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RESEARCH ARTICLE

Electromagnetic waves in the Kerr–Schild metric

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Abstract A short electromagnetic wavelength approximation is used to obtain general solutions to the Maxwell equations for electromagnetic radiation in a Kerr–Schild plane gravitational wave metric. Their properties are then investigated for the specific case of a light beam in a weak, harmonic gravitational wave.

Keywords Electromagnetic waves · Kerr–Schild

1 Introduction

If V is a unit timelike vector field whose integral curves correspond with the worldlines of an observer, the electric and magnetic fields relative to V are then

$$E = \widetilde{\mathrm{i}_V F} \qquad B = -\widetilde{\mathrm{i}_V * F} \tag{1}$$

where, following the notation in [1], i_V is the interior derivative with respect to V, F is the electromagnetic 2-form, $*$ is the Hodge map and the tilde indicates a metric dual. The Maxwell equations in a vacuum are the elegant pair of equations

$$\mathrm{d}F = 0 \qquad \mathrm{d} * F = 0 \tag{2}$$

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where d indicates the exterior derivative. The presence of the Hodge map indicates that background spacetime curvature influences the behaviour of electromagnetic radiation, and this phenomenon has been widely studied ever since it was first exploited in the early tests of general relativity. Various techniques have been adopted, such as treating the gravitational field as a medium with constitutive relations derived from the metric, distilling transport properties from contractions of Maxwell's equations, exploiting Killing symmetries or using Green's function methods, fluid descriptions, $3+1$ decompositions and geometrical optics approximations [2–13].

A great deal of interest has also been aroused by the various schemes proposed to detect experimentally the presence of gravitational wave-type metric perturbations (see, for example, [14,15]). It is thus unsurprising that the behaviour of an electromagnetic wave in a gravitational wave background has been an active subject for decades [16–22].

A linearly polarised, plane gravitational wave can be simulated by adopting the Kerr–Schild metric [18,23–26]

$$g = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y - \frac{1}{2}(\mathrm{d}\mu \otimes \mathrm{d}\nu + \mathrm{d}\nu \otimes \mathrm{d}\mu) + H\mathrm{d}\mu \otimes \mathrm{d}\mu \tag{3}$$

with

$$H = h(\mu)\left((x^2 - y^2)\cos 2\theta + 2xy\sin 2\theta\right) \tag{4}$$

where the null co-ordinates are $\mu = t - z$, $\nu = t + z$, the scalar $h(\mu)$ gives the wave's longitudinal profile and θ is a polarisation constant. If $\theta = 0$, then the wave is said to have $x^2 - y^2$ polarisation (sometimes known as + polarisation). If $\theta = \frac{\pi}{4}$, the wave has $2xy$ (or $\times$) polarisation. Setting $h(\mu) = 0$ recovers Minkowski space. The metric disturbance (4) propagates in the positive z-direction. A wavefront $\mathcal{W}$ is a subspace of constant μ. A vector V will lie in $\mathcal{W}$ if $V(\mu) = 0$ which can also be written $g(V, \frac{\partial}{\partial\nu}) = 0$ [23]. $\mathcal{W}$ is therefore spanned by the basis $\left\{\frac{\partial}{\partial\nu}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. Throughout this paper, we follow [24–26] by considering an ideal observer whose worldline is an integral curve of the unit timelike vector field

$$X_0 = \frac{\partial}{\partial\mu} + (1 + H)\frac{\partial}{\partial\nu} \tag{5}$$

which reduces in Minkowski space Cartesian co-ordinates to $\frac{\partial}{\partial t}$. The observer's rest space is the space of vectors orthogonal to X_0, which has the basis

$$X_1 = \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial y} \qquad X_3 = -\frac{\partial}{\partial\mu} + (1 - H)\frac{\partial}{\partial\nu} \tag{6}$$

Here X_3 reduces to $\frac{\partial}{\partial z}$ in Minkowski space. The intersection of the observer rest space with the wavefront $\mathcal{W}$ is simply the 2D surface spanned by $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. Thus, the wavefronts in the observer rest frame are parallel to the (x, y) plane. Define an observer's neighbour as a spacelike vector field on the observer worldline. It can be shown [23] that the neighbour's relative acceleration is orthogonal to the propagation direction of the wave and sits in the observer's measured wavefront.

Thus, an observer travelling along an integral curve of X_0 in the metric (3) will see planar wavefronts, and its neighbours will experience a transverse tidal acceleration. It is therefore valid, at least locally, to model a gravitational plane wave using this choice of metric and observer frame.[1]

Griffiths [18] investigated exact solutions to the Maxwell and Einstein equations for electromagnetic radiation travelling in a class of plane gravitational wave metric similar to (3). This treatment, which was restricted to homogeneous plane electromagnetic waves, revealed interesting phenomena resulting from the choice of metric, the dynamical interaction between the gravitational and electromagnetic fields, and the nonlinearity of the Einstein equations.

However, difficulties lie in extending this treatment to electromagnetic waves without complete plane symmetry. In such situations, it is normal practice to work in a linearised weak-field gravitational wave metric that is quite different from (3). Working in such a metric, Cooperstock [19] developed a perturbation scheme which he applied to plane electromagnetic waves bouncing between parallel conducting walls in a gravitational wave, while [20–22] obtained solutions to Maxwell equations corresponding to light beams.

In common with [20] and [21], this paper also uses a geometrical optics approximation to obtain the electromagnetic field, but, in contrast to existing treatments, the Kerr–Schild metric (3) is retained throughout.[2] The contrast between the background metrics may affect the predicted behaviour of the electromagnetic radiation.

The aims of this paper will be twofold. First, we describe electromagnetic waves with specified Minkowski space polarisations and transverse amplitude profiles that satisfy Maxwell's equations in a plane gravitational wave metric of arbitrary profile. Then, we use these results to examine the effect of a harmonic gravitational wave on the paths of light rays and the field magnitudes in a beam of light. When applying the equations, we assume that the positive, dimensionless gravitational wave strain satisfies $h_0 \ll 1$, in order to obtain explicit values for the metric functions and to simplify the final equations. However, none of the expressions derived in the first section require any such assumptions and are valid for intense gravitational waves as well as weak ones, provided the electromagnetic wavelength is sufficiently small.

We use a WKB-style geometrical optics approximation. Following treatments in [12] and [13], an electromagnetic wave can be considered 'locally plane' if its wavelength is extremely short compared with the typical radius of curvature of the spacetime and the length scale over which properties such as polarisation change. This does not necessarily imply a constant amplitude across the entire wavefront: a transversely

[1] Owing to their simplicity and symmetry, metrics such as (3) are commonly used to describe gravitational plane wave spacetimes [23–26, 18]. However, one should remember that this represents an idealisation. In particular, the Bondi news function for this class of metrics has been found to be zero, suggesting that the Kerr–Schild metric cannot globally describe gravitational radiation from bounded sources [27,28].

[2] We will occasionally transform to an alternative co-ordinate chart to facilitate intermediate calculations but the results will ultimately be presented in the Kerr–Schild metric.

bounded beam of light can be described with this approximation if its wavelength is sufficiently small. Such a wave can be written in terms of a real phase (or Eikonal) function S, a real scalar amplitude α and a complex unit polarisation 1-form $\mathcal{P}$, orthogonal to dS, so that

$$g(\mathcal{P}, \overline{\mathcal{P}}) = 1 \tag{7}$$

$$\mathrm{i}_{\widetilde{\mathrm{d}S}}\mathcal{P} = 0 \tag{8}$$

where $\overline{\mathcal{P}}$ indicates the complex conjugate of $\mathcal{P}$. Equation (7) is the normalisation condition for the polarisation 1-form, while (8) ensures it is transverse.

An electromagnetic 2-form given by

$$F = \Re\left\{\alpha e^{iS}\mathrm{d}S \wedge \mathcal{P}\right\} \tag{9}$$

(where, throughout this paper, $\Re$ indicates the real part of an expression) is an approximate solution to the Maxwell equations provided S, α and $\mathcal{P}$ obey

$$\mathrm{i}_{\widetilde{\mathrm{d}S}}\mathrm{d}S = 0 \tag{10}$$

$$\nabla_{\widetilde{\mathrm{d}S}}\alpha = \frac{1}{2}(*\mathrm{d} * \mathrm{d}S)\alpha \tag{11}$$

$$\nabla_{\widetilde{\mathrm{d}S}}\mathcal{P} = 0 \tag{12}$$

∇_X indicates covariant differentiation with respect to the vector field X. $\widetilde{\mathrm{d}S}$ is called the wave vector. Equation (10), sometimes known as the Eikonal equation, ensures that the wave vector is null. It also implies that $\nabla_{\widetilde{\mathrm{d}S}}\mathrm{d}S = 0$, so that the light rays, which are integral curves of the vector field dS, are null geodesics. Equations (11) and (12) describe the transport along the light ray of the amplitude profile and polarisation 1-form, respectively.

In the next section, we calculate allowed phase, amplitude and polarisations for the electromagnetic wave in a Kerr–Schild gravitational plane wave metric. We seek values of S, α and P that satisfy (7)–(8) and (10)–(12) in the metric given by (3) where the longitudinal profile of the gravitational wave is unspecified. Finally, these equations will be used to investigate the behaviour of a beam of light in a low-strain, harmonic gravitational wave.

2 General expressions

We now implement the approximation scheme described in (7)–(12). The first step is to fix S by finding solutions to the Eikonal equation (10). In the metric (3), this becomes:[3]

[3] This equation appears in a different form in [31]. For a photon with wave vector k^i, one can write an Eikonal function ψ such that $k_i = \frac{\partial\psi}{\partial x^i}$, which satisfies $g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0$. In a strong gravitational wave, this gives a dispersion relation that, after relabelling the gravitational wave functions and changing the signature of the metric, is identical to our Eq. (13).

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = 4\frac{\partial S}{\partial \nu}\left(\frac{\partial S}{\partial \mu} + H\frac{\partial S}{\partial \nu}\right) \tag{13}$$

This S is then used obtain the polarisations and amplitudes. We will look at three different solutions to the Eikonal equation, corresponding to electromagnetic and gravitational waves propagating in the same direction, in opposite directions, and meeting at an angle.

2.1 Parallel propagation

A simple solution to equation (13) is

$$S = \Omega\mu \tag{14}$$

This corresponds to a wave of frequency Ω travelling in the positive z-direction, the same direction as the gravitational wave. This propagation direction is encoded in the wave vector $\widetilde{\mathrm{d}S} = \frac{\partial}{\partial \nu}$. Thus, equation (11) for the amplitude is satisfied when α is any function of the co-ordinates x, y and μ.

Equation (8) demands that the polarisation 1-form contains no $\mathrm{d}\nu$ components. By inspection of (9), any $\mathrm{d}\mu$ component in $\mathcal{P}$ will not show up in the final electromagnetic 2-form. Thus, our polarisation 1-form can be written

$$\mathcal{P} = P_x \mathrm{d}x + P_y \mathrm{d}y. \tag{15}$$

Equation (12) is satisfied by any functions P_x and P_y that are functions of x, y and μ. provided they are normalised according to (7) so that

$$P_x\overline{P_x} + P_y\overline{P_y} = 1. \tag{16}$$

No $h(\mu)$ terms appear in the expressions for the phase, amplitude or polarisation, so this solution is not coupled to the gravitational wave. We can therefore conclude that a gravitational wave is transparent to an electromagnetic wave travelling in the same direction.

2.2 Antiparallel propagation

The Kerr–Schild chart, with its single, explicit gravitational wave term (4), describing plane-fronted gravitational radiation causing transverse tidal acceleration in the neighbourhood of an inertial observer is the most suitable arena for interpreting the effect of a gravitational wave on electromagnetic radiation. However, a temporary chart transformation often facilitates calculations. Our plane gravitational wave spacetime can be expressed locally in Rosen co-ordinates [24,26]. While the Kerr–Schild chart has one Killing vector, $\frac{\partial}{\partial \nu}$, the Rosen chart possesses five, and this high degree of symmetry renders our equations more easily tractable. The Eikonal equation and the equations of motion are solved in the Rosen chart and then the final expressions rewritten in terms of Kerr–Schild co-ordinates. This is detailed in Appendix A.1. The phase function is given by

$$S = \Omega V \tag{17}$$

where Ω is the frequency of the electromagnetic wave, and

$$\begin{aligned} V = \nu &- \frac{a'(\mu)}{a(\mu)}(x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta) \\ &- \frac{b'(\mu)}{b(\mu)}(x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta) \end{aligned} \tag{18}$$

$a(\mu)$ and $b(\mu)$ are arbitrary metric functions constrained by the ordinary differential equations

$$a''(\mu) = h(\mu)a(\mu) \tag{19}$$

$$b''(\mu) = -h(\mu)b(\mu). \tag{20}$$

These metric functions depend on the gravitational wave profile $h(\mu)$ given by (4). In Minkowski space, $a(\mu)$ and $b(\mu)$ are taken to equal unity, the phase reduces to $S = \Omega\nu = \Omega(t+z)$ and the ray paths will be integral curves of $\frac{\partial}{\partial\nu}$. In general, the phase and ray paths will depend on the gravitational wave profile.

The amplitude is

$$\alpha = \frac{C(V, X_0, Y_0)}{\sqrt{a(\mu)b(\mu)}} \tag{21}$$

where $C(V, X_0, Y_0)$ is an arbitrary function of V, X_0 and Y_0 with

$$X_0 = \frac{1}{a(\mu)}(x\cos^2\theta + y\sin\theta\cos\theta) + \frac{1}{b(\mu)}(x\sin^2\theta - y\sin\theta\cos\theta) \tag{22}$$

$$Y_0 = \frac{1}{a(\mu)}(x\cos\theta\sin\theta + y\sin^2\theta) + \frac{1}{b(\mu)}(-x\sin\theta\cos\theta + y\cos^2\theta) \tag{23}$$

X_0 and Y_0 reduce in the Minkowski limit to x and y. However, in gravitational wave space it not generally possible to write x or y as a linear combination of X_0, Y_0 and V. A Minkowski space transverse profile $C(x, y)$ will be replaced by $C(X_0, Y_0)$. Thus, a gravitational wave distorts the transverse amplitude profile of an electromagnetic wave. The $\sqrt{a(\mu)b(\mu)}$ factor in (21) can also cause distortion.

Finally, the polarisation 1-form is given by

$$\begin{aligned} \mathcal{P} = P_x\mathrm{d}x + P_y\mathrm{d}y &- \left(\frac{a'(\mu)}{a(\mu)}(P_x\cos\theta + P_y\sin\theta)(x\cos\theta + y\sin\theta)\right. \\ &\left. + \frac{b'(\mu)}{b(\mu)}(-P_x\sin\theta + P_y\cos\theta)(-x\sin\theta + y\cos\theta)\right)\mathrm{d}\mu \end{aligned} \tag{24}$$

where P_x and P_y are arbitrary functions of X_0, Y_0 and V. An interesting feature of this solution is the P_μ term. The electric and magnetic fields are given in terms of the electromagnetic 2-form F by (1). In the g-orthonormal basis given by (5) and (6), the electric field will now have a longitudinal component given by $-2\Re\left\{e^{iS}\frac{\partial S}{\partial \nu}P_\mu\right\}X_3$. The magnetic field also picks up a longitudinal term.

2.3 Intersection at an angle

Further exact solutions of (7), (8) and (10)–(12) are derived in the Rosen chart in Appendix B. These correspond to the electromagnetic and gravitational waves intersecting at an angle. The equations are kept simple by assuming that the light does not travel in the x-direction in the Minkowski limit, and that the gravitational wave has $x^2 - y^2$ polarisation. In principle, the analysis could be extended to other directions and gravitational wave polarisations by using the full Eikonal function at the start of Appendix B, defining a second angle of incidence and altering the set of basis functions used to derive α and $\mathcal{P}$. In the Kerr–Schild chart, the Eikonal function is given by

$$S = \Omega\left(\frac{y\sin\gamma}{b(\mu)} + \frac{1+\cos\gamma}{2}\left(\nu - \frac{a'(\mu)}{a(\mu)}x^2 - \frac{b'(\mu)}{b(\mu)}y^2\right) + \frac{B(\mu)(1-\cos\gamma)}{2}\right) \tag{25}$$

where

$$B(\hat{\mu}) = \int \frac{\mathrm{d}(\hat{\mu})}{b(\hat{\mu})^2} \tag{26}$$

When $\gamma = 0$ the antiparallel values of the previous section are recovered. In the Minkowski limit, when $a(\mu) = b(\mu) = 1$, the Eikonal reduces in non-null co-ordinates to

$$S = \Omega(t + z\cos\gamma + y\sin\gamma) \tag{27}$$

Ω can therefore be identified as the Minkowski space frequency of the wave and γ is the angle that the ray paths, which are integral curves of $\widetilde{\mathrm{d}S}$, make with the z-axis.

The amplitude is found to be

$$\alpha = \frac{C(X_\gamma, Y_\gamma, S)}{\sqrt{a(\mu)b(\mu)}} \tag{28}$$

We can select an arbitrary amplitude profile C provided it is a function of X_γ, Y_γ and S where S is the Eikonal function and

$$X_\gamma = \frac{x}{a(\mu)} \tag{29}$$

$$Y_\gamma = \frac{y\cos\gamma}{b(\mu)} + \frac{1}{2}\left(B(\mu) - \nu + x^2\frac{a'(\mu)}{a(\mu)} + y^2\frac{b'(\mu)}{b(\mu)}\right)\sin\gamma. \tag{30}$$

In the Minkowski limit, X_γ becomes x, while Y_γ reduces to $y\cos\gamma - z\sin\gamma$, which is the y co-ordinate rotated through an angle γ. To facilitate comparison with Minkowski space behaviour, it is convenient to use Cartesian co-ordinates rotated through an angle α in the (y, z) plane. This produces a natural z' axis along which the electromagnetic rays travel in the Minkowski limit, as well as a transverse y' axis. However, until approximations are considered, the equations for S, α and $\mathcal{P}$ become unwieldy in this chart due to the μ dependent gravitational wave terms. We will therefore keep the co-ordinate frame aligned with the gravitational wave for the duration of this section. The basis of functions is normalised so that $\mathrm{i}_{\widetilde{\mathrm{d}X_\gamma}}\mathrm{d}X_\gamma = \mathrm{i}_{\widetilde{\mathrm{d}Y_\gamma}}\mathrm{d}Y_\gamma = 1$. $\mathrm{d}S$ is null and orthogonal to $\mathrm{d}X_\gamma$ and $\mathrm{d}Y_\gamma$. Finally, the polarisation is

$$\mathcal{P} = P_X \mathrm{d}x + P_Y \mathrm{d}y + \left(P_Y \left(\frac{\sin\gamma}{1+\cos\gamma}\frac{1}{b(\mu)} - y\frac{b'(\mu)}{b(\mu)} \right) - P_X \frac{a'(\mu)}{a(\mu)} \right) \mathrm{d}\mu \quad (31)$$

where P_X and P_Y are arbitrary functions of X_γ, Y_γ and S normalised so that

$$P_X \overline{P_X} + P_Y \overline{P}_Y = 1. \quad (32)$$

As the electromagnetic 2-form F is unchanged by the adding multiples of $\mathrm{d}S$ to $\mathcal{P}$, the polarisation can be equivalently rewritten

$$\mathcal{P} = a(\mu) P_X \mathrm{d}X + b(\mu) P_Y \mathrm{d}Y. \quad (33)$$

This form of the polarisation allows us to find gravitational wave versions of given Minkowski space polarisation states. This is demonstrated in Sect. 2.4.

2.4 Application

In order to make contact with standard notation and describe a given Minkowski space electromagnetic waveform in a gravitational wave metric, the co-ordinate chart should first be aligned with the electromagnetic wave by rotating through γ:

$$z' = z\cos\gamma + y\sin\gamma \qquad y' = y\cos\gamma - z\sin\gamma \quad (34)$$

In the Minkowski limit, these rotated co-ordinates simplify our phase, amplitude and polarisation, so that $S = \Omega(z'+t)$, $\alpha = C(x, y', z')$ and $\mathcal{P} = P_X \mathrm{d}x + P_Y \mathrm{d}y'$. With F given by (9), the electric field given by (1) becomes

$$E = \Re\left\{ e^{i\Omega(z'+t)} \alpha\Omega P_X \right\} \frac{\partial}{\partial x} + \Re\left\{ e^{i\Omega(z'+t)} \alpha\Omega P_Y \right\} \frac{\partial}{\partial y'} \quad (35)$$

and we have a similar expression for B.

An elliptically polarised electromagnetic wave in Minkowski space travelling in the negative z-direction is conventionally expressed [29]

$$E = \Re\left\{ e^{i\Omega(z'+t)} E_0 \cos\phi \right\} \frac{\partial}{\partial x} + \Re\left\{ e^{i\Omega(z'+t)} E_0 \sin\phi e^{i\chi} \right\} \frac{\partial}{\partial y'} \quad (36)$$

where the polarisation ellipse is inclined at an angle ϕ with respect to the x-axis and the x- and y'-components differ in phase by χ. Thus, the wave (36) is expressed in our formalism by

$$P_X = \cos\phi \qquad P_Y = \sin\phi e^{i\chi} \qquad \alpha = \frac{E_0}{\Omega}. \tag{37}$$

Setting $\chi = 0$ gives pure linear polarisation, while setting $\phi = \frac{\pi}{4}$, $\chi = \frac{\pi}{2}$ gives pure circular polarisation. In order to describe the behaviour of this wave when entering a plane gravitational wave at an angle γ, the amplitude and polarisation are calculated by substituting (37) into (28) and (33), and then fed into (9) using (25) to give F. A given transverse profile in Minkowski space is generalised to gravitational wave space by replacing x and y with the basis functions X_γ and Y_γ. Thus, a circularly polarised Gaussian light beam with width parameter σ would be given by

$$F = \Re\left\{\frac{\alpha_0}{\Omega\sqrt{2a(\mu)b(\mu)}} e^{iS} e^{-\sigma^2(X_\gamma^2+Y_\gamma^2)}(a(\mu)\mathrm{d}X_\gamma + ib(\mu)\mathrm{d}Y_\gamma)\right\} \tag{38}$$

where α_0 is a constant and S, X_γ and Y_γ are given by (25), (29) and (30).

3 Harmonic gravitational waves

All the solutions to the Maxwell equations derived in the previous section include the gravitational wave dependence implicitly through the metric functions $a(\mu)$ and $b(\mu)$, given by (19) and (20). The next stage is to specify a gravitational wave profile so that these functions can be determined and the electromagnetic 2-form can be written explicitly. Take a harmonic gravitational wave profile given by

$$h(\mu) = H_0 \cos\omega\mu. \tag{39}$$

The solutions to Maxwell's equations derived in the previous section do not rely on the weak field limit. For certain gravitational wave profiles, including the square "shock wave" described in [26], there exist exact, analytic solutions to (19) and (20), and the electromagnetic 2-form can then be applied to gravitational waves of any intensity, provided the electromagnetic wavelength is made sufficiently small to justify the "locally plane waves" approximation.

However, for the harmonic gravitational wave, there is no simple, analytic solution to (19) and (20), and we must resort to approximation. It is straightforward to verify that

$$a(\mu) = e^{-\frac{H_0}{\omega^2}\cos\omega\mu} \tag{40}$$

$$b(\mu) = e^{\frac{H_0}{\omega^2}\cos\omega\mu} \tag{41}$$

is correct to first order in $\frac{H_0}{\omega^2}$. In the literature (see, for example, the chapter by Thorne in [30]), the gravitational wave term in the weak-field metric is usually written as $h_0\cos\omega\mu$, where h_0 is a small, dimensionless strain. To first order in h_0, a comparison of the curvature tensors in each metric gives the relation

$$H_0 = \omega^2 h_0 \tag{42}$$

Having set a realistic value for H_0 and evaluated $a(\mu)$ and $b(\mu)$, we can now trace the paths of the light rays.

3.1 Paths of electromagnetic rays travelling antiparallel to gravitational wave

The paths taken by the electromagnetic rays are integral curves of the wave vector $\widetilde{\mathrm{d}S}$. These are derived in Appendix A.2 using the Rosen chart, transformed to the Kerr–Schild chart and written in terms of x, y and z to first order in the strain h_0, giving:

$$x \simeq x_o - h_o \cos(2\omega z)\,(x_o \cos 2\theta + y_o \sin 2\theta) \tag{43}$$

$$y \simeq y_o + h_o \cos(2\omega z)\,(y_o \cos 2\theta - x_o \sin 2\theta). \tag{44}$$

Thus, the rays of light experience oscillatory deflections when travelling antiparallel to a harmonic gravitational wave. The deflections are proportional to the transverse position of the rays as well as the gravitational wave strain. This is due to the quadrupole nature of the metric perturbation: a ray travelling through the central axis, for example, will not be deflected. In this first approximation, the frequency of the light does not affect the ray paths.

It would be desirable to plot a group of ray paths at different points in the (x, y) plane, as the gravitational wave's polarisation will cause rays at different points to have distinct behaviour. However, the maximum transverse distance each ray is deflected is equal to its transverse position multiplied by the (extremely small) gravitational wave strain. The deflections will thus be too small to show up in such a plot. We can, however, produce a schematic plot where the deflection is 'scaled up' in order to produce a qualitative picture of the effect of a ray's starting position on its behaviour. Such plots are shown in Figs. 1 and 2 for a gravitational wave with $x^2 - y^2$ polarisation. If the gravitational wave has $2xy$ polarisation, the entire set of curves will be rotated by $\frac{\pi}{4}$ radians, reflecting the fact that one can transform between the gravitational wave polarisation states by rotating the co-ordinate chart. This is described in more detail in Appendix A.2.

3.2 Paths of light rays entering gravitational wave at an angle

The analysis detailed in Sect. 3.1 and Appendix A.2 was repeated for electromagnetic waves entering the gravitational wave at an angle of γ. Parametrised integral curves of (25) were calculated as Taylor series around $h_0 = 0$. Rotating the co-ordinates according to (34) allowed the parameter to be eliminated, so that the paths can be expressed

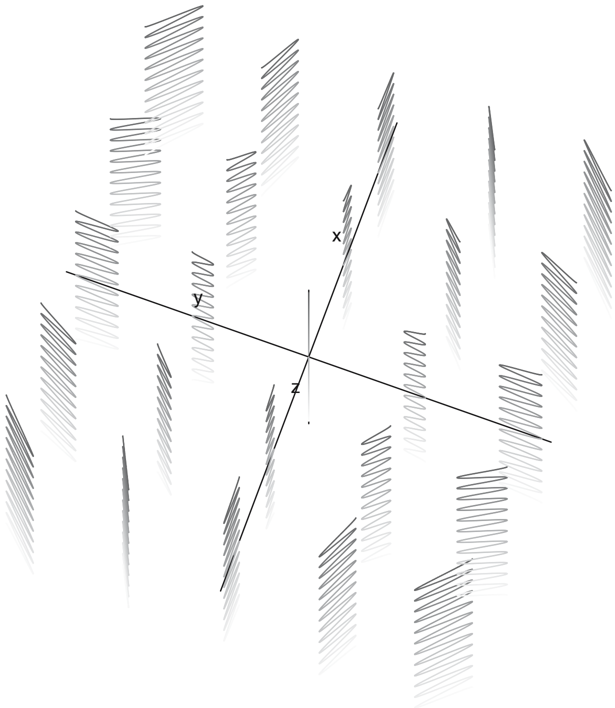

Fig. 1 Integral curves of the wave vector $\widetilde{\mathrm{d}S}$ give the ray paths for light. Light rays travelling antiparallel to a gravitational wave of $x^2 - y^2$ polarisation will additionally undergo oscillatory motion in the transverse direction. The size and direction of these oscillations depends on the transverse position of the ray and stems from the quadrupole nature of the gravitational wave. Here, the rays are traced out for a section of space near the co-ordinate origin. A ray passing through the origin simply travels undeflected along the z-axis

$$x = x_0 \left(1 - h_0 \cos\left(\omega z(1 + \cos\gamma)\right)\right) \tag{45}$$

$$\begin{aligned} y' = {y_0}' + h_0 \Big\{ & ({y_0}' \cos\gamma + z' \sin\gamma) \cos\gamma \cos\left(\Omega z'(1 + \cos\gamma)\right) \\ & + \sin\left(\Omega z'(1 + \cos\gamma)\right) \sin\gamma \left[\frac{-x_0 + {y_0}' \cos\gamma + z' \sin\gamma}{2} - \frac{1}{\omega(1 + \cos\gamma)}) \right] \Big\}. \end{aligned} \tag{46}$$

In the Minkowski limit, the rays travel along parallel to the z'-axis, while for $\gamma = 0$ the antiparallel values of the previous section are recovered.

3.3 Magnitudes of the electric and magnetic fields

The gravitational wave will also affect the electric and magnetic field magnitudes $g(E, E)$ and $g(B, B)$.[4] The metric dual of the electric field is the 1-form:

[4] In index notation, this is written $E_a E^a$ and $B_a B^a$.

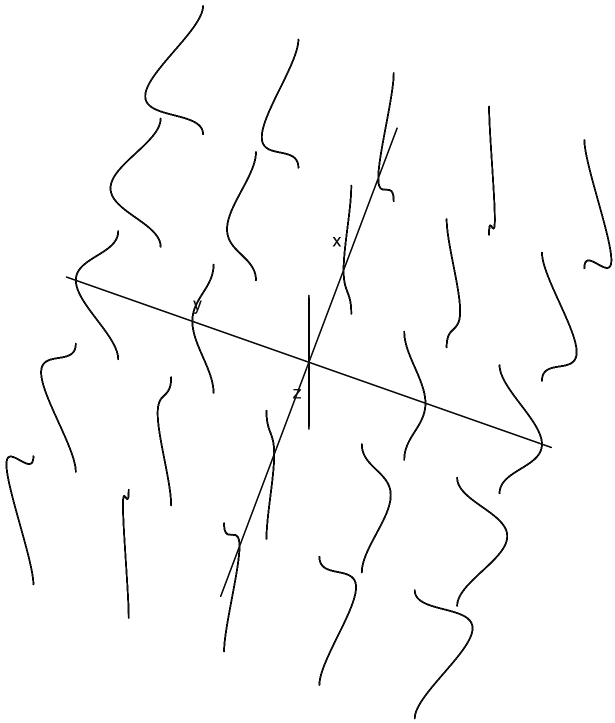

Fig. 2 One oscillation cycle is shown for the integral curves of Fig. 1. When $z = 0$, the set of rays is closest to the y-axis and furthest away from the x-axis. Half a cycle later, this has been reversed

$$\widetilde{E} = \Re\{\alpha e^{iS} \mathrm{i}_{X_0}(\mathrm{d}S \wedge \mathcal{P})\} \tag{47}$$

where X_0 is given by (5). We adopt the Gaussian amplitude profile described in Sect. 2.4:[5]

$$\alpha = \frac{\alpha_0}{\Omega\sqrt{a(\mu)b(\mu)}} e^{-\sigma^2(X_\gamma^2 + Y_\gamma^2)} \tag{48}$$

The fields are evaluated in Appendix C for a harmonic gravitational wave. A general expression is derived for light of arbitrary, constant polarisation. This is then evaluated for linearly and circularly polarised light.

3.3.1 Linear polarisation

The electric field amplitude $g(E, E)$ of linearly polarised light in a gravitational wave is derived in Appendix C.1. The resulting complicated expression can be simplified by removing all but the dominant terms in the regime of high electromagnetic wave frequency (Ω) and low gravitational wave frequency (ω) for which the WKB

[5] Such a profile can be used, for example, to model a laser beam, which can be accurately described by the WKB approximation in the weak gravitational wave limit due to its short wavelength.

approximation is valid. This gives

$$g(E, E) \approx \alpha_o^2 e^{-2\sigma^2(x^2+y'^2)} \left\{ \cos(\Omega(t+z')) + h_o \frac{\Omega}{\omega} \sin(2\Omega(t+z')) \right. \\ \left. \times \sin(\omega(t + y' \sin\gamma - z' \cos\gamma))(1-\cos\gamma) \right\} \tag{49}$$

provided $\gamma \neq 0$. The change in $g(E, E)$ caused by the gravitational wave is the term proportional to h_0. When the angle of incidence $\gamma = 0$, so that the electromagnetic beam is antiparallel to the gravitational wave, this dominant gravitational wave term vanishes. The field magnitude perturbations will be much smaller, and will be determined by the second largest part of the general equation given in Appendix C.1, giving

$$g(E, E) \approx \alpha_o^2 e^{-2\sigma^2(x^2+y^2)} \left\{ \cos(\Omega(t+z)) + h_o (x^2 - y^2) \right. \\ \left. \times \Big(\Omega\omega \sin(\omega(t-z)) \sin(2\Omega(t+z)) - 4\sigma^2 \cos^2(\Omega(t+z)) \cos(\omega(t-z)) \Big) \right\}. \tag{50}$$

The $4\sigma^2 \cos^2(\Omega(t+z)) \cos(\omega(t-z))$ term comes from the beam's amplitude profile (and would vanish for a plane electromagnetic wave where $\sigma = 0$), while the $\Omega\omega \sin(\omega(t-z)) \sin(2\Omega(t+z))$ term arises from the phase. The $x^2 - y^2$ part indicates that, for antiparallel propagation, the electric field perturbation takes on the quadrupole properties of the gravitational wave. Equation (50) can be adapted to consider a gravitational wave of arbitrary polarisation 2θ by rotating the (x, y) plane through an angle θ, giving

$$g(E, E) \approx \alpha_o^2 e^{-2\sigma^2(x^2+y^2)} \left\{ \cos(\Omega(t+z)) + h_o \Big((x^2 - y^2)\cos(2\theta) - 2xy \sin(2\theta) \Big) \right. \\ \left. \times \Big(\Omega\omega \sin(\omega(t-z)) \sin(2\Omega(t+z)) - 4\sigma^2 \cos^2(\Omega(t+z)) \cos(\omega(t-z)) \Big) \right\}.$$

Equations (49), (50) and (51) show that, to first approximation, the behaviour of the electromagnetic beam does not depend on the direction of its linear polarisation. The magnetic field is given by

$$\widetilde{B} = -\Re \left\{ \alpha e^{iS} \mathrm{i}_{X_0} * (\mathrm{d}S \wedge \mathcal{P}) \right\}. \tag{51}$$

Inspection of the derivation in Appendix C.1 reveals that the dominant terms of the magnetic field magnitude will experience the same gravitational wave induced perturbations as the electric field.

3.3.2 Circular polarisation

The electric field magnitude for circularly polarised light entering a gravitational wave at an angle γ is derived in Appendix C.2. In the high Ω, low ω regime, we can write

$$g(E,E) \approx {\alpha_o}^2 e^{-2\sigma^2(x^2+y^2)} \left\{ \frac{1}{2} - \frac{4\sigma^2 h_o}{\omega} y' \sin(\omega(t + y' \sin\gamma - z' \cos\gamma)) \sin\gamma \right\} \tag{52}$$

provided the angle of incidence γ is not close to zero. This expression is quite different to the linearly polarised value (49). In particular, the $\sigma^2 y'$ factor replacing the Ω factor will have implications for the size of the perturbation and its transverse variation.

For the antiparallel case, when $\gamma = 0$, the gravitational wave part of (52) vanishes. For a low gravitational wave frequency, the equation derived in Appendix C.2 now reduces to

$$g(E,E) \approx h_o \alpha_o e^{-2\sigma(x^2+y^2)} \left(\frac{1}{2} - 4h_o\sigma^2 \cos^2(\Omega(t+z)) \cos(\omega(t-z)) \right) (x^2 - y^2). \tag{53}$$

The analogue to (53) for linearly polarised light was (50). The gravitational wave induced perturbation in (53) matches the amplitude profile part of in (50). Thus, for large σ and small ω, a beam of light travelling in the opposite direction to the gravitational wave will be affected in the same way whether it is linearly or circularly polarised. In the homogeneous plane wave limit, where $\sigma = 0$, the gravitational wave part of (53) vanishes. A rotation of the (x, y) plane generalises (53) to describe arbitrary gravitational wave polarisations, giving:

$$\begin{aligned} g(E,E) = e^{-2\sigma^2(x^2+y^2)} &\left(\frac{1}{2} - 4h_0\sigma^2 \cos^2(\Omega(z+t)) \cos(\omega(z-t)) \right) \\ &\times \left(\left((x^2 - y^2)\cos(2\theta) - 2xy\sin(2\theta) \right) \right). \end{aligned} \tag{54}$$

4 Conclusions

Expressions were obtained in Sect. 2 for an electromagnetic wave in the Kerr–Schild plane gravitational wave metric with given Minkowski space phase, amplitude and polarisation. The electromagnetic wavelength was taken to be short compared with the radius of curvature of spacetime and the length scale over which the fields change properties, but no other assumptions about the strength of the gravitational wave were required. The gravitational wave dependence of these expressions stems from the metric functions $a(\mu)$ and $b(\mu)$ defined by ordinary differential equations (19) and (20) which contain the longitudinal gravitational wave profile. Where these ODEs have exact, analytic solutions, for example in the case of the gravitational shock wave, then the Maxwell equations can be satisfied to an arbitrary degree of accuracy by constraining the electromagnetic wavelength.

Section 2.1 considered electromagnetic and gravitational waves propagating in the same direction. The gravitational wave was not found to affect the electromagnetic radiation. Section 2.2 gave expressions for the amplitude, polarisation and phase (and hence wave vector and propagation direction) for electromagnetic waves travelling in the opposite direction to the gravitational wave, while Sect. 2.3 considered the electromagnetic wave approaching the gravitational wave at an angle. In each case, the amplitude, polarisation and phase were found to contain gravitational wave dependent terms, while the polarisation picks up an extra component that is longitudinal with respect to the average propagation direction.

In Sect. 3, these general results were explored for the case of harmonic gravitational waves. The paths of light rays travelling in the opposite direction to gravitational waves were found to oscillate sinusoidally according to (43) and (44). The directions of the paths were indicated in Figs. 1 and 2. Rays intersecting at an angle were found to travel in paths given by (45) and (46).

The electric and magnetic fields also experience gravitational wave induced changes in magnitude, described in Sects. 3.3.1 and 3.3.2. The field magnitudes for a Gaussian beam were obtained to first approximation in the short electromagnetic, long gravitational wavelength limit. The perturbation induced by the gravitational wave is largest when it meets the light beam at an angle. When the angle of incidence is zero, the perturbation is much smaller, and takes on the quadrupole properties of the gravitational wave.

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Appendix

A Antiparallel propagation

A.1 Calculation of S, α and $\mathcal{P}$

In this section, we derive the phase, amplitude and polarisation presented in Sect. 2.2. Solutions to (7), (8) and (10)–(12) corresponding to electromagnetic waves travelling in the opposite direction to the gravitational wave are obtained by means of a chart transformation.

The first step is to simplify (4) by rotating the (x, y) axis through an angle of θ, so that

$$\overline{x} = x\cos\theta + y\sin\theta \tag{55}$$
$$\overline{y} = -x\sin\theta + y\cos\theta. \tag{56}$$

A gravitational wave term given by (4) will now become $h(\mu)(\overline{x}^2 - \overline{y}^2)$. This implies that the $2xy$ and $x^2 - y^2$ states differ only by a rotation of $\frac{\pi}{4}$ radians. Next, transform to the Rosen chart, which is related to the Kerr–Schild chart by [24,26]:

$$\hat{\mu} = \mu \tag{57}$$

$$\hat{x} = \frac{\bar{x}}{a(\mu)} = \frac{1}{a(\mu)}(x\cos\theta + y\sin\theta) \tag{58}$$

$$\hat{y} = \frac{\bar{y}}{b(\mu)} = \frac{1}{b(\mu)}(-x\sin\theta + y\cos\theta) \tag{59}$$

$$\begin{aligned}\hat{\nu} &= \nu - \bar{x}^2\frac{a'(\mu)}{a(\mu)} - \bar{y}^2\frac{b'(\mu)}{b(\mu)} \\ &= \nu - \frac{a'(\mu)}{a(\mu)}(x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta) \\ &\quad -\frac{b'(\mu)}{b(\mu)}(x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta)\end{aligned} \tag{60}$$

where $a(\mu)$ and $b(\mu)$ are given by (19) and (20). The metric (3) becomes

$$g = -\frac{1}{2}(\mathrm{d}\hat{\mu}\otimes\mathrm{d}\hat{\nu} + \mathrm{d}\hat{\nu}\otimes\mathrm{d}\hat{\mu}) + a(\mu)^2\mathrm{d}\hat{x}\otimes\mathrm{d}\hat{x} + b(\mu)^2\mathrm{d}\hat{y}\otimes\mathrm{d}\hat{y}. \tag{61}$$

In this chart, the Eikonal equation (13) becomes

$$\frac{1}{a(\hat{\mu})^2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{b(\hat{\mu})^2}\left(\frac{\partial S}{\partial y}\right)^2 = 4\left(\frac{\partial S}{\partial\hat{\mu}}\right)\left(\frac{\partial S}{\partial\hat{\nu}}\right). \tag{62}$$

A simple solution to this equation is

$$S = \Omega\hat{\nu}. \tag{63}$$

This phase function describes an electromagnetic wave of frequency Ω travelling in the opposite direction to the gravitational wave. Equation (11) for the amplitude is satisfied by

$$\alpha = \frac{C(\hat{x},\hat{y},\hat{\nu})}{\sqrt{a(\mu)b(\mu)}}. \tag{64}$$

The integration constant $C(\hat{x},\hat{y},\hat{\nu})$ is an arbitrary function of $\hat{x}$, $\hat{y}$ and $\hat{\nu}$. As with the parallel case, (8) and (9) imply that the polarisation 1-form will not contain $\mathrm{d}\hat{\mu}$ or $\mathrm{d}\hat{\nu}$ components, and is therefore taken to be

$$\mathcal{P} = \hat{P}_x\mathrm{d}\hat{x} + \hat{P}_y\mathrm{d}\hat{y}. \tag{65}$$

Equation (12) is then satisfied by

$$\mathcal{P} = a(\hat{\mu})q_x(\hat{x},\hat{y},\hat{\nu})\mathrm{d}\hat{x} + b(\hat{\mu})q_y(\hat{x},\hat{y},\hat{\nu})\mathrm{d}\hat{y} \tag{66}$$

where q_x, q_y and q_ν are all arbitrary functions of $\hat{x}$, $\hat{y}$ and $\hat{\nu}$, normalised according to (7) to ensure

$$|q_x(\hat{x},\hat{y},\hat{\nu})|^2 + |q_y(\hat{x},\hat{y},\hat{\nu})|^2 = 1. \tag{67}$$

The phase, amplitude and polarisation, given by (63), (64) and (66) can now be converted to Kerr–Schild co-ordinates, yielding equations (17)–(24) presented in the main body of the paper. Due to the co-ordinate rotation, the polarisation can be simplified by rewriting the arbitrary functions q_x and q_y as the linear combination

$$q_x = P_x \cos\theta + P_y \sin\theta; \qquad q_y = P_y \cos\theta - P_x \sin\theta. \tag{68}$$

In Rosen co-ordinates, C, P_x and P_y are arbitrary functions of $\hat{x}$, $\hat{y}$ and $\hat{\nu}$. When transforming back to the Kerr–Schild chart, the independent variables $\hat{x}$ and $\hat{y}$ can be replaced with the linear combinations X_0 and Y_0, which reduce to x and y in the Minkowski limit regardless of θ. V is $\hat{\nu}$ written in the Kerr–Schild chart. X_0, Y_0 and V are defined in terms of x, y and ν in equations (18), (22) and (23).

A.2 Ray paths

Working initially in the Rosen chart, and then transforming to Kerr–Schild co-ordinates, we derive parametrised paths of light rays travelling antiparallel to the gravitational wave correct to first order in h_0. We can then eliminate the parameter to write the transverse deflections of the rays in terms of the longitudinal co-ordinate.

In the Rosen chart,

$$\widetilde{\mathrm{d}S} = -2\Omega \frac{\partial}{\partial \hat{\mu}}. \tag{69}$$

Thus, the rays are parametrised as follows:

$$\hat{\mu}(\tau) = -2\Omega\tau + C_1, \qquad \hat{\nu}(\tau) = C_2, \qquad \hat{x}(\tau) = C_3, \qquad \hat{y}(\tau) = C_4 \tag{70}$$

where C_1, C_2, C_3 and C_4 are constants. Now, taking $C_1 = C_2 = 0$, transform to Kerr–Schild co-ordinates using (57)–(60):

$$\mu(\tau) = \hat{\mu}(\tau) = -2\omega\tau \tag{71}$$

$$\begin{aligned}\nu(\tau) = {} & {C_3}^2 \frac{H_o}{\omega} \sin(\omega(-2\Omega\tau)) e^{-2\frac{H_o}{\omega}\cos(\omega(-2\Omega\tau))} \\ & - {C_4}^2 \frac{H_o}{\omega} \sin(\omega(-2\Omega\tau)) e^{2\frac{H_o}{\omega^2}\cos(\omega(-2\Omega\tau))}\end{aligned} \tag{72}$$

$$x(\tau) = C_3 e^{-\frac{H_o}{\omega^2}\cos(2\Omega\omega\tau)} \cos\theta - C_4 e^{\frac{H_o}{\omega^2}\cos(2\Omega\omega\tau)} \sin\theta \tag{73}$$

$$y(\tau) = C_3 e^{-\frac{H_o}{\omega^2}\cos(2\Omega\omega\tau)} \sin\theta + C_4 e^{\frac{H_o}{\omega^2}\cos(2\Omega\omega\tau)} \cos\theta. \tag{74}$$

The ray paths begin at $\tau = 0$ at the initial co-ordinates $(\mu, \nu, x, y) = (0, 0, x_0, y_0)$. By inverting (73) and (74) at $\tau = 0$ we can rewrite the constants C_2 and C_3 in terms of x_0 and y_0. Writing the parametrised curves in terms of the gravitational wave strain using (42) and expressing as a series about $h_0 = 0$ truncated after the first term gives

$$x(\tau) \simeq x_o - h_o \cos(2\omega\Omega\tau)\,(x_o \cos 2\theta + y_o \sin 2\theta) \tag{75}$$

$$y(\tau) \simeq y_o + h_o \cos(2\omega\Omega\tau)\,(y_o \cos 2\theta - x_o \sin 2\theta) \tag{76}$$

$$z(\tau) \simeq \Omega\tau + \frac{1}{2}h_o\omega\sin(2\omega\Omega\tau)\left[(x_0^2 - y_0^2)\cos 2\theta - 2x_0 y_0 \sin 2\theta\right] \qquad (77)$$

where $z(\tau) = \frac{1}{2}(\nu(\tau) - \mu(\tau))$. Rewriting (75) and (76) in terms of z, rather than τ gives a clearer picture of the paths of the rays in (x, y, z) space. τ only appears in the first order gravitational wave terms in these two equations so need only be evaluated to zero order in h_0. From (77), this is given by

$$\tau \simeq \frac{z}{\Omega}. \qquad (78)$$

Substituting this into (75) and (76) gives the ray paths (43) and (44) in Sect. 3.1.

B Propagation at an angle: calculation of S, α and $\mathcal{P}$

In this section, we repeat the calculation in Sect. 2.2 to obtain the equations for the phase, polarisation and amplitude of an electromagnetic wave that meets a gravitational wave at an angle γ. The resulting equations are presented in Sect. 2.3.

The Rosen metric Eikonal equation (62) is exactly satisfied by:

$$S = k_1\hat{x} + k_2\hat{y} + k_3\hat{\nu} + \frac{1}{4k_3}\int\left(\frac{{k_1}^2}{a(\hat{\mu})^2} + \frac{{k_2}^2}{b(\hat{\mu})^2}\right)\mathrm{d}\hat{\mu}. \qquad (79)$$

It will eventually be shown that this reduces to a plane wave in Minkowski space. The constants k_1, k_2 and k_3 are related to the wave's frequency and direction of propagation. For simplicity, assume that the light does not travel in the x-direction. We therefore set $k_1 = 0$, and can rename the other constants:

$$k_2 = \Omega\sin\gamma \qquad k_3 = \frac{\Omega}{2}(1 + \cos\gamma) \qquad (80)$$

so that the Eikonal function becomes:

$$S = \Omega\left(\hat{y}\sin\gamma + \frac{\hat{\nu}}{2}(1 + \cos\gamma) + \frac{B(\hat{\mu})}{2}(1 - \cos\gamma)\right) \qquad (81)$$

with $B(\hat{\mu})$ given by (26). (Although this is expressed in Kerr–Schild co-ordinates, μ is the same in both charts.) For the Eikonal function in (81), (11) becomes

$$\mathrm{d}\alpha(\widetilde{\mathrm{d}S}) = \frac{\Omega}{2}(1 + \cos\gamma)\left(\frac{a'(\hat{\mu})}{a(\hat{\mu})} + \frac{b'(\hat{\mu})}{b(\hat{\mu})}\right)\alpha. \qquad (82)$$

To obtain a general solution to (82), set up a basis $\{\mathrm{d}X_\gamma, \mathrm{d}Y_\gamma, \mathrm{d}\varphi, \mathrm{d}S\}$, where $\mathrm{d}\phi$ and $\mathrm{d}\psi$ are spacelike and $\mathrm{d}\varphi$ is null, so that

$$\mathrm{i}_{\widetilde{\mathrm{d}S}}\mathrm{d}X_\gamma = \mathrm{i}_{\widetilde{\mathrm{d}S}}\mathrm{d}Y_\gamma = 0 \qquad (83)$$

and

$$\mathrm{i}_{\widetilde{\mathrm{d}S}}\mathrm{d}\varphi \neq 0 \tag{84}$$

(83) can be satisfied by

$$X_\gamma = \hat{x} \tag{85}$$

$$Y_\gamma = \hat{y}\cos\gamma + \frac{1}{2}\left(B(\hat{\mu}) - \hat{\nu}\right)\sin\gamma \tag{86}$$

while (84) is satisfied by assuming that φ is a function of $\hat{\mu}$ only. Take

$$\alpha = \varphi(\hat{\mu})C(X_\gamma, Y_\gamma, S) \tag{87}$$

where C is our arbitrary amplitude profile, which is a function of X_γ, Y_γ and S. Equation (82) becomes

$$\varphi'(\mu) = -\frac{1}{2}\left(\frac{a'(\hat{\mu})}{a(\hat{\mu})} + \frac{b'(\hat{\mu})}{b(\hat{\mu})}\right)\varphi(\hat{\mu}) \tag{88}$$

which is satisfied with

$$\varphi(\hat{\mu}) = \frac{1}{\sqrt{a(\hat{\mu})b(\hat{\mu})}}. \tag{89}$$

Now for the polarisation 1-form $\mathcal{P}$. (8) will be satisfied by setting

$$\mathcal{P} = f_S\mathrm{d}S + f_X\mathrm{d}X_\gamma + f_Y\mathrm{d}Y_\gamma. \tag{90}$$

In general, a non-zero f_S is required to satisfy (12). Once a suitable $\mathcal{P}$ has been identified, the form of (9) allows this polarisation to be simplified by adding multiples of $\mathrm{d}S$.

Any function of x, y, μ and ν can be rewritten in our new basis as a function of X_γ, Y_γ, S and μ. It is convenient to rewrite our coefficients $f_n \in \{f_S, f_X, f_Y\}$ as $f_n = P_n(X_\gamma, Y_\gamma, S)\zeta_n(\mu)$. Then, (83) will ensure that $\nabla_{\widetilde{\mathrm{d}S}} f_n = P_n(X, Y, X)\mathrm{i}_{\widetilde{\mathrm{d}S}}\mathrm{d}\zeta_n$, simplifying (12), which is then solved for ζ_n. After some algebra, we find that (12) is satisfied by the polarisation

$$\mathcal{P} = a(\mu)P_X(X_\gamma, Y_\gamma, S)\mathrm{d}x + b(\mu)P_Y(X_\gamma, Y_\gamma, S)\mathrm{d}\left(y + \frac{B(\mu)\sin\gamma}{1 + cos\gamma}\right). \tag{91}$$

Subtracting $\frac{\sin\gamma}{1+\cos\gamma}P_Y(X, Y, S)b(\mu)\mathrm{d}S$ will not affect F, and allows the polarisation to be rewritten

$$\mathcal{P} = a(\mu)P_X(X_\gamma, Y_\gamma, S)\mathrm{d}X_\gamma + b(\mu)P_Y(X_\gamma, Y_\gamma, S)\mathrm{d}Y_\gamma \tag{92}$$

where P_X and P_Y are normalised according to (7). Transforming back to Kerr–Schild co-ordinates then yields the results stated in Sect. 2.3.

C Field magnitudes

Here, we derive expressions, correct to first order in the gravitational wave strain h_0, for the electric and magnetic field magnitudes of a Gaussian light beam with arbitrary polarisation. We then evaluate these expressions for circularly and linearly polarised light in Sects. C.1 and C.2. The results presented here are then described in Part 3 of the main text in the low ω, high Ω regime.

We adopt the g-orthonormal co-basis

$$\begin{aligned} e^0 &= \frac{\mathrm{d}\nu}{2} + (1-H)\frac{\mathrm{d}\mu}{2} \\ e^1 &= \mathrm{d}x \\ e^2 &= \cos\gamma \mathrm{d}y + \frac{\sin\gamma}{2}((1+H)\mathrm{d}\mu - \mathrm{d}\nu)) \\ e^3 &= \sin\gamma \mathrm{d}y + \frac{\cos\gamma}{2}(\mathrm{d}\nu - (1+H)\mathrm{d}\mu)\,. \end{aligned} \tag{93}$$

This is dual to the basis $\{X_a\}$ given in (5) and (6), rotated through the angle γ so that it is aligned with the electromagnetic beam. In Minkowski space, this would become $\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y', \mathrm{d}z'\}$. We calculate the metric dual of the electric field, given by (47). The Eikonal function S is given by equation (25). Expand about $h_o = 0$. To first order, we have

$$S \simeq \Omega(z' + t) + h_o \delta_S \tag{94}$$

where

$$\delta_S = \Omega \left\{ \cos\Omega M \sin\gamma (y'\cos\gamma + z'\sin\gamma) + \sin\omega M \left(\frac{1-\cos\gamma}{\omega} + \frac{\omega}{2}(1+\cos\gamma)((y'\cos\gamma + z'\sin\gamma)^2 - x^2) \right) \right\}. \tag{95}$$

The laser beam's Gaussian amplitude profile is given by (48). X_γ and Y'_γ can be rewritten as truncated Taylor series:

$$X_\gamma \simeq x + h_o \Delta_x \tag{96}$$
$$Y'_\gamma \simeq y' + h_o \Delta_y \tag{97}$$

where

$$\Delta_x = \cos\omega M \tag{98}$$

$$\Delta'_y = \sin\omega M \sin\gamma \left(\frac{\omega x^2 - \omega(y'\cos\gamma + z'\sin\gamma)^2}{2} - \frac{1}{\omega} \right) - \cos\omega M \cos\gamma (y'\cos\gamma + z'\sin\gamma) \tag{99}$$

so that the amplitude is approximated by

$$\alpha = \frac{\alpha_o}{\Omega} e^{-\sigma^2(x^2+y'^2)} \Big(1 - h_o\sigma^2(2x\Delta_x + 2y'\Delta'_y)\Big). \tag{100}$$

To evaluate the remaining part of (47), use the polarisation in (31). This gives

$$\mathrm{i}_{X_0}(\mathrm{d}S \wedge \mathcal{P}) = \Omega\left((P_x + h_o\delta_1)e^1 + (P_y + h_o\delta_2)e^2 + h_0\delta_3 e^3\right) \tag{101}$$

where

$$\begin{aligned}
\delta_1 &= P_x \cos(\omega M)(\cos\gamma - 1) + \omega \sin(\omega M)\sin\gamma(xP_y + yP_x) && (102)\\
\delta_2 &= 2P_y \cos(\omega M)(\cos\gamma - 1)\omega \sin(\omega M)\sin\gamma(P_y(z'\sin\gamma + y'\cos\gamma) - xP_x) && (103)\\
\delta_3 &= \omega\sin(\omega M)(1+\cos\gamma)(xP_x - y'P_y\cos\gamma - z'P_y\sin\gamma)\\
&\quad +2P_y\cos(\omega M)\sin\gamma\\
M &= t + y'\sin\gamma - z'\cos\gamma. && (104)
\end{aligned}$$

The analogous expression for the magnetic field is

$$\mathrm{i}_{X_0} * (\mathrm{d}S \wedge \mathcal{P}) = \Omega\left((-P_y + h_o\delta_4)e^1 + (P_x + h_o\delta_5)e^2 + h_0\delta_6 e^3\right) \tag{105}$$

where

$$\begin{aligned}
\delta_4 &= 2q_y\cos(\omega M)(1-\cos\gamma) + \omega\sin\gamma\sin(\omega M)(q_x - q_y y'\cos\gamma - q_y\sin\gamma)\\
\delta_5 &= q_x\cos(\Omega M)(\cos\gamma - 1) + \sin(\omega M)\sin\gamma(q_x y'\cos\gamma + q_x z'\sin\gamma + q_y x)\\
\delta_6 &= q_x\sin\gamma\cos(\omega M) - \omega\cos(\omega M)(1+\cos\gamma)(xq_y + y'q)x\cos\gamma + z'q_x\sin\gamma.
\end{aligned}$$

C.1 Linear polarisation

From Sect. 2.4, a linearly polarised wave with constant polarisation angle ϕ has the components

$$P_X = \cos\phi, \quad P_Y = \sin\phi. \tag{106}$$

As these quantities are real, the phase part of the electric field, $\Re\{e^{iS}\}$ will contribute a $\cos^2 S$ term to $g(E, E)$. Considering (94), this can be rewritten

$$\cos^2 S = \cos^2(\Omega(z'+t)) - h_o\sin(2\Omega(z'+t))\delta_S \tag{107}$$

giving

$$g(E,E) = \alpha_o^2 e^{-2\sigma^2(x^2+y'^2)} \Big\{ \cos(\Omega(t+z')) + h_o \Big(-\sin(2\Omega(t+z'))\delta_S + \cos^2(\Omega(t+z')) \Big(2\delta_1 \cos\phi + 2\delta_2 \sin\phi - 4\sigma^2(x\Delta_x + y\Delta_{y'})\Big)\Big)\Big\} \tag{108}$$

where Δ_x, Δ'_y, δ_S, δ_1 and δ_2 are defined in (98), (99), (95), (102) and (103).

C.2 Circular polarisation

Equation (37) implies that $P_X = \frac{1}{\sqrt{2}}$, $P_Y = \frac{i}{\sqrt{2}}$ corresponds to circularly polarised light. The electric field is again described by (100) and (101). Split δ_1 and δ_2 into real and imaginary parts, viz.

$$\delta_1 \equiv R_1 + iI_1 \qquad \delta_2 \equiv R_2 + iI_2. \tag{109}$$

Then, leaving α and S undefined for the moment and dropping the e^3 component which will not affect $g(E,E)$ to first order in h_0, we have

$$E = \alpha \Big\{ \Big(\frac{1}{\sqrt{2}} \cos S + h_o(R_1 \cos S - I_1 \sin S) \Big) e^1 + \Big(-\frac{1}{\sqrt{2}} \sin S + h_o(R_2 \cos S - I_2 \sin S) \Big) e^2 \Big\} \tag{110}$$

so that, considering (100),

$$g(E,E) = \alpha_o^2 e^{-2\sigma^2(x^2+y^2)} \Big\{ \frac{1}{2} + h_o \Big[-4\sigma^2(x\Delta_x + y'\Delta_{y'}) + \frac{1}{\sqrt{2}}(\cos^2(S)R_1 - \sin(S)\cos(S)(I_1+R_2) + \sin^2(S)I_2) \Big]\Big\} \tag{111}$$

where

$$\begin{aligned} R_1 &= \frac{1}{\sqrt{2}}(\cos(\omega M)(\cos\gamma - 1) + \omega y' \sin(\omega M)\sin\gamma) \\ I_1 &= \frac{1}{\sqrt{2}}(\omega x \sin(\omega M)\sin\gamma) \\ R_2 &= \frac{\omega x}{\sqrt{2}} \sin(\omega M)\sin\gamma \\ I_2 &= \frac{1}{\sqrt{2}}(2\cos(\omega\mu)(\cos\gamma - 1) + \omega\sin(\omega\mu)\sin\gamma(z'\sin\gamma + y'\cos\gamma)) \end{aligned} \tag{112}$$

Δ_x, Δ_y and M are given by (98), (99) and (104). As all these perturbed quantities are already first order in h_o, S can be given its zero order value of $z' + t$.

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