The Electrodynamics of Charged Continua

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Abstract

The dynamic behaviour of a distribution of charged particles is explored in terms of a self-permeable continuum model interacting self-consistently with the Maxwell field in vacuo. The model is developed using intrinsic tensor field theory and exploits to the full the relativistic structure of Minkowski spacetime. The model predicts the dynamic formation of domains that separate multi-component currents. To determine the location of such domains one is confronted with a new type of electrodynamic problem in which the number of charged current components is indefinite and the state of a finite bunch of charge may approach a highly mixed configuration reminiscent of turbulence. In this paper a formalism is established to describe such a multi-component system in terms of a flow map between 4−manifolds. This map inter-relates a complex Euler description of electrodynamics on spacetime with a computational Lagrangian scheme on a 4-dimensional body-time manifold, the domain of the flow map.

1 Introduction

Modern theories regard matter as being composed of interacting particles. A fruitful way to formulate these interactions is in terms of fields whose sources are related to the particles themselves [1], [2]. In most classical and quantum descriptions the fields and sources are sections of bundles over spacetime that fulfil the requirements placed on them by physical laws compatible with observation. Although the notion of a point particle is probably no more than a convenient idealisation in classical physics it forms the conceptual basis of many models of charged sources in electrodynamics [3], [5]. However the physical laws of electrodynamics sit uneasily in such a framework and require awkward manoeuvres to eliminate the self-interaction effects attributed to the fields produced by particles acting on themselves [4]. Models of matter that adopt charged continua as fundamental concepts can evade these issues [6], [7]. They have the advantage that notions of continuity and differentiability can then be controlled mathematically in the field equations that determine the dynamics of such continua. Furthermore by regarding the motion of charged continua as a subset of spacetime on which a smooth 4-velocity vector field is defined [8], the notion of a particle history can be recovered by identifying it with a particular parameterised integral curve of such a vector field. The distribution of integral curves can be specified by a measure (the proper charge density) obtained in principle by solving Maxwell's equations in conjunction with a force law based on the vanishing of the divergence of the total stress-energy-momentum tensor of the complete system.

Although this program leads to a well defined differential system for the electromagnetic fields, source density and velocity field it is rare that initial conditions exist leading to a smooth vector field on spacetime. The existence of crossing integral curves after a finite time means that the premise on which the model is based breaks down. This is a common occurrence in many fluid models for flow fields. In neutral gas dynamics such occurrences are identified with the formation of shocks and appeal is made to dissipative effects to ameliorate singularities that arise as a result. Although energy dissipation can arise in many dynamic configurations of charged continua appeal to a similar amelioration is not available for systems controlled solely by electrodynamic forces and a new physical scenario must be accommodated in the model.

The approach adopted here is to regard a charged continuum subject to purely electrodynamic forces as a self-permeable structure that permits self-penetration. It may be described as the continuum limit of a collection of dynamically interacting but non-colliding particles. Alternatively it may be regarded as a multi-component continuum described by a collection of vector fields on subsets of spacetime that have supports determined by the global dynamics of the entire system. At the interface of such subsets the measure describing the smooth source proper charge density may degenerate from a volume charge density to a surface or even line charge density. To determine the location of and interaction between these lower dimensional sources means that one is confronted with a new type of electrodynamic problem in which the number of charged current components is indefinite and the state of a finite bunch of charge may approach a highly mixed configuration reminiscent of turbulence.

In this paper a formalism is established to describe such a multi-component system in terms of a flow map whose properties follow from a generalisation of the single component scheme outlined above. This map inter-relates a complex Euler description of electrodynamics on spacetime with a computational Lagrangian scheme on a 4-dimensional body-time manifold, the domain of the flow map.

2 Fields over maps

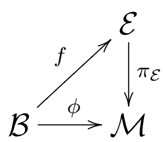
To establish notation the reader is reminded about the notion of a section over a map. Let $\phi : \mathcal{B} \to \mathcal{M}$ be a continuous map between manifolds $\mathcal{B}$ and $\mathcal{M}$ (assumed orientable), and let $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}$ be a bundle over $\mathcal{M}$. The notation

$$f \in \Gamma(\phi, \mathcal{E})$$

means that

$$f : \mathcal{B} \to \mathcal{E} \qquad \text{and} \qquad \pi_{\mathcal{E}} \circ f = \phi \tag{1}$$

i.e. the following diagram commutes



(2)

The map f may be referred to as a *section of $\mathcal{E}$ over ϕ* or an *$\mathcal{E}$ field over ϕ*. Scalar fields over $\mathcal{B}$ may also be regarded as scalar fields over ϕ i.e.

$$\Gamma\Lambda^0\mathcal{B} = \Gamma(\phi, \Lambda^0\mathcal{M}) \tag{3}$$

where $\Lambda^p\mathcal{M}$ is the bundle of exterior $p-$forms over $\mathcal{M}$ and in general $\Gamma\mathcal{E}$ denotes the space of sections of $\mathcal{E}$.

Let $\mathcal{M}$ be (Minkowski) $4-$dimensional spacetime with metric tensor g and associated Hodge map $\star$. The canonical measure on $\mathcal{M}$ is taken as $\star 1$. A general point in $\mathcal{M}$ will be written $p \in \mathcal{M}$. Denote $\mathcal{B} = \mathbb{R} \times \underline{\mathcal{B}}$ as the four dimensional *body-time* manifold, where $\underline{\mathcal{B}}$ is a three dimensional oriented *body manifold*. A general point in $\mathcal{B}$ will be written $P = (\tau, \underline{P}) \in \mathcal{B}$. Since $\mathcal{B} = \mathbb{R} \times \underline{\mathcal{B}}$, there exist projection maps

$$\tau : \mathcal{B} \to \mathbb{R}\ , \qquad \tau(\tau', \underline{P}) = \tau' \tag{4}$$

and

$$\underline{\pi} : \mathcal{B} \to \underline{\mathcal{B}}\ , \qquad \underline{\pi}(\tau', \underline{P}) = \underline{P} \tag{5}$$

These give rise to a preferred vector field $\mathcal{T} \in \Gamma T\mathcal{B}$ which may be written

$$\mathcal{T} = \partial_\tau \tag{6}$$

The model under consideration is constructed in terms of two fundamental fields: the flow field

$$C : \mathcal{B} \to \mathcal{M} \tag{7}$$

and the electromagnetic field

$$F \in \Gamma\Lambda^2\mathcal{M}, \tag{8}$$

each assumed to be continuous with degrees of differentiability as required.

3 Generic and Critical points

In general the flow map $C : \mathcal{B} \to \mathcal{M}$ is assumed neither surjective (onto) nor injective (one-to-one). As a result, for any point $p \in \mathcal{M}$, there may exists none, one, many or even an infinite number of roots $P \in \mathcal{B}$ which solve the equation $C(P) = p$. The set inverse of C is defined as

$$C^{-1}(U) = \big\{ P \in \mathcal{B} \,\big|\, C(P) \in U \big\} \qquad \text{for} \qquad U \subset \mathcal{M} \tag{9}$$

Thus

$$C^{-1}(\{p\}) = \big\{ P \in \mathcal{B} \,\big|\, C(P) = p \big\} \tag{10}$$

Let the function

$$N : \mathcal{M} \to \{0, 1, 2, \ldots\} \cup \{\infty\}\ ; \qquad N(p) = \text{number of elements of } C^{-1}(\{p\}) \tag{11}$$

be the number of roots of C. If the number of roots is finite write

$$C^{-1}(\{p\}) = \big\{ P_{[1]}, \ldots, P_{[N(p)]} \big\} \tag{12}$$

Square bracketed subscripts are used to label the roots and any structure associated with the root $P_{[i]}$.

Let $p \in \mathcal{M}$ and $\{P_{[1]}, \dots, P_{[N(p)]}\} = C^{-1}(\{p\})$ with $0 < N(p) < \infty$. The point p is said to be *generic* if there exist open disjoint neighbourhoods $U^{\mathcal{B}}_{[1]}, \dots, U^{\mathcal{B}}_{[N(p)]} \subset \text{interior}(\mathcal{B})$ and $U^{\mathcal{M}} \subset \mathcal{M}$ such that $P_{[i]} \in U^{\mathcal{B}}_{[i]}$, $p \in U^{\mathcal{M}}$,

$$\bigcup_{i=1}^{N(p)} U^{\mathcal{B}}_{[i]} = C^{-1}(U^{\mathcal{M}}) \tag{13}$$

and such that the maps

$$C_{[i]} = C|_{U^{\mathcal{B}}_{[i]}} : U^{\mathcal{B}}_{[i]} \to U^{\mathcal{M}} \tag{14}$$

are diffeomorphisms, i.e. $C_{[i]}$ is differentiable and invertible and $C^{-1}_{[i]}$ is differentiable.

For $p \in \mathcal{M}$ such that $N(p) = 0$, p is defined as generic if N is continuous at p. Recall that for integer valued functions, being continuous at a point is equivalent to being constant about that point.

It may be shown that a point $p \in \mathcal{M}$ is generic if and only if it obeys the following four conditions

- The set $\{P_{[1]}, \dots, P_{[N(p)]}\} = C^{-1}(\{p\})$ is finite (15)
- The Jacobian of C at $P_{[i]}) \neq 0$ (16)
- $P_{[i]} \notin \partial\mathcal{B}$ (17)
- $N : \mathcal{M} \to \{0, 1, 2, \dots\} \cup \{\infty\}$ is continuous at p (18)

All points which are not generic are called *critical*. Denote the set of all critical points $p \in \mathcal{M}$ by $\mathcal{M}_{\text{crit}}$ and the set of all generic points $p \in \mathcal{M}$ by $\mathcal{M}_{\text{gen}}$. From the definition of generic points the function N is continuous on $\mathcal{M}_{\text{gen}}$ but in general is not continuous on $\mathcal{M}$. For a given neighbourhood $U^{\mathcal{M}}$, all $p' \in U^{\mathcal{M}}$ have the same number of pre-images, $N(p')$, so $N(U^{\mathcal{M}}) = N(p')$ can be defined. The point $P \in \mathcal{B}$ will be called critical if it is the pre-image of a critical point $p \in \mathcal{M}$. Thus

$$P \quad \text{is critical in } \mathcal{B} \text{ if} \quad C(P) \quad \text{is critical in } \mathcal{M} \tag{19}$$

yielding the corresponding sets $\mathcal{B}_{\text{crit}}$ and $\mathcal{B}_{\text{gen}}$. The sets $\mathcal{M}_{\text{gen}} \subset \mathcal{M}$ and $\mathcal{B}_{\text{gen}} \subset \mathcal{B}$ are open sets and $\mathcal{M}_{\text{crit}} \subset \mathcal{M}$ and $\mathcal{B}_{\text{crit}} \subset \mathcal{B}$ are closed sets.

4 Differential equations for the flow field

The map (7) and the preferred vector field $\mathcal{T}$ give rise to the vector valued map

$$\dot{C} \in \Gamma(C, T\mathcal{M}) \,; \qquad \dot{C}(P) = C_\star(\mathcal{T}|_P) \in T_{C(P)}\mathcal{M} \tag{20}$$

This is the push forward, under $C_\star$, of the vector $\mathcal{T}_P \in T_P\mathcal{B}$ to the tangent fibre $T_{C(P)}\mathcal{M}$. In terms of coordinate maps $(x^\mu, \dot{x}^\mu)$ for $T\mathcal{M}$ the field $\dot{C}$ over C is given by

$$x^\mu(\dot{C}(P)) = x^\mu(C(P)) \qquad \text{and} \qquad \dot{x}^\mu(\dot{C}(P)) = \mathcal{T}(x^\mu \circ C)|_P \tag{21}$$

The field C also gives rise to a collection of curves given by

$$C_{\underline{P}} : \mathbb{R} \to \mathcal{M} \quad \text{for} \quad \underline{P} \in \underline{\mathcal{B}} \,, \qquad C_{\underline{P}}(\tau) = C(\tau, \underline{P}) \tag{22}$$

The total derivative of $C_{\underline{P}}$ is given by

$$\dot{C}_{\underline{P}} \in \Gamma(C_{\underline{P}}, T\mathcal{M}) \ , \qquad \dot{C}_{\underline{P}}(\tau') = \frac{dC_{\underline{P}}}{d\tau}(\tau') = C_{\underline{P}\star}(\partial_\tau|_{\tau'}) = \dot{C}(\tau', \underline{P}) \tag{23}$$

The coordinate τ is chosen so that $\dot{C}$ is a unit *timelike* field

$$g(\dot{C}, \dot{C}) = -1 \tag{24}$$

Furthermore let it be related to the Maxwell 2-form F on $\mathcal{M}$ by the equation of motion

$$\nabla_{\dot{C}} \dot{C} = \widetilde{i_{\dot{C}} F} \tag{25}$$

where for any vector field X on $\mathcal{M}$, the 1−form $\tilde{X} \equiv g(X, -)$.

The above yields an ordinary differential system for the curves $C_{\underline{P}}$ given by

$$\nabla_{\dot{C}_{\underline{P}}} \dot{C}_{\underline{P}} = \widetilde{i_{\dot{C}_{\underline{P}}} F} \tag{26}$$

In the coordinates $(x^\mu, \dot{x}^\mu)$ on $T\mathcal{M}$ this becomes

$$\ddot{C}^\mu_{\underline{P}}(\tau) + \Gamma^\mu{}_{\alpha\beta}(p) \dot{C}^\alpha_{\underline{P}}(\tau) \dot{C}^\beta_{\underline{P}}(\tau) = F^{\mu\alpha}(p) \dot{C}^\beta_{\underline{P}}(\tau) g_{\alpha\beta}(p) \tag{27}$$

where $p = C(\tau, \underline{P})$, $C^\mu_{\underline{P}}(\tau) = x^\mu(C_{\underline{P}}(\tau))$

$$\dot{C}^\mu_{\underline{P}}(\tau) = \dot{x}^\mu(C_{\underline{P}}(\tau)) = \frac{dC^\mu_{\underline{P}}(\tau)}{d\tau}, \quad \ddot{C}^\mu_{\underline{P}}(\tau) = \frac{d\dot{C}^\mu_{\underline{P}}(\tau)}{d\tau} = \frac{d^2 C^\mu_{\underline{P}}(\tau)}{d\tau^2}$$

and where $\Gamma^\mu{}_{\alpha\beta}(p)$ denote the coordinate components of the Levi-Civita connection ∇ at p.

5 Proper Charge Density and the map Δ

A Lagrangian current 3-form $\mathcal{J}$ on $\mathcal{B}$ will be identified with a measure (non-vanishing 3−form) $\underline{\mathcal{J}} \in \Gamma\Lambda^3\underline{\mathcal{B}}$ on $\underline{\mathcal{B}}$

$$\mathcal{J} = \underline{\pi}^\star \underline{\mathcal{J}} \in \Gamma\Lambda^3\mathcal{B} \tag{28}$$

This induces a natural measure on $\mathcal{B}$ given by

$$d\tau \wedge \mathcal{J} \in \Gamma\Lambda^4\mathcal{B} \tag{29}$$

Note that $d\mathcal{J} = d\underline{\pi}^\star\underline{\mathcal{J}} = \underline{\pi}^\star d\underline{\mathcal{J}} = 0$ and $i_{\mathcal{T}}\mathcal{J} = i_{\mathcal{T}}\underline{\pi}^\star\underline{\mathcal{J}} = \underline{\pi}^\star i_{\underline{\pi}_\star \mathcal{T}}\underline{\mathcal{J}} = 0$. The measure (29) enables us to define the map Δ related to the Jacobian of the flow field C

$$\Delta \in \Gamma\Lambda^0\mathcal{B} = \Gamma(\phi, \Lambda^0\mathcal{M}) \ , \qquad \Delta\, d\tau \wedge \mathcal{J} = C^\star(\star 1) \tag{30}$$

Furthermore since τ is used to define the unit timelike field $C_\star\mathcal{T}$ this can be identified as the inverse of the partial proper charge density scalar, given by

$$\rho : \mathcal{B} \to \mathbb{R} \cup \{\infty\} \ , \qquad \rho = \frac{1}{|\Delta|} \tag{31}$$

It will be shown below that the pull back by C of $i_{\dot{C}} \star 1$ is the pull back by C of the total electric current 3−form $\star\tilde{J}$ on $\mathcal{M}$. Regions on $\mathcal{B}$ where $\Delta = 0$ and hence $\rho = \infty$ may be identified with loci having a finite surface or line charge density.

6 Example

Before considering a coupled problem in which the flow field depends on F through Maxwell's equations it is of interest to examine an artificial but non-trivial flow field that exhibits features that may be expected to arise in the coupled situation.

Let $\underline{\mathcal{B}} = I \times \mathbb{R}^2$ with coordinates (σ, Y, Z), where $\sigma \in I \subset \mathbb{R}$ is the closed-open interval $I = \{\sigma | 0 \leq \sigma < 1\}$. This interval will demonstrate the various types of critical points that can arise[1]. Let spacetime $\mathcal{M}$ have Cartesian coordinates (t, x, y, z) with metric $g = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz$ and choose the measure on $\underline{\mathcal{B}}$ to be

$$\underline{\mathcal{J}} = K(\sigma) d\sigma \wedge dY \wedge dZ \tag{32}$$

where $K(\sigma) > 0$. Define the flow map by

$$\begin{aligned} (t, x, y, z) = C(\tau, \sigma, Y, Z) &= (\hat{t}(\tau, \sigma), \hat{x}(\tau, \sigma), Y, Z) \\ \text{where} \qquad \hat{t}(\tau, \sigma) = \sinh\tau + \sigma \,, &\quad \hat{x}(\tau, \sigma) = \cosh\tau \end{aligned} \tag{33}$$

The map Δ then follows from

$$\begin{aligned} \Delta\, d\tau \wedge \pi^\star \underline{\mathcal{J}} &= \Delta\, d\tau \wedge K(\sigma) d\sigma \wedge dY \wedge dZ = C^\star(\star 1) \\ &= (\cosh\tau\, d\tau + d\sigma) \wedge (\sinh\tau\, d\tau) \wedge dY \wedge dZ \end{aligned}$$

Hence

$$\Delta = -\frac{\sinh\tau}{K(\sigma)} \qquad \text{and} \qquad \rho = \frac{K(\sigma)}{|\sinh\tau|} \tag{34}$$

The map C possesses various types of critical points which may be written

$$\mathcal{M}_{\text{crit}} = \mathcal{M}^{\text{degen}}_{\text{crit}} \cup \mathcal{M}^{\text{closed}}_{\text{crit}} \cup \mathcal{M}^{\text{open}}_{\text{crit}}$$

The set $\mathcal{M}^{\text{degen}}_{\text{crit}}$ correspond to the points where $\Delta = 0$ i.e. $\tau = 0$ and hence $x = 1$:

$$\mathcal{M}^{\text{degen}}_{\text{crit}} = \big\{(t, x, y, z) \in \mathcal{M} \,\big|\, x = 1 \text{ and } 0 \leq t \leq 1\big\}$$

The set $\mathcal{M}^{\text{closed}}_{\text{crit}}$ is the image of $\partial\mathcal{B} = \{(\tau, 0) \in \mathcal{B}\}$

$$\mathcal{M}^{\text{closed}}_{\text{crit}} = \big\{(t, x, y, z) \in \mathcal{M} \,\big|\, x^2 - t^2 = 1 \text{ and } x \geq 1\big\}$$

The set $\mathcal{M}^{\text{open}}_{\text{crit}}$ must include the remaining critical points, i.e. those points where N changes but which are not in $\mathcal{M}^{\text{degen}}_{\text{crit}}$ or $\mathcal{M}^{\text{closed}}_{\text{crit}}$.

$$\mathcal{M}^{\text{open}}_{\text{crit}} = \Big\{\lim_{\sigma\to 1}(C(\tau, \sigma, Y, Z)) \,\Big|\, \tau, Y, Z \in \mathbb{R}\Big\} = \big\{(t, x, y, z) \in \mathcal{M} \,\big|\, x^2 - (t-1)^2 = 1 \text{ and } x \geq 1\big\}$$

Some of the points in $\mathcal{M}^{\text{open}}_{\text{crit}}$ have pre-images (e.g. given by $\mathcal{B}^{\text{open}}_{\text{crit}}$ below) while others do not. All critical points in $\mathcal{M}_{\text{crit}}$ are indicated on the right of figure 1.

The generic points $\mathcal{M}_{\text{gen}}$ are then the remaining open sets. There are 5 disconnected components of $\mathcal{M}_{\text{gen}}$ shown in figure 1 labelled

$$\mathcal{M}_{\text{gen}} = U^{\mathcal{M}}_{[0,\text{left}]} \cup U^{\mathcal{M}}_{[0,\text{right}]} \cup U^{\mathcal{M}}_{[1,\text{low}]} \cup U^{\mathcal{M}}_{[1,\text{high}]} \cup U^{\mathcal{M}}_{[2,\text{cent}]}$$

[1] Domains of this type are useful to accommodate fields which would otherwise have singularities in their domains of definition.

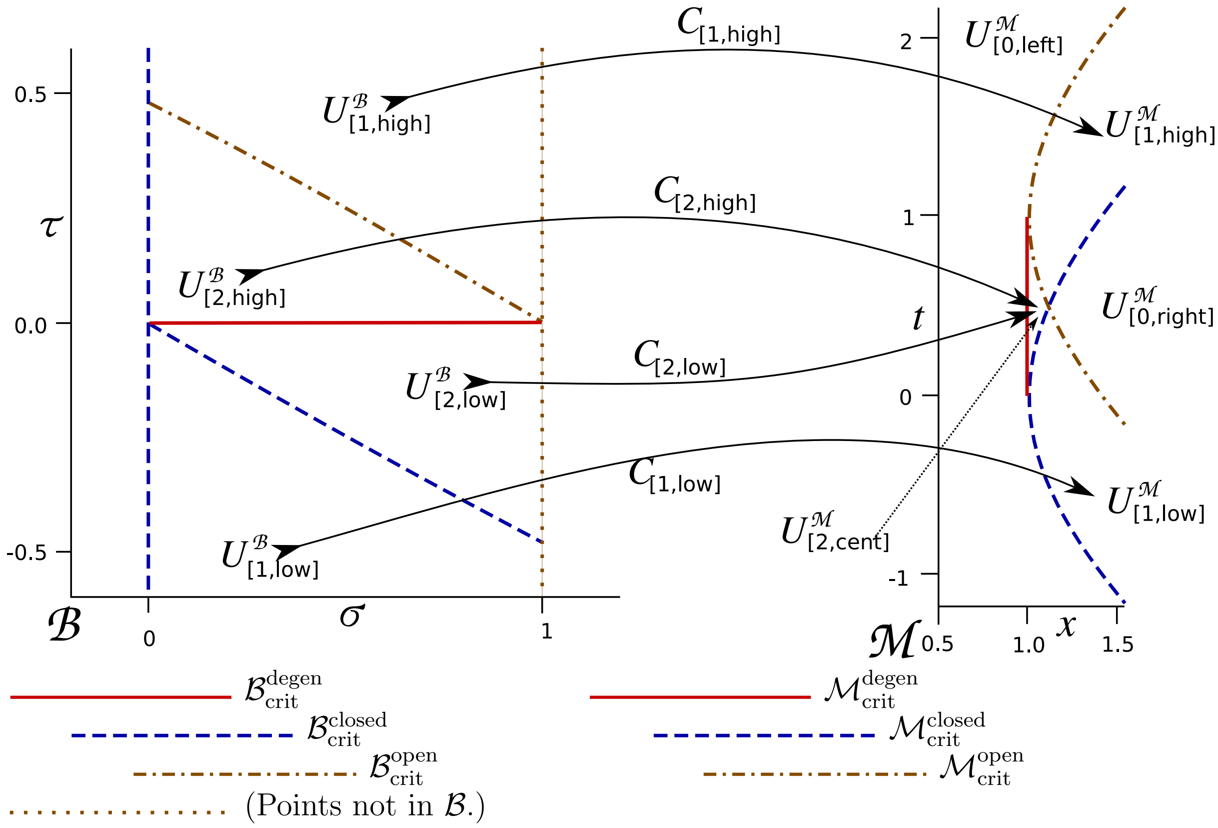


Figure 1: Anatomy of $\mathcal{B}$ and $\mathcal{M}$ with the map C between them. Coordinates Y, Z and y, z are suppressed.

The number in each case refers to the number of pre-images.

$$\begin{aligned}
U^{\mathcal{M}}_{[2,\text{cent}]} &= \left\{(t,x,y,z)\in\mathcal{M}\,\middle|\, x<\sqrt{1+t^2}\,,\ x<\sqrt{1+(t-1)^2}\,,\ x>1 \text{ and } 0<t<1\right\}\,,\\
U^{\mathcal{M}}_{[0,\text{left}]} &= \left\{(t,x,y,z)\in\mathcal{M}\,\middle|\, x<\sqrt{1+t^2}\,,\ x<\sqrt{1+(t-1)^2} \text{ and } (t,x,y,z)\notin U^{\mathcal{M}}_{[2,\text{cent}]}\right\}\,,\\
U^{\mathcal{M}}_{[0,\text{right}]} &= \left\{(t,x,y,z)\in\mathcal{M}\,\middle|\, x>\sqrt{1+t^2}\,,\ x>\sqrt{1+(t-1)^2}\right\}\,,\\
U^{\mathcal{M}}_{[1,\text{high}]} &= \left\{(t,x,y,z)\in\mathcal{M}\,\middle|\, x<\sqrt{1+t^2}\,,\ x>\sqrt{1+(t-1)^2}\right\}\,,\\
U^{\mathcal{M}}_{[1,\text{low}]} &= \left\{(t,x,y,z)\in\mathcal{M}\,\middle|\, x>\sqrt{1+t^2}\,,\ x<\sqrt{1+(t-1)^2}\right\}
\end{aligned}$$

The critical points on $\mathcal{B}$ are given by

$$\mathcal{B}_{\text{crit}} = \mathcal{B}^{\text{degen}}_{\text{crit}} \cup \mathcal{B}^{\text{closed}}_{\text{crit}} \cup \mathcal{B}^{\text{open}}_{\text{crit}}$$

where

$$\begin{aligned}
\mathcal{B}^{\text{degen}}_{\text{crit}} &= \left\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\middle|\,\tau=0\right\}\,,\\
\mathcal{B}^{\text{closed}}_{\text{crit}} &= \left\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\middle|\,\sigma=0\right\}\cup\left\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\middle|\,\sigma=-2\sinh\tau\right\}\,,\\
\mathcal{B}^{\text{open}}_{\text{crit}} &= \left\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\middle|\,\sigma=1-2\sinh\tau\right\}
\end{aligned}$$

These are shown on the left of figure 1. Thus the generic points of $\mathcal{B}$ are given by

$$\mathcal{B}_{\text{gen}} = U^{\mathcal{B}}_{[1,\text{high}]} \cup U^{\mathcal{B}}_{[1,\text{low}]} \cup U^{\mathcal{B}}_{[2,\text{high}]} \cup U^{\mathcal{B}}_{[2,\text{low}]}$$

where

$$U^{\mathcal{B}}_{[1,\mathrm{high}]} = \big\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\big|\,\sigma > 1-2\sinh\tau\big\}\ ,$$
$$U^{\mathcal{B}}_{[1,\mathrm{low}]} = \big\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\big|\,\sigma < -2\sinh\tau\big\}\ ,$$
$$U^{\mathcal{B}}_{[2,\mathrm{high}]} = \big\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\big|\,\sigma < 1-2\sinh\tau \text{ and } \tau>0\big\}\ ,$$
$$U^{\mathcal{B}}_{[2,\mathrm{low}]} = \big\{(\tau,\sigma,X,Y)\in\mathcal{B}\,\big|\,\sigma > -2\sinh\tau \text{ and } \tau<0\big\}$$

The diffeomorphisms from the components of $\mathcal{B}_{\mathrm{gen}}$ to the components of $\mathcal{M}_{\mathrm{gen}}$ are given by

$$\begin{aligned} &C_{[1,\mathrm{high}]} : U^{\mathcal{B}}_{[1,\mathrm{high}]} \to U^{\mathcal{M}}_{[1,\mathrm{high}]}\ , \qquad &&C_{[1,\mathrm{low}]} : U^{\mathcal{B}}_{[1,\mathrm{low}]} \to U^{\mathcal{M}}_{[1,\mathrm{low}]}\ ,\\ &C_{[2,\mathrm{high}]} : U^{\mathcal{B}}_{[2,\mathrm{high}]} \to U^{\mathcal{M}}_{[2,\mathrm{cent}]}\ , \qquad &&C_{[2,\mathrm{low}]} : U^{\mathcal{B}}_{[2,\mathrm{low}]} \to U^{\mathcal{M}}_{[2,\mathrm{cent}]} \end{aligned} \tag{35}$$

If the Maxwell field generated by sources defined by C is ignored one can readily find a background electromagnetic field that generates this flow. The C given in (33) obeys the equations of motion with a prescribed constant electric field in the above frame. This follows since

$$\dot{C}(\tau,\sigma) = \big(\cosh\tau\,\partial_t + \sinh\tau\,\partial_x\big)\big|_{C(\tau,\sigma)} \tag{36}$$

so that C generates a normalised timelike velocity field (24) and furthermore setting the prescribed (external) electromagnetic field to

$$F_{\mathrm{ext}} = dt\wedge dx \tag{37}$$

confirms that

$$\ddot{C}|_{C(\tau,\sigma)} = (\sinh\tau\,\partial_t + \cosh\tau\,\partial_x)\big|_{C(\tau,\sigma)} = i_{\widetilde{\dot{C}(\tau,\sigma)}}F_{\mathrm{ext}}$$

Thus C is a flow field in the background external electromagnetic field (37).

7 Equations for the electromagnetic field.

In the coupled situation, where there is no external applied electromagnetic field on $\mathcal{M}$ then F in a domain containing generic points obeys Maxwell's equations

$$dF = 0 \tag{38}$$

and

$$d\star F = -\star\widetilde{J} \tag{39}$$

with a current vector field $J\in\Gamma T\mathcal{M}_{\mathrm{gen}}$. At a generic point $p_0\in\mathcal{M}_{\mathrm{gen}}$ let $C_{[i]} : U^{\mathcal{B}}_{[i]} \to U^{\mathcal{M}}$ be the diffeomorphism given in (14). The partial current $J_{[i]}$ associated with C is defined by

$$J_{[i]}\in\Gamma TU^{\mathcal{M}}\ , \qquad J_{[i]} = \big(\rho\circ C^{-1}_{[i]}\big)\big(\dot{C}\circ C^{-1}_{[i]}\big) \tag{40}$$

that is for all $p\in U^{\mathcal{M}}$,

$$J_{[i]}|_p = \rho(P_{[i]})\dot{C}(P_{[i]}) \quad \text{where} \quad P_{[i]} = C^{-1}_{[i]}(p) \tag{41}$$

The total current J associated with C at all generic points is defined to be the sum of the partial currents

$$J|_{U^{\mathcal{M}}} = \sum_{i=1}^{N(U^{\mathcal{M}})} J_{[i]} \quad \in \Gamma T U^{\mathcal{M}} \tag{42}$$

The field F is then given at generic points by the Maxwell system above. Continuity conditions of F must be used to define F at critical points.

For an equivalent definition of the partial current, let $p \in \mathcal{M}_{\text{gen}}$, $P_{[i]} = C_{[i]}^{-1}(p)$ and observe that from (30)

$$\Delta(P_{[i]})\,(d\tau \wedge \mathcal{J})|_{P_{[i]}} = C^{\star}_{P_{[i]}}(\star 1)$$

where $C^{\star}_{P} : \Lambda^q_{C(P)}\mathcal{M} \to \Lambda^q_P\mathcal{B}$ is the pointwise pull back. Contracting with $i_{\mathcal{T}|_{P_{[i]}}}$ gives

$$\begin{aligned}\Delta(P_{[i]})\,\mathcal{J}|_{P_{[i]}} &= i_{\mathcal{T}|_{P_{[i]}}} C^{\star}_{P_{[i]}}(\star 1) = C^{\star}_{P_{[i]}}(i_{C_{[i]\star}(\mathcal{T}|_{P_{[i]}})} \star 1) = C^{\star}_{P_{[i]}}(i_{\dot{C}(P_{[i]})} \star 1)\\ &= C^{\star}_{P_{[i]}}(\star\widetilde{\dot{C}(P_{[i]})}) = C^{\star}_{P_{[i]}}\Big(\star\,\widetilde{\dot{C}(P_{[i]})}\rho(P_{[i]})\,\big|\Delta(P_{[i]})\big|\Big) = C^{\star}_{P_{[i]}}(\star\widetilde{J_{[i]}}|_p)\,\big|\Delta(P_{[i]})\big|\end{aligned}$$

Since $C_{[i]}$ is a diffeomorphism, so that $(C^{\star}_{[i]})^{-1} = (C^{-1}_{[i]})^{\star}$, one has the equivalent form

$$\star\widetilde{J_{[i]}}|_p = \operatorname{sign}(\Delta(P_{[i]}))\, C^{-1\star}_{P_{[i]}}(\mathcal{J}|_{P_{[i]}}) \tag{43}$$

Since (43) is true for all $p \in U^{\mathcal{M}}$ write (43) in terms of the pull back $C^{-1\star}_{[i]} : \Gamma\Lambda^3 U^{\mathcal{B}}_{[i]} \to \Gamma\Lambda^3 U^{\mathcal{M}}$

$$\star\widetilde{J_{[i]}}|_{U^{\mathcal{M}}} = \operatorname{sign}(\Delta|_{U^{\mathcal{B}}_{[i]}})\, C^{-1\star}_{[i]}(\mathcal{J}) \tag{44}$$

From (44) it follows that the source of F is closed:

$$d \star \widetilde{J_{[i]}}|_{U^{\mathcal{M}}} = \operatorname{sign}(\Delta|_{U^{\mathcal{B}}_{[i]}})\, dC^{-1\star}_{[i]}(\mathcal{J}) = \operatorname{sign}(\Delta|_{U^{\mathcal{B}}_{[i]}})\, C^{-1\star}_{[i]}(d\mathcal{J}) = 0$$

hence at generic points

$$d \star \widetilde{J} = 0 \tag{45}$$

There is also an integral relation which inter-relates (39), (42) and (44). Integral formulae offer a practical method to implement numerical discretisations of the above dynamical equations.

If $\phi : \mathcal{B} \to \mathcal{M}$ is a diffeomorphism, $S \subset \mathcal{M}$ a hypersurface of dimension s and $\omega \in \Gamma\Lambda^s\mathcal{M}$, then the theory of integration gives

$$\int_S \omega = \kappa \int_{\phi^{-1}S} \phi^{\star}\omega \tag{46}$$

where $\kappa = 1$ if ϕ^{-1} preserves the orientation of S and $\kappa = -1$ otherwise.

Given a 3−dimensional spatial hypersurface $S \subset \mathcal{M}$ such that the set $S \cap \mathcal{M}_{\text{crit}}$ has measure zero let $U^{\mathcal{B}}_{[i]}$ be one of the open sets in (13). Then from (46) and (44) the partial electric charge

$$Q_{[i]}[S \cap U^{\mathcal{M}}] = \int_{S\cap U^{\mathcal{M}}} \star\widetilde{J_{[i]}} = \kappa \int_{C^{-1}_{[i]}(S\cap U^{\mathcal{M}})} C^{\star}_{[i]}(\star\widetilde{J_{[i]}}) = \operatorname{sign}(\Delta|_{U^{\mathcal{B}}_{[i]}})\kappa \int_{C^{-1}_{[i]}(S)\cap U^{\mathcal{B}}_{[i]}} \mathcal{J}|_{U^{\mathcal{B}}_{[i]}}$$

Since $Q_{[i]}[S \cap U^{\mathcal{M}}]$ cannot change sign under evolution one must choose

$$\kappa = \operatorname{sign}(\Delta|_{U^{\mathcal{B}}_{[i]}})$$

and so

$$Q_{[i]}[S \cap U^{\mathcal{M}}] = \int_{C^{-1}_{[i]}(S) \cap U^{\mathcal{B}}_{[i]}} \mathcal{J}|_{U^{\mathcal{B}}_{[i]}} = \int_{C^{-1}_{[i]}(S) \cap U^{\mathcal{B}}_{[i]}} \mathcal{J}$$

Summing over all partial currents gives the total charge on $S \cap U^{\mathcal{M}}$

$$\int_{S \cap U^{\mathcal{M}}} \star \widetilde{J} = \sum_{i=1}^{N(U^{\mathcal{M}})} \int_{S \cap U^{\mathcal{M}}} \star \widetilde{J_{[i]}} = \sum_{i=1}^{N(U^{\mathcal{M}})} Q_{[i]}[S \cap U^{\mathcal{M}}] = \sum_{i=1}^{N(U^{\mathcal{M}})} \int_{C^{-1}_{[i]}(S) \cap U^{\mathcal{B}}_{[i]}} \mathcal{J}$$

Since the disjoint union

$$\bigcup_{i=1}^{N(U^{\mathcal{M}})} \left(C^{-1}_{[i]}(S) \cap U^{\mathcal{B}}_{[i]} \right) = C^{-1}(S \cap U^{\mathcal{M}})$$

one has $\int_{S \cap U^{\mathcal{M}}} \star \widetilde{J} = \int_{C^{-1}(S \cap U^{\mathcal{M}})} \mathcal{J}$. Taking the union of all the $U^{\mathcal{M}}$ yields $\int_{S \cap \mathcal{M}_{\text{gen}}} \star \widetilde{J} = \int_{C^{-1}(S \cap \mathcal{M}_{\text{gen}})} \mathcal{J}$ and since $S \cap \mathcal{M}_{\text{crit}}$ has measure zero $\int_S \star \widetilde{J} = \int_{C^{-1}(S)} \mathcal{J}$. But from (39) $\int_{\partial S} \star F = \int_S d \star F = - \int_S \star \widetilde{J}$. Hence

$$\int_{\partial S} \star F = - \int_{C^{-1}(S)} \mathcal{J} \tag{47}$$

This is a global identification of the total electric charge (associated with C) with the integral of $\star F$ over a regular $3-$dimensional spacelike hypersurface $S \subset \mathcal{M}$.

8 Example Continued

It is of interest to compute from Maxwell's equations the field F for the flow field prescribed in the example in section 6 above. This of course ignores the back reaction of the electromagnetic field on the source which is taken into account in the fully coupled system.

The inverses of the maps (35) are given by

$$\begin{aligned}
C^{-1}_{[1,\text{high}]}(t,x,y,z) &= (\tau = \operatorname{arccosh}(x), \sigma = t - \sqrt{x^2-1}, Y = y, Z = z)\,, \\
C^{-1}_{[1,\text{low}]}(t,x,y,z) &= (\tau = -\operatorname{arccosh}(x), \sigma = t + \sqrt{x^2-1}, Y = y, Z = z)\,, \\
C^{-1}_{[2,\text{high}]}(t,x,y,z) &= (\tau = \operatorname{arccosh}(x), \sigma = t - \sqrt{x^2-1}, Y = y, Z = z)\,, \\
C^{-1}_{[2,\text{low}]}(t,x,y,z) &= (\tau = -\operatorname{arccosh}(x), \sigma = t + \sqrt{x^2-1}, Y = y, Z = z)
\end{aligned} \tag{48}$$

The partial current $J_{[2,\text{high}]} \in \Gamma T U^{\mathcal{M}}_{[2,\text{cent}]}$ is given by (41), (48), (34) and (36) as

$$\begin{aligned}
J_{[2,\text{high}]}|_{(t,x,y,z)} &= \rho(\tau,\sigma,Y,Z)\dot{C}(\tau,\sigma,Y,Z) \\
&= \rho(\operatorname{arccosh}(x), t - \sqrt{x^2-1}, Y, Z)\dot{C}(\operatorname{arccosh}(x), t - \sqrt{x^2-1}, Y, Z) \\
&= \frac{K(t - \sqrt{x^2-1})}{\sinh(\operatorname{arccosh}(x))}(\cosh(\operatorname{arccosh}(x))\partial_t + \sinh(\operatorname{arccosh}(x))\partial_x) \\
&= K(t - \sqrt{x^2-1}) \left(\frac{x}{\sqrt{x^2-1}}\partial_t + \partial_x \right)
\end{aligned}$$

and likewise $J_{[2,\text{low}]} \in \Gamma TU^{\mathcal{M}}_{[2,\text{cent}]}$ is given by $J_{[2,\text{low}]}|_{(t,x,y,z)} = K(t+\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t - \partial_x\right)$.

Also $J_{[1,\text{high}]} \in \Gamma TU^{\mathcal{M}}_{[1,\text{high}]}$ is $J_{[1,\text{high}]}|_{(t,x,y,z)} = K(t-\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t + \partial_x\right)$ and $J_{[1,\text{low}]} \in \Gamma TU^{\mathcal{M}}_{[1,\text{low}]}$ is $J_{[1,\text{low}]}|_{(t,x,y,z)} = K(t+\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t - \partial_x\right)$.

Summing the partial currents in the five domains of $\mathcal{M}_{\text{gen}}$ gives

$$J|_{(t,x,y,z)} = \begin{cases} K(t-\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t + \partial_x\right) + K(t+\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t - \partial_x\right) & \\ \qquad\qquad \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[2,\text{cent}]} & \\ K(t-\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t + \partial_x\right) & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[1,\text{high}]} \\ K(t+\sqrt{x^2-1})\left(\frac{x}{\sqrt{x^2-1}}\partial_t - \partial_x\right) & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[1,\text{low}]} \\ 0 & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[0,\text{left}]} \\ 0 & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[0,\text{right}]} \end{cases}$$

Maxwell's equations (38) and (39) are solved with

$$F = E(t,x)dt \wedge dx \tag{49}$$

where

$$E(t,x) = \begin{cases} k(t+\sqrt{x^2-1}) - k(t-\sqrt{x^2-1}) + E_{-\infty} & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[2,\text{cent}]} \\ k(t+\sqrt{x^2-1}) + E_{-\infty} & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[1,\text{high}]} \\ k(1) - k(t-\sqrt{x^2-1}) + E_{-\infty} & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[1,\text{low}]} \\ E_{-\infty} & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[0,\text{left}]} \\ E = k(1) + E_{-\infty} & \text{if } (t,x,y,z) \in U^{\mathcal{M}}_{[0,\text{right}]} \end{cases}$$

with

$$k(\sigma) = \int_0^\sigma K(\sigma)$$

and $E_{-\infty}$ is a constant.

9 The Spherically Symmetric Coupled System

In this section the coupled system (24), (25), (38), (39) is explored where (40) and (42) define the dynamic sources. A spherically symmetric distribution of charge is considered to simplify the analysis. In spacetime $\mathcal{M}$ with standard spherical coordinates (t, r, θ, ϕ) and metric $g = -dt\otimes dt + dr\otimes dr + r^2 d\theta\otimes d\theta + r^2(\sin\theta)^2 d\phi\otimes d\phi$, write the electromagnetic field

$$F|_{(t,r,\theta,\phi)} = \frac{\mathcal{Q}(t,r)}{r^2} dt \wedge dr \tag{50}$$

for all $(t, r, \theta, \phi) \in \mathcal{M}$ with $r > 0$.

Let $\underline{\mathcal{B}} = I \times S^2$ where $I \subseteq \mathbb{R}_+$, with coordinates (σ, Θ, Φ). By spherical symmetry the solution can be described in terms of fields on and maps between 2-dimensional manifold. These will be shown in bold font.

Let $\boldsymbol{\mathcal{B}} = \mathbb{R} \times I$, coordinated by (τ, σ), be the projected body-time manifold and $\boldsymbol{\mathcal{M}} = \mathbb{R} \times \mathbb{R}_+$ coordinated by (t, r) be the projected spacetime manifold.

Let $\underline{\boldsymbol{\pi}} : \boldsymbol{\mathcal{B}} \to I$ be the projection and $\underline{\boldsymbol{\mathcal{J}}} \in \Gamma\Lambda^1 I$ be the choice of measure, so that $d\tau \wedge \boldsymbol{\mathcal{J}} \in \Gamma\Lambda^2\boldsymbol{\mathcal{B}}$ is a measure on $\boldsymbol{\mathcal{B}}$ where $\boldsymbol{\mathcal{J}} = \underline{\boldsymbol{\pi}}^\star \underline{\boldsymbol{\mathcal{J}}}$.

On $\boldsymbol{\mathcal{M}}$ the induced metric is $\boldsymbol{g} = -dt \otimes dt + dr \otimes dr$. This induces the flat Levi-Civita connection $\boldsymbol{\nabla}$ and Hodge map $\star$ with $\star 1 = dt \wedge dr$.

The projected flow map is

$$\boldsymbol{C} : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{M}}\,; \qquad \boldsymbol{C}(\tau,\sigma) = (\hat{t}(\tau,\sigma), \hat{r}(\tau,\sigma)) \tag{51}$$

Thus the 4-dimensional spherically symmetrical flow map C is given by

$$\begin{aligned} &t \circ C(\tau,\sigma,\Theta,\Phi) = \hat{t}(\tau,\sigma)\,, \quad r \circ C(\tau,\sigma,\Theta,\Phi) = \hat{r}(\tau,\sigma)\,, \\ &\theta \circ C(\tau,\sigma,\Theta,\Phi) = \Theta \quad \text{and} \quad \phi \circ C(\tau,\sigma,\Theta,\Phi) = \Phi \end{aligned} \tag{52}$$

Substituting (50) and (52) into the equations of motion (24) and (25) yields

$$\boldsymbol{g}(\dot{\boldsymbol{C}}, \dot{\boldsymbol{C}}) = -1 \tag{53}$$

$$\boldsymbol{\nabla}_{\dot{\boldsymbol{C}}} \tilde{\dot{\boldsymbol{C}}}|_{(\tau,\sigma)} = \frac{\mathcal{Q}(C(\tau,\sigma))}{\hat{r}(\tau,\sigma)^2} \star \tilde{\dot{\boldsymbol{C}}}(\tau,\sigma) \tag{54}$$

where

$$\dot{\boldsymbol{C}}(\tau,\sigma) = C_\star(\partial_\tau|_{(\tau,\sigma)}) \in \Gamma(\boldsymbol{C}, T\boldsymbol{\mathcal{M}}) \tag{55}$$

Here (53) is an equation over $\Gamma\Lambda^0\boldsymbol{\mathcal{B}}$ and (54) is an equation over $\Gamma(\boldsymbol{C}, T\boldsymbol{\mathcal{M}})$. In terms of the component maps $\hat{t}_\sigma(\tau) = \hat{t}(\tau,\sigma)$ and $\hat{r}_\sigma(\tau) = \hat{r}(\tau,\sigma)$, (54) gives the ordinary differential system

$$\ddot{\hat{t}}_\sigma(\tau) = \frac{\mathcal{Q}(\hat{t}_\sigma(\tau), \hat{r}_\sigma(\tau))}{\hat{r}_\sigma(\tau)^2}\,\dot{\hat{r}}_\sigma(\tau) \qquad \text{and} \qquad \ddot{\hat{r}}_\sigma(\tau) = \frac{\mathcal{Q}(\hat{t}_\sigma(\tau), \hat{r}_\sigma(\tau))}{\hat{r}_\sigma(\tau)^2}\,\dot{\hat{t}}_\sigma(\tau) \tag{56}$$

where $\dot{} = d/d\tau$.

Maxwell's equations yield on $\boldsymbol{\mathcal{M}}$

$$d\mathcal{Q} = \sum_{i=1}^{N(p)} \boldsymbol{C}_{[i]}^{-1\star}(\boldsymbol{\mathcal{J}}) \tag{57}$$

For a spherically symmetric charge distribution, the integral representation (47) reduces to

$$\int_{\partial \boldsymbol{S}} \mathcal{Q} = \int_{\boldsymbol{C}^{-1}(\boldsymbol{S})} \boldsymbol{\mathcal{J}} \tag{58}$$

where $\boldsymbol{S} \subset \boldsymbol{\mathcal{M}}$ is a curve and $\partial\boldsymbol{S}$ are its end points.

For $(t,r) \in \boldsymbol{\mathcal{M}}$ let $\boldsymbol{S}(t,r) = \{(t,r') \in \boldsymbol{\mathcal{M}}\,|\, 0 < r' < r\}$ then

$$\mathcal{Q}(t,r) - \mathcal{Q}(t,0) = \int_{\boldsymbol{C}^{-1}(\boldsymbol{S}(t,r))} \boldsymbol{\mathcal{J}}$$

$S(t,r)$ represents a spherically symmetric ball of radius r at time t. Since $\boldsymbol{\mathcal{J}}$ is closed in regular domains $\mathcal{Q}(t,0) = \mathcal{Q}_0$ must be independent of t and hence

$$\mathcal{Q}(t,r) = \int_{\boldsymbol{C}^{-1}(\boldsymbol{S}(t,r))} \boldsymbol{\mathcal{J}} + \mathcal{Q}_0 \tag{59}$$

If $\mathcal{Q}_0$ is non zero one has a point charge fixed at the centre of the ball. For currents that are smooth in regular domains $\mathcal{Q}_0 = 0$. Since (59) involves a field E at time t one must express (56) as a system of o.d.e's with evolution parameter t. Since $\dot{\boldsymbol{C}}$ is required to be unit future timelike, $\hat{t}_\sigma$ is strictly increasing so set

$$\check{\tau}_\sigma = (\hat{t}_\sigma)^{-1} \qquad \text{and} \qquad \check{r}_\sigma = \hat{r}_\sigma \circ \check{\tau}_\sigma \tag{60}$$

Then, with $\prime = d/dt$

$$\begin{aligned} &\dot{\hat{t}}_\sigma(\check{\tau}_\sigma(t)) = \frac{1}{\check{\tau}'_\sigma(t)}\,, \quad \ddot{\hat{t}}_\sigma(\check{\tau}_\sigma(t)) = -\frac{\check{\tau}''_\sigma(t)}{(\check{\tau}'_\sigma(t))^3}\,, \\ &\dot{\hat{r}}_\sigma(\check{\tau}_\sigma(t)) = \frac{\check{r}'_\sigma(t)}{\check{\tau}'_\sigma(t)} \quad \text{and} \quad \ddot{\hat{r}}_\sigma(\check{\tau}_\sigma(t)) = \frac{\check{r}''_\sigma(t)}{(\check{\tau}'_\sigma(t))^2} - \frac{\check{r}'_\sigma(t)\check{\tau}''_\sigma(t)}{(\check{\tau}'_\sigma(t))^3} \end{aligned} \tag{61}$$

and substituting (61) into (56) yields ordinary differential equations for $\check{\tau}_\sigma(t)$ and $\check{r}_\sigma(t)$. These equations involve $\mathcal{Q}$ so must be solved in conjunction with (59). To express (59) in terms of $\check{\tau}_\sigma(t)$ and $\check{r}_\sigma(t)$ observe that

$$\boldsymbol{C}^{-1}(\boldsymbol{S}(t, \check{r}_\sigma(t))) = \left\{(\tau, \sigma')\middle|\hat{t}_{\sigma'}(\tau) = t \quad \text{and} \quad \hat{r}_{\sigma'}(\tau) < \check{r}_\sigma(t)\right\} = \left\{(\check{\tau}(t), \sigma')\middle|\check{r}_{\sigma'}(t) < \check{r}_\sigma(t)\right\}$$

and, since $\underline{\boldsymbol{\pi}}$ is injective on the set $\left\{(\check{\tau}(t), \sigma')\middle|\check{r}_{\sigma'}(t) < \check{r}_\sigma(t)\right\}$

$$\begin{aligned} \int_{\boldsymbol{C}^{-1}(\boldsymbol{S}(t,\check{r}_\sigma(t)))} \boldsymbol{\mathcal{J}} &= \int_{\{(\check{\tau}(t),\sigma')|\check{r}_{\sigma'}(t)<\check{r}_\sigma(t)\}} \boldsymbol{\mathcal{J}} = \int_{\underline{\boldsymbol{\pi}}\{(\check{\tau}(t),\sigma')|\check{r}_{\sigma'}(t)<\check{r}_\sigma(t)\}} \underline{\boldsymbol{\pi}}^{-1\star}(\boldsymbol{\mathcal{J}}) \\ &= \int_{\{(\sigma')|\check{r}_{\sigma'}(t)<\check{r}_\sigma(t)\}} \underline{\boldsymbol{\mathcal{J}}} \end{aligned}$$

Therefore

$$\mathcal{Q}(t, \check{r}_\sigma(t)) = \int_{\{\sigma'|\check{r}_{\sigma'}(t)<\check{r}_\sigma(t)\}} \underline{\boldsymbol{\mathcal{J}}} + \mathcal{Q}_0 \tag{62}$$

Equations (56), (61) and (62) can now be integrated numerically by discretising σ and t. Discretise I by choosing $\sigma'_0 < \sigma_1 < \sigma'_1 < \sigma_2 < \sigma'_2 < \cdots < \sigma_m < \sigma'_m \in I$ with $\sigma'_0 = \inf(I)$ and $\sigma'_m = \sup(I)$. Let

$$Q_i = \int_{\sigma'_{i-1}}^{\sigma'_i} \underline{\boldsymbol{\mathcal{J}}}$$

Furthermore, since $\mathcal{Q}(t, r)$ will change with t, the o.d.e system generated from (56) and (61) will be integrated numerically in a series of time bands given by $t_0 < t_1 < \cdots < t_\lambda < \cdots < t_{\max}$. This yields the curves $\check{\tau}_i(t) \approx \check{\tau}_{\sigma_i}(t)$ and $\check{r}_i(t) \approx \check{r}_{\sigma_i}(t)$ for some initial conditions for $\check{\tau}_i(t_0)$, $\check{\tau}'_i(t_0)$, $\check{r}_i(t_0)$ and $\check{r}'_i(t_0)$, where for each t_λ and σ_i, $\mathcal{Q}(t, \check{r}(t))$ for the time interval $t_\lambda < t < t_{\lambda+1}$ is replaced by

$$\mathcal{Q}_i(t_\lambda) = \mathcal{Q}_0 + \sum_{\{j|\check{r}_j(t_\lambda)<\check{r}_i(t_\lambda)\}} Q_j$$

10 Conclusion

A formalism has been established for the description of the motion of electric charge under the influence of both external and self electromagnetic fields. The laws of classical covariant electrodynamics have been expressed in terms of a flow map between a structured body-time manifold $\mathcal{B}$ and Minkowski spacetime. By assuming that this map is not necessarily either surjective or injective, distinct domains in spacetime may be associated with possibly more than one pre-image in the body-time manifold. These pre-images in turn give rise to a complex of electric currents that determine the structure of the flow map via Maxwell's equations. The total proper charge density is a dynamic scalar related to the Jacobian of the flow map and a Lagrangian measure on a $3-$dimensional body manifold on $\mathcal{B}$.

A simple example of a non-trivial flow map is explicitly constructed corresponding to the plane symmetric motion of charge in a prescribed constant laboratory electric field. It is also demonstrated how Maxwell's equations are treated in the presence of a prescribed source corresponding to this non-trivial flow map. Finally a fully coupled system is considered in terms of the evolution of a spherically symmetric ball of charge from rest with an initially smooth gaussian distribution of charge. The evolution is calculated numerically by digitising the coupled equations of motion and Maxwell's equations. The results of this simulation (figure 2) indicate that the integral curves $\boldsymbol{C}_\sigma$ cross and that the initial crossing occurs within the charge distribution. As expected the ball of charge explodes outward but with some of the inner spheres of charge overlapping the outer ones. An interesting feature of these solutions is that although F is continuous across regions in spacetime where $\Delta = 0$ (and hence $\rho = \infty$) it is not in general differentiable. This is a general property of solutions where the sources can change discontinuously during the evolution of the coupled system.

The techniques established here have immediate application in accelerator science particularly in devices where charged bunches with large laboratory charge densities in ultra-relativistic motion are demanded [9]. They extend naturally to multi-component continua such as plasmas where the phenomena of "wave breaking" in wake-field accelerators and bubble regimes may benefit from an analysis in terms of relativistic flow maps with properties analogous to those presented in this paper.

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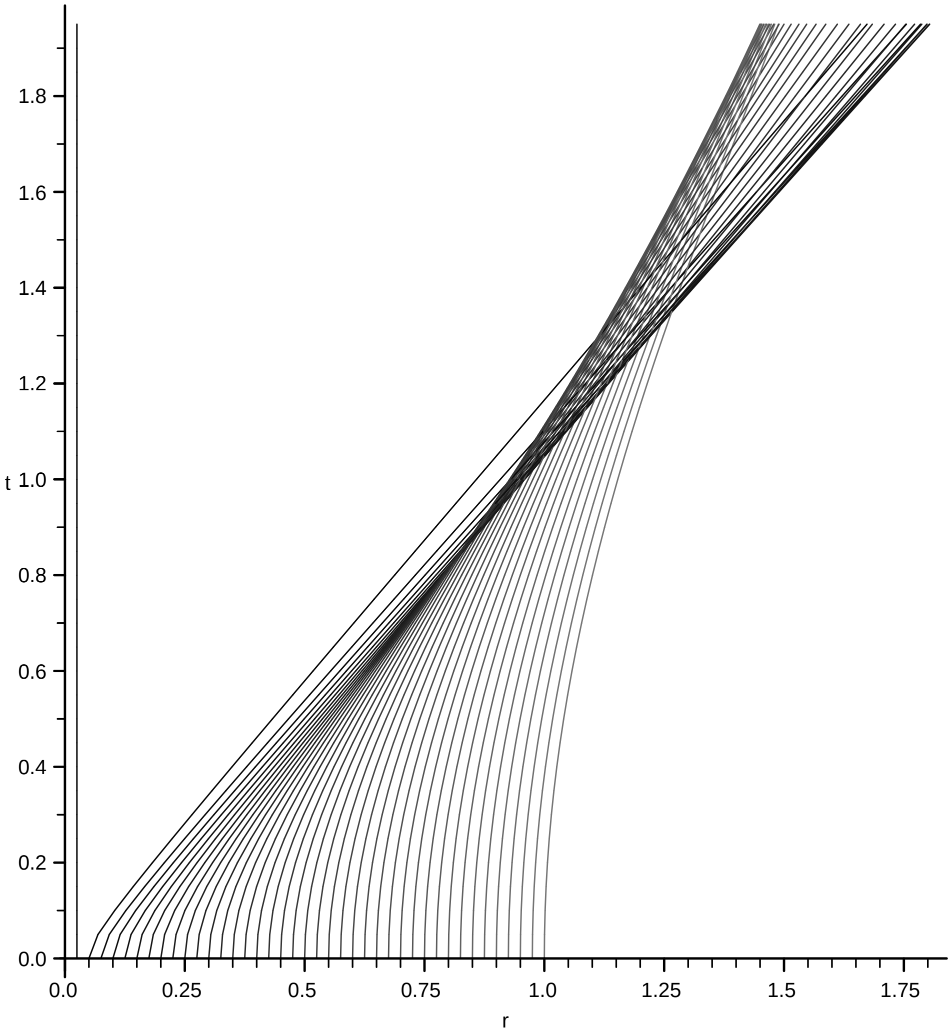

Figure 2: History of a spherically symmetric Gaussian ball of charge evolving from rest. The discretisation used $\sigma_i = i/40, i = 1 \ldots 40$, with $Q_i = 0.05 \exp(-(5\sigma_i)^2)$, $\mathcal{Q}_0 = 0$, $t_\lambda = \lambda/20, \lambda = 0 \ldots 40$. $\check{\tau}_i(t_0) = 0$, $\check{\tau}_i'(t_0) = 1$, $\check{r}_i(t_0) = \sigma_i$ and $\check{r}_i'(t_0) = 0$.
Evidence for the multi-component nature of the evolution is clearly visible as charge initially closer to the centre overtakes more slowly moving charge in the expanding ball. The curve furthest left at $r = 0.025$, is vertical since it corresponds to the innermost digitised shell inside of which there is no charge.