

# Irrotational Momentum Fluctuations Conditioning the Quantum Nature of Physical Processes

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Starting from a simple classical framework and employing some stochastic concepts, the basic ingredients of the quantum formalism are recovered. It has been shown that the traditional axiomatic structure of quantum mechanics can be rebuilt, so that the quantum mechanical framework resembles to a large extent that of the classical statistical mechanics and hydrodynamics. The main assumption used here is the existence of a random irrotational component in the classical momentum. Various basic elements of the quantum formalism (calculation of expectation values, the Heisenberg uncertainty principle, the correspondence principle) are recovered by applying traditional techniques, borrowed from classical statistical mechanics.

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## I. INTRODUCTION

One of the most intriguing achievement of the 20-th century physics is the foundation of quantum mechanics and its basic tool the Schrödinger equation. It is one of the most studied equation in contemporary physics both from mathematical point of view, as well as from the perspective of the enormous number of its important applications. Various links have been proposed between classical and quantum picture, in order to overcome the main difficulties due to the difference of formalisms, and to better understand their interplay and possible connection.

A formal approach is represented by the Wigner-Weyl transformation [1, 2], based on the reformulation of quantum mechanics into phase space by means of a continuous map. Although having some drawbacks, such as the fact that the Wigner distribution function is not strictly positive, this approach has proved an important tool in many areas of quantum physics and chemistry. In particular, the semi-classical limit of quantum mechanics can be recovered by using the Wigner-Weyl transformation.

While the Wigner-Weyl approach can be considered as an extension of quantum mechanics in a classical domain (phase space), the Koopman-von Neumann approach [3–5] is a reformulation of classical mechanics into the Hilbert-space language. This quantum-like theory, although being always classical has the advantage that it can be directly compared to quantum mechanics, at least in terms of the underlying formalism.

The two guidelines of development mentioned above are actually the two sides of the same subject, namely to find a formal mathematical language capable to handle both classical and quantum mechanics. There exists a

third attempt fundamentally different compared to those, which is known as the Bohm-Fényes-Nelson approach [6–11]. This approach is centered on the physical footings and interpretation of quantum mechanics. The basic idea is that quantum phenomena are conditioned by stochastic effects, which take place in a classical framework, so that the notion of equations of motion and hence the notion of trajectory (although a random one) remains valid. The Planck's constant  $\hbar$  plays now the role of a measure of the strength of the stochastic effects. This is an interesting point of view exhibiting a number of remarkable properties and links to the theory of deformation quantization. Many articles [12–15] are devoted to the study of the relation between the phase space representation of quantum states (quasi-distributions) in the deformation quantization method, and the Bohm distributions in phase space in the framework of the Bohm-Fényes-Nelson approach.

Recently, it has been shown [16] that an exact uncertainty principle can be formulated, which provides the key argument in the transition from the dynamical description of a classical ensemble to that of a quantum ensemble. Another interesting derivation of the equations of nonrelativistic quantum mechanics is based on the use of the principle of minimum Fisher information [17].

The point of view pursued in the present paper has been formulated previously in a different context [18]. Its cornerstone is the assumption that the particle velocity, being an infinitesimal quantity (derivative of the position with respect to time), is random and consists of a mean part and an irrotational fluctuation. The mean part is the actual classical velocity (momentum), while the fluctuation represents a measure of the uncertainty with which it can be determined. Since the fluctuation is irrotational, it does not affect the averaged Hamilton equations of motion, however it yields an additional second-order correlation term in a picture, where an ideal particle lo-

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calization is not valid. Moreover, as it will become clear in the sequel, the strength of the momentum fluctuation depends functionally on the measure of particle delocalization, namely on the density distribution  $\varrho(\mathbf{x}; t)$ .

The paper is organized as follows. In the next Section a brief parallel between the Hamiltonian and Liouvillean description is outlined. In Section III, a mapping of the classical kinetic balance equations onto a Schrödinger equation is derived. The second-order correlation tensor of the random, irrotational velocity field is derived in Section IV. Further, it is shown that the hydrodynamic equations thus obtained, are in fact the Madelung equations [19], known to be formally equivalent to the Schrödinger equation. Section V presents a further elaboration aimed to recover some basic rules of the quantum formalism. Finally, Section VI is devoted to discussion and conclusions.

## II. LIOUVILLEAN DESCRIPTION

To start with, we consider a  $n$ -dimensional Hamiltonian system, whose dynamics is governed by a smooth Hamiltonian function  $H(\mathbf{x}, \mathbf{p}; t)$ . Here  $\mathbf{x}$  and  $\mathbf{p}$  are  $n$ -dimensional vectors. It is well-known that the description of the dynamical system in terms of equations of motion (Hamilton's equations) is formally equivalent to the description in terms of a phase space density specified by a distribution function  $f(\mathbf{x}, \mathbf{p}; t)$ . The latter satisfies the Liouville equation

$$\frac{\partial f}{\partial t} + \{f, H\} = 0, \quad (\text{II.1})$$

where

$$\begin{aligned} \{F, G\} &= \sum_{k=1}^n \left( \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} - \frac{\partial G}{\partial x_k} \frac{\partial F}{\partial p_k} \right) \\ &= (\nabla_x F) \cdot (\nabla_p G) - (\nabla_x G) \cdot (\nabla_p F), \end{aligned} \quad (\text{II.2})$$

is the standard Poisson bracket. This equivalence is however subtle. Actually, it is a formal mathematical property, which is evidently incomplete in a physical sense. Note for example that, unlike the Hamilton's equations of motion

$$\frac{d\mathbf{x}}{dt} = \nabla_p H, \quad \frac{d\mathbf{p}}{dt} = -\nabla_x H, \quad (\text{II.3})$$

where  $\mathbf{x} = \mathbf{x}(t)$  and  $\mathbf{p} = \mathbf{p}(t)$ , in equation (II.1)  $\mathbf{x}$  and  $\mathbf{p}$  are independent variables and they do not depend on  $t$ . Moreover, the Hamilton equations correspond to the ideal case of a perfectly localized particle

$$f(\mathbf{x}, \mathbf{p}; t) = \delta[\mathbf{x} - \mathbf{X}(t)]\delta[\mathbf{p} - \mathbf{P}(t)], \quad (\text{II.4})$$

where  $(\mathbf{X}(t), \mathbf{P}(t))$  represents their explicit solution. Expression (II.4) also satisfies the Liouville equation, however in addition, it admits a solution of the form (IV.15).

The latter implies that for a given value of  $\mathbf{x}$ , the classical momentum  $\mathbf{p}$  is uniquely determined according to the expression [see equation (IV.34)]

$$\mathbf{p} = \nabla_x \mathcal{S}(\mathbf{x}; t). \quad (\text{II.5})$$

In addition, the function  $\mathcal{S}(\mathbf{x}; t)$  satisfies the Hamilton-Jacobi equation and represents a family of classical trajectories. Evidently, the density distribution  $\varrho(\mathbf{x}; t)$  is a new element in general. It can be regarded as a generalization of the delta-distribution in the Hamiltonian description, and subsequently as a measure of particle delocalization.

Let us define the characteristic function

$$\mathcal{G}(\mathbf{x}, \mathbf{s}; t) = \int d\mathbf{p} f(\mathbf{x}, \mathbf{p}; t) \exp\left(\frac{i}{\lambda} \mathbf{s} \cdot \mathbf{p}\right), \quad (\text{II.6})$$

with a Fourier inverse

$$f(\mathbf{x}, \mathbf{p}; t) = \frac{1}{(2\pi\lambda)^n} \int d\mathbf{s} \mathcal{G}(\mathbf{x}, \mathbf{s}; t) \exp\left(-\frac{i}{\lambda} \mathbf{s} \cdot \mathbf{p}\right), \quad (\text{II.7})$$

where the variable  $\mathbf{s}$  is chosen such that to have the same dimension as the coordinate  $\mathbf{x}$ , and  $\lambda$  is a formal parameter with dimension of action. It is introduced in order to make the argument under the exponent dimensionless. In fact, equation (II.6) [or equivalently (II.7)] represents the definition of the Wigner-Weyl transformation [2]. For the time being, we avoid any reference to the latter, apart from formal similarity and coincidence. In Section V, we will establish a firm link between the characteristic functional and the Wigner-Weyl representation.

Before we proceed further, let us point out a couple of important features of the characteristic function. It is known that in the limit  $\mathbf{s} \rightarrow 0$ , the characteristic function yields the moments of the distribution function  $f(\mathbf{x}, \mathbf{p}; t)$  after a proper marginalization with respect to the conjugate momentum variable  $\mathbf{p}$  is performed. For example,

$$\lim_{\mathbf{s} \rightarrow 0} \mathcal{G}(\mathbf{x}, \mathbf{s}; t) = \frac{\varrho(\mathbf{x}; t)}{m\mathcal{N}}, \quad (\text{II.8})$$

where  $m$  is the particle mass, and

$$\mathcal{N} = \lim_{N, V \rightarrow \infty} \frac{N}{V}, \quad (\text{II.9})$$

is the particle number density in the thermodynamic limit. Here  $N$  is the total number of particles in the system and  $V$  is the volume occupied by the system. Further, we have

$$\frac{\lambda}{i} \lim_{\mathbf{s} \rightarrow 0} \nabla_s \mathcal{G}(\mathbf{x}, \mathbf{s}; t) = \frac{1}{N} \varrho(\mathbf{x}; t) \mathbf{v}(\mathbf{x}; t), \quad (\text{II.10})$$

and so on.

It is therefore very instructive to derive an equation for the characteristic function and try to manipulate it in a suitable manner.

### III. ANALYSIS OF THE LIOUVILLE EQUATION AND MAPPING ONTO A SCHRÖDINGER EQUATION

In this Section, we will work out in detail the simplest case, where

$$H(\mathbf{x}, \mathbf{p}; t) = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}; t). \quad (\text{III.1})$$

For each term in the Liouville equation (II.1), we have subsequently

$$\frac{\partial f}{\partial t} = \frac{1}{(2\pi\lambda)^n} \int d\mathbf{s} \frac{\partial \mathcal{G}}{\partial t} \exp\left(-\frac{i}{\lambda} \mathbf{s} \cdot \mathbf{p}\right), \quad (\text{III.2})$$

$$\frac{\mathbf{p}}{m} \cdot \nabla_x f = \frac{-i\lambda}{m(2\pi\lambda)^n} \int d\mathbf{s} \nabla_s \cdot \nabla_x \mathcal{G} \exp\left(-\frac{i}{\lambda} \mathbf{s} \cdot \mathbf{p}\right), \quad (\text{III.3})$$

$$\nabla_x U \cdot \nabla_p f = \frac{-i}{\lambda(2\pi\lambda)^n} \int d\mathbf{s} (\mathbf{s} \cdot \nabla_x U) \mathcal{G} \exp\left(-\frac{i}{\lambda} \mathbf{s} \cdot \mathbf{p}\right). \quad (\text{III.4})$$

Combining all the terms, the sought-for equation can be written in the form

$$\frac{\partial \mathcal{G}}{\partial t} - \frac{i\lambda}{m} \nabla_s \cdot \nabla_x \mathcal{G} + \frac{i}{\lambda} (\mathbf{s} \cdot \nabla_x U) \mathcal{G} = 0. \quad (\text{III.5})$$

We would like now to diagonalize the differential operator, encountered in the second term of equation (III.5). This is achieved by a simple linear change of variables

$$\mathbf{x}_1 = \mathbf{x} + \frac{\mathbf{s}}{2}, \quad \mathbf{x}_2 = \mathbf{x} - \frac{\mathbf{s}}{2}. \quad (\text{III.6})$$

Taking into account the identities

$$\nabla_x = \nabla_1 + \nabla_2, \quad \nabla_s = \frac{1}{2}(\nabla_1 - \nabla_2), \quad (\text{III.7})$$

where  $\nabla_{1,2}$  denotes the differential operator taken with respect to the variables  $\mathbf{x}_{1,2}$  respectively, we note that equation (III.5) can be rewritten as

$$\frac{\partial \mathcal{G}}{\partial t} - \frac{i\lambda}{2m} (\nabla_1^2 - \nabla_2^2) \mathcal{G} + \frac{i}{\lambda} (\mathbf{s} \cdot \nabla_x U) \mathcal{G} = 0. \quad (\text{III.8})$$

The last equation suggests the ansatz

$$\mathcal{G}(\mathbf{x}_1, \mathbf{x}_2; t) = \Psi_1(\mathbf{x}_1; t) \Psi_2(\mathbf{x}_2; t). \quad (\text{III.9})$$

In addition, the yet unknown complex valued functions  $\Psi_k$  ( $k = 1, 2$ ) can be represented in the form

$$\Psi_k(\mathbf{x}_k; t) = \mathcal{R}_k(\mathbf{x}_k; t) \exp\left[\frac{i}{\lambda} \mathcal{S}_k(\mathbf{x}_k; t)\right], \quad k = 1, 2. \quad (\text{III.10})$$

Since  $\mathcal{G}(\mathbf{x}, \mathbf{s} = 0; t)$  must be real, it follows immediately that

$$\mathcal{S}(\mathbf{x}; t) = \mathcal{S}_1(\mathbf{x}; t) = -\mathcal{S}_2(\mathbf{x}; t). \quad (\text{III.11})$$

Therefore, the characteristic function can be written according to the relation

$$\mathcal{G}(\mathbf{x}, \mathbf{s}; t) = F(\mathbf{x}, \mathbf{s}; t) \exp\left[\frac{i}{\lambda} \mathcal{G}(\mathbf{x}, \mathbf{s}; t)\right], \quad (\text{III.12})$$

where

$$F(\mathbf{x}, \mathbf{s}; t) = \mathcal{R}_1\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right) \mathcal{R}_2\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right), \quad (\text{III.13})$$

$$\mathcal{G}(\mathbf{x}, \mathbf{s}; t) = \mathcal{S}\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right) - \mathcal{S}\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right). \quad (\text{III.14})$$

From equation (III.14) it becomes clear that the phase  $\mathcal{G}(\mathbf{x}, \mathbf{s}; t)$  is an odd function of the variable  $\mathbf{s}$ . Let us now substitute the characteristic function represented by (III.12) into equation (III.5) and separate the real and imaginary part. We obtain

$$m \frac{\partial F}{\partial t} + \nabla_x \cdot (F \nabla_s \mathcal{G}) + (\nabla_s F) \cdot (\nabla_x \mathcal{G}) = 0, \quad (\text{III.15})$$

$$F \frac{\partial \mathcal{G}}{\partial t} + \frac{F}{m} (\nabla_s \mathcal{G}) \cdot (\nabla_x \mathcal{G}) = -F \mathbf{s} \cdot \nabla_x U + \frac{\lambda^2}{m} \nabla_s \cdot \nabla_x F. \quad (\text{III.16})$$

In what follows, we will analyze equations (III.15) and (III.16) in zero and first order with respect to the  $\mathbf{s}$ -variable. The zero-order reads as

$$\frac{\partial F^{(0)}}{\partial t} + \frac{1}{m} \nabla_x \cdot (F^{(0)} \nabla_x \mathcal{S}) = 0, \quad (\text{III.17})$$

$$\mathcal{R}_2 \nabla_x \mathcal{R}_1 - \mathcal{R}_1 \nabla_x \mathcal{R}_2 = 0, \quad (\text{III.18})$$

where  $F^{(0)}(\mathbf{x}; t) = \mathcal{R}_1(\mathbf{x}; t) \mathcal{R}_2(\mathbf{x}; t)$ . Equation (III.18) simply implies that  $\mathcal{R}_1$  and  $\mathcal{R}_2$  must be equal up to a multiplicative constant that can be normalized. Thus, without loss of generality, we can write

$$\mathcal{R}(\mathbf{x}; t) = \mathcal{R}_1(\mathbf{x}; t) = \mathcal{R}_2(\mathbf{x}; t). \quad (\text{III.19})$$

Therefore, the amplitude  $F(\mathbf{x}, \mathbf{s}; t)$  is an even function of the  $\mathbf{s}$ -variable

$$F(\mathbf{x}, \mathbf{s}; t) = \mathcal{R}\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right) \mathcal{R}\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right), \quad (\text{III.20})$$

In first order, we obtain

$$\nabla_x \left( \frac{\partial \mathcal{S}}{\partial t} \right) + \nabla_x \frac{(\nabla_x \mathcal{S})^2}{2m} = -\nabla_x U + \frac{\lambda^2}{2m} \nabla_x \left( \frac{\nabla_x^2 \mathcal{R}}{\mathcal{R}} \right), \quad (\text{III.21})$$

which integrated once yields

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{(\nabla_x \mathcal{S})^2}{2m} = -U + \frac{\lambda^2}{2m} \frac{\nabla_x^2 \mathcal{R}}{\mathcal{R}}. \quad (\text{III.22})$$

The final step is to introduce the complex wave function  $\psi(\mathbf{x}; t)$  according to the de Broglie ansatz

$$\psi(\mathbf{x}; t) = \mathcal{R}(\mathbf{x}; t) \exp\left[\frac{i}{\lambda} \mathcal{S}(\mathbf{x}; t)\right]. \quad (\text{III.23})$$

Combination of equations (III.17) and (III.22) yields the following result

$$i\lambda \frac{\partial \psi}{\partial t} = -\frac{\lambda^2}{2m} \nabla_x^2 \psi + U\psi. \quad (\text{III.24})$$

By identification of  $\lambda$  with the Planck's constant  $\hbar$ , equation (III.24) transforms into the Schrödinger equation. In addition, equations (III.17) and (III.22) coincide with the system of equations, describing the properties of the Madelung fluid [19].

The above considerations can be repeated for the case of nonrelativistic motion of a spinless particle in electromagnetic field, governed by the Hamiltonian

$$H(\mathbf{x}, \mathbf{p}; t) = \frac{1}{2m} [\mathbf{p} - e\mathbf{A}(\mathbf{x}; t)]^2 + eU(\mathbf{x}; t). \quad (\text{III.25})$$

It is worthwhile to mention that the approach based on the characteristic function yields the Schrödinger equation up to first order in the variable  $\mathbf{s}$ . In this sense it can be considered as an infinitesimal mapping of the classical kinetic balance equations onto the Schrödinger equation as pointed out previously [20, 21]. However, a drawback of this method is evident since ansatz (III.9) restricts a possible class of states, while some classical distributions are not described with it.

#### IV. IRROTATIONAL MOMENTUM FLUCTUATIONS

Our basic assumption concerns the equitability of position and momentum, which are obviously not on the same footing. We assume position to be a fundamental variable, while momentum being proportional to the infinitesimal variation of position respective to an infinitesimal variation of time cannot be determined exactly. The physical argument for such assumption is the following. If an object is perfectly localized, there is no reason for the impossibility to determine its velocity accurately. If however, a probability assignment in configuration space strongly violating particle localization is at hand, there must be some uncertainty in the specification of the infinitesimal variation of the particle "position" in the course of time.

Following [18] instead of (III.1), we consider a dynamical system described by the Hamiltonian

$$H(\mathbf{x}, \mathbf{p}; t) = \frac{1}{2m} [\mathbf{p} + \mathbf{A}(\mathbf{x}; t)]^2 + U(\mathbf{x}; t), \quad (\text{IV.1})$$

where  $\mathbf{A}(\mathbf{x}; t)$  is yet unspecified fluctuating part of the classical momentum with vanishing mean value

$$\langle \mathbf{A}(\mathbf{x}; t) \rangle = 0. \quad (\text{IV.2})$$

Defining the new variable

$$\mathbf{P} = \mathbf{p} + \mathbf{A}, \quad (\text{IV.3})$$

we can write the Hamilton equations of motion as follows

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{P}}{m}, \quad \frac{d\mathbf{P}}{dt} = \frac{\partial \mathbf{A}}{\partial t} - \nabla_x U - \frac{\mathbf{P}}{m} \times \nabla_x \times \mathbf{A}. \quad (\text{IV.4})$$

Suppose now that  $\mathbf{A}$  is irrotational, that is

$$\nabla_x \times \mathbf{A} = 0, \quad (\text{IV.5})$$

which also implies

$$\mathbf{A} = -\nabla_x \Phi. \quad (\text{IV.6})$$

With these observations, it follows that the Hamilton equations (IV.4) can be obtained from a new Hamiltonian

$$\mathcal{H}(\mathbf{x}, \mathbf{P}; t) = \frac{\mathbf{P}^2}{2m} + U(\mathbf{x}; t) + \frac{\partial \Phi(\mathbf{x}; t)}{\partial t}, \quad (\text{IV.7})$$

Next, we perform a polynomial marginalization if the distribution function  $f(\mathbf{x}, \mathbf{P}; t)$ . This is done by initially multiplying the stochastic Liouville equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{P}}{m} \cdot \nabla_x f + \mathbf{F} \cdot \nabla_{\mathbf{P}} f = 0 \quad (\text{IV.8})$$

by various powers  $P_1^{k_1} P_2^{k_2} \dots P_n^{k_n}$  and then formally integrating over the momentum variable. Here, the random force  $\mathbf{F}$  is given by the expression

$$\mathbf{F} = \frac{\partial \mathbf{A}}{\partial t} - \nabla_x U. \quad (\text{IV.9})$$

The equations for the first two moments can be written as

$$\frac{\partial \varrho}{\partial t} + \nabla_x \cdot (\varrho \mathbf{V}) = 0, \quad (\text{IV.10})$$

$$\frac{\partial (\varrho V_n)}{\partial t} + \frac{\partial \Pi_{kn}}{\partial x_k} = \frac{\varrho F_n}{m}, \quad (\text{IV.11})$$

where

$$\varrho(\mathbf{x}; t) = m\mathcal{N} \int d\mathbf{P} f(\mathbf{x}, \mathbf{P}; t), \quad (\text{IV.12})$$

$$\varrho(\mathbf{x}; t) \mathbf{V}(\mathbf{x}; t) = \mathcal{N} \int d\mathbf{P} \mathbf{P} f(\mathbf{x}, \mathbf{P}; t), \quad (\text{IV.13})$$

$$\Pi_{kl}(\mathbf{x}; t) = \frac{\mathcal{N}}{m} \int d\mathbf{P} P_k P_l f(\mathbf{x}, \mathbf{P}; t). \quad (\text{IV.14})$$

It is a simple matter to verify that the Liouville equation (IV.8) possesses an exact solution of the form

$$f(\mathbf{x}, \mathbf{P}; t) = \frac{\varrho(\mathbf{x}; t)}{m\mathcal{N}} \delta[\mathbf{P} - m\mathbf{V}(\mathbf{x}; t)], \quad (\text{IV.15})$$

which is also known as the classical Bohm distribution, usually interpreted as the classical limit of quantum pure

states [15]. Substitution of the classical Bohm distribution into equation (IV.14) yields the expression

$$\Pi_{kl} = \varrho V_k V_l, \quad (\text{IV.16})$$

for the stress tensor  $\Pi_{kl}$ . Hence the system (IV.10) and (IV.11) represents an exact closure of hydrodynamic equations, fully equivalent to the Liouville equation.

We now take into account the fact that the momentum  $\mathbf{P}$  and hence the current velocity  $\mathbf{V}$  consists of a classical mean part  $\mathbf{v}$  (corresponding to  $\mathbf{p}$ ) and a fluctuation  $\tilde{\mathbf{V}}$  (corresponding to  $\mathbf{A}$ ). Averaging equations (IV.10) and (IV.11), we obtain

$$\frac{\partial \varrho}{\partial t} + \nabla_x \cdot (\varrho \mathbf{v}) = 0, \quad (\text{IV.17})$$

$$\frac{\partial v_n}{\partial t} + v_k \frac{\partial v_n}{\partial x_k} = -\frac{1}{m} \frac{\partial U}{\partial x_n} - \frac{1}{\varrho} \frac{\partial}{\partial x_k} (\varrho \mathcal{C}_{kn}), \quad (\text{IV.18})$$

where

$$\mathcal{C}_{kn} = \langle \tilde{V}_k \tilde{V}_n \rangle. \quad (\text{IV.19})$$

Recall that it was initially assumed that the momentum fluctuations are irrotational. This also implies irrotationality of the second term on the right-hand-side of equation (IV.18). It can be written in the form

$$\frac{\partial \mathcal{Z}}{\partial x_n} = \frac{\partial \mathcal{C}_{kn}}{\partial x_k} + \mathcal{C}_{kn} \frac{\partial R}{\partial x_k}, \quad (\text{IV.20})$$

where

$$R = \ln \varrho. \quad (\text{IV.21})$$

Taking curl of both sides of equation (IV.20), we obtain

$$\epsilon_{lmn} \frac{\partial}{\partial x_m} \left( \frac{\partial \mathcal{C}_{kn}}{\partial x_k} + \mathcal{C}_{kn} \frac{\partial R}{\partial x_k} \right) = 0, \quad (\text{IV.22})$$

where as usual,  $\epsilon_{lmn}$  denotes the fully antisymmetric third-rank unit tensor. Multiplication by  $\epsilon_{lqp}$  and summation on  $l$  in the last identity, yields a second order linear partial differential equation for the unknown correlation tensor  $\mathcal{C}_{kn}$

$$\left( \frac{\partial}{\partial x_k} + \frac{\partial R}{\partial x_k} \right) \left( \frac{\partial \mathcal{C}_{kn}}{\partial x_m} - \frac{\partial \mathcal{C}_{km}}{\partial x_n} \right) + \mathcal{C}_{kn} \frac{\partial^2 R}{\partial x_k \partial x_m} - \mathcal{C}_{km} \frac{\partial^2 R}{\partial x_k \partial x_n} = 0. \quad (\text{IV.23})$$

Note that the left-hand-side of equation (IV.23) is antisymmetric with respect to the indices  $m$  and  $n$ . This restricts considerably the number of its solutions. On the other hand, equation (IV.23) is a linear equation with respect to  $\mathcal{C}_{kn}$ , so that its general solution can be written as a linear combination of particular solutions. Since the correlation tensor is symmetric, it can be represented in diagonal form. To find a particular solution, suppose that equation (IV.23) is written in a reference frame in which the correlation tensor is diagonal. Then, all elements must be equal, which is the only possibility for this particular solution [represented by the second term in equation (IV.24)]. Another solution can be represented in the form of a Hessian matrix of a generic function. A simple and straightforward verification shows that these two particular solutions exhaust all possibilities, and  $\mathcal{C}_{kn}$  can be written as follows

$$\mathcal{C}_{kn} = \alpha \frac{\partial^2 \Gamma}{\partial x_k \partial x_n} + \beta \delta_{kn} \mathcal{F}, \quad (\text{IV.24})$$

where  $\alpha$  and  $\beta$  are constant coefficients, and the scalar functions  $\mathcal{F}(\mathbf{x}; t)$  and  $\Gamma(\mathbf{x}; t)$  are some functions of  $\mathbf{x}$  and  $t$  that remain to be determined. Direct substitution of the solution (IV.24) into equation (IV.23) shows that  $\mathcal{F}$  is an arbitrary function of  $R$

$$\mathcal{F}(\mathbf{x}; t) = F(R), \quad (\text{IV.25})$$

while  $\Gamma$  is equal to  $R$  up to a multiplicative constant, which (without loss of generality) can be set equal to unity

$$\Gamma(\mathbf{x}; t) = R(\mathbf{x}; t). \quad (\text{IV.26})$$

Hence, equation (IV.20) can be rewritten as follows

$$\begin{aligned} \nabla_x \mathcal{Z} &= \alpha \nabla_x \left[ \nabla_x^2 R + \frac{1}{2} (\nabla_x R)^2 \right] + \nabla_x \mathcal{F} + \mathcal{F} \nabla_x R \\ &= 2\alpha \nabla_x \left( \frac{\nabla_x^2 \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \nabla_x \mathcal{F} + \mathcal{F} \nabla_x R. \end{aligned} \quad (\text{IV.27})$$

We would like now to show that the arbitrary constant  $\alpha$  must be negative ( $\alpha < 0$ ). In analogy to the definitions (IV.12)–(IV.14), we can introduce the kinetic energy density according to the relation

$$E(\mathbf{x}; t) = \frac{\mathcal{N}}{2m} \int d\mathbf{P} \mathbf{P}^2 f(\mathbf{x}, \mathbf{P}; t) = \frac{\varrho \mathbf{V}^2}{2}. \quad (\text{IV.28})$$

Averaging equation (IV.28), we obtain

$$\langle E(\mathbf{x}; t) \rangle = \frac{\varrho \mathbf{V}^2}{2} + \frac{\varrho}{2} \text{Tr} \mathcal{C}, \quad (\text{IV.29})$$

where  $\text{Tr}\mathcal{C}$  denotes the trace of the correlation tensor (IV.19). The second term on the right-hand-side of equation (IV.29) represents the density of the internal energy, which is due to the fluctuating part of the current velocity. The total internal energy is given by the expression

$$\mathcal{E} = \frac{1}{2} \int d\mathbf{x} \varrho \text{Tr}\mathcal{C} = \frac{n}{2} \int d\mathbf{x} \varrho \mathcal{F}(R) - \frac{\alpha}{2} \int d\mathbf{x} \frac{(\nabla_x \varrho)^2}{\varrho}, \quad (\text{IV.30})$$

where integration by parts and taking into account of vanishing integrals has been performed in the second term on the right-hand-side. Since the total internal energy must be positive for any choice of the arbitrary function  $\mathcal{F}$  (including  $\mathcal{F} = 0$ ), the free parameter  $\alpha$  must be negative. Remarkably enough, the second term on the right-hand-side of equation (IV.30) is proportional to the Fisher information [22]

$$\mathcal{I} = \int d\mathbf{x} \frac{(\nabla_x \varrho)^2}{\varrho}. \quad (\text{IV.31})$$

The above equation (IV.31) represents a direct link between quantum mechanics and Fisher information theory. It is remarkable that in the simplest case, where  $\mathcal{F} = 0$ , the Fisher information measure  $\mathcal{I}$  has a very transparent meaning as being simply a measure of the internal energy of the quantum system.

First of all, we would like to explore the simplest case, where  $\mathcal{F} = 0$ . Identifying the parameter  $\alpha$  as

$$\alpha = -\frac{\hbar^2}{4m^2}, \quad (\text{IV.32})$$

equation (IV.18) can be written accordingly

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_x \mathbf{v} = -\frac{\nabla_x U}{m} + \frac{\hbar^2}{2m^2} \nabla_x \left( \frac{\nabla_x^2 \sqrt{\varrho}}{\sqrt{\varrho}} \right). \quad (\text{IV.33})$$

If we further define

$$\mathbf{v} = \frac{1}{m} \nabla_x \mathcal{S}, \quad (\text{IV.34})$$

the de Broglie ansatz (III.23) with

$$\lambda = \hbar, \quad \mathcal{R} = \sqrt{\varrho}, \quad (\text{IV.35})$$

yields immediately the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_x^2 \psi + U \psi. \quad (\text{IV.36})$$

Another simple but nontrivial example is the case where  $\mathcal{F} = b/m = \text{const.}$  As a result, we obtain the Schrödinger equation with logarithmic nonlinearity [23]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_x^2 \psi + \left( U + b \ln |\psi|^2 \right) \psi. \quad (\text{IV.37})$$

Note that the constant  $b$  can be positive, as well as negative. From the requirement of the positivity of the internal energy, it is easy to obtain the upper bound for the case  $b < 0$

$$|b| < \frac{\hbar^2}{4nm^2N} \mathcal{I}. \quad (\text{IV.38})$$

Concluding this Section, it is worthwhile to reiterate that the Schrödinger equation has been derived by the sole use of purely classical stochastic arguments. In addition, it should be pointed out that the linear Schrödinger equation is not the unique possibility.

## V. THE QUANTUM PICTURE

First of all, we note that  $\varrho$ , which has the meaning of mass density [see equation (IV.12)] can be rescaled  $\varrho \rightarrow mN\varrho$ , such that it becomes normalized

$$\int d\mathbf{x} \varrho(\mathbf{x}; t) = 1. \quad (\text{V.1})$$

This implies that the wave function is normalized as well

$$\int d\mathbf{x} |\psi(\mathbf{x}; t)|^2 = \int d\mathbf{x} \varrho(\mathbf{x}; t) = 1. \quad (\text{V.2})$$

From (III.23) and (IV.15) for the expectation value of an arbitrary function  $F(\mathbf{x})$  of position, we obtain

$$\langle F(\mathbf{x}) \rangle = \int d\mathbf{x} d\mathbf{P} F(\mathbf{x}) f(\mathbf{x}, \mathbf{P}; t) = \int d\mathbf{x} F(\mathbf{x}) |\psi(\mathbf{x}; t)|^2. \quad (\text{V.3})$$

The expectation value of momentum can be found in a similar manner

$$\begin{aligned} \langle \mathbf{P} \rangle &= \int d\mathbf{x} d\mathbf{P} \mathbf{P} f(\mathbf{x}, \mathbf{P}; t) = m \int d\mathbf{x} \langle \mathbf{V} \rangle \varrho(\mathbf{x}; t) \\ &= -i\hbar \int d\mathbf{x} \psi^* \nabla_x \psi. \end{aligned} \quad (\text{V.4})$$

Further, we have (for the case, where  $\mathcal{F} = 0$ )

$$\begin{aligned} \langle \mathbf{P}^2 \rangle &= m^2 \int d\mathbf{x} \varrho(\mathbf{x}; t) (\mathbf{v}^2 + \text{Tr}\mathcal{C}) \\ &= \hbar^2 \int d\mathbf{x} (\nabla_x \psi^*) \cdot (\nabla_x \psi). \end{aligned} \quad (\text{V.5})$$

The expectation value for the energy is represented by the expression (for the case, where  $\mathcal{F} = 0$ )

$$\langle \mathcal{H} \rangle = \int d\mathbf{x} \left[ \frac{\hbar^2}{2m} (\nabla_x \psi^*) \cdot (\nabla_x \psi) + U(\mathbf{x}; t) |\psi|^2 \right]. \quad (\text{V.6})$$

Thus, we recover the basic quantum rules to calculate expectation values of observables, which are not higher than quadratic in momentum. This implies that not only expectation values, but also uncertainties for position and momentum can be calculated so that they correspond to the standard quantum expression. Similar to the observations made by M.J.W. Hall and M. Reginatto [16, 17], we find that the Heisenberg uncertainty principle can be obtained solely by using classical statistical mechanics formalism with stochastic ingredients added.

Since only the mean value of the current velocity  $\mathbf{v}$  and the correlation tensor  $\mathcal{C}_{mn}$  have been specified to this end, it is not immediately clear how one can proceed with calculation of expectation values of an arbitrary function of momentum. A possible approach to this problem will be outlined in the sequel.

The definition (II.6) of the characteristic function can be generalized into a characteristic functional

$$\langle \mathcal{G}(\mathbf{x}, \mathbf{s}; t) \rangle = |\psi(\mathbf{x}; t)|^2 \left\langle \exp \left( \frac{im}{\hbar} \mathbf{s} \cdot \mathbf{V} \right) \right\rangle. \quad (\text{V.7})$$

On the other hand, from (III.9) we would obtain

$$\langle \mathcal{G}(\mathbf{x}, \mathbf{s}; t) \rangle = \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right), \quad (\text{V.8})$$

which implies

$$\begin{aligned} & |\psi(\mathbf{x}; t)|^2 \left\langle \exp \left( \frac{im}{\hbar} \mathbf{s} \cdot \mathbf{V} \right) \right\rangle \\ &= \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right). \end{aligned} \quad (\text{V.9})$$

The last equation represents a strong condition to be imposed on the stochastic properties of the random velocity field. Clearly, higher order correlation functions of the random velocity field  $\tilde{\mathbf{V}}$  must be specified accordingly in order to satisfy equation (V.9). The consequences are analyzed in the Appendix. There, it is shown that equation (V.9) remains valid to second order in the variable  $\mathbf{s}$ . In addition, an expression for the third-order correlation function  $\langle \tilde{V}_k \tilde{V}_i \tilde{V}_n \rangle$  of the random velocity field  $\tilde{\mathbf{V}}$  is derived.

Let us define the Fourier transform of the wave function according to the well-known relation

$$\psi(\mathbf{x}; t) = \frac{1}{(2\pi\hbar)^{n/2}} \int d\mathbf{p} \hat{\psi}(\mathbf{p}; t) \exp \left( \frac{i}{\hbar} \mathbf{x} \cdot \mathbf{p} \right). \quad (\text{V.10})$$

In order to cast a parallel with the discussion in Section II, we integrate equation (V.8) over  $\mathbf{x}$ . As a result, we

---


$$\begin{aligned} &= \int d\mathbf{x} (-i\hbar \nabla_{sk}) \left[ \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right) (-i\hbar \nabla_{xk})^n \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \right] \Big|_{\mathbf{s}=0} \\ &= \frac{1}{2} \int d\mathbf{x} i\hbar \nabla_{xk} \psi^*(\mathbf{x}; t) (-i\hbar \nabla_{xk})^n \psi(\mathbf{x}; t) + \frac{1}{2} \int d\mathbf{x} \psi^*(\mathbf{x}; t) (-i\hbar \nabla_{xk})^{n+1} \psi(\mathbf{x}; t) \end{aligned} \quad (\text{V.17})$$

$$= \int d\mathbf{x} \psi^*(\mathbf{x}; t) (-i\hbar \nabla_{xk})^{n+1} \psi(\mathbf{x}; t), \quad (\text{V.18})$$

where integration by parts have been performed in the

obtain

$$\bar{\mathcal{G}}(s) = \int d\mathbf{x} \langle \mathcal{G}(\mathbf{x}, \mathbf{s}; t) \rangle = \int d\mathbf{p} \left| \hat{\psi}(\mathbf{p}; t) \right|^2 \exp \left( \frac{i}{\hbar} \mathbf{s} \cdot \mathbf{p} \right). \quad (\text{V.11})$$

According to expressions (II.8) and (II.10) and their obvious generalization, the expectation value of an arbitrary function of momentum  $F(\mathbf{p})$  can be written in the form

$$\langle F(\mathbf{p}) \rangle = \hat{F}(-i\hbar \nabla_s) \bar{\mathcal{G}}(s) \Big|_{\mathbf{s}=0} = \int d\mathbf{p} F(\mathbf{p}) \left| \hat{\psi}(\mathbf{p}; t) \right|^2. \quad (\text{V.12})$$

In coordinate representation, we have

$$\langle F(\mathbf{p}) \rangle = \int d\mathbf{x} \psi^*(\mathbf{x}; t) \hat{F}(-i\hbar \nabla_x) \psi(\mathbf{x}; t). \quad (\text{V.13})$$

To see that, it suffices to verify the following identity

$$\langle p_k^n \rangle = \int d\mathbf{x} \psi^*(\mathbf{x}; t) (-i\hbar \nabla_{xk})^n \psi(\mathbf{x}; t). \quad (\text{V.14})$$

From expressions (V.8) and (V.11) it follows that

$$\langle p_k^n \rangle = \int d\mathbf{x} (-i\hbar \nabla_{sk})^n \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right) \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \Big|_{\mathbf{s}=0}. \quad (\text{V.15})$$

Suppose that

$$\begin{aligned} & \int d\mathbf{x} (-i\hbar \nabla_{sk})^n \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right) \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \Big|_{\mathbf{s}=0} \\ &= \int d\mathbf{x} \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right) (-i\hbar \nabla_{xk})^n \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \Big|_{\mathbf{s}=0} \\ &= \int d\mathbf{x} \psi^*(\mathbf{x}; t) (-i\hbar \nabla_{xk})^n \psi(\mathbf{x}; t), \end{aligned} \quad (\text{V.16})$$

holds true. By induction, we have

$$\int d\mathbf{x} (-i\hbar \nabla_{sk})^{n+1} \psi^* \left( \mathbf{x} - \frac{\mathbf{s}}{2}; t \right) \psi \left( \mathbf{x} + \frac{\mathbf{s}}{2}; t \right) \Big|_{\mathbf{s}=0}$$

---

first term of equation (V.17) to yield the final expression (V.18). This completes the proof of equation (V.13).

A natural generalization is now in order

$$\langle F(\mathbf{x}, \mathbf{p}) \rangle = \int d\mathbf{x} \psi^*(\mathbf{x}; t) \widehat{F}(\mathbf{x}, -i\hbar\nabla_x) \psi(\mathbf{x}; t), \quad (\text{V.19})$$

where as usual, an appropriate operator ordering must be specified.

$$\langle f(\mathbf{x}, \mathbf{p}; t) \rangle = \frac{1}{(2\pi\hbar)^n} \int d\mathbf{s} \psi\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right) \psi^*\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right) \exp\left(-\frac{i}{\hbar} \mathbf{s} \cdot \mathbf{p}\right), \quad (\text{V.20})$$

This implies that  $|\psi|^2 \langle \delta(\mathbf{p} - m\mathbf{V}) \rangle$  is the Wigner function. Leaving more speculations aside on the fact that the latter is a quasi-distribution which is not always positive, we note that equation (V.20) represents a relation between the averaged classical Bohm distribution and the Wigner function.

## VI. MOYAL BRACKET

The Moyal bracket is a useful tool when one wishes to determine a semiclassical limit to wave mechanics. Moyal [24] elaborated on the theory of Wigner [2] on how to describe quantum systems in phase space in a way which is formally analogous to the dynamics of classical distributions. The Moyal bracket provides a semiclassical limit to quantum mechanical commutation relations, which is what is of interest to us, and we consider this in some detail in the present Section.

First of all, let us introduce the characteristic dynamical variable defined as

$$\mathcal{C}(\mathbf{x}, \mathbf{p}; \mathbf{k}, \mathbf{s}) = \exp\left[\frac{i}{\hbar}(\mathbf{k} \cdot \mathbf{x} + \mathbf{s} \cdot \mathbf{p})\right]. \quad (\text{VI.1})$$

The rule to calculate expectation values according to expression (V.19) suggests the introduction of the corresponding characteristic operator [25]

$$\widehat{\mathcal{C}}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}; \mathbf{k}, \mathbf{s}) = \exp\left[\frac{i}{\hbar}(\mathbf{k} \cdot \widehat{\mathbf{x}} + \mathbf{s} \cdot \widehat{\mathbf{p}})\right], \quad (\text{VI.2})$$

where  $\widehat{\mathbf{x}}$  implies  $\mathbf{x}$ , while  $\widehat{\mathbf{p}} = -i\hbar\nabla_x$ . Since, we are interested in calculating expectation values of various dynamical variables (the characteristic dynamical variable included), it is natural to define the characteristic function as

$$\widetilde{\mathcal{C}}(\mathbf{k}, \mathbf{s}) = \langle \mathcal{C}(\mathbf{x}, \mathbf{p}; \mathbf{k}, \mathbf{s}) \rangle. \quad (\text{VI.3})$$

According to equation (V.19) and taking into account the Campbell-Baker-Hausdorff identity,

$$e^{\widehat{A} + \widehat{B}} = e^{\widehat{A}} e^{\widehat{B}} e^{-\frac{1}{2}[\widehat{A}, \widehat{B}]}, \quad (\text{VI.4})$$

We end this Section by emphasizing another remarkable link between the formalism developed here and the Wigner-Weyl approach. Using the compatibility condition in the form of (V.8), we can rewrite equation (II.7) as

when the commutator commutes with both  $\widehat{A}$  and  $\widehat{B}$  (which is the case for operators proportional to  $\widehat{\mathbf{x}}$  and  $\widehat{\nabla}_x$ ), we obtain

$$\widetilde{\mathcal{C}}(\mathbf{k}, \mathbf{s}) = \int d\mathbf{x} \psi\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right) \psi^*\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right) \exp\left(\frac{i}{\hbar} \mathbf{k} \cdot \mathbf{x}\right). \quad (\text{VI.5})$$

Taking into account the inverse Fourier transform of expression (VI.5) and equation (V.20), it immediately follows that the characteristic function is a double Fourier transform of the Wigner function

$$W(\mathbf{x}, \mathbf{p}; t) = \langle f(\mathbf{x}, \mathbf{p}; t) \rangle$$

$$= \frac{1}{(2\pi\hbar)^{2n}} \int d\mathbf{k} d\mathbf{s} \widetilde{\mathcal{C}}(\mathbf{k}, \mathbf{s}) \exp\left[-\frac{i}{\hbar}(\mathbf{k} \cdot \mathbf{x} + \mathbf{s} \cdot \mathbf{p})\right], \quad (\text{VI.6})$$

or

$$\widetilde{\mathcal{C}}(\mathbf{k}, \mathbf{s}) = \int d\mathbf{x} d\mathbf{p} W(\mathbf{x}, \mathbf{p}; t) \exp\left[\frac{i}{\hbar}(\mathbf{k} \cdot \mathbf{x} + \mathbf{s} \cdot \mathbf{p})\right]. \quad (\text{VI.7})$$

Equation (VI.7) implies also that the expectation value of the characteristic dynamical variable (VI.1) is a result of integration of its product with the Wigner function over all of phase space.

Further, a generic dynamical variable  $A(\mathbf{x}, \mathbf{p})$  can be represented by the Fourier integral

$$A(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^{2n}} \int d\mathbf{k} d\mathbf{s} \mathcal{A}(\mathbf{k}, \mathbf{s}) \mathcal{C}(\mathbf{x}, \mathbf{p}; \mathbf{k}, \mathbf{s}), \quad (\text{VI.8})$$

while its corresponding operator in terms of  $\widehat{\mathbf{x}}$  and  $\widehat{\mathbf{p}}$  specified by the characteristic operator (VI.2) can be written as

$$\widehat{A}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) = \frac{1}{(2\pi\hbar)^{2n}} \int d\mathbf{k} d\mathbf{s} \mathcal{A}(\mathbf{k}, \mathbf{s}) \widehat{\mathcal{C}}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}; \mathbf{k}, \mathbf{s}). \quad (\text{VI.9})$$

The dynamical variable (VI.8) is usually called a phase function of the operator (VI.9).

Since the physically measurable, and therefore, feasible characteristic of a dynamical variable is its expectation

value, it is important to emphasize that by virtue of the rule (V.19) to calculate expectation values, the model employed in Section IV and V directly implies the use of operator description. We can take the Fourier transform of the phase function  $A(\mathbf{x}; \mathbf{p})$ , which returns  $\mathcal{A}(\mathbf{k}; \mathbf{s})$ , and substitute it back into equation (VI.9) for the corresponding operator. This results in

$$\widehat{A}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) = \frac{1}{(2\pi\hbar)^{2n}} \int d\mathbf{x}d\mathbf{p}d\mathbf{k}d\mathbf{s}A(\mathbf{x}; \mathbf{p}) \times \exp \left\{ -\frac{i}{\hbar} [\mathbf{k} \cdot (\mathbf{x} - \widehat{\mathbf{x}}) + \mathbf{s} \cdot (\mathbf{p} - \widehat{\mathbf{p}})] \right\}. \quad (\text{VI.10})$$

Taking into account expression (VI.7) for the expectation value of the characteristic dynamical variable, we obtain

$$\langle A(\mathbf{x}, \mathbf{p}) \rangle = \langle \widehat{A}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) \rangle = \int d\mathbf{x}d\mathbf{p}A(\mathbf{x}, \mathbf{p})W(\mathbf{x}, \mathbf{p}; t). \quad (\text{VI.11})$$

This describes the expectation value of a dynamical variable (an operator observable in quantum mechanical sense) as being the result of integrating its product

(the product of its corresponding phase function) with the Wigner function over all of phase space. In this way the Wigner function acts much like a joint probability distribution over position and momentum.

Expression (V.19) shows that the expectation value of the product of two dynamical variables [corresponding operators in the sense of equation (VI.9)] depends on the order of the multipliers, and in this sense they do not commute. We now wish to determine the measure of non commutativity of two arbitrary operators

$$i\hbar\widehat{D}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) = [\widehat{A}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}), \widehat{B}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})], \quad (\text{VI.12})$$

which is expected to be proportional to  $\hbar$ .

Note that one could repeat the subsequent treatment by directly comparing the quantities  $\langle A(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})B(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) \rangle$  and  $\langle B(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})A(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}) \rangle$  without the use of the operator framework introduced above. Since by virtue of expression (VI.11) we would obtain the same result, we prefer to work within the operator description. We begin by substituting in the general expression of equation (VI.9) for each of the operators  $\widehat{A}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})$ ,  $\widehat{B}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})$  and  $\widehat{D}(\widehat{\mathbf{x}}, \widehat{\mathbf{p}})$ . This brings us to

$$i\hbar \int d\mathbf{k}d\mathbf{s}\mathcal{D}(\mathbf{k}, \mathbf{s})\widehat{\mathcal{C}}(\mathbf{k}, \mathbf{s}) = \frac{1}{(2\pi\hbar)^{2n}} \int d\mathbf{k}_1d\mathbf{s}_1d\mathbf{k}_2d\mathbf{s}_2\mathcal{A}(\mathbf{k}_1, \mathbf{s}_1)\mathcal{B}(\mathbf{k}_2, \mathbf{s}_2) [\widehat{\mathcal{C}}(\mathbf{k}_1, \mathbf{s}_1), \widehat{\mathcal{C}}(\mathbf{k}_2, \mathbf{s}_2)], \quad (\text{VI.13})$$

where for the sake of simplicity the explicit dependence on the operators  $\widehat{\mathbf{x}}$  and  $\widehat{\mathbf{p}}$  has been dropped. Taking into account the Campbell-Baker-Hausdorff identity (VI.4), and the fact that the commutator

$$[\mathbf{k}_1 \cdot \widehat{\mathbf{x}} + \mathbf{s}_1 \cdot \widehat{\mathbf{p}}, \mathbf{k}_2 \cdot \widehat{\mathbf{x}} + \mathbf{s}_2 \cdot \widehat{\mathbf{p}}] = -i\hbar(\mathbf{k}_2 \cdot \mathbf{s}_1 - \mathbf{k}_1 \cdot \mathbf{s}_2), \quad (\text{VI.14})$$

is a scalar quantity it is straightforward to verify

$$\begin{aligned} & [\widehat{\mathcal{C}}(\mathbf{k}_1, \mathbf{s}_1), \widehat{\mathcal{C}}(\mathbf{k}_2, \mathbf{s}_2)] \\ &= 2i\widehat{\mathcal{C}}(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{s}_1 + \mathbf{s}_2) \sin \frac{\mathbf{k}_2 \cdot \mathbf{s}_1 - \mathbf{k}_1 \cdot \mathbf{s}_2}{2\hbar}. \end{aligned} \quad (\text{VI.15})$$

We now determine the expectation value on both sides of equation (VI.13), resulting in characteristic functions on each side of the resulting equation. Substituting expression (VI.7) for the characteristic function and using the representation (VI.8), we obtain

$$\hbar \int d\mathbf{x}d\mathbf{p}D(\mathbf{x}, \mathbf{p})W(\mathbf{x}, \mathbf{p}; t)$$

$$\begin{aligned} &= \frac{2}{(2\pi\hbar)^{4n}} \int d\mathbf{x}d\mathbf{p}d\mathbf{k}_1d\mathbf{s}_1d\mathbf{k}_2d\mathbf{s}_2W(\mathbf{x}, \mathbf{p}; t)\mathcal{A}(\mathbf{k}_1, \mathbf{s}_1) \\ &\quad \times \mathcal{B}(\mathbf{k}_2, \mathbf{s}_2) \sin \frac{\mathbf{k}_2 \cdot \mathbf{s}_1 - \mathbf{k}_1 \cdot \mathbf{s}_2}{2\hbar} \mathcal{C}(\mathbf{k}_1, \mathbf{s}_1)\mathcal{C}(\mathbf{k}_2, \mathbf{s}_2). \end{aligned} \quad (\text{VI.16})$$

Next, we note that the argument of the sine on the right-hand-side of equation (VI.16) can be represented as

$$\begin{aligned} & (\mathbf{k}_2 \cdot \mathbf{s}_1 - \mathbf{k}_1 \cdot \mathbf{s}_2)\mathcal{C}(\mathbf{k}_1, \mathbf{s}_1)\mathcal{C}(\mathbf{k}_2, \mathbf{s}_2) \\ &= \hbar^2(\nabla_{x1} \cdot \nabla_{p2} - \nabla_{p1} \cdot \nabla_{x2})\mathcal{C}(\mathbf{k}_1, \mathbf{s}_1)\mathcal{C}(\mathbf{k}_2, \mathbf{s}_2), \end{aligned} \quad (\text{VI.17})$$

where the subscripts on the differential operators refer to which function they operate on, i.e. either  $\mathcal{C}(\mathbf{k}_1, \mathbf{s}_1)$  or  $\mathcal{C}(\mathbf{k}_2, \mathbf{s}_2)$ , never both. Bearing in mind that the functions  $\mathcal{A}(\mathbf{k}_1, \mathbf{s}_1)$  and  $\mathcal{B}(\mathbf{k}_2, \mathbf{s}_2)$  are independent of  $\mathbf{x}$  and  $\mathbf{p}$ , we can take the differential operators outside the integrals over  $\mathbf{k}_1$ ,  $\mathbf{s}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{s}_2$ . As a result, we obtain

$$\hbar \int d\mathbf{x}d\mathbf{p} D(\mathbf{x}, \mathbf{p}) W(\mathbf{x}, \mathbf{p}; t) = 2 \int d\mathbf{x}d\mathbf{p} W(\mathbf{x}, \mathbf{p}; t) \widehat{\sin} \frac{\hbar}{2} (\nabla_{x_A} \cdot \nabla_{p_B} - \nabla_{p_A} \cdot \nabla_{x_B}) A(\mathbf{x}, \mathbf{p}) B(\mathbf{x}, \mathbf{p}), \quad (\text{VI.18})$$

where again the subscripts on the differential operators refer to the corresponding function they operate on, i.e. either  $A(\mathbf{x}, \mathbf{p})$  or  $B(\mathbf{x}, \mathbf{p})$ , never both. Comparing the non-Wigner function terms inside the integrals immediately implies

$$\begin{aligned} & \frac{\hbar}{2} D(\mathbf{x}, \mathbf{p}) \\ &= \widehat{\sin} \frac{\hbar}{2} (\nabla_{x_A} \cdot \nabla_{p_B} - \nabla_{p_A} \cdot \nabla_{x_B}) A(\mathbf{x}, \mathbf{p}) B(\mathbf{x}, \mathbf{p}), \end{aligned} \quad (\text{VI.19})$$

which is the final result. Considering only the lowest order term in  $\hbar$ , this reduces to

$$D(\mathbf{x}, \mathbf{p}) = \frac{\partial A(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}} \cdot \frac{\partial B(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} - \frac{\partial A(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \cdot \frac{\partial B(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}. \quad (\text{VI.20})$$

A common shorthand notation for equation (VI.19) is

$$\frac{i\hbar}{2} D(\mathbf{x}, \mathbf{p}) = \{A(\mathbf{x}, \mathbf{p}), B(\mathbf{x}, \mathbf{p})\}_{MB}, \quad (\text{VI.21})$$

where the subscript  $MB$  stands for Moyal bracket, associating this expression with both the Poisson bracket and the commutator bracket. It describes how initially commuting usual product of two classical dynamical variables (phase-space functions) transforms into a non-commutative star-product, whenever one proceeds with calculation of the expectation value with the classical Bohm distribution (IV.15).

Let us finally examine how the Poisson bracket

$$P(\mathbf{x}, \mathbf{p}) = \frac{\partial A(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}} \cdot \frac{\partial B(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} - \frac{\partial A(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \cdot \frac{\partial B(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}, \quad (\text{VI.22})$$

transforms when the expectation value is taken on both sides of equation (VI.22). In a manner similar to the described above, one obtains

$$P(\mathbf{x}, \mathbf{p}) = (\nabla_{x_A} \cdot \nabla_{p_B} - \nabla_{p_A} \cdot \nabla_{x_B}) A(\mathbf{x}, \mathbf{p}) * B(\mathbf{x}, \mathbf{p}), \quad (\text{VI.23})$$

where

$$\begin{aligned} & A(\mathbf{x}, \mathbf{p}) * B(\mathbf{x}, \mathbf{p}) \\ &= \widehat{\exp} \left[ \frac{i\hbar}{2} (\nabla_{x_A} \cdot \nabla_{p_B} - \nabla_{p_A} \cdot \nabla_{x_B}) \right] A(\mathbf{x}, \mathbf{p}) B(\mathbf{x}, \mathbf{p}), \end{aligned} \quad (\text{VI.24})$$

is the Moyal star product.

It is worthwhile to note that in the limit if  $\hbar \rightarrow 0$  (which actually implies that the random part of the current velocity in the hydrodynamic picture of Section IV

vanishes), the Moyal star product transforms into the usual commuting product of two dynamical variables. In this limit any two dynamical variables are commutative [see equation (VI.19)], and the Poisson bracket is unaffected by the rule of taking expectation values. In this sense, the Moyal star product can be considered as a quantum non-commutative deformation of the classical product of two dynamical variables.

## VII. DISCUSSION AND CONCLUSIONS

Starting from a simple classical framework and employing some stochastic concepts, the basic ingredients characterizing the quantum nature of physical processes are recovered. It has been shown that the traditional axiomatic structure of quantum mechanics can be rebuilt, so that the quantum mechanical framework resembles to a large extent that of the classical statistical mechanics.

The main assumption used in the present paper is the existence of a random irrotational component in the classical momentum. The physical grounds for such assumption are that an ideal particle localization is not feasible. Hence, provided a probability density in configuration space is prescribed, the infinitesimal variation of the particle "position" in the course of time (i.e., the particle velocity) cannot be determined precisely. Therefore, there is always some uncertainty in the specification of particle momentum, which should strongly depend on the degree of particle delocalization in configuration space. The approach pursued here is by no means an attempt to build a realistic model of the underlying momentum fluctuations, however some hints concerning their higher-order correlation properties are presented. In particular, the current velocity fluctuations are shown to be related to the turbulent fluctuations in the standard picture of Reynolds turbulence. The latter represents an interesting and promising guideline for further investigations.

As a result of the investigation performed, various basic elements of the quantum formalism (calculation of expectation values, the Heisenberg uncertainty principle, the correspondence principle) are recovered by applying traditional techniques, borrowed from classical statistical mechanics.

Finally, it is worthwhile to mention that the link between the formalism used in the deformation quantization method and the usual techniques of classical statistical mechanics appears quite natural in our approach.

## APPENDIX A: ANALYSIS OF THE COMPATIBILITY CONDITION (V.9)

To verify the validity of the compatibility condition (V.9), we expand its both sides in a power series in the variable  $\mathbf{s}$ . To third order, we have

$$\begin{aligned} \left\langle \exp\left(\frac{im}{\hbar}\mathbf{s}\cdot\mathbf{V}\right) \right\rangle &= 1 + \frac{is_k}{\hbar}\partial_k\mathcal{S} - \frac{s_k s_n}{2\hbar^2}\partial_k\mathcal{S}\partial_n\mathcal{S} \\ &\quad - \frac{m^2 s_k s_n}{2\hbar^2}\mathcal{C}_{kn} - \frac{im^3 s_k s_l s_n}{6\hbar^3} \\ &\quad \times (v_k v_l v_n + v_k \mathcal{C}_{ln} + v_l \mathcal{C}_{nk} + v_n \mathcal{C}_{kl} + \mathcal{D}_{kln}) + \dots, \end{aligned} \quad (\text{A.1})$$

where

$$\mathcal{D}_{kln} = \left\langle \tilde{V}_k \tilde{V}_l \tilde{V}_n \right\rangle, \quad (\text{A.2})$$

is the yet unknown third order correlation function of the random velocity field. Similarly, for the right-hand-side of equation (V.9), we obtain

$$\begin{aligned} \psi\left(\mathbf{x} + \frac{\mathbf{s}}{2}; t\right)\psi^*\left(\mathbf{x} - \frac{\mathbf{s}}{2}; t\right) &= \varrho + \frac{s_k}{2}(\psi^*\partial_k\psi - \psi\partial_k\psi^*) \\ &+ \frac{s_k s_n}{8}(\psi^*\partial_k\partial_n\psi - \partial_k\psi^*\partial_n\psi - \partial_n\psi^*\partial_k\psi^* + \psi\partial_k\partial_n\psi^*) \\ &+ \frac{s_k s_l s_n}{48}(\psi^*\partial_k\partial_l\partial_n\psi - \psi\partial_k\partial_l\partial_n\psi^*) \\ &+ \frac{s_k s_l s_n}{48}[\partial_k\psi\partial_l\partial_n\psi^* - \partial_k\psi^*\partial_l\partial_n\psi + \text{Sym}(k, l, n)], \end{aligned} \quad (\text{A.3})$$

where  $\text{Sym}(k, l, n)$  denotes four additional terms obtained by cyclic permutation of the indices  $k, l$  and  $n$ .

Taking into account the explicit expression (IV.24) for the correlation tensor  $\mathcal{C}_{kn}$  with  $\mathcal{F} = 0$ , and the amplitude-phase representation (III.23) of the wave function, it is a simple matter to verify in a straightforward manner that equation (V.9) is satisfied up to second order in the expansion variable  $\mathbf{s}$ . To satisfy the third order however, one needs to specify the third-order correlator of the random part of the current velocity. We obtain the following expression

$$\mathcal{D}_{kln} = -\frac{\hbar^2}{12m^2}(\partial_l\partial_n v_k + \partial_n\partial_k v_l + \partial_k\partial_l v_n). \quad (\text{A.4})$$

No attempt is made neither to interpret, nor to provide a physical model (which beyond doubt should be nonlocal) of a possible source, underlying the velocity (momentum) fluctuations. However, from expression (A.4), it is clear that these fluctuations are far from being Gaussian. A further insight on the relation between the current velocity fluctuations and the standard picture of Reynolds turbulence is given in Appendix B.

## APPENDIX B: TURBULENT FLUCTUATIONS AND COMPATIBILITY CONDITION (V.9)

To show that the results from Appendix A are consistent with the model presented in Section IV, we rewrite equations (IV.10) and (IV.11) as

$$\frac{\partial\varrho}{\partial t} + \nabla_x \cdot (\varrho\mathbf{V}) = 0, \quad (\text{B.1})$$

$$\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla_x \mathbf{V} = \frac{\mathbf{F}}{m}, \quad (\text{B.2})$$

From the assumption that the density  $\varrho$  does not fluctuate, it follows that the fluctuating part of the continuity equation (B.1) reduces to

$$\nabla_x \cdot \tilde{\mathbf{V}} = -\tilde{\mathbf{V}} \cdot \nabla_x R. \quad (\text{B.3})$$

Averaging the equation for momentum balance (B.2) and taking into account relation (B.3), we readily obtain equation (IV.18).

Let us further write

$$\frac{\partial}{\partial t}(V_n V_s) + V_k \frac{\partial}{\partial x_k}(V_n V_s) = \frac{1}{m}(F_n V_s + F_s V_n). \quad (\text{B.4})$$

which follows directly from the equation for momentum balance (B.2). Averaging the last equation (B.4) and taking into account equation (IV.18), we obtain

$$\begin{aligned} \frac{\partial\mathcal{C}_{ns}}{\partial t} + v_k \frac{\partial\mathcal{C}_{ns}}{\partial x_k} + \mathcal{C}_{nk} \frac{\partial v_s}{\partial x_k} + \mathcal{C}_{sk} \frac{\partial v_n}{\partial x_k} + \frac{\partial\mathcal{D}_{nsk}}{\partial x_k} + \mathcal{D}_{nsk} \frac{\partial R}{\partial x_k} \\ = \frac{1}{m} \left\langle V_s \frac{\partial A_n}{\partial t} + V_n \frac{\partial A_s}{\partial t} \right\rangle. \end{aligned} \quad (\text{B.5})$$

Having already determined the correlation tensor  $\mathcal{C}_{ns}$  in the form (IV.24), we can manipulate the first term on the left-hand-side of equation (B.5) using the continuity equation (IV.17). We again consider the case, where  $\mathcal{F} = 0$ . The result is

$$\begin{aligned} \mathcal{C}_{sk} \left( \frac{\partial v_n}{\partial x_k} - \frac{\partial v_k}{\partial x_n} \right) + \mathcal{C}_{nk} \left( \frac{\partial v_s}{\partial x_k} - \frac{\partial v_k}{\partial x_s} \right) \\ - \alpha \left( \frac{\partial^3 v_k}{\partial x_n \partial x_s \partial x_k} + \frac{\partial^2 v_k}{\partial x_n \partial x_s} \frac{\partial R}{\partial x_k} \right) + \frac{\partial\mathcal{D}_{nsk}}{\partial x_k} \\ + \mathcal{D}_{nsk} \frac{\partial R}{\partial x_k} = \frac{1}{m} \left\langle V_s \frac{\partial A_n}{\partial t} + V_n \frac{\partial A_s}{\partial t} \right\rangle. \end{aligned} \quad (\text{B.6})$$

If the current velocity  $\mathbf{v}$  is irrotational as specified by (IV.34), the first two terms on the left-hand-side of equation (B.6) vanish. Assuming also that the current velocity fluctuation  $\tilde{\mathbf{V}}$  is uncorrelated with the random part of the force  $\mathbf{F}$ , we readily arrive (after symmetrization) at the expression (A.4) for the triple correlation tensor  $\mathcal{D}_{nsk}$ .

Using equations (B.2) and (B.4) one can proceed in calculation of the fourth-order correlator of the current velocity fluctuation  $\tilde{\mathbf{V}}$ . This procedure can be continued further and all higher order correlators can be found in principle.

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