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Linear coupling in a storage ring is conveniently analyzed in terms of transformations that put the single-turn map into block-diagonal form. Such a transformation allows us to define new variables, in which the dynamics are uncoupled. In principle, a similar approach may be taken to nonlinear coupling; we discuss such an approach in this paper, giving some simple illustrations of the ideas, based on the well-known techniques of normal form analysis.

INTRODUCTION

Coupling between dynamical degrees of freedom in accelerator beamlines arises from many sources, for example, from the tilt of normal quadrupoles around the beam axis, or from a vertical offset of the beam in a sextupole. The presence of coupling complicates the description of the dynamics. For example, consider the phase space picture drawn by a particle making multiple turns through a storage ring. If the motion is uncoupled, the phase space coordinates of the particle in any plane lie on a smooth ellipse; the area of each ellipse corresponds to a conserved quantity of the motion. The existence of such conserved quantities in symplectic systems is an expression of Liouville's theorem. With coupling, the ellipses become irregular, and the conserved quantities no longer correspond to simple geometric quantities. The dynamics may be simplified by making a coordinate transformation in which the motion becomes uncoupled. Symplecticity is preserved if the coordinate transformation is canonical. If the motion is fully decoupled by the transformation, the phase space trajectory of a particle describes a smooth ellipse in each plane as before, and the conserved quantities required by Liouville's theorem again correspond to the areas of the ellipses.

In the case of linear coupling, it is well known how to find a canonical tranformation to new coordinates in which the motion is uncoupled. For nonlinear coupling, the question is a little more complicated, and new features are introduced. In this paper, we consider the use of Lie algebra methods for constructing decoupling transformations in the linear and nonlinear cases. The results provide some insight into the nature of the dynamics in systems with nonlinear coupling.

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LINEAR COUPLING

Consider a linear map given in the form of a Lie transformation [1]:

$$M = \exp(:\mathbf{x}^{\mathrm{T}}\bar{\mathbf{m}}\mathbf{x}:) \tag{1}$$

where $\bar{\mathbf{m}}$ is a symmetric matrix (we use a bar to indicate a matrix that appears in the generator of a Lie transformation), and the components of the vector $\mathbf{x}$ are the phase space variables. In n degrees of freedom, $\bar{\mathbf{m}}$ is a $2n \times 2n$ matrix. The motion described by M is coupled if $\bar{\mathbf{m}}$ has components outside the 2×2 block-diagonals: a decoupling transformation is one that removes the off block-diagonal components of $\bar{\mathbf{m}}$.

A linear canonical transformation F can be written in Lie operator form as:

$$F \cdot \mathbf{x} = \exp(:\mathbf{x}^{\mathrm{T}}\bar{\mathbf{f}}\mathbf{x}:) \cdot \mathbf{x} = \mathbf{f}\,\mathbf{x} \tag{2}$$

where $\mathbf{f}$ is a matrix. Under F, $M \to \tilde{M}$ where:

$$\tilde{M} = F{\cdot}M{\cdot}F^{-1} = \exp(:F{\cdot}\mathbf{x}^{\mathrm{T}}\bar{\mathbf{m}}\mathbf{x}:) = \exp(:\mathbf{x}^{\mathrm{T}}\mathbf{f}^{\mathrm{T}}\bar{\mathbf{m}}\,\mathbf{f}\,\mathbf{x}:) \tag{3}$$

To decouple the motion, we need to find a matrix $\mathbf{f}$ that diagonalizes $\bar{\mathbf{m}}$. Such a matrix can be constructed from the eigenvectors of $\mathbf{S}\bar{\mathbf{m}}$, where $\mathbf{S}$ is the usual $2n \times 2n$ antisymmetric matrix with block-diagonals:

$$\mathbf{S}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

Let $\mathbf{e}_1$ be the matrix of the eigenvectors of $\mathbf{S}\bar{\mathbf{m}}$, ordered and normalized so that:

$$\mathbf{e}_1^{\mathrm{T}}\mathbf{S}\,\mathbf{e}_1 = i\mathbf{S} \tag{5}$$

Then $\mathbf{f}$ can be written:

$$\mathbf{f} = \mathbf{e}_1\mathbf{T} \tag{6}$$

where $\mathbf{T}$ is a block-diagonal matrix constructed from:

$$\mathbf{T}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \tag{7}$$

The generator $\bar{\mathbf{f}}$ of F can be written:

$$\bar{\mathbf{f}} = \frac{1}{2}\mathbf{S}\,\mathbf{e}_2\,(\ln\mathbf{\Lambda})\,\mathbf{e}_2^{-1} \tag{8}$$

where $\mathbf{e}_2$ is the matrix of the eigenvectors of $\mathbf{f}$, and $(\ln\mathbf{\Lambda})$ is a diagonal matrix whose components are the logarithms of the eigenvalues of $\mathbf{f}$.

If the original map in matrix form is given by $\mathbf{m}$:

$$M \cdot \mathbf{x} = \mathbf{m}\mathbf{x} \tag{9}$$

then under the action of F, $\mathbf{m}$ transforms as:

$$\mathbf{m} \rightarrow \tilde{\mathbf{m}} = \mathbf{f}^{-1}\mathbf{m}\mathbf{f} \tag{10}$$

and $\mathbf{f}^{-1}\mathbf{m}\mathbf{f}$ is block diagonal: it is the decoupled map.

Fig. 1 shows the results of "tracking" 1000 turns, using a map representing a coupled linear lattice. The points obtained by applying the coupled map are plotted in black; the same points, after applying a linear decoupling transformation, are plotted in red. As expected, the decoupling transformation completely eliminates the scatter.

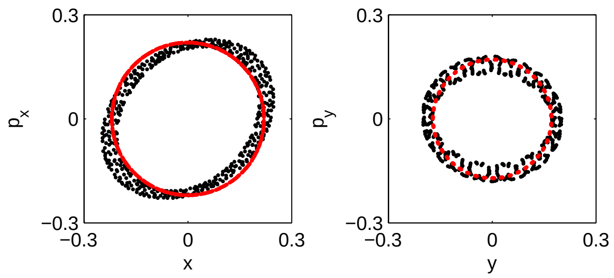

Figure 1: Linear decoupling.

NONLINEAR COUPLING

The linear analysis presented above generalizes to the nonlinear case. For convenience, we now work in action-angle rather than cartesian variables. As a first step, we normalize the map, in the sense that we eliminate any dependence on the angle variables. In general, coupling terms involving only the action variables remain. We will show how to complete the decoupling transformation by removing these terms using time-dependent generating functions.

Techniques for constructing a transformation to remove dependencies of a nonlinear map on the angle variables are familiar from normal-form theory [2]. As an example, consider the map:

$$M = R \cdot \exp\left(:\bar{m}_2:\right) \tag{11}$$

where R is a linear rotation, and

$$\bar{m}_2 = J_x J_y e^{2i\phi_x} e^{2i\phi_y} \tag{12}$$

In general, the linear part of the map represented by R may include coupling; however, the linear map may be decoupled using the techniques of the previous section. To address the nonlinear part, we apply a canonical transformation F_2 that is second-order in the action variables:

$$M \rightarrow \tilde{M} = F_2 \cdot M \cdot F_2^{-1} \tag{13}$$

where

$$\begin{aligned} \tilde{M} &= R \cdot R^{-1} \cdot F_2 \cdot R \cdot \exp\left(:\bar{m}_2:\right) \cdot F_2^{-1} \\ &= R \cdot \exp\left(:R^{-1}\bar{f}_2:\right) \cdot \exp\left(:\bar{m}_2:\right) \cdot \exp\left(:-\bar{f}_2:\right) \\ &= R \cdot \exp\left(:\bar{m}_2' + O(3):\right) \end{aligned} \tag{14}$$

where $\bar{f}_2$ is the generator of the transformation F_2, and

$$\bar{m}_2' = R^{-1}\bar{f}_2 + J_x J_y e^{2i\phi_x} e^{2i\phi_y} - \bar{f}_2 \tag{15}$$

To eliminate the second-order term (in other words, to set $\bar{m}_2' = 0$), the generator $\bar{f}_2$ must satisfy:

$$\left(1 - R^{-1}\right)\bar{f}_2 = J_x J_y e^{2i\phi_x} e^{2i\phi_y} \tag{16}$$

or:

$$\bar{f}_2 = \frac{J_x J_y e^{2i\phi_x} e^{2i\phi_y}}{1 - e^{-2i\theta_x} e^{-2i\theta_y}} \tag{17}$$

where θ_x and θ_y are the rotation angles in R.

In eliminating the unwanted second-order terms, we introduce additional third-order (and higher) terms. However, any terms not invariant under rotations may be removed by applying the same procedure. Assuming convergence (which may not necessarily occur, particularly in regions where the motion appears chaotic), the normalization proceeds order-by-order until the desired accuracy is achieved.

Nonlinear Decoupling of Hamiltonians

Clearly, terms in the generator that are invariant under rotations cannot be removed by the above procedure: coupling terms such as $J_x J_y$ may remain. To find a means of eliminating these terms, let us consider the analogous situation in Hamiltonian dynamics; this has been treated by Schräpel [3]. We will take as the Hamiltonian:

$$H = \mu_x J_x + \mu_y J_y + \kappa J_x J_y \tag{18}$$

The independent variable will be t, which does not appear explicitly in the Hamiltonian. We now write down a generating function of the second kind [4]:

$$F_2 = \phi_x \tilde{J}_x + \phi_y \tilde{J}_y - \kappa \tilde{J}_x \tilde{J}_y t \tag{19}$$

where $\tilde{J}_x$ and $\tilde{J}_y$ are the new action variables. Note the explicit appearance of t in the generating function. The old action variables are given in terms of the new variables by:

$$J_{x(y)} = \frac{\partial F_2}{\partial \phi_{x(y)}} = \tilde{J}_{x(y)} \tag{20}$$

and the new angle variables are given by:

$$\tilde{\phi}_{x(y)} = \frac{\partial F_2}{\partial \tilde{J}_{x(y)}} = \phi_{x(y)} - \kappa \tilde{J}_{y(x)} t = \phi_{x(y)} - \kappa J_{y(x)} t \tag{21}$$

The transformation takes us into a rotating frame of reference, in which the rate of rotation in either plane depends on the action in other plane. The transformed Hamiltonian $\tilde{H}$ is given by:

$$\tilde{H} = H + \frac{\partial F_2}{\partial t} = \mu_x \tilde{J}_x + \mu_y \tilde{J}_y \tag{22}$$

In the new variables, the motion is uncoupled: we have achieved our goal by means of a generating function in which the independent variable t appears explicitly.

Nonlinear Decoupling of Maps

We now return to the case of a map with nonlinear coupling. We assume that we have decoupled the linear part, and normalized the nonlinear part so that it contains terms only in J_x and J_y. As an explicit example, consider:

$$M = R \cdot \exp\left(: \kappa J_x J_y :\right) \tag{23}$$

The nonlinear part cannot be decoupled using a time-independent canonical transformation; however, let us consider a time-dependent canonical transformation, $F(t)$. Under the map M and under the transformation $F(t)$, the variables transform as follows:

$$\mathbf{J}_{t+1} = M \cdot \mathbf{J}_t \tag{24}$$

$$\tilde{\mathbf{J}}_t = F(t) \cdot \mathbf{J}_t \tag{25}$$

It follows that under $F(t)$, M transforms as:

$$M \to \tilde{M} = F(t+1) \cdot M \cdot F^{-1}(t) \tag{26}$$

An appropriate form for the generating function is:

$$F(t) = \exp\left(: -\kappa J_x J_y t :\right) \tag{27}$$

in which case the fully decoupled map is simply:

$$\tilde{M} = R \tag{28}$$

Numerical Example

We consider a map of the form:

$$M = R \cdot \exp\left[: \kappa J_x J_y \cos(2\phi_x + 2\phi_y) :\right] \tag{29}$$

The normalizing transformation (to second order) has the form $F_2 = \exp\left(: \bar{f}_2 :\right)$ where

$$\bar{f}_2 = \frac{1}{2}\kappa J_x J_y \frac{\sin\left(\theta_x + \theta_y + 2\phi_x + 2\phi_y\right)}{\sin\left(\theta_x + \theta_y\right)} \tag{30}$$

The transformed map is:

$$\tilde{M} = R \cdot \exp\left(: \bar{m}_3' + O(4) :\right) \tag{31}$$

where, from the Baker-Campbell-Hausdorff formula, the third order generator term $\bar{m}_3'$ is given by:

$$\bar{m}_3' = \frac{1}{4}\kappa^2 J_x J_y \left(J_x + J_y\right) \frac{\sin\left[2\left(\theta_x + \theta_y\right)\right]}{\sin\left(\theta_x + \theta_y\right)^2} \tag{32}$$

Observe that $\bar{m}_3'$ is invariant under rotations, so it must be removed using a time-dependent transformation, with generator:

$$\bar{f}_3(t) = -\bar{m}_3' t \tag{33}$$

The results of tracking 25 turns (with $\kappa = 0.85$, and linear rotation angles $\theta_x = 100°$ and $\theta_y = 65°$) are shown in Fig. 2. The blue and black points show the results of applying the map M to particles with two different initial vertical amplitudes. The second-order time-independent normalization reduces the scatter observed in the full nonlinear map; the symplectic condition is now evident as Liouville's theorem applied in each plane, with the conserved quantities corresponding to the areas of the ellipses. However, the motion is still coupled since the horizontal phase advance depends on the vertical amplitude (and vice-versa). After a third-order time-dependent normalization, the phase advance in each plane is independent of the amplitude in the other plane: the motion is almost fully decoupled.

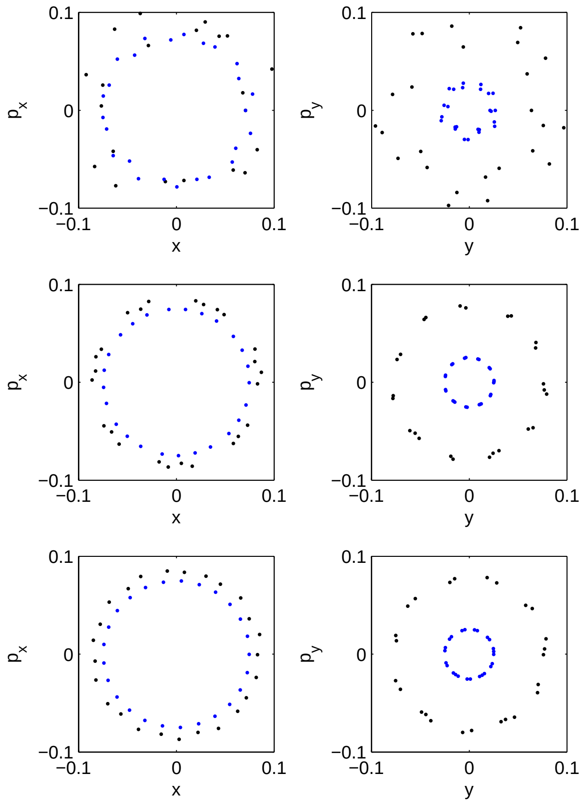

Figure 2: Nonlinear decoupling. Top: nonlinear map. Middle: map normalized using second-order time-independent canoncical transformation. Bottom: map further normalized using third-order time-dependent canonical transformation.

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