Stimulated Brillouin Scattering Driven By White Light

B. Brandão and L. O. Silva*

GoLP/Instituto de Plasmas e Fusão Nuclear – Laboratório Associado,

Instituto Superior Técnico, 1049-001 Lisboa, Portugal

J. E. Santos

Department of Applied Mathematics and Theoretical Physics,

Centre for Mathematical Sciences, University of Cambridge,

Wilberforce Road, Cambridge CB3 0WA, U.K.

R. Bingham

Rutherford Appleton Laboratory, Didcot, U.K.

Abstract

A generalized Wigner-Moyal statistical theory of radiation is used to obtain a general dispersion relation for Stimulated Brillouin Scattering (SBS) driven by a spatially stationary radiation field with arbitrary statistics. The monochromatic limit is recovered from our general result, reproducing the classic monochromatic dispersion relation. The behavior of the growth rate of the instability as a simultaneous function of the bandwidth of the pump wave, the intensity of the incident field and the wave number of the scattered wave is further explored by numerically solving the dispersion relation. Our results show that the growth rate of SBS can be reduced by 1/3 for a bandwidth of 0.3 nm, for typical NIF parameters.

*Electronic address: `luis.silva@ist.utl.pt`

I. INTRODUCTION

Due to the increasing attention dedicated to nonlinear and collective effects, the study of parametric instabilities, present at the onset of these scenarios, has a central role in many fields of science [1–4]. Standard methods use a coherent wave description to study this problem, but the externally induced incoherence or the partial coherence of most systems render this approach incomplete.

The use of the Wigner-Moyal statistical theory has proven to be quite powerful in studying this kind of instabilities, with some of the most important developments using this technique carried out in nonlinear optics. With the derivation of a statistical description of a partially incoherent electromagnetic wave propagating in a nonlinear medium [5], it became clear that a stabilization of the modulational instability is possible as a result of an effect similar to Landau damping, and caused by random phase fluctuations of the propagating wave, which is equivalent to the broadening of the Wigner spectrum. In similar studies [6, 7], focusing on the onset of the transverse modulational instability in nonlinear media in the presence of partially incoherent light, the Wigner distribution was once more confirmed as a suitable approach. This faster progress in nonlinear optics is partially justified by the validity of the paraxial wave approximation, which justifies a forward propagating *ansatz* for the evolution of electromagnetic waves in dispersive nonlinear media. In plasma physics, this is clearly a limitation; many critical scenarios in ICF, fast ignition and several applications in laser-plasma and astrophysical scenarios demand a detailed analysis of the backscattered radiation.

The inclusion of bandwidth or incoherence effects in laser driven parametric instabilities has been studied extensively, but a full self-consistent theoretical approach capable of dealing with the multitude of scenarios associated with laser driven parametric instabilities was still lacking. The addition of small random deflections to the phase of a plane wave was shown to significantly suppress the three-wave decay instability [8], which was one of the first suggestions to decrease the laser coherence as a mean to avoid its deleterious effects. The threshold values for some electrostatic instabilities can also be effectively increased either by applying a random amplitude modulation to the laser or by the inclusion of a finite bandwidth of the pump wave [9, 10]. A method for the inclusion of finite bandwidth effects on parametric instabilities, allowing arbitrary fluctuations of any group velocity, has also been developed [11, 12]. As far as the Stimulated Raman Scattering instability is concerned, it became clear that, although it may seriously decollimate a coherent laser beam, laser bandwidth is an effective way to suppress the instability [13].

A statistical description of light can be achieved through the Wigner-Moyal formalism of quantum mechanics, which provides, in its original formulation, a one-mode description of systems ruled by Schrödinger-like equations. In order to address other processes apart from the direct forward scattering, a generalization of Photon Kinetics theory (GPK) was recently developed [14]. This new formulation is completely equivalent to the full Klein-Gordon equation for propagation of light in plasmas, and was readily employed to derive a general dispersion relation for stimulated Raman scattering driven by white light [15]. These results leveraged on previous developments that have considered the Wigner-Moyal description in plasma physics [16, 17] as a way to represent Schrödinger light propagation *i.e.* only in the forward scattering approximation and discarding the role of the backscattered radiation.

In this paper, we focus on the study of the analytical and numerical properties of another white light parametric instability occurring in a plasma, stimulated Brillouin scattering (SBS), where light couples with ion acoustic waves. The full dispersion relation is derived for an arbitrary spatially stationary pump field. The suppression of the growth rate of the instability as a result of the inclusion of bandwidth in the pump wave is qualitatively and quantitatively verified for realistic ICF parameters, and in particular for NIF parameters. This paper is organized as follows. In section II , we employ GPK to derive a general dispersion relation for SBS driven by a spatially stationary field with arbitrary statistics. We perform a detailed analytical study of different regimes of SBS and compare it with classical references dealing with the monochromatic limit of the instability. The whole domain of unstable wave numbers is numerically explored for a wide range of bandwidth choices. Finally, in section III, we state the conclusions.

II. BROADBAND STIMULATED BRILLOUIN SCATTERING

In the following we use normalized units, where length is normalized to c/ω_{p0}, with c the velocity of light in vacuum and $\omega_{p0} = (4\pi e^2 n_{e0}/m_e c^2)^{1/2}$ the electron plasma frequency, time to $1/\omega_{p0}$, mass and absolute charge to those of the electron, respectively, m_e and e, with $e > 0$. The plasma is modeled as an interpenetrating fluid of both electrons and ions, with n_{e0} and n_{i0} their equilibrium (zeroth order) particle densities, respectively. Densities are also normalized to the equilibrium electron density, so we have $n_{e0} = 1$ and $n_{i0} = 1/Z$, where Z is the electric charge of the ions in units of e. As detailed in Ref. [15], we use $\mathbf{a}_p(\mathbf{r}, t) = 2^{-1/2}(\hat{z} + i\hat{y})a_0 \int d\mathbf{k}A(\mathbf{k})\exp[i(\mathbf{k}.\mathbf{r} - (\mathbf{k}^2 + 1)^{1/2}t)]$ as the normalized vector potential of the circularly polarized pump field, $\mathbf{a}_p = e\mathbf{A}_p/m_e c^2$, where $(\mathbf{k}^2 + 1)^{1/2} \equiv \omega(\mathbf{k})$ is the monochromatic dispersion relation in a uniform plasma. We also allow for a stochastic component in the phase of the vector potential $A(\mathbf{k}) = \hat{A}(\mathbf{k})\exp[i\psi(\mathbf{r}, t)]$ such that $\left\langle \mathbf{a}_p^*(\mathbf{r} + \mathbf{y}/2, t).\mathbf{a}_p(\mathbf{r} - \mathbf{y}/2)\right\rangle = a_0^2 m(\mathbf{y})$ is independent of $\mathbf{r}$ with $m(0) = 1$ and $|m(\mathbf{y})|$ is bounded between 0 and 1, which means that the field is spatially stationary. In this section, $\tilde{q}$ denotes the first-order component of a generic quantity q. Unless specifically stated, the same notation for the functions and their Fourier transforms is used, as the argument of such functions (either $(\mathbf{r}, t)$ or $(\mathbf{k}, \omega)$) avoids any confusion. To obtain a dispersion relation for SBS we must couple the typical plasma response to an independently derived driving term, obtained within the GPK framework.

A. Plasma response and driving term

Combining the continuity equation and the force equation for each species and closing the system with an isothermal equation of state, we can readily present, without more details, the low frequency plasma response to the propagation of a light wave $\mathbf{a}_p$, beating with its scattered component $\tilde{\mathbf{a}}$, to produce the ponderomotive force of the laser [1]

$$\left(\frac{\partial^2}{\partial t^2} - 2\tilde{\nu}\partial t - c_S^2\nabla^2\right)\tilde{n} = \frac{Z}{M}\nabla^2\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}], \tag{1}$$

where $c_S \equiv \sqrt{\frac{Z\theta_e}{M}}$ is the ion sound speed, M is the mass of the ions, θ_e is the electron temperature and $\tilde{\nu}$ an integral operator whose Fourier transform is $\nu|\mathbf{k}_L|c_S$ that accounts for the damping mechanisms associated with the ion acoustic waves, namely Landau damping, and where the right hand side of eq.(1) is the term responsible for driving the instability.

In Appendix IV A we perform the detailed derivation of what we denote as the driving term, which describes how the incident radiation is affected by the propagation in our dispersive medium. Although this term is usually described through the wave equation for the vector potential [1, 24], the study of white light parametric instabilities is not possible with this approach. The driving term obtained within the framework of GPK is

$$W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]} = \frac{1}{2}\tilde{n}\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right], \tag{2}$$

where $D_s^{\pm} = \omega_L^2 \mp \left[\mathbf{k}.\mathbf{k}_L - \omega_L\omega\left(\mathbf{k}\mp\frac{\mathbf{k}_L}{2}\right)\right]$ and ω_L $(\mathbf{k}_L)$ represents the instability frequency (wave vector), and where $W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]}$ denotes the Wigner transform of $\mathrm{Re}\left[\mathbf{a}_p.\tilde{\mathbf{a}}\right]$, such that $W_{\mathbf{f}\cdot\mathbf{g}}(\mathbf{k},\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}.\mathbf{y}}\mathbf{f}^*\left(\mathbf{r}+\frac{\mathbf{y}}{2}\right)\cdot\mathbf{g}\left(\mathbf{r}-\frac{\mathbf{y}}{2}\right)d\mathbf{y}$ (f^* denotes the complex conjugate of the function f), and $\rho_0\left(\mathbf{k}\right)$ is the zero-order photon distribution functions, as defined in [15] and Appendix IV A, is given by $\rho_0\left(\mathbf{k}\right) = W_{\mathrm{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}_p}]}$

B. General dispersion relation for Stimulated Brillouin Scattering and classical monochromatic limit

Performing time and space Fourier transforms on the plasma response, eq.(1), ($\partial t \to i\omega_L, \nabla_{\mathbf{r}} \to -i\mathbf{k}_L$) yields

$$\mathcal{F}[\tilde{n}] = \frac{Z}{M}\frac{k_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2}\mathcal{F}\left[\mathrm{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}]\right], \tag{3}$$

while the Fourier transform of the driving term, eq.(49) gives

$$\mathcal{F}\left[W_{\mathrm{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}]}\right] = \frac{1}{2}\mathcal{F}[\tilde{n}]\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right], \tag{4}$$

with $D_s^{\pm} = \omega_L^2 \mp \left[\mathbf{k}.\mathbf{k}_L - \omega_L\omega\left(\mathbf{k}\mp\frac{\mathbf{k}_L}{2}\right)\right]$.

Using one of the properties of the Wigner function [18, 21] *viz.*

$$\int W_{f.g}d\mathbf{k} = f^*g \Rightarrow \int \frac{W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]}}{\mathrm{Re}\left[\mathbf{a}_p.\tilde{\mathbf{a}}\right]}d\mathbf{k} = 1$$

and combining eqs.(3,4) the dispersion relation is given by

$$1 = \frac{\omega_{pi}^2}{2}\frac{\mathbf{k}_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2}\int\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right]d\mathbf{k}, \tag{5}$$

with $\omega_{pi} = \sqrt{Z/M}$ being the ion plasma frequency in normalized units. By making an appropriate change of variables, our general dispersion relation can be written in a more compact form as

$$1 = \frac{\omega_{pi}^2}{2}\frac{\mathbf{k}_L^2}{\omega_L^2 + 2i\nu\omega_L|\mathbf{k}_L|c_S - c_S^2\mathbf{k}_L^2}\int\rho_0(\mathbf{k})\left(\frac{1}{D^+} + \frac{1}{D^-}\right)d\mathbf{k}, \tag{6}$$

with $D^{\pm}(\mathbf{k}) = [\omega(\mathbf{k}) \pm \omega_L]^2 - (\mathbf{k} \pm \mathbf{k}_L)^2 - 1$, the main result of this section. We note that this dispersion relation is valid for all angles and thus can be used to study some of the scenarios recently explored in [22, 23].

We first show that the standard results are readily obtained from this formalism, and thus we explore our general dispersion relation in the more simple and common scenario of a pump plane wave of wave vector $\mathbf{k}_0$; in this case $\rho_0(\mathbf{k}) = a_0^2 \delta(\mathbf{k} - \mathbf{k}_0)$. With the purpose of the following comparisons, we drop the Landau damping contribution from our discussion. The dispersion relation then reduces to

$$1 = \frac{\omega_{pi}^2}{2} \frac{a_0^2 \mathbf{k}_L^2}{\omega_L^2 - c_S^2 \mathbf{k}_L^2} \left\{ \frac{1}{[\omega(\mathbf{k}_0) + \omega_L]^2 - (\mathbf{k}_0 + \mathbf{k}_L)^2 - 1} + \frac{1}{[\omega(\mathbf{k}_0) - \omega_L]^2 - (\mathbf{k}_0 - \mathbf{k}_L)^2 - 1} \right\}. \tag{7}$$

This result recovers the dispersion relation in [1], which considers the case of a pump wave $\mathbf{A}_L = \mathbf{A}_{L0} \cos(\mathbf{k}_0 . \mathbf{r} - \omega_0 t)$, if we account for the difference in polarization and use $\omega_0 = \omega(\mathbf{k}_0)$. All the conclusions in [1], based on this dispersion relation, are then consistent with the predictions of GPK.

C. 1D waterbag zero-order photon distribution function

We now take advantage of the general dispersion relation (6) to examine the role of the bandwidth of the pump wave on SBS. In order to obtain analytical results we consider a bandwidth profile for the pump wave, represented by the one-dimensional waterbag zero-order distribution function,

$$\rho_0(\mathbf{k}) = \frac{a_0^2}{\sigma_1 + \sigma_2} [\theta(k - k_0 + \sigma_1) - \theta(k - k_0 - \sigma_2)], \tag{8}$$

where $\theta(k)$ is the Heaviside function and σ_1 (σ_2) represents the spectral bandwidth to the left (right) of the central wave number, k_0. For this distribution function, the autocorrelation function of the random phase $\psi(x)$ satisfies

$$\left\langle \exp\left[-i\psi\left(x + \frac{y}{2}\right) + i\psi\left(x - \frac{y}{2}\right) \right] \right\rangle = e^{-iy\tilde{\sigma}} \frac{\sin(y\bar{\sigma})}{y\bar{\sigma}}, \tag{9}$$

where $\tilde{\sigma} \equiv (\sigma_2 - \sigma_1)/2$ and $\bar{\sigma} \equiv (\sigma_1 + \sigma_2)/2$. The correlation length of this distribution is $\approx \pi/\sqrt{2}\bar{\sigma}$.

The dispersion relation for the waterbag distribution function of Eq. (8) can be derived (see Appendix IV B for the full derivation) yielding

$$1 = \frac{a_0^2\omega_{pi}^2}{8\bar{\sigma}} \frac{k_L}{\omega_L^2 - c_S^2 k_L^2} \left[\frac{k_L^2}{k_L^2 - \omega_L^2} \log\left(\frac{D_1^- D_2^+}{D_1^+ D_2^-} \right) + \frac{2\omega_L k_L}{\sqrt{Q_0}} (\text{arctanh } b^+ + \text{arctanh } b^-) \right], \tag{10}$$

with $\omega_{0i} = \sqrt{[k_0 + (-1)^i\sigma_i]^2 + 1}$, $D_i^\pm = \omega_L^2 - k_L^2 \pm 2[(k_0 + (-1)^i\sigma_i)k_L - \omega_{0i}\omega_L]$, $Q_0 = (k_L^2 - \omega_L^2)(k_L^2 - \omega_L^2 + 4)$, $Q^\pm = \prod_{i=1}^2 [D_i^\pm + (k_L - \omega_L)(\omega_L \mp 2\omega_{0i})]$ and $b^\pm = 2k_L^2(\omega_L + k_L)\sqrt{Q_0}(2\bar{\sigma} + \omega_{01} - \omega_{02}) / \left[Q^0 k_L^2 - Q^\pm(\omega_L + k_L)^2\right]$.

We are interested in an expression for the maximum growth rate of the Brillouin instability. Analytical results can be obtained in the case where all the photons of the distribution propagate in an underdense medium, which implies that $k_0 + (-1)^i\sigma_i \gg 1$. This also guarantees that $k_0 > \sigma_1$, which assures that $\rho_0(k)$ represents a broadband source of forward propagating photons. From this condition, the approximations $\omega_{0i} \approx k_0 + (-1)^i\sigma_i$ and $b^\pm \approx 0$ are also valid. The dispersion relation (10) can then be approximated by

$$1 = \frac{a_0^2\omega_{pi}^2}{8\bar{\sigma}} \frac{k_L^3}{\omega_L^2 - c_S^2 k_L^2} \frac{1}{k_L^2 - \omega_L^2} \left\{ \ln\left[\frac{2(k_0 - \sigma_1) + (\omega_L + k_L)}{2(k_0 - \sigma_1) - (\omega_L + k_L)} \right] + \ln\left[\frac{2(k_0 + \sigma_2) - (\omega_L + k_L)}{2(k_0 + \sigma_2) + (\omega_L + k_L)} \right] \right\}. \tag{11}$$

The resonance condition for SBS can be expressed as $\omega_L \sim k_L c_S$, with $c_S \ll 1$ [1]. Furthermore, the backscattering regime of stimulated Brillouin scattering (SBBS) is known to provide the highest growth rates [1], so we make one of the terms D_i^+ resonant (corresponding to the contribution of the downshifted photons of the distribution function). By making the D_1^+ term resonant ($D_1^+ = 0 \Rightarrow k_{L_{SBBS}}^m \approx \frac{2}{c_S+1}(k_0 - \sigma_1)$), we are considering the contribution of the photons of the lowest wave number, while with D_2^+ ($D_2^+ = 0 \Rightarrow k_{L_{SBBS}}^M \approx \frac{2}{c_S+1}(k_0 + \sigma_2)$) we are searching for those of the highest wave number. This means that k_L is of the order of k_0 and the range of unstable wave numbers is then given by

$$k_L \in \left[\frac{2}{c_S + 1}(k_0 - \sigma_1), \frac{2}{c_S + 1}(k_0 + \sigma_2) \right]. \tag{12}$$

We consider the upper limit case (as we will later see, the growth rate of the instability is within the same order of magnitude for the whole range of unstable wave numbers) and we note that $\omega_L \sim k_L c_S$, with $c_S \ll 1$, implies that both $\omega_L \ll k_L$ and $\omega_L \ll k_0$. To determine the growth rate of the instability, we now write $\omega = k_L c_S + i\Gamma$, where Γ is the (real valued) growth rate of the instability and $|\Gamma| \ll k_L c_S$ (which corresponds to the weak field limit).

The dispersion relation (11) can then be rewritten in the form $1 = A \ln B$, where

$$A = \frac{a_0^2 \omega_{pi}^2}{8\bar{\sigma}} \frac{k_L^3}{\omega_L^2 - c_S^2 k_L^2} \frac{1}{k_L^2 - \omega_L^2} \approx \frac{a_0^2 \omega_{pi}^2 (k_0 + \sigma_2)}{4i(\sigma_1 + \sigma_2)\Gamma c_S k_L}, \tag{13}$$

$$B = \frac{2(k_0 - \sigma_1) + (\omega_L + k_L)}{2(k_0 - \sigma_1) - (\omega_L + k_L)} \frac{2(k_0 + \sigma_2) - (\omega_L + k_L)}{2(k_0 + \sigma_2) + (\omega_L + k_L)} \approx \frac{2k_0 - \sigma_1 + \sigma_2}{2(\sigma_1 + \sigma_2) + i\Gamma} \frac{i\Gamma}{2(k_0 + \sigma_2)}. \tag{14}$$

We now take the imaginary part of the dispersion relation, working with a real valued Γ and using the fact that, for a complex $Z = \rho e^{i\theta}$, with real ρ and θ, then $\ln Z = \ln \rho + i\theta$, we obtain

$$\Gamma c_S k_L = \frac{a_0^2 \omega_{pi}^2 (k_0 + \sigma_2)}{4(\sigma_1 + \sigma_2)} \arctan \left[\frac{2(\sigma_1 + \sigma_2)}{\Gamma} \right]. \tag{15}$$

From eq.(15), we first examine the well-know limits for the growth rate of SBS corresponding to the plane wave limit [1], which is obtained in the limits $\sigma_{1,2} \to 0$. In this limit, and from eq.(12), the lower bound of the unstable region of wavenumber $k_{L_{SBBS}}^m \approx \frac{2}{c_S+1}(k_0 - \sigma_1)$, and the upper bound $k_{L_{SBBS}}^M \approx \frac{2}{c_S+1}(k_0 + \sigma_2)$, which implies that, in the monochromatic limit, $k_{L_{SBBS}}^{m,pw} = k_{L_{SBBS}}^{M,pw} \equiv k_{L_{SBBS}}^{pw} = \frac{2}{c_S+1} k_0 \approx 2k_0(1 - c_S) = 2k_0 - 2\omega_0 c_S$, because $\omega_0 \equiv \omega_{01}(\sigma_1 = 0) = \omega_{02}(\sigma_2 = 0) \approx k_0$, where we have also considered that the ion acoustic velocity is much smaller than the speed of light. This recovers the result in ref. [1] for the wave number that maximizes the growth rate.

To recover the maximum growth rate in the weak field (wf) scenario from eq.(15) in the monochromatic scenario, we consider the limit $\sigma_1, \sigma_2 \to 0$ and make use of $\arctan x \sim x$ when $x \to 0$ to obtain

$$\Gamma_{SBBSwf}^{pw,\max} = \frac{a_0 \omega_{pi}}{2\sqrt{c_S}}, \tag{16}$$

which also coincides with the monochromatic result in [1], also taking into account the already discussed correction for the polarization.

We now examine the main consequences of the general case associated with eq.(15), and work in the limit $(\sigma_1 + \sigma_2) \gg \Gamma$, such that the approximation $\arctan x \sim \frac{\pi}{2} - \frac{1}{x}$ when $x \to \infty$ can be used, yielding

$$\Gamma_{SBBSwf}^{\max} = \frac{\pi a_0^2 \omega_{pi}^2}{16 c_S k_0} \frac{k_0 + \sigma_2}{\sigma_1 + \sigma_2} \frac{1}{1 + \frac{a_0^2 \omega_{pi}^2}{16 c_S k_0} \frac{k_0 + \sigma_2}{(\sigma_1 + \sigma_2)^2}}. \tag{17}$$

The corresponding saturation value for large bandwidths is

$$\Gamma_{SBBSwf}^{\max,sat} = \frac{\pi a_0^2 \omega_{pi}^2}{16 c_S k_0}. \tag{18}$$

We observe that eq.(15) is also valid in the strong field limit, i.e., in cases such that $|\omega_L| \gg k_L c_S$. Again we consider the underdense limit, as in the weak field case, so that the range of unstable wave numbers still holds, and $k_L \approx 2(k_0 + \sigma_2)$ for the wave number for maximum growth, which means that k_L is still of the order of k_0 We also neglect $|\omega_L|$ when compared to k_0, which establishes the scale $k_L c_S \ll |\omega_L| \ll k_L \approx k_0$, consistent with $c_S \ll 1$. This means that we are not neglecting the magnitude of the imaginary part of ω_L when compared to its real part. With these assumptions, we now expand $\omega_L = \alpha + i\beta$, with real α and β, so that the dispersion relation yields (see Appendix IV C)

$$\omega_L = \left(\frac{k_L a_0^2 \omega_{pi}^2}{2}\right)^{1/3} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right), \tag{19}$$

which is, once more, the result presented in [1] with the usual polarization considerations. The maximum growth rate in the strong field limit is then

$$\Gamma_{SBBSsf}^{pw,\max} = \frac{\sqrt{3}}{2} \left(\frac{k_L a_0^2 \omega_{pi}^2}{2}\right)^{1/3}. \tag{20}$$

D. Numerical solution of the full dispersion relation

The numerical solution of the full dispersion relation (10) illustrates the evolution of the strength of the instability as a function of the bandwidth and the wave number of the scattered wave itself. In Fig. 1 we show the maximum growth rate of the Brillouin instability as a function of the bandwidth parameter, σ_2, with σ_1 kept fixed. As expected, Eq. (17) is a good approximation to the complete solution for large bandwidths. The difference between the approximate and the numerical solutions increases as bandwidth (σ_2) decreases. As σ_2 approaches k_0, the results start to agree and Eq. (17) can be used. As we approach the monochromatic limit, only the numerical solution should be considered, as the choice of $\sigma_1 = 0.1k_0$ still accounts for a considerable difference between $\Gamma_{\max}(\sigma_2 = 0)$ and the maximum growth rate in the monochromatic limit, $\Gamma_{\max}(\sigma_1, \sigma_2 = 0)$, expressed by Eq. (16). It is clear that a bandwidth as small as 10% can still cause a reduction of the growth rate of the instability by a factor of more than 100, which is quite significant.

Figure 2 shows the same results for the case of $\sigma_2 \approx 0$. As in the previous scenario, the approximation of Eq. (17) agrees with the numerical solution as σ_2 approaches k_0. The monochromatic limit of Eq. (16) can also be confirmed at the origin of the plot, as expected.

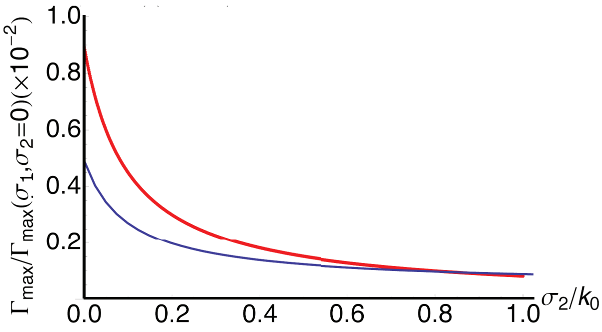

FIG. 1: Maximum growth rate of SBBS as a function of bandwidth - $a_0 = 0.1$, $k_0 = 80.0$, $\sigma_1 = 0.1k_0$, $c_S = 0.01$, $\omega_{pi} = 0.1$. Red line - numerical solution; blue line - analytical limit for $\Gamma \ll (\sigma_1 + \sigma_2)$ of Eq. (17)

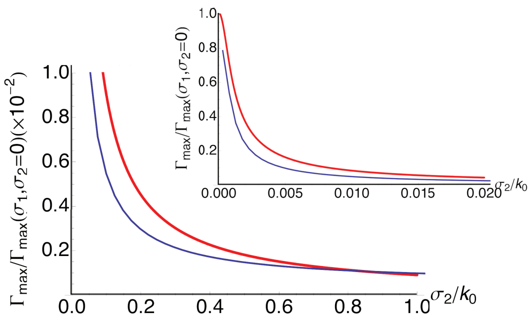

FIG. 2: Maximum growth rate of SBBS as a function of bandwidth - $a_0 = 0.1$, $k_0 = 80.0$, $\sigma_1 \approx 0$, $c_S = 0.01$, $\omega_{pi} = 0.1$. Red line - numerical solution; blue line - analytical limit for $\Gamma \ll (\sigma_1 + \sigma_2)$. In the inset the growth rate is shown for the regime where $\sigma_2/k_0 \ll 1$

We now study the behavior of the growth rate of the instability as a function of the wave number of the scattered wave. In Fig. 3, we plot the growth rate for a set of bandwidths and express it as a function of the wave number of the instability. We observe a very good agreement with the range of unstable wave numbers predicted by Eq. (12): the lower limit does not depend on σ_2 and remains fixed as we increase bandwidth, while the upper bound linearly grows as we increase the value of σ_2.

We should also note that the flat structure observed indicates that the magnitude of the growth rate is within the same order for the full range of unstable wave numbers, meaning that the instability can grow on a wide range of wave numbers and lead to significant

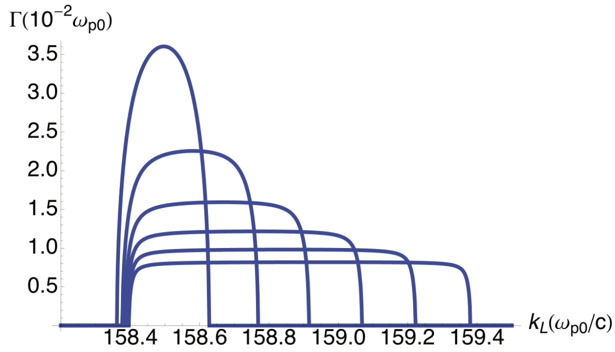

FIG. 3: Growth rate of SBBS as a function of the wave number of the scattered wave for different bandwidths of the waterbag (from the left to the right: $\sigma_2 = 0.1k_0, 0.2k_0, 0.3k_0, 0.4k_0, 0.5k_0, 0.6k_0$, with $a_0 = 0.1$, $k_0 = 80.0$, $\sigma_1 \approx 0$, $c_S = 0.01$ and $\omega_{pi} = 0.1$)

phenomena of turbulence. This is valid for relatively small bandwidths, as it is clear for $\sigma_2 > 0.1k_0$.

A two-dimensional representation of the variation of the growth rate of SBBS as a continuous function of both the bandwidth of the pump and the instability wave number is presented in Fig. 4, providing a global picture of the instability. As expected, we observe a strong dependence of the instability on the bandwidth of the radiation used as a driver. For a bandwidth of just 1% in k_0, the instability is already reduced to 10% of the plane wave limit, which justifies the use of bandwidth as a means of significantly reducing the strength of the instability. For fixed k_0, a_0 and σ_1, the growth rate for SBBS scales with $\propto 1/\sigma_2$, similarly to other distribution functions (e.g., asymmetric Lorentzian or Gaussian distribution of photons [15]). Both the wave number for maximum growth and the upper bound of the unstable wave numbers domain depend linearly on σ_2.

III. CONCLUSIONS

A general dispersion relation for stimulated Brillouin scattering, driven by a partially coherent pump field, has been derived, using the GPK formalism [15] which is formally equivalent to the coupling of the full wave equation with the plasma fluid equations. After having retrieved the monochromatic limit of the instability, we have used a one-dimensional waterbag profile for the incident field to model broadband effects. The analysis has revealed

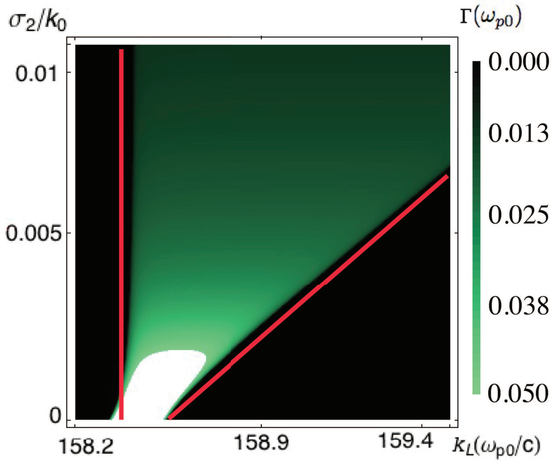

FIG. 4: Growth rate of SBBS as a function of the wave number of the scattered wave and the bandwidth of the waterbag: $a_0 = 0.1$, $k_0 = 80.0$, $\sigma_1 \approx 0$, $c_S = 0.01$, $\omega_{pi} = 0.1$ (2D representation). The red lines illustrate the theoretical range of unstable wave numbers, as predicted from eq.(12).

a growth rate dependence on the coherence width σ of the radiation field which scales with $1/\sigma$ for backscattering, typical of 3-wave processes [15]. Furthermore, a numerical analysis of the growth rate of the instability has been performed as a function of the intensity of the incident field and the wave number of the scattered wave, confirming the theoretical predictions for the domain of unstable wave numbers.

The possibility of an accurate estimate of the growth rate of the instability, for a wide range of parameters, not only stresses the important role of bandwidth in the suppression of the instability, but also suggests a comparison with particle-in-cell simulations [27], to be presented elsewhere.

In this paper, we have focused on the backscattering regime of SBS, but the general dispersion relation we have derived (Eq. (6)) may be readily applied to different regimes. A detailed comparison with previous models for SBS pumped by a wave with finite bandwidth [22–26] can then be performed [27] and will be presented in the future, along with particle-in-cell simulations of parametric instabilities pumped by broadband radiation [27]. A prediction of the suppression of SBS by the experimental mechanism of polarization smoothing [28] can also be be readily obtained through GPK and will be presented elsewhere.

IV. APPENDIX

For the sake of completeness the most important derivations of this paper are presented in this appendix.

A. Derivation of the driving term using Generalized Photon Kinetics

GPK, first proposed in [14, 15], can deal with the two mode problem of electromagnetic wave propagation, describing the vector potential of the radiation field $\mathbf{a}$ by two auxiliary fields $\phi, \chi = (\mathbf{a} \pm i\partial_t \mathbf{a})/2$, thus allowing for a formally equivalent representation of the full wave equation in terms of two coupled Schrödinger equations for the auxiliary fields. With the introduction of four real phase-space densities

$$W_0 = W_{\phi\phi} - W_{\chi\chi} \tag{21}$$

$$W_1 = 2\mathrm{Re}[W_{\phi\chi}] \tag{22}$$

$$W_2 = 2\mathrm{Im}[W_{\phi\chi}] \tag{23}$$

$$W_3 = W_{\phi\phi} + W_{\chi\chi} \tag{24}$$

and the usual definition for the Wigner transform $W_{\mathbf{f}.\mathbf{g}}(\mathbf{k}, \mathbf{r}, t) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\mathbf{k}.\mathbf{y}} \mathbf{f}^* \left(\mathbf{r} + \frac{\mathbf{y}}{2}\right) . \mathbf{g} \left(\mathbf{r} - \frac{\mathbf{y}}{2}\right) d\mathbf{y}$ as in Refs. [18–21], the coupled equations for ϕ, χ (and, therefore, the complete Klein-Gordon equation) are shown [14] to be equivalent to the following set of transport equations for the W_i, $i = 0, ..., 3$:

$$\partial_t W_0 + \hat{\mathcal{L}}(W_2 + W_3) = 0 \tag{25}$$

$$\partial_t W_1 - \hat{\mathcal{G}}(W_2 + W_3) - 2W_2 = 0 \tag{26}$$

$$\partial_t W_2 - \hat{\mathcal{L}} W_0 + \hat{\mathcal{G}} W_1 + 2W_1 = 0 \tag{27}$$

$$\partial_t W_3 + \hat{\mathcal{L}} W_0 - \hat{\mathcal{G}} W_1 = 0 \tag{28}$$

with the definitions for the operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{G}}$:

$$\hat{\mathcal{L}} \equiv \mathbf{k}.\vec{\nabla}_{\mathbf{r}} - n \sin\left(\frac{1}{2} \overleftarrow{\nabla}_{\mathbf{r}} . \overrightarrow{\nabla}_{\mathbf{k}}\right) \tag{29}$$

$$\hat{\mathcal{G}} \equiv \left(\mathbf{k}^2 - \frac{\vec{\nabla}_{\mathbf{r}}^2}{4}\right) + n \cos\left(\frac{1}{2} \overleftarrow{\nabla}_{\mathbf{r}} . \overrightarrow{\nabla}_{\mathbf{k}}\right) \tag{30}$$

where the arrows denote the direction of the operator and the trigonometric functions represent the equivalent series expansion of the operators.

We first evaluate the zeroth order terms of each W_i, $i = 0, ..., 3$, so we use $\mathbf{a} = \mathbf{a}_p$. It can be easily shown that

$$W^{(0)}_{\phi\phi} = \frac{\rho_0(\mathbf{k})}{4}[1 + \omega^2(\mathbf{k}) + 2\omega(\mathbf{k})] \tag{31}$$

$$W^{(0)}_{\chi\chi} = \frac{\rho_0(\mathbf{k})}{4}[1 + \omega^2(\mathbf{k}) - 2\omega(\mathbf{k})] \tag{32}$$

$$W^{(0)}_{\phi\chi} = \frac{\rho_0(\mathbf{k})}{4}[1 - \omega^2(\mathbf{k})] = -\frac{\rho_0(\mathbf{k})}{4}\mathbf{k}^2 \tag{33}$$

where $\rho_0(\mathbf{k}) \equiv W_{\mathbf{a}_p.\mathbf{a}_p}$ can be interpreted as the equilibrium distribution function of the photons. We can immediately write

$$W^{(0)}_0 = W^{(0)}_{\phi\phi} - W^{(0)}_{\chi\chi} = \rho_0(\mathbf{k})\omega(\mathbf{k}) \tag{34}$$

$$W^{(0)}_1 = 2\mathrm{Im}\left[W^{(0)}_{\phi\chi}\right] = 0 \tag{35}$$

$$W^{(0)}_2 = 2\mathrm{Re}\left[W^{(0)}_{\phi\chi}\right] = -\frac{\rho_0(\mathbf{k})}{2}\mathbf{k}^2 \tag{36}$$

$$W^{(0)}_3 = W^{(0)}_{\phi\phi} + W^{(0)}_{\chi\chi} = \rho_0(\mathbf{k})\left(1 + \frac{\mathbf{k}^2}{2}\right) \tag{37}$$

where we have taken into account the real valueness conditions of the Wigner function [18–21].

We now explore the first order perturbative term of the transport equations (25,26,27,28), since the zeroth order terms either return trivial results or the dispersion relation for plane circularly polarized monochromatic waves in a uniform plasma, $\omega(\mathbf{k}) = (\mathbf{k}^2 + 1)^{1/2}$, as expected. The first transport equation (25) yields, in first order,

$$\partial_t \tilde{W}_0 + \mathbf{k}.\vec{\nabla}_{\mathbf{r}}(\tilde{W}_2 + \tilde{W}_3) - \tilde{n}\sin\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{r}}.\overrightarrow{\nabla}_{\mathbf{k}}\right)\rho_0(\mathbf{k}) = 0 \tag{38}$$

We now perform time and space Fourier transforms ($\partial t \to i\omega_L, \nabla_{\mathbf{r}} \to -i\mathbf{k}_L$), leading to

$$i\omega_L \tilde{W}_0 - i\mathbf{k}.\mathbf{k}_L(\tilde{W}_2 + \tilde{W}_3) + \tilde{n}\sin\left(\frac{i}{2}\mathbf{k}_L.\vec{\nabla}_k\right)\rho_0(\mathbf{k}) = 0 \tag{39}$$

We note that we can write $\sin\hat{\mathcal{A}} = \frac{e^{i\hat{\mathcal{A}}} - e^{-i\hat{\mathcal{A}}}}{2i}$, for any operator $\hat{\mathcal{A}}$. Similarly, $\cos\hat{\mathcal{A}} = \frac{e^{i\hat{\mathcal{A}}} + e^{-i\hat{\mathcal{A}}}}{2}$. Making use of these relations, we have

$$e^{\mathbf{A}.\nabla_k} f(\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(\mathbf{A}.\nabla_k)^n}{n!} f(\mathbf{k}) = f(\mathbf{k} + \mathbf{A}) \tag{40}$$

The first transport equation can then be reduced to

$$\omega_L \tilde{W}_0 - \mathbf{k}.\mathbf{k}_L(\tilde{W}_2 + \tilde{W}_3) - \tilde{n}\frac{\rho_0\left(\mathbf{k} - \frac{\mathbf{k}_L}{2}\right) - \rho_0\left(\mathbf{k} + \frac{\mathbf{k}_L}{2}\right)}{2} = 0 \tag{41}$$

We proceed analogously with the other three transport equations (26,27,28), leading to a system of four independent first order equations for the four variables $\tilde{W}_i$. We also note that

$$W_2 + W_3 = W_{\phi\phi} + W_{\chi\chi} + 2\text{Re}[W_{\phi\chi}] = W_{\mathbf{a}.\mathbf{a}} \tag{42}$$

In zeroth order, as expected, it is straigthforward to obtain

$$W_2^{(0)} + W_3^{(0)} = W_{\mathbf{a}_p.\mathbf{a}_p} = \rho_0(\mathbf{k}) \tag{43}$$

while in first order

$$\tilde{W}_2 + \tilde{W}_3 = W_{\mathbf{a}_p.\tilde{\mathbf{a}}} + W_{\tilde{\mathbf{a}}.\mathbf{a}_p} = 2W_{\mathbf{a}_p.\tilde{\mathbf{a}}} \tag{44}$$

where we have used the simmetry property of the Wigner distribution function that can be immediately derived from the fact that the Wigner distribution function is real valued *i.e.* $W_{\mathbf{f}\cdot\mathbf{g}} = W_{\mathbf{g}\cdot\mathbf{f}}$.

We are also only interested in a real valued electron density, so we take the real part of the right-hand side of the plasma response equation. Similarly, we write

$$\tilde{W}_2 + \tilde{W}_3 = 2W_{\text{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}]} \tag{45}$$

We solve eq. (45) together with the four independent equations for each $\tilde{W}_i$. The calculations are a bit lengthy but straightforward and yield

$$W_{\text{Re}[\mathbf{a}_p\cdot\tilde{\mathbf{a}}]} = \frac{1}{2}\tilde{n}\left[\frac{\rho_0\left(\mathbf{k} + \frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k} - \frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right] \tag{46}$$

with

$$\frac{1}{D_s^{\mp}} = \frac{1 \pm \frac{2\mathbf{k}.\mathbf{k}_L}{\omega_L^2} \pm \frac{2\omega\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{\omega_L}}{\omega_L^2 - 4\mathbf{k}_L^2 - \mathbf{k}_L^2 + 4\frac{(\mathbf{k}.\mathbf{k}_L)^2}{\omega_L^2} - 4} \tag{47}$$

The expression for $D_s^{\mp}$ can be greatly simplified to

$$D_s^{\pm} = \frac{(\omega_L^2 \mp 2\mathbf{k}\mathbf{k}_L)^2 - \left[2\omega_L\omega\left(\mathbf{k} \mp \frac{\mathbf{k}_L}{2}\right)\right]^2}{\omega_L^2 \mp 2\mathbf{k}.\mathbf{k}_L \mp 2\omega_L\omega\left(\mathbf{k} + \frac{\mathbf{k}_L}{2}\right)}, \tag{48}$$

providing the final expression for the driving term:

$$W_{\mathrm{Re}[\mathbf{a}_p.\tilde{\mathbf{a}}]} = \frac{1}{2}\tilde{n}\left[\frac{\rho_0\left(\mathbf{k}+\frac{\mathbf{k}_L}{2}\right)}{D_s^-} + \frac{\rho_0\left(\mathbf{k}-\frac{\mathbf{k}_L}{2}\right)}{D_s^+}\right], \tag{49}$$

with

$$D_s^{\pm} = \omega_L^2 \mp \left[\mathbf{k}.\mathbf{k}_L - \omega_L\omega\left(\mathbf{k}\mp\frac{\mathbf{k}_L}{2}\right)\right]. \tag{50}$$

B. Dispersion relation derivation for the one-dimensional waterbag distribution function

Let us consider the zero order distribution of photons $\rho_0(\mathbf{k}) = \frac{a_0^2}{\sigma_1+\sigma_2}[\theta(k-k_0+\sigma_1) - \theta(k-k_0-\sigma_2)]$, where $\theta(k)$ is the Heaviside function, in the generalized dispersion relation (6), which gives

$$1 = \frac{\omega_{pi}^2}{2}\frac{k_L^2}{\omega_L^2 - c_S^2 k_L^2}\frac{a_0^2}{\sigma_1+\sigma_2}\int_{k_0-\sigma_1}^{k_0+\sigma_2}\left[\frac{1}{D^+(k)} + \frac{1}{D^-(k)}\right]dk \tag{51}$$

with $D^{\pm}(k) = [\omega(k) \pm \omega_L]^2 - (k \pm k_L)^2 - 1$, and $c_S = \sqrt{\frac{Z\theta_e}{M}}$.

The integral of (51) can be performed through the substitution $y = k - k_0$, so

$$\int_{k_0-\sigma_1}^{k_0+\sigma_2}\left[\frac{1}{D^+(k)} + \frac{1}{D^-(k)}\right]dk =$$

$$\frac{1}{2k_L}\int_{-\sigma_1}^{\sigma_2}\left[\frac{1}{y + \frac{2k_0k_L - (k_L^2-\omega_L^2)}{2k_L} - \frac{\omega_L}{k_L}\sqrt{(y+k_0)^2+1}} - \frac{1}{y + \frac{2k_0k_L + (k_L^2-\omega_L^2)}{2k_L} - \frac{\omega_L}{k_L}\sqrt{(y+k_0)^2+1}}\right]dy$$

We are then left with

$$I^{\pm} \equiv \int_{-\sigma_1}^{\sigma_2}\frac{1}{y + b^{\pm} + k\sqrt{(y+a)^2+1}}dy \tag{52}$$

with $b^{\pm} \equiv k_0 \pm (\omega_L^2 - k_L^2)/(2k_L)$, $k \equiv -\omega_L/k_L$ and $a \equiv k_0$. To solve this integral, we start with the substitution $\sqrt{(y+a)^2+1} = y + t$, from which we get that $y = \frac{a^2+1-t^2}{2(t-a)}$, $\frac{dy}{dt} = \frac{-4t(t-a)-2(a^2+1-t^2)}{4(t-a)^2}$ and $\sqrt{(y+a)^2+1} = \frac{1+(t-a)^2}{2(t-a)}$. The integral then becomes

$$\int_{\sqrt{(a-\sigma_1)^2+1}+\sigma_1}^{\sqrt{(a+\sigma_2)^2+1}-\sigma_2}\frac{-4t(t-a) - 2(a^2+1-t^2)}{4(t-a)^2\left[\frac{a^2+1-t^2}{2(t-a)} + b^{\pm} + k\frac{1+(t-a)^2}{2(t-a)}\right]}dt \tag{53}$$

One last transformation is performed, $t - a = z$, which yields for the integral

$$-\int_{(\sigma_1-a)-\sqrt{(a-\sigma_1)^2+1}}^{-(\sigma_2+a)+\sqrt{(a+\sigma_2)^2+1}}\frac{1+z^2}{z[(k+1) + 2(b^{\pm}-a)z + (k-1)z^2]}dz \tag{54}$$

The problem has now been reduced to the computation of an integral of a rational function, for which a primitive can be explicitly obtained

$$I^{\pm} = -\left\{\frac{2(a-b^{\pm})k}{(k^2-1)\sqrt{(k^2-1)-(a-b^{\pm})^2}}\arctan\left[\frac{-(a-b^{\pm})+(k-1)z}{\sqrt{(k^2-1)-(a-b^{\pm})^2}}\right]+\right.$$
$$\left.+\frac{\ln z}{k+1}+\frac{\ln[(k+1)+2(b^{\pm}-a)z+(k-1)z^2]}{k^2-1}\right\}_{\sqrt{(a-\sigma_1)^2+1}-(a-\sigma_1)}^{\sqrt{(a+\sigma_2)^2+1}-(a+\sigma_2)} \tag{55}$$

If we define the quantity $Q^0 \equiv (k_L^2-\omega_L^2)(k_L^2-\omega_L^2+4)$ and recall the property $\operatorname{arctanh}(x) = -i\arctan(ix)$, we can rewrite $I^{\pm}$ as

$$I^{\pm} = \left\{\mp\frac{2\omega_L k_L}{\sqrt{Q^0}}\operatorname{arctanh}\left[\frac{\pm(k_L^2-\omega_L^2)+2(k_L+\omega_L)z}{\sqrt{Q^0}}\right]+\frac{k_L}{\omega_L-k_L}\ln z+\right.$$
$$\left.+\frac{k_L^2}{k_L^2-\omega_L^2}\ln[(k_L-\omega_L)\mp((k_L^2-\omega_L^2)z-(k_L+\omega_L)z^2]\right\}_{L_1}^{L_2} \tag{56}$$

where $L_1 \equiv \sqrt{(k_0-\sigma_1)^2+1}-(k_0-\sigma_1)$ and $L_2 \equiv \sqrt{(k_0+\sigma_2)^2+1}-(k_0+\sigma_2)$.

We now study the terms of the integral one by one. The second term $(I_2^{\pm})$ may be neglected since the contributions for the dispersion relation exactly cancel (the term does not depend on $b^{\pm} \Rightarrow I_2^+ = I_2^-$). As for the third term, we write the argument of the logarithm, with $s=\pm 1$, as

$$-(k_L+\omega_L)z^2 - s(k_L^2-\omega_L^2)z + (k_L-\omega_L) = -(k_L+\omega_L)(z-z_{01})(z-z_{02}) \tag{57}$$

where z_{01} and z_{02} are the roots of the argument and $s=+1$ for b^+ (first contribution) and $s=-1$ for b^- (second contribution). So we have

$$z_{01,2} = -\frac{s(k_L^2-\omega_L^2)\pm\sqrt{Q^0}}{2(k_L+\omega_L)} \tag{58}$$

The third contribution to the dispersion relation is of the form $\frac{k_L^2}{k_L^2-\omega_L^2}\ln D$, where

$$D \equiv \frac{\left[Z_2+\frac{(k_L^2-\omega_L^2)+\sqrt{Q^0}}{2(k_L+\omega_L)}\right]\left[Z_2+\frac{(k_L^2-\omega_L^2)-\sqrt{Q^0}}{2(k_L+\omega_L)}\right]}{\left[Z_1+\frac{(k_L^2-\omega_L^2)+\sqrt{Q^0}}{2(k_L+\omega_L)}\right]\left[Z_1+\frac{(k_L^2-\omega_L^2)-\sqrt{Q^0}}{2(k_L+\omega_L)}\right]}\times$$
$$\times\frac{\left[Z_1-\frac{(k_L^2-\omega_L^2)-\sqrt{Q^0}}{2(k_L+\omega_L)}\right]\left[Z_1-\frac{(k_L^2-\omega_L^2)+\sqrt{Q^0}}{2(k_L+\omega_L)}\right]}{\left[Z_2-\frac{(k_L^2-\omega_L^2)-\sqrt{Q^0}}{2(k_L+\omega_L)}\right]\left[Z_2-\frac{(k_L^2-\omega_L^2)+\sqrt{Q^0}}{2(k_L+\omega_L)}\right]} \tag{59}$$

where $Z_i \equiv \omega_{0i} - [k_0 + (-1)\sigma_i]$. We focus on each fraction individually and write them as in the following example

$$\frac{2[\omega_{02} - (k_0 + \sigma_2)](k_L + \omega_L) + (k_L^2 - \omega_L^2) + \sqrt{Q^0}}{2[\omega_{02} - (k_0 + \sigma_2)](k_L + \omega_L) - (k_L^2 - \omega_L^2) - \sqrt{Q^0}} \equiv \frac{A_1 + B_1}{A_2 + B_2} \tag{60}$$

where $A_{1,2} \equiv \mp(\omega_L^2 - k_L^2) + 2\omega_{02}\omega_L - 2(k_0 + \sigma_2)k_L$ and $B_{1,2} \equiv 2\omega_{02}k_L - 2(k_0 + \sigma_2)\omega_L \pm \sqrt{Q^0}$. It can now be easily shown that $A_1A_2 = B_1B_2 \iff \frac{A_1+B_1}{A_2+B_2} = \frac{A_1}{B_2}$, such that

$$\begin{aligned}\frac{2[\omega_{02} - (k_0 + \sigma_2)](k_L + \omega_L) + (k_L^2 - \omega_L^2) + \sqrt{Q^0}}{2[\omega_{02} - (k_0 + \sigma_2)](k_L + \omega_L) - (k_L^2 - \omega_L^2) - \sqrt{Q^0}} &= \\ = \frac{-(\omega_L^2 - k_L^2) + 2\omega_{02}\omega_L - 2(k_0 + \sigma_2)k_L}{2\omega_{02}k_L - 2(k_0 + \sigma_2)\omega_L - \sqrt{Q^0}}\end{aligned} \tag{61}$$

The second fraction may be written as $\frac{A_1+B_2}{A_2+B_1} = \frac{B_2}{A_2}$, so the product of the first two fractions becomes

$$\frac{A_1}{B_2}\frac{B_2}{A_2} \equiv -\frac{D_2^+}{D_2^-} \tag{62}$$

where $D_2^\pm = \omega_L^2 - k_L^2 \pm 2[(k_0 + \sigma_2)k_L - \omega_{02}\omega_L]$. Proceeding similarly with the second group of two fractions, the total contribution to the dispersion relation is

$$I_2^+ - I_2^- = \frac{k_L^2}{k_L^2 - \omega_L^2} \log\left(\frac{D_1^- D_2^+}{D_1^+ D_2^-}\right) \tag{63}$$

where $D_i^\pm \equiv \omega_L^2 - k_L^2 \pm 2[(k_0 + (-1)^i\sigma_i)k_L - \omega_{0i}\omega_L]$.

Finally, the first contribution is the sum of two terms of the form

$$\begin{aligned}\mp\frac{2\omega_L k_L}{\sqrt{Q^0}}\Bigg\{&\text{arctanh}\left[\frac{\pm(k_L^2 - \omega_L^2) + 2(k_L + \omega_L)[\omega_{02} - (k_0 + \sigma_2)]}{\sqrt{Q^0}}\right] - \\ &- \text{arctanh}\left[\frac{\pm(k_L^2 - \omega_L^2) + 2(k_L + \omega_L)[\omega_{01} - (k_0 - \sigma_1)]}{\sqrt{Q^0}}\right]\Bigg\}\end{aligned} \tag{64}$$

We make use of the property $\text{arctanh}(x) - \text{arctanh}(y) = \text{arctanh}\left(\frac{x-y}{1-xy}\right)$ and write $\bar{\sigma} = \frac{\sigma_1+\sigma_2}{2}$, so the contribution becomes

$$I_3^+ - I_3^- = \frac{2\omega_L k_L}{\sqrt{Q_0}}(\text{arctanh } b^+ + \text{arctanh } b^-) \tag{65}$$

with $b^\pm = \frac{2k_L^2(\omega_L+k_L)\sqrt{Q_0}(2\bar{\sigma}+\omega_{01}-\omega_{02})}{Q^0k_L^2 - Q^\pm(\omega_L+k_L)^2}$ and $Q^\pm = \prod_{i=1}^2[D_i^\pm + (k_L - \omega_L)(\omega_L \mp 2\omega_{0i})]$.

Collecting all the terms, we get the final dispersion relation for the waterbag zero-order photon distribution

$$1 = \frac{a_0^2\omega_{pi}^2}{8\bar{\sigma}}\frac{k_L}{\omega_L^2 - c_S^2 k_L^2}\left[\frac{k_L^2}{k_L^2-\omega_L^2}\log\left(\frac{D_1^- D_2^+}{D_1^+ D_2^-}\right) + \right. \\ \left. + \frac{2\omega_L k_L}{\sqrt{Q_0}}(\text{arctanh } b^+ + \text{arctanh } b^-)\right] \tag{66}$$

with $\omega_{0i} = \sqrt{[k_0 + (-1)^i\sigma_i]^2+1}$, $D_i^\pm = \omega_L^2 - k_L^2 \pm 2[(k_0 + (-1)^i\sigma_i)k_L - \omega_{0i}\omega_L]$, $Q_0 = (k_L^2 - \omega_L^2)(k_L^2 - \omega_L^2 + 4)$, $Q^\pm = \prod_{i=1}^2 [D_i^\pm + (k_L - \omega_L)(\omega_L \mp 2\omega_{0i})]$ and $b^\pm = 2k_L^2(\omega_L + k_L)\sqrt{Q_0}(2\bar{\sigma} + \omega_{01} - \omega_{02})/\left[Q^0 k_L^2 - Q^\pm(\omega_L + k_L)^2\right]$.

C. Derivation of the growth rate in the strong field limit

Applying the expansion $\omega_L = \alpha + i\beta$, for real α and β, to the complete dispersion relation of Eq. (10), we obtain

$$1 = \frac{a_0^2\omega_{pi}^2}{4(\sigma_1+\sigma_2)}\frac{k_L}{\alpha^2-\beta^2+i2\alpha\beta}\ln\left\{\frac{2k_0 - \sigma_1 + \sigma_2}{2(k_0+\sigma_2)\left[(\alpha + 2(\sigma_1+\sigma_2))^2 + \beta^2\right]}\cdot \right. \\ \left. \cdot\left[\alpha^2+\beta^2+2\alpha(\sigma_1+\sigma_2) + i2\beta(\sigma_1+\sigma_2)\right]\right\} \tag{67}$$

It is necessary both the real and imaginary parts of this equation, from which we obtain the following system of equations

$$2\alpha\beta = \frac{a_0^2\omega_{pi}^2}{4(\sigma_1+\sigma_2)}k_L \arctan\left[\frac{2\beta(\sigma_1+\sigma_2)}{\alpha^2+\beta^2+2\alpha(\sigma_1+\sigma_2)}\right] \tag{68}$$

$$\alpha^2 - \beta^2 = \frac{a_0^2\omega_{pi}^2}{4(\sigma_1+\sigma_2)}k_L \ln\left\{\frac{2k_0-\sigma_1+\sigma_2}{2(k_0+\sigma_2)}\cdot \right. \\ \left. \cdot\frac{\left[(\alpha^2+\beta^2+2\alpha(\sigma_1+\sigma_2))^2 + (2\beta(\sigma_1+\sigma_2))^2\right]^{1/2}}{[(\alpha+2(\sigma_1+\sigma_2))^2+\beta^2]}\right\} \tag{69}$$

These equations can be numerically solved for α and β to obtain the maximum growth rate $\Gamma = \mathrm{Im}(\omega_L) = \beta$. However, we focus on the plane wave limit and analytically derive the maximum growth rate of SBBS, for which we have a classical result [1]. The equation for the imaginary part becomes

$$\alpha(\alpha^2+\beta^2) = \frac{a_0^2\omega_{pi}^2}{2}k_0 \tag{70}$$

where we have used $\arctan x \sim x$ when $x \to 0$. The equation for the real part is more complicated and we work under the conditions $\sigma_1 = 0$ and $\sigma_2 \to 0$. Neglecting terms of

$\mathcal{O}(\sigma_2^2)$ in the arguments of the logarithms, the following approximation for the equation is valid

$$\beta^2 - \alpha^2 \approx \frac{a_0^2 \omega_{pi}^2}{4} \frac{4k_0\alpha + \alpha^2 + \beta^2}{\alpha^2 + \beta^2} \tag{71}$$

where we have used the expansion $\ln(1+x) \sim x$ for $x \to 0$. Plugging the result for the imaginary part into this last equation, we obtain

$$\beta = \sqrt{3}\alpha \tag{72}$$

Using the equation for the imaginary part again, we get

$$\alpha = \frac{1}{2}\left(\frac{k_L a_0^2 \omega_{pi}^2}{2}\right)^{1/3} \tag{73}$$

and ω_L can finally be written as

$$\omega_L = \left(\frac{k_L a_0^2 \omega_{pi}^2}{2}\right)^{1/3} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \tag{74}$$

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