Impedance of Ultrarelativistic Charged Distributions in Tapering Geometries

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Abstract

We develop a scheme for obtaining the impedance of a gradually tapered, axisymmetric geometry containing a bunch of arbitrary profile travelling at the speed of light parallel to the axis of the taper. Coordinate-free expressions for Maxwell's equations are 2+2-split in a coordinate system adapted to the particle beam and the taper and, using an asymptotic expansion for a gradual taper, a coupled hierarchy of Poisson equations is obtained. Applications of the scheme are presented.

INTRODUCTION

The design of accelerator components such as collimators relies on understanding the consquences of passing an ultrarelativistic charged beam through a waveguide with a gradual taper. This is currently studied using a combination of experiment and computer simulation. However, various analytical methods have also been developed to estimate impedances (see, for example, [1]). Most recently, Stupakov [2] developed a process for evaluating the impedance up to the second order of iteration for low frequency beams travelling at $v = c$ in perfectly conducting waveguides of arbitrary cross-section. We shall use a similar method, restricted to axially symmetric confining geometries [3]. However, our approach, using auxiliary potentials, enables us to relax Stupakov's low-frequency condition and produce a hierarchy of equations that can be solved to arbitrary order.

MAXWELL EQUATIONS AND BOUNDARY CONDITIONS

The spacetime metric g is given in cylindrical polar coordinates by[1]

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}z \otimes \mathrm{d}z + \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta \tag{1}$$

and the transverse, cross-sectional domain $\mathcal{D}$ at fixed t and z has the induced metric

$$g_\perp = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta \tag{2}$$

In spacetime, the Hodge map and exterior derivative are denoted $\star$ and d, and in the transverse domain, they are denoted by $\#_\perp$ and $\mathrm{d}_\perp$. The transverse co-derivative $\delta_\perp$ is defined as

$$\delta_\perp = \#_\perp^{-1} \mathrm{d}_\perp \#_\perp \eta \tag{3}$$

[1]We work in the MKS system, with units in which the speed of light $c = 1$.

where $\eta\omega = (-1)^p \omega$ for any p-form ω and $\#_\perp 1 = r\mathrm{d}r \wedge \mathrm{d}\theta$.
The source, moving in the positive z-direction at the speed of light, has charge density ρ and 4-velocity field

$$V = \partial_t + \partial_z \tag{4}$$

The vacuum Maxwell equations for the spacetime 2-form F are given by

$$\mathrm{d}F = 0, \quad \mathrm{d} \star F = -\frac{\rho}{\varepsilon_0} \star \tilde{V} \tag{5}$$

where $\tilde{V} = g(V, -)$ and ε_0 is the permittivity of free space. In terms of new co-ordinates

$$u := z - t, \quad \zeta := z \tag{6}$$

the metric and volume 4-form $\star 1 = \mathrm{d}t \wedge \mathrm{d}z \wedge \#_\perp 1$ become

$$g = \mathrm{d}\zeta \otimes \mathrm{d}u + \mathrm{d}u \otimes \mathrm{d}\zeta - \mathrm{d}u \otimes \mathrm{d}u + g_\perp \tag{7}$$

$$\star 1 = \mathrm{d}\zeta \wedge \mathrm{d}u \wedge \#_\perp 1 \tag{8}$$

and the velocity 1-form $\tilde{V}$ and its Hodge dual are

$$\tilde{V} = \mathrm{d}u, \quad \star\tilde{V} = \mathrm{d}u \wedge \#_\perp 1 \tag{9}$$

Taking the exterior derivative of the second Maxwell equation implies that the charge density is independent of ζ and can thus be written $\rho(r, \theta, u)$. One may uniquely express F in terms of 0-forms $\Phi(r, \theta, \zeta, u)$ and $\Psi(r, \theta, \zeta, u)$, and 1-forms $\alpha_\perp(r, \theta, \zeta, u)$ and $\beta_\perp(r, \theta, \zeta, u)$ which are independent of $\mathrm{d}\zeta$ and $\mathrm{d}u$:

$$F = \Phi \mathrm{d}\zeta \wedge \mathrm{d}u + \mathrm{d}u \wedge \alpha_\perp + \mathrm{d}\zeta \wedge \beta_\perp + \Psi \#_\perp 1 \tag{10}$$

Furthermore, one may write [4]

$$\alpha_\perp = \mathrm{d}_\perp A + \#_\perp \mathrm{d}_\perp a, \quad \beta_\perp = \mathrm{d}_\perp B + \#_\perp \mathrm{d}_\perp b \tag{11}$$

for 0-forms $A(r, \theta, \zeta, u)$, $a(r, \theta, \zeta, u)$, $B(r, \theta, \zeta, u)$ and $b(r, \theta, \zeta, u)$ provided A and B vanish on the boundary $\partial\mathcal{D}$. This condition is compatible with the perfectly conducting boundary conditions that will be imposed on F below.
Without loss of generality, it proves expedient to re-write the form of F in terms of six new fields W, X, $\mathcal{H}^B$, $\mathcal{H}^b$, $\mathcal{H}^\Phi$ and $\mathcal{H}^\varphi$ that will facilitate our subsequent analysis;

$$\begin{aligned}
A &= \partial_u W + \partial_\zeta W - \mathcal{H}^B, \quad B = \mathcal{H}^B - \partial_\zeta W \\
\Phi &= \partial_u \mathcal{H}^\Phi + \partial_u \mathcal{H}^B + \partial_\zeta \mathcal{H}^B - 2\partial^2_{u\zeta} W - \partial^2_{\zeta\zeta} W \\
a &= \partial_u X, \quad b = \partial_\zeta X - \mathcal{H}^\varphi \\
\Psi &= \partial_\zeta \mathcal{H}^\varphi + \partial_u \mathcal{H}^\varphi + \mathcal{H}^b - 2\partial^2_{u\zeta} X - \partial^2_{\zeta\zeta} X
\end{aligned} \tag{12}$$

Thus, the Maxwell equations then reduce to the following relations:

$$\begin{aligned}
&\delta_\perp \mathrm{d}_\perp \mathcal{H}^B = 0 \\
&\mathrm{d}_\perp \mathcal{H}^b = \#_\perp \mathrm{d}_\perp \left(\partial_u \mathcal{H}^B\right), \quad \mathrm{d}_\perp \mathcal{H}^\varphi = \#_\perp \mathrm{d}_\perp \mathcal{H}^\Phi \\
&\delta_\perp \mathrm{d}_\perp W - 2\partial^2_{u\zeta} W - \partial^2_{\zeta\zeta} W \\
&\qquad + \partial_u \mathcal{H}^\Phi + \partial_u \mathcal{H}^B + \partial_\zeta \mathcal{H}^B = P(r,\theta,u) \\
&\delta_\perp \mathrm{d}_\perp X - 2\partial^2_{u\zeta} X - \partial^2_{\zeta\zeta} X + \partial_\zeta \mathcal{H}^\varphi + \partial_u \mathcal{H}^\varphi + \mathcal{H}^b = 0
\end{aligned} \tag{13}$$

where $\partial_u P(r,\theta,u) = \frac{\rho(r,\theta,u)}{\varepsilon_0}$. The second equation in (13) implies the harmonic equations

$$\delta_\perp \mathrm{d}_\perp \mathcal{H}^b = \delta_\perp \mathrm{d}_\perp \mathcal{H}^\varphi = \delta_\perp \mathrm{d}_\perp \mathcal{H}^\Phi = 0 \tag{14}$$

The waveguide wall is the spacelike hypersurface

$$f := r - R(\zeta) = 0 \tag{15}$$

for some smooth function $R(\zeta)$. We assume a perfectly conducting boundary condition for F:

$$\mathrm{d}f \wedge F = 0 \quad \text{at} \quad f = 0 \tag{16}$$

Equation (16) can be satisfied by setting

$$W = \partial_r X = 0, \quad \mathcal{H}^B = \partial_\zeta W, \quad \mathcal{H}^\Phi = -R'(\zeta)\frac{1}{r}\partial_\theta X \tag{17}$$

on the boundary $f = 0$.

GRADUALLY TAPERING WAVEGUIDE

Consider first a regular cylindrical waveguide with constant radius $R(\zeta) = R_0$. As $\partial_\zeta \rho = 0$, the source and confining geometry are both symmetric with respect to translations in the ∂_ζ direction. The simplest solution to the Maxwell system (13) with the boundary conditions (17) is then

$$X_0 = \mathcal{H}^b_0 = \mathcal{H}^B_0 = \mathcal{H}^\Phi_0 = \mathcal{H}^\varphi_0 = \partial_\zeta W_0 = 0 \tag{18}$$

$$\delta_\perp \mathrm{d}_\perp W_0 = P(r,\theta,u) \tag{19}$$

with $W_0 = 0$ on the boundary.

A waveguide is defined to be gradually tapering if

$$f := r - \check{R}(\epsilon\zeta) = 0 \tag{20}$$

where ϵ is a small, dimensionless parameter. The fields will then vary slowly with ζ. Introduce a 'slow' longitudinal coordinate

$$s = \epsilon\zeta \tag{21}$$

and rewrite all the potentials in terms of s, using the notation

$$\chi(r,\theta,\zeta,u) = \check{\chi}(r,\theta,s,u) \tag{22}$$

where $\check{\chi} \in \left\{\check{W}, \check{X}, \check{\mathcal{H}}^B, \check{\mathcal{H}}^b, \check{\mathcal{H}}^\Phi, \check{\mathcal{H}}^\varphi\right\}$. Express the potentials in the form of asymptotic series in ϵ:

$$\check{\chi} = \sum_{n=0}^{\infty} \epsilon^n \check{\chi}_n \tag{23}$$

Note $\partial_\zeta \chi = \epsilon \check{\chi}'$ (where, from now on, a prime denotes differentiation with respect to s). The Maxwell equations (13) with boundary conditions (17) decouple to yield a hierarchical set of 2-dimensional Laplace and Poisson equations for every order n, and the boundary conditions on $\check{\mathcal{H}}^B_n$ and $\check{\mathcal{H}}^\Phi_n$ depend on $(n-1)$-order potentials. This leads to a straightforward procedure for calculating the potentials order-by-order. For $n = 0$, the only non-zero potential is $\hat{W}_0$ which is a solution to $\delta_\perp \mathrm{d}_\perp W_0 = P(r,\theta,u)$ and vanishes at $r = R(s)$.

For every subsequent order of n:

1. Calculate the harmonic potential $\check{\mathcal{H}}^B_n$ by solving the 2-dimensional Laplace equation

$$\delta_\perp \mathrm{d}_\perp \check{\mathcal{H}}^B_n = 0 \tag{24}$$

subject to the boundary condition[2]

$$\check{\mathcal{H}}^B_n = \check{W}'_{n-1} \quad \text{at } r = \check{R}(s) \tag{25}$$

2. Calculate $\check{\mathcal{H}}^b_n$ from [3]

$$\mathrm{d}_\perp \check{\mathcal{H}}^b_n = \partial_u \#_\perp \mathrm{d}_\perp \check{\mathcal{H}}^B_n \tag{26}$$

3. Calculate the harmonic potential $\check{\mathcal{H}}^\Phi_n$ by solving the 2-dimensional Laplace equation

$$\delta_\perp \mathrm{d}_\perp \check{\mathcal{H}}^\Phi_n = 0 \tag{27}$$

subject to $\check{\mathcal{H}}^\Phi_n = -\check{R}'(s)\frac{1}{r}\partial_\theta \check{X}_{n-1}$ at $r = \check{R}(s)$

4. Calculate $\check{\mathcal{H}}^\varphi_n$ from

$$\mathrm{d}_\perp \check{\mathcal{H}}^\varphi_n = \#_\perp \mathrm{d}_\perp \check{\mathcal{H}}^\Phi_n \tag{28}$$

5. Calculate the potential W_n by solving the 2-dimensional Poisson equation

$$\begin{aligned}
\delta_\perp \mathrm{d}_\perp \check{W}_n = \check{W}''_{n-2} + 2\partial_u \check{W}'_{n-1} \\
- \partial_u \check{\mathcal{H}}^\Phi_n - \partial_u \check{\mathcal{H}}^B_n - \check{\mathcal{H}}^{B\prime}_{n-1}
\end{aligned} \tag{29}$$

where $\check{W}_n$ vanishes at $r = \check{R}(s)$.

6. Calculate the potential $\check{X}_n$ by solving the 2-dimensional Poisson equation

$$\delta_\perp \mathrm{d}_\perp \check{X}_n = \check{X}''_{n-2} + 2\partial_u \check{X}'_{n-1} - \check{\mathcal{H}}^b_n - \partial_u \check{\mathcal{H}}^\varphi_n - \check{\mathcal{H}}^{\varphi\prime}_{n-1} \tag{30}$$

with $\partial_r \check{X}_n = 0$ at $r = \check{R}(s)$

[2] Throughout this section, we are dealing with the *transverse* Laplacian $\delta_\perp \mathrm{d}_\perp$. When considering the boundary conditions, s can thus be treated as a parameter.

[3] As $\check{\mathcal{H}}^B_n$ is harmonic, the converse of Poincaré's Lemma guarantees that a solution exists to (26). $\check{\mathcal{H}}^b_n$ is thus defined up to arbitrary functions of s and u. These are subsequently constrained to zero by the boundary condition on $\hat{X}_n$. By an analogous argument, a unique value for $\mathcal{H}^\varphi$ can be obtained from $\mathcal{H}^\Phi$ using (28).

EXAMPLE

The method can be used to replicate and extend the longitudinal impedance[4] calculation in [2] for a harmonic Fourier component of a transverse delta-function beam offset from the central axis. In our notation, the source term and impedance formula are

$$\rho_\omega(r,\theta,u) = \lambda_\omega e^{i\omega u}\frac{1}{r}\delta(r-r_0)\delta(\theta) \tag{31}$$

$$Z_\parallel(\omega,r,\theta,u) = -Z_0\frac{\varepsilon_0}{\lambda_\omega}\int_{-\infty}^{\infty} e^{-i\omega u}\Phi \mathrm{d}\zeta \tag{32}$$

$$= -Z_0\frac{\varepsilon_0}{\lambda_\omega}\frac{1}{\epsilon}\int_{-\infty}^{\infty} e^{-i\omega u}\check{\Phi}\mathrm{d}s \tag{33}$$

where λ_ω is the linear charge density, Z_0 is the impedance of free space and Φ is given by (12). First, $\check{W}_0$ is obtained by solving $\delta_\perp \mathrm{d}_\perp \check{W}_0 = -\frac{i}{\omega}\rho_\omega(r,\theta,u)$ subject to Dirichlet boundary condition at $r=\check{R}(s)$. The solution is

$$\check{W}_0 = -\frac{i}{\omega}p(u)\Bigg\{\ln\left(\frac{r^2r_0^2}{\check{R}(s)^2}+\check{R}(s)^2-2rr_0\cos\theta\right) - \ln\left(r^2+r_0^2-2rr_0\cos\theta\right)\Bigg\} \tag{34}$$

where $p(u) := \frac{\lambda_\omega e^{i\omega u}}{4\pi\varepsilon_0}$. Furthermore,

$$\check{W}_0' = -\frac{2i}{\omega}p(u)\frac{\check{R}'(s)}{\check{R}(s)}\left\{1+2\sum_{m=1}^{\infty}\Upsilon_m(r,\theta,s)\right\} \tag{35}$$

where $\Upsilon_m(r,\theta,s) := \left(\frac{r_0 r}{\check{R}(s)^2}\right)^m \cos m\theta$. Evaluating the potentials according to the procedure in the previous section gives $\check{\mathcal{H}}_1^B = W_0'$, $\check{\mathcal{H}}_1^\Phi = \check{\mathcal{H}}_1^\varphi = 0$ and

$$\check{\mathcal{H}}_1^b = 4p(u)\frac{\check{R}'(s)}{\check{R}(s)}\sum_{m=1}^{\infty}\left(\frac{r_0 r}{\check{R}(s)^2}\right)^m \sin m\theta \tag{36}$$

$$\check{W}_1 = \frac{p(u)}{2}\frac{\check{R}'(s)}{\check{R}(s)}\left(\check{R}(s)^2-r^2\right)\left[1+\sum_{m=1}^{\infty}\frac{2\Upsilon_m(r,\theta,s)}{1+m}\right] \tag{37}$$

$$\check{X}_1 = p(u)\frac{\check{R}'(s)}{\check{R}(s)}\sum_{m=1}^{\infty}\frac{1}{1+m}\left(\frac{r_0 r}{\check{R}(s)^2}\right)^m \times\left(r^2-\frac{m+2}{m}\check{R}(s)^2\right)\sin m\theta \tag{38}$$

$$\check{\mathcal{H}}_2^B = p(u)\check{R}'(s)^2\left[1+\sum_{m=1}^{\infty}\frac{2\Upsilon_m(r,\theta,s)}{1+m}\right] \tag{39}$$

$$\check{\mathcal{H}}_2^\Phi = 2p(u)\check{R}'(s)^2\sum_{m=1}^{\infty}\frac{\Upsilon_m(r,\theta,s)}{1+m} \tag{40}$$

As can be seen from the equation for Φ in (12), $\check{\mathcal{H}}_2^b$, $\check{\mathcal{H}}_2^\varphi$, $\check{W}_2$ and $\check{X}_2$ are not required in order to evaluate the impedance to second order. The longitudinal electric field at this approximation is

$$\check{\Phi} = p(u)\Bigg\{2\epsilon\frac{\check{R}'(s)}{\check{R}(s)}\left[1+2\sum_{m=1}^{\infty}\Upsilon_m(r,\theta,s)\right] + i\omega\epsilon^2\Bigg[\left(\frac{\check{R}'(s)}{\check{R}(s)}\left(r^2-\check{R}(s)^2\right)\left[1+\sum_{m=1}^{\infty}\frac{2\Upsilon_m(r,\theta,s)}{1+m}\right]\right)' + \check{R}'(s)^2\left(1+4\sum_{m=1}^{\infty}\frac{\Upsilon_m(r,\theta,s)}{1+m}\right)\Bigg]\Bigg\} \tag{41}$$

The longitudinal impedance follows from (33). If the waveguide aproaches constant radii R_1 as $s\to-\infty$ and R_2 as $s\to\infty$, then $R'(s)=0$ at $s=\pm\infty$ and the second line of (41) will not contribute to the integral. Thus, to this approximation,

$$Z_\parallel = Z_0\frac{\varepsilon_0}{4\pi\lambda_\omega}\Bigg\{2\ln\frac{R_1}{R_2} - i\omega\epsilon\int_{-\infty}^{\infty}\check{R}'(s)^2\mathrm{d}s + \sum_{m=1}^{\infty}\left(\Upsilon_m\Big|_{R(s)=R_1}^{R(s)=R_2} - \frac{4i\omega\epsilon}{1+m}\int_{-\infty}^{\infty}\Upsilon_m\check{R}'(s)^2\mathrm{d}s\right)\Bigg\} \tag{42}$$

After changing variable from s to ζ and truncating the series at $m=1$, the second-order impedance (42) is identical to the tapered cylinder result in [2]. Evaluating $\check{W}_2$ and $\check{X}_2$ and repeating the procedure of the previous section for $n=3,4,\ldots$ yields higher order correction terms. The third order correction turns out to be zero for an asymptotically cylindrical pipe. The fourth order correction is

$$Z_{\parallel 4} = \frac{Z_0\varepsilon_0}{4\pi\lambda_\omega}i\omega\epsilon^3\int_{-\infty}^{\infty}(\Lambda_1\check{R}'(s)^4+\Lambda_2\check{R}''(s)^2\check{R}(s)^2)\mathrm{d}s \tag{43}$$

where

$$\Lambda_1 = \frac{5}{24}+\sum_{m=1}^{\infty}\frac{\Upsilon_m\kappa_{1m}}{3}\Big(2m^2+6m+1 - \omega^2(4m-3)\kappa_{2m}\check{R}(s)^2\Big)$$

$$\Lambda_2 = \frac{3}{24}-\frac{\omega^2}{12}\check{R}(s)^2+\sum_{m=1}^{\infty}\Upsilon_m\kappa_{1m}\left(1-\omega^2\kappa_{2m}\check{R}(s)^2\right)$$

$$\kappa_{1m} = \frac{2}{m(m+1)(m+2)},\quad \kappa_{2m} = \frac{3m^2+8m+6}{m(m+1)^2(m+3)}$$

while the fifth order contribution is zero.

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