioners with $S_0 = S$ and the null-space preconditioners with $N_0 = N$ for the CUTEst set matrices from Example 4.2. * stands for did not converge after 1000 iterations.

slightly higher for CONT-101 and CONT-200 than the theory predicts. However, for these matrices N and S are quite ill-conditioned.

In a further test we consider the simplest approximation to the matrices S and N, namely the identity matrix of appropriate size. These results are given in Table 4.2. Since the identity is generally a poor approximation of S and N we find that, similarly to Example 4.1, the size of $n-m$ relative to m plays an important role in determining whether the null-space-based preconditioners are more, or less, effective than the Schur-complement-based preconditioners. In particular, when m is large the Schur-complement preconditioned iterative methods do not always converge to the desired tolerance within 1000 iterations but the null-space preconditioners are somewhat more robust. Both the central-null and central-Schur preconditioners tend to require twice as many iterations as the other preconditioners.

Considering now the null-space preconditioners in more detail, we find that the upper-null preconditioner performs better than the lower-null preconditioner for some problems, but the difference tends to be small. The constraint preconditioner $\mathcal{P}_{con}$ also requires higher iteration counts for some problems, and is not effective for DUAL1–DUAL4, since for these problems $\mathcal{P}_{con}$ is very ill-conditioned. More generally, our experiments indicate that this constraint preconditioner may be less effective when the approximation to the null-space matrix is poor, although it does have the benefit of being an exact constraint preconditioner. Additionally, the non-standard inner product CG method can require more iterations than using the lower-, upper- or constraint-preconditioners with GMRES. On the other hand, the CG method uses

short-term recurrences.

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
AUG3DC	35	37	71	35	35	87	88	166	91	100
AUG3DCQP	36	38	73	36	36	87	88	166	91	100
CONT-050	*	*	*	*	*	16	16	30	15	16
CONT-100	*	*	*	*	*	21	21	41	21	21
CONT-101	*	*	*	*	*	21	21	40	31	*
CONT-200	*	*	*	*	*	28	56	55	28	29
CVXQP3_S	82	83	150	83	*	26	26	44	26	29
DPKLO1	51	53	102	51	88	24	24	50	25	30
DTOC3	*	*	*	*	*	6	5	10	6	7
DUAL1	2	2	3	2	1	66	67	71	*	*
DUAL2	2	2	3	2	1	59	63	66	*	81
DUAL3	2	2	3	2	1	59	62	64	*	71
DUAL4	2	2	3	2	1	36	38	39	*	36
GOULDQP3	20	21	39	19	19	40	40	71	41	38
HUES-MOD	3	4	4	3	2	3	3	4	9	2
HUESTIS	3	4	4	3	2	3	3	4	11	3
LASER	65	66	130	65	67	2	2	3	2	1
LISWET1	*	*	*	*	*	3	3	5	4	2
LOTSCHD	7	8	14	7	7	6	6	11	6	5
MOSARQP1	464	467	927	464	550	15	15	29	15	14
MOSARQP2	444	447	889	444	614	17	17	38	17	15
PRIMAL1	77	79	154	77	137	40	41	79	41	71
QPCSTAIR	247	249	490	247	551	51	53	93	53	69
STCQP2	267	269	528	267	615	85	94	95	93	93
YAO	*	*	*	*	*	3	3	5	4	2

Table 4.3: Iteration counts for the Schur complement preconditioners with $S_0 = I$ and the null-space preconditioners with $N_0 = I$ for the CUTEst set matrices from Example 4.2. * stands for did not converge after 1000 iterations.

Finally, we give results (Table 4.4) for the same tests with a more accurate approximation of S or N, namely the incomplete Cholesky factorization described at the start of this section. Generally, using these better approximations of N and S improves the iteration counts for the Schur-complement preconditioners and the null-space preconditioners. Again, the null-space preconditioners are more robust than their Schur-complement counterparts, and for no problems do we see the high iteration counts that the Schur-complement preconditioners give for CONT-100, CONT-101 and COND-200.

4.3. $\mathcal{F}-$matrices. Let $\mathcal{A}$ be a saddle point matrix of the form (1.1) where A is symmetric positive definite and B is a gradient matrix, i.e., B has at most two entries per row, and if there are two entries they sum to zero; we call such a matrix $\mathcal{A}$ an $\mathcal{F}-$matrix [47]. Such matrices arise naturally in, e.g., discretizations of fluid-flow [3], or in electrical networks [45].

Due to the special structure of B it is possible to find an invertible sub-block B_1 without performing any arithmetic—see, e.g., [39, 45]. This property makes $\mathcal{F}-$matrices an ideal candidate for null-space preconditioning. We test our preconditioners for a number of $\mathcal{F}-$matrices[1], listed in Table 4.5. We set $maxit = 1000$.

When we use the cheap, but inaccurate, approximations $S_0 = I$ and $N_0 = I$ (see Table 4.6) the iteration counts can be quite high for all preconditioners. However,

[1]We would like to thank Miroslav Tůma for providing these test matrices.

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
AUG3DC	10	11	21	9	10	16	16	33	16	16
AUG3DCQP	10	11	21	9	9	16	16	33	16	16
CONT-050	85	86	170	85	86	18	18	34	17	17
CONT-100	376	377	752	376	417	40	40	59	39	44
CONT-101	418	419	837	418	536	26	26	48	25	*
CONT-200	*	*	*	*	*	24	57	43	23	23
CVXQP3_S	7	9	14	7	7	6	6	33	5	5
DPKLO1	15	17	30	15	16	8	8	28	7	7
DTOC3	6	9	12	6	7	5	5	10	5	4
DUAL1	2	2	3	1	1	9	8	12	8	8
DUAL2	2	2	3	1	1	8	7	10	7	7
DUAL3	2	2	3	1	1	9	8	11	8	8
DUAL4	2	2	3	1	1	17	16	18	16	16
GOULDQP3	7	7	13	6	6	6	7	27	6	6
HUES-MOD	1	2	2	1	1	8	7	9	7	7
HUESTIS	1	2	2	1	1	8	7	10	7	*
LASER	10	12	20	10	10	2	2	3	1	1
LISWET1	6	6	9	4	6	2	2	4	1	1
LOTSCHD	3	4	6	3	3	4	4	11	3	3
MOSARQP1	25	26	51	25	25	8	7	22	7	7
MOSARQP2	26	28	53	26	26	7	7	19	6	6
PRIMAL1	4	5	8	4	4	13	13	25	12	12
QPCSTAIR	10	11	20	10	10	20	20	40	19	20
STCQP2	13	16	26	13	13	21	21	22	20	20
YAO	5	5	9	4	4	2	2	5	1	1

Table 4.4: Iteration counts for the Schur complement preconditioners and the null-space preconditioners, with incomplete Cholesky preconditioners for S_0 and N_0, for the CUTEst set matrices from Example 4.2. * stands for did not converge after 1000 iterations.

Matrix	n	m	Matrix	n	m
DORT	13360	9607	M3P	2160	1584
DORT2	7515	5477	S3P	270	207
L3P	17280	12384	dan2	63750	46661

Table 4.5: Problem sizes, $\mathcal{F}$-matrices in Example 4.3.

the null-space based preconditioners consistently give lower iteration counts; this can partly be explained by the dimensions of the problems, since in general $n - m$ is significantly smaller than m. As in the previous example, the non-standard CG method with $\mathcal{P}_{ls}$ or $\mathcal{P}_{ln}$ seems to be less robust than right-preconditioned GMRES, while the upper-null preconditioner performs slightly better than the lower-null preconditioner. Similarly to other examples in this section, the central-Schur and central-null preconditioners tend to take twice as many iterations as the other preconditioners. When N_0 and S_0 are replaced by incomplete Cholesky preconditioners, the iteration counts drop for all preconditioners, but the same trends are evident. In particular, the null-space preconditioners are particularly well-suited to these $\mathcal{F}-$matrices.

4.4. University of Florida matrices. Finally, we examine a subset of problems from the University of Florida sparse matrix collection [13] (see Table 4.8). For these problems we set $maxit = 1000$.

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
DORT	*	*	*	*	*	750	763	*	750	*
DORT2	*	*	*	*	*	473	481	946	473	*
L3P	359	360	717	373	375	218	223	441	217	288
M3P	204	205	407	224	207	89	91	177	89	108
S3P	113	114	225	126	115	36	36	65	36	34
dan2	*	*	*	*	*	*	*	*	*	*

Table 4.6: Iteration counts for the Schur complement preconditioners with $S_0 = I$ and the null-space preconditioners with $N_0 = I$ for the $\mathcal{F}$-matrices in Example 4.3. * stands for did not converge after 1000 iterations.

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
DORT	121	121	237	120	125	12	14	32	12	12
DORT2	117	117	233	116	120	8	10	30	8	8
L3P	44	44	87	43	43	16	18	36	15	15
M3P	24	24	47	23	23	12	15	31	11	11
S3P	12	12	23	11	11	10	12	28	9	9
dan2	7	7	13	6	6	9	11	48	8	8

Table 4.7: Iteration counts for the Schur complement preconditioners and the null-space preconditioners, with incomplete Cholesky preconditioners for S_0 and N_0, for the $\mathcal{F}$-matrices in Example 4.3.

We see from Tables 4.9 and 4.10 similar results to those in previous examples. For all problems except qpband, $n - m < m$, so that it is unsurprising that the null-space preconditioners perform better in general. However, even for qpband, the null-space preconditioners are competitive. Similarly to previous examples, the central-null and central-Schur preconditioners require approximately twice as many iterations as the other preconditioners, while the constraint preconditioner $\mathcal{P}_{con}$ is less robust when a poor approximation to N is used.

5. Conclusion. We have presented a new paradigm for preconditioning based on a null-space factorization. By dropping, or approximating, different terms in the null-space factorization, in a similar manner to standard Schur complement preconditioners, we arrived at four different null-space preconditioners.

We have given eigenvalue bounds for these preconditioners, and have shown that the eigenvalues of the upper-null, lower-null and constraint preconditioners are clustered when a good approximation to the null-space matrix can be found. Additionally, two of the preconditioners, although indefinite and non-symmetric, can be applied with a Krylov method with a short term recurrence.

Finally, we investigated the effectiveness of these preconditioners at reducing the number of iterations of Krylov subspace methods. We found that the preconditioners were more robust than equivalent Schur-complement based preconditioners, and were more effective when a reasonable approximation to the null-space matrix was available or when the dimension of $n - m$ was small.

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Matrix	n	m	Matrix	n	m
brainpc2	13807	13800	tuma1	13360	9607
mario001	23130	15304	tuma2	7515	5477
qpband	15000	5000			

Table 4.8: Problem sizes, University of Florida matrices in Example 4.4.

Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
brainpc2	*	*	*	*	*	8	8	13	8	10
mario001	*	*	*	*	*	296	296	571	*	781
qpband	3	3	5	3	2	6	6	9	6	5
tuma1	*	*	*	*	*	744	755	*	*	*
tuma2	*	*	*	*	*	458	465	917	*	*

Table 4.9: Iteration counts for the Schur complement preconditioners with $S_0 = I$ and the null-space preconditioners with $N_0 = I$ for the University of Florida matrices in Example 4.4. * stands for did not converge after 1000 iterations.

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Matrix	$\mathcal{P}_{us}$	$\mathcal{P}_{ls}$	$\mathcal{P}_{cs}$	$\mathcal{P}_{cons}$	$\mathcal{P}_{ls}$ (NSCG)	$\mathcal{P}_{un}$	$\mathcal{P}_{ln}$	$\mathcal{P}_{cn}$	$\mathcal{P}_{conn}$	$\mathcal{P}_{ln}$ (NSCG)
brainpc2	93	121	183	92	*	4	5	7	4	4
mario001	5	6	11	5	5	14	14	33	13	13
qpband	2	2	3	1	1	2	2	7	1	1
tuma1	122	122	240	121	125	13	14	33	12	12
tuma2	115	116	223	115	118	9	10	30	8	9

Table 4.10: Iteration counts for the Schur complement preconditioners and the null-space preconditioners, with incomplete Cholesky preconditioners for S_0 and N_0, for the University of Florida matrices in Example 4.4. * stands for did not converge after 1000 iterations.

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