

Microscopic Connection between Bose-Einstein Condensation and the Two Fluid Model of Liquid ^4He

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Abstract

It is shown that a function $\psi(\vec{r})$ of the position co-ordinate \vec{r} can be defined in terms of the many particle Schrödinger wave function Ψ . It is shown that for the ideal Bose gas and correlated basis functions, $\psi(\vec{r})$ can be split into two parts in the presence of Bose Einstein condensation. One part, $\psi_S(\vec{r})$, is completely phase coherent over macroscopic length scales, while the other part, $\psi_N(\vec{r})$, has only short range phase coherence, with a phase which varies randomly over atomic length scales. It is postulated that this is also true in superfluid ^4He . It is shown that in a sufficiently large system ψ_S and ψ_N contribute separately to the fluid density, particle flow, the momentum distribution, the dynamic structure factor $S(\vec{q}, \omega)$ and the static structure factor $S(\vec{q})$. This is so only in a Bose condensed system and not when BEC is absent. It is shown that the two components satisfy the standard two fluid equations of liquid ^4He . The identification of ψ_S with the superfluid and ψ_N with the normal fluid provides a direct link between BEC and two fluid behaviour. The additional postulate that $\psi_S(\vec{r})$ is identical to $\psi(\vec{r})$ in the ground state, leads to the predictions that the condensate fraction f and the intensity of sharp peaks in $S(\vec{q}, \omega)$ are both exactly proportional to the superfluid fraction, in good agreement with experiment. The postulates also provide a simple explanation of the anomalous decreases in density and spatial order as the temperature is lowered in superfluid ^4He .

1. Introduction

According to the two fluid model¹, superfluid ^4He behaves as if it were a mixture of two fluids, freely intermingling, without any viscous interaction. These two fluids are termed the normal fluid and the superfluid and have temperature dependent densities. It is widely recognised that Bose-Einstein condensation (BEC) is fundamentally linked to two fluid behaviour in liquid ^4He . For example from the assumption that BEC is present, the two fluid equations can be derived using field theoretical methods^{2,3,4}. Further theoretical evidence for the link has been provided by path integral Monte Carlo calculations, which show that the development of BEC coincides with the appearance of a finite superfluid fraction^{5,6}. There is also strong experimental evidence for the presence of BEC in superfluid ^4He ^{7,8,9,10,11} with the most recent measurements^{9,11} giving a condensate fraction between 6% and 9% at temperatures below 1K, in good agreement with calculations using a variety of methods^{5,12,13,14}.

However the nature of the link between BEC and the two fluid model is not entirely clear. For example there is no theoretical basis for any link between the size of the superfluid fraction and the condensate fraction. The unique response of liquid helium to neutrons and X-rays is also still unexplained. Below the superfluid transition the dynamic structure factor, $S(\vec{q}, \omega)$, in superfluid ^4He contains sharp peaks, with amplitudes proportional to the superfluid fraction^{15, 16}. The reason for the appearance of these peaks and the temperature dependence of their amplitude is not well understood. Furthermore measurements of the static structure factor $S(\vec{q})$ show that in contrast to other liquids, superfluid ^4He becomes more spatially disordered^{17,18,19} as the temperature is lowered. The explanation of this behaviour has been a matter of discussion for many years^{20,21,22,23,24,25}. Another unique feature of superfluid ^4He is that the liquid becomes less dense as the temperature is reduced. Again this behaviour has no microscopic explanation.

In this paper, these questions are investigated using the “conditional wave function”^{26,27,28,29,30}. In section 2 the conditional wave function ψ is defined in terms of the many particle wave function and the properties of ψ are reviewed. In section 3 the concept of long range phase coherence (LRPC) is introduced and a quantitative measure of the “degree of long range phase coherence” D in ψ is defined. It is shown in section 4, that for ideal Bose gas and correlated basis functions, ψ can be divided into a completely coherent component ψ_S , with $D = 1$ and a completely incoherent component ψ_N , with $D = 0$. The basic postulate of the paper, presented in section 5, is that this is also true in superfluid ^4He . It is shown in section 5 that in a sufficiently large system, the incoherent and coherent components behave as non-interacting fluids, if BEC is present but not if BEC is absent. The coherent and incoherent

components are identified with the superfluid and normal fluid components respectively. In section 6 it is shown that the components satisfy the two fluid equations. The assumption that $\psi_s(\vec{r})$ is identical to $\psi(\vec{r})$ in the ground state, as in the IBG and CBF models, implies that the superfluid fraction should be exactly proportional to the condensate fraction, in good agreement with experiment. In section 7 it is shown that the dynamic structure factor can also be divided into independent contributions from the normal fluid and superfluid and that the intensity of sharp peaks in $S(\vec{q}, \omega)$ should be exactly proportional to the superfluid fraction. This is again in good agreement with experimental measurements. An explanation of the unique behaviour of the density and static structure factor is given in section 8. Section 9 contains a summary and discussion. The appendix contains a description of a one-dimensional model, which is used to illustrate the physical implications of the formalism throughout the paper.

2 Summary of Previous Work

The basic formalism of the conditional wave function approach is first summarised. The results of this section are derived in more detail in a previous paper³⁰. We consider a system of N particles, which occupy a volume V and take $\hbar = 1$. Periodic boundary conditions are used, so that the allowed momentum states are discrete and occupy a volume $1/V$ in momentum space. Sums over momentum states can be replaced by an integral, using the identity

$$\frac{1}{V} \sum_{\vec{p}} = \int d\vec{p} \quad (2.1)$$

2a Bose Einstein Condensation

The probability that a particle occupies a particular momentum state \vec{p} is defined as $n(\vec{p})/N$. It follows that the average number of particles $N_{\vec{p}}$ in a single momentum state is

$$N_{\vec{p}} = n(\vec{p}) \times N \quad (2.2)$$

In the absence of BEC, $N_{\vec{p}}$ is independent of system size for all \vec{p} , whereas when BEC is present, the number of particles in one of the momentum states, which for simplicity we take as the $\vec{p} = 0$ state, is proportional to N ,

$$N_0 = fN \quad (2.3)$$

where f is the condensate fraction. In the presence of BEC $n(\vec{p})$ has a sharp peak occupying a volume $1/V$ in momentum space and with a total weight f . In the limit $V \rightarrow \infty$, this peak is a Dirac delta function and the momentum distribution takes the form,

$$n(\vec{p}) = (1 - f)n_U(\vec{p}) + f\delta(\vec{p}) \quad (2.4)$$

where $n_U(\vec{p})$ is the momentum distribution of “uncondensed” atoms, i.e. atoms not in the $\vec{p} = 0$ state. The momentum distributions $n(\vec{p})$ and $n_U(\vec{p})$ are normalised functions.

$$\int n(\vec{p})d\vec{p} = \int n_U(\vec{p})d\vec{p} = 1 \quad (2.5)$$

2b The conditional wave function

The Schrödinger wave function corresponding to a particular many particle state i with energy E_i is denoted as

$\Psi_i(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_{N-1}, \vec{r}_N)$. The co-ordinates of an arbitrarily chosen particle are denoted as \vec{r} . Which particle is chosen is irrelevant due to the Bose symmetry of the wave function. The $N - 1$ co-ordinates of all other particles are denoted by \vec{s} , i.e. $\Psi_i(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_{N-1}, \vec{r}_N) = \Psi_i(\vec{r}, \vec{s})$. We consider a particular state i and particular configuration \vec{s} and define the conditional wave function as

$$\psi_i(\vec{r} | \vec{s}) = \Psi_i(\vec{r}, \vec{s}) / \sqrt{P(\vec{s})} \quad (2.6)$$

“Conditional” means that the detailed microscopic dependence of $\psi_i(\vec{r} | \vec{s})$ on \vec{r} , is determined by the particular i and \vec{s} chosen. $P(\vec{s})$ is the total probability that a particular configuration \vec{s} of the other $N - 1$ atoms will occur.

$$P(\vec{s}) = \int |\Psi_i(\vec{r}, \vec{s})|^2 d\vec{r} \quad (2.7)$$

$\psi_i(\vec{r} | \vec{s})$ is a normalised function of \vec{r} and

$$P_i(\vec{r} | \vec{s}) = |\psi_i(\vec{r} | \vec{s})|^2 \quad (2.8)$$

is the conditional probability distribution in space³¹ of the chosen particle. The conditional momentum distribution is

$$n_i(\vec{p} | \vec{s}) = \left| \int \psi_i(\vec{r} | \vec{s}) \exp(i\vec{p} \cdot \vec{r}) d\vec{r} \right|^2 \quad (2.9)$$

Observable properties are obtained by averaging these expressions over i and \vec{s} . For example the total probability that the particle is at \vec{r} is

$$P(\vec{r}) = \sum_i \eta_i(T) \int P(\vec{s}) d\vec{s} |\psi_i(\vec{r} | \vec{s})|^2 \quad (2.10)$$

where

$$\eta_i(T) = \frac{1}{Z} \exp\left(-\frac{E_i}{kT}\right) \quad (2.11)$$

is the Boltzman factor for the probability that a state i is occupied at temperature T and

$$Z = \sum_i \exp\left(-\frac{E_i}{kT}\right) \quad (2.12)$$

is the partition function. The observable momentum distribution is

$$n(\vec{p}) = \sum_i \eta_i(T) \int P(\vec{s}) d\vec{s} n_i(\vec{p} | \vec{s}) \quad (2.13)$$

2c Fluctuations in the Condensate

The condensate fraction is

$$f = \frac{n(0)}{V} = \sum_i \eta_i \int P(\vec{s}) d\vec{s} \frac{1}{V} \left| \int \psi_i(\vec{r} | \vec{s}) d\vec{r} \right|^2 \quad (2.14)$$

The mean square deviation of f from its mean is

$$\Delta f^2 = \sum_i \eta_i \int P(\vec{s}) d\vec{s} \left[\frac{1}{V} \left| \int \psi_i(\vec{r} | \vec{s}) d\vec{r} \right|^2 - f \right]^2. \quad (2.15)$$

Δf^2 determines the size of fluctuations in f . Quantum fluctuations occur due to the fact that \vec{s} is distributed according to the probability distribution $P(\vec{s})$, while thermal fluctuations are due to the distribution of i values according to $\eta_i(T)$. It is widely accepted that the superfluid order parameter has amplitude proportional to \sqrt{f} ³². Since the amplitude of the order parameter must be a thermodynamic quantity, this implies that in superfluid ⁴He fluctuations in f must be

$$\Delta f \sim 1/\sqrt{N} \quad (2.16)$$

In a sufficiently large system equation 2.15 and 2.16 imply that all \vec{s} and i must give the same f . In general, macroscopic observable quantities can be calculated to within terms $\sim 1/\sqrt{N}$, by choosing one particular \vec{s} and i from the possible values in the distributions $P(\vec{s})$ and $\eta_i(T)$.

2d The function $\psi(\vec{r})$

To simplify notation, explicit reference to \vec{s} and i is dropped and $\psi_i(\vec{r} | \vec{s})$ is denoted as $\psi(\vec{r})$, although the conditional nature of $\psi(\vec{r})$ should be borne in mind for physical interpretation. The function $\psi(\vec{r})$ has a phase and amplitude

$$\psi(\vec{r}) = |\psi(\vec{r})| \exp[i\phi(\vec{r})] \quad (2.17)$$

Observable quantities are calculated using standard single particle expressions³⁰. For example the momentum distribution is

$$n(\vec{p}) = \left| \int \psi(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2 \quad (2.18)$$

The condensate fraction is

$$f = \frac{1}{V} \left| \int \psi(\vec{r}) d\vec{r} \right|^2 \quad (2.19)$$

The particle flow is

$$F(\vec{r}) = \frac{1}{M} \int |\psi(\vec{r})|^2 \nabla \phi(\vec{r}) d\vec{r} \quad (2.20)$$

The total energy is

$$E = \int \psi^*(\vec{r}) \hat{H} \psi(\vec{r}) d\vec{r} \quad (2.21)$$

where

$$\hat{H} = N \frac{-\hbar^2}{2M} \nabla^2 + V(\vec{r}) \quad (2.22)$$

and $V(\vec{r})$ is the potential energy of the system at fixed \vec{s} ³⁰. The order parameter $\bar{\psi}$ can be expressed as the average value of $\psi(\vec{r})$ over \vec{r} .

$$\bar{\psi} = \frac{1}{V} \int \psi(\vec{r}) d\vec{r} = \frac{1}{V} \int \psi_s(\vec{r}) d\vec{r} = \sqrt{\frac{f}{V}} \quad (2.23)$$

2e Physical Interpretation

The physical nature of $\psi(\vec{r})$ in ⁴He can be understood by considering a many particle wave function Ψ_0 of the Feynman type⁴⁴. In the Feynman approximation atoms are treated as hard spheres of diameter a , which cannot overlap. Ψ_0 is zero if the centres of any two spheres lie within a distance a of each other and Ψ_0 has a constant value for all other configurations. This is reasonable approximation in some respects to the ground state of liquid ⁴He. The phase of the ground state wave function is independent of \vec{r} in any Bose system³³ and the Feynman approximation gives a condensate fraction of ~8%, close to the experimental value³⁴. The conditional ground state wave function, $\psi_0(\vec{r})$, corresponding to the Feynman model, is only non-zero providing that \vec{r} does not lie within a distance a from the centre of any of the other $N - 1$ atom. In the region where it is non-zero $\psi_0(\vec{r})$ has uniform amplitude and phase and the total volume of this region is exactly equal to fV ³⁰. This is illustrated in one dimension in Figure 1. Figure 1a shows a configuration of non-overlapping ‘atoms’, with positions that were randomly generated, subject to the

constraint that no two atom centres lie within a distance a . Figure 1b shows the corresponding $\psi_0(\vec{r})$ which is either zero or equal to $1/\sqrt{fV}$.

The only significant difference between $\psi_0(\vec{r})$ in the ground state of liquid ^4He and $\psi_0(\vec{r})$ for the Feynman model is that the amplitude of $\psi_0(\vec{r})$ varies more smoothly in ^4He . One consequence of this is that, rather than occupying a volume exactly equal to fV , $\psi_0(\vec{r})$ in ^4He must occupy a volume $\geq fV$. However in both the model and liquid ^4He , $\psi_0(\vec{r})$ occupies a macroscopic total volume, which lies in the spaces between the other particles and in both cases the phase of $\psi_0(\vec{r})$ is independent of \vec{r} ³³. The physical origin of the small quantum fluctuations in f with \vec{s} , is that f is determined essentially by the amount of space $\sim fV$ in the structure of the liquid. In the presence of BEC this amounts to a macroscopic volume, which in a large system is independent of the precise microscopic configuration \vec{s} of the other particles³⁵. Equation 2.23 implies that not only the amplitude, but also the average phase of $\psi(\vec{r})$ over macroscopic volumes of space is independent of i, \vec{s} . This is related to the presence of long range phase coherence in $\psi(\vec{r})$, as discussed in the following section.

3. Long Range Phase Coherence

3a. Definition of LRPC

The autocorrelation function of $\psi(\vec{r})$ is³⁶

$$R_\psi(\vec{r}) = \int \psi^*(\vec{r}')\psi(\vec{r}' + \vec{r})d\vec{r}' \quad (3.2)$$

It follows from the Wiener-Kintchine theorem³⁶ that $R_\psi(\vec{r})$ is the Fourier transform of the momentum distribution defined in equation 2.18.

$$R_\psi(\vec{r}) = \int n(\vec{p})\exp(-i\vec{p}\cdot\vec{r})d\vec{p} \quad (3.3)$$

The presence of the term $f\delta(\vec{p})$ in the momentum distribution of equation 2.4 implies that

$$R_\psi(\vec{r}) \rightarrow f \text{ as } r \rightarrow \infty \quad (3.4)$$

Equations 3.2 and 3.4 implies that³⁰:

(1) $|\psi(\vec{r})|$ is non-zero within a *macroscopic* volume which is at least a fraction f of the total volume V .

(2) The phases $\phi(\vec{r})$ and $\phi(\vec{r}')$ at two points \vec{r} and \vec{r}' are not randomly oriented as $|\vec{r} - \vec{r}'| \rightarrow \infty$.

These two conditions are collectively known as long range phase coherence (LRPC). The criterion for LRPC in equation 3.4 is very similar to that for off diagonal long range order (ODLRO) in the one particle density matrix^{34,37}. BEC, LRPC and ODLRO are equivalent statements of the same phenomenon.

Above the superfluid transition temperature T_λ , BEC is absent and either condition 1 or 2 is not satisfied. Whether or not the first condition is satisfied is determined by the amount of space in the liquid structure. Crudely speaking, $\psi(\vec{r})$ is non zero wherever there is a sufficient space to insert another atom between the atoms already present. At $T = 0$ in ^4He , these spaces must comprise a fraction $\sim f \sim 10\%$ of the total liquid volume. Above T_λ one would expect that many spaces still exist in the structure of liquid ^4He and that condition 1 is still satisfied. Assuming that this is so, the absence of BEC above the superfluid transition is due to the absence of long range phase correlations in $\psi(\vec{r})$. It is assumed that this is the case in the rest of the paper³⁸.

3b Degree of Long Range Phase Coherence

A quantitative measure of the degree of LRPC in $\psi(\vec{r})$ can be given in an analogous way to the discussion of phase coherence in optics³⁹. The coherence function is defined as

$$\gamma(\vec{r}) = \left| \frac{1}{V} \int \exp[i\phi(\vec{r}')] \exp[-i\phi(\vec{r}' + \vec{r})] d\vec{r}' \right| \quad (3.5)$$

The degree of long range phase coherence D is defined as

$$D = \gamma(\vec{r}) \quad \text{as } r \rightarrow \infty \quad (3.6)$$

The coherence length r_c is defined as the distance at which $\gamma(\vec{r}_c)$ falls to zero from its maximum value $\gamma(0) = 1$.

r_c measures the distance $|\vec{r} - \vec{r}'|$ at which correlations between the phases $\phi(\vec{r})$ and $\phi(\vec{r}')$ disappear. The coherence volume $\sim r_c^3$ is denoted as V_c .

Three types of phase coherence in $\psi(\vec{r})$ can be distinguished.

- (1) If $D = 1$, then the phase is a pure plane wave $\exp[i\phi(\vec{r})] = \exp(i\vec{p} \cdot \vec{r})$ and $\psi(\vec{r})$ is said to be “*completely coherent*”. In this case $\gamma(\vec{r}) = 1$ for all \vec{r} . The ground state is a special case of complete coherence, with $p = 0$.
- (2) If $D = 0$, $\psi(\vec{r})$ is said to be “*completely incoherent*”. In this case, the phases $\phi(\vec{r})$ and $\phi(\vec{r}')$ at two points \vec{r} and \vec{r}' are randomly oriented as $|\vec{r} - \vec{r}'| \rightarrow \infty$.

(3) If $0 < D \leq 1$, $\psi(\vec{r})$ is said to be “*partially coherent*”.

In cases 1 and 3 the coherence length r_c is infinite and BEC is present, whereas in case 2, r_c is finite and BEC cannot be present.

4. Ideal Bose Gas and Correlated Basis Functions

The ideal Bose gas (IBG) has the many particle wave function⁴⁰

$$\Phi_i(\vec{r}_N) = \sqrt{\frac{N_1!N_2!\dots}{N!V^N}} \sum_P \exp(ik_1.\vec{r}_1) \exp(ik_2.\vec{r}_2) \dots \exp(ik_N.\vec{r}_N) \quad (4.1)$$

where $\vec{r}_N = \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ denotes the positions of the N particles and the sum contains $N!/(N_1!N_2!\dots)$ terms. These terms are permutations⁴¹ of the N particle indices in the occupied plane wave states, with N_j particles in the plane wave state \vec{k}_j etc. When $|\Phi_i|^2$ is integrated over $\vec{r}_1, \dots, \vec{r}_N$, the squared modulus of each term in the sum contributes V^N to the integral and the products of different terms make no contribution. Thus the normalisation constant in equation 4.1 ensures that

$$\int |\Phi_i(\vec{r}_N)|^2 d\vec{r}_N = 1 \quad (4.2)$$

The momentum distribution $n_i(\vec{p} | \vec{s})$ of equation 2.11 is calculated by (for example) choosing $\vec{r}_1 = \vec{r}$ and grouping the terms in equation 4.1 according to the momentum state occupied by \vec{r}_1 . Terms containing $\exp(i\vec{k}_j.\vec{r}_1)$ each contribute $1/V$ to $n(\vec{k}_j)$. The number of terms that contain $\exp(i\vec{k}_j.\vec{r}_1)$ is a fraction N_j / N of the total number of terms⁴² and hence equation 4.1 gives the probability that a particle occupies the state \vec{k}_j as N_j / N , as expected.

The conditional wave function for the IBG is denoted as $\phi(\vec{r})$ and can be divided into two parts.

$$\phi(\vec{r}) = A + B(\vec{r}) \quad (4.3)$$

A contains only terms of equation 4.1 in which $\vec{r} = \vec{r}_1$ appears in the zero momentum state. Hence A is independent of \vec{r} and completely phase coherent. $B(\vec{r})$ is of the form

$$B(\vec{r}) = \sum_{m \neq 0} B_m \exp(i\vec{k}_m.\vec{r}) \quad (4.4)$$

Each coefficient B_m is the sum of a very large number⁴³ of terms, which are each a product of $N - 1$ phase factors of the form $\exp(i\vec{k}_l.\vec{r}_j)$, with $j \neq 1$. Randomness exists in the directions and magnitudes of the thermally generated

states \vec{k}_l and in the positions \vec{r}_j of the atoms. Thus one would expect that different B_m are randomly phased with respect to each other. Assuming that this is so, $B(\vec{r})$ is completely incoherent, consisting of random noise, as illustrated in Figure 1c.

A more accurate approximation to liquid ${}^4\text{He}$ is provided by correlated basis functions (CBF's), which have been used very successfully to calculate the form of the excitation spectrum in ${}^4\text{He}$ ^{44,45,46}. In the CBF approach the basis set for the representation of ${}^4\text{He}$ wave functions, is essentially a product of the exact many particle ground state, with the IBG wave function.

$$\Psi_i(\vec{r}, \vec{s}) = \Phi_i(\vec{r}, \vec{s})\Psi_0(\vec{r}, \vec{s}) \quad (4.5)$$

Equations 4.3 and 4.5 imply that $\psi(\vec{r})$ for CBF wave functions is of the form,

$$\psi(\vec{r}) = \phi(\vec{r})\psi_0(\vec{r}) = A\psi_0(\vec{r}) + B(\vec{r})\psi_0(\vec{r}) \quad (4.6)$$

Again $\psi(\vec{r})$ can be separated into two parts: a completely coherent part, $A\psi_0(\vec{r})$, which is proportional to the ground state and a completely incoherent part $B(\vec{r})\psi_0(\vec{r})$. The physical nature of the incoherent part is illustrated in Figure 1d and consists of randomly phased regions of finite amplitude. These regions occur throughout the entire fluid volume, wherever there is sufficient space to insert another atom into the liquid structure.

5. Two Fluid Behaviour

5a. Basic Postulates

The basic postulates of the paper are that superfluid ${}^4\text{He}$ shares the following properties with the IBG and CBF wave functions:

- (1) $\psi(\vec{r})$ can be split into a completely coherent part $\psi_S(\vec{r})$ and a completely incoherent part $\psi_N(\vec{r})$.

$$\psi(\vec{r}) = \alpha_S\psi_S(\vec{r}) + \alpha_N\psi_N(\vec{r}) \quad (5.1)$$

- (2) The completely coherent part of $\psi(\vec{r})$ is identical to the ground state conditional wave function at all temperatures.

$$\psi_S(\vec{r}) = \psi_0(\vec{r}) \quad (5.2)$$

The three types of phase coherence discussed in section 3b occur as follows:

- (1) Complete phase coherence occurs at $T = 0$ when $\alpha_N = 0$ and $D = 1$.
- (2) Complete incoherence occurs for $T \geq T_\lambda$ when $\alpha_S = 0$ and $D = 0$

(3) Partial coherence occurs for $0 < T < T_\lambda$, when $\alpha_S, \alpha_N \neq 0$ and $0 < D \leq 1$.

The coefficients α_S, α_N are independent of \vec{r} , but temperature dependent. In the IBG and CBF models, the temperature dependence of α_S and α_N arises through the occupation numbers N_n , which depend upon temperature via the Bose distribution function. The coherence length of $\psi_N(\vec{r})$ is denoted in the rest of the paper as r_C . In the IBG and CBF models, $r_C \sim 1/k_M$ where k_M is the maximum value of \vec{k}_m in equation 4.4. For these models r_C is of the order of the spacing between atoms. More generally one would expect that $r_C \sim 2\pi / \Delta p_U$ where Δp_U is the width of the momentum distribution of the uncondensed atoms, $n_U(\vec{p})$ in equation 2.5. This implies that in ^4He , where $\Delta p_U \sim 2 \text{ \AA}^{-1}$, r_C is again comparable to the distance between atoms in the fluid. The coherence volume of $\psi_N(\vec{r})$ is thus

$$V_C \sim V / N \quad (5.3)$$

5b. Probability density in space

The functions $\psi_S(\vec{r})$ and $\psi_N(\vec{r})$ are normalised and contribute separately to the probability integral over \vec{r} .

$$\int |\psi(\vec{r})|^2 d\vec{r} = \int |\alpha_S \psi_S(\vec{r})|^2 d\vec{r} + \int |\alpha_N \psi_N(\vec{r})|^2 d\vec{r} = |\alpha_S|^2 + |\alpha_N|^2 = 1 \quad (5.4)$$

The cross terms between $\psi_S(\vec{r})$ and $\psi_N(\vec{r})$ average to zero providing that the integration volume is sufficiently large. This can be seen by dividing space into M regions of size comparable to the coherence volume V_C . We define

$$A_m = \frac{1}{V_C} \int_{V_m} \psi_S^* \psi_N d\vec{r} \quad (5.5)$$

The integral is over a volume $V_m = V_C$ centred at \vec{r}_m . Then the integral can be expressed as the sum

$$\int \psi_S^* \psi_N d\vec{r} = \sum_m A_m \quad (5.6)$$

Due to the complete phase incoherence of ψ_N the terms $A_m, A_{m'}$ are randomly phased when $|\vec{r}_m - \vec{r}_{m'}| \gg r_C$. The calculation of $\sum_m A_m$ is similar to the problem of a random walk in two dimensions. One would expect that the sum has a mean value of zero⁴⁷, with a Gaussian distribution about the mean. The standard deviation of this distribution will be proportional to \sqrt{M} , where M is the number of randomly phased regions. Since the other terms in 5.4 are proportional to M , the relative size of terms such as those in equation 5.6 is $\sim 1/\sqrt{M}$. It follows from equation 5.3 that the number of regions M is $\sim N$. Hence cross terms should be proportional to $1/\sqrt{N}$ and negligible in the limit

of a very large system⁴⁸. Equation 5.4 implies that $|\alpha_S|^2$ can be interpreted as the weight of the coherent part and $|\alpha_N|^2$ as the weight of the incoherent part of the wave function⁴⁹.

5b Momentum Distribution

A similar argument can be used to show that the momentum distribution is the sum of independent contributions from $\psi_S(\vec{r})$ and $\psi_N(\vec{r})$.

$$n(\vec{p}) = \left| \int \psi_S(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2 + \left| \int \psi_N(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2 = n_S(\vec{p}) + n_N(\vec{p}) \quad (5.7)$$

Equation 5.7 also follows directly from equation 5.4 and Parseval's theorem for Fourier transform pairs. The weight of $n_S(\vec{p})$ is

$$\int n_S(\vec{p}) d\vec{p} = |\alpha_S|^2 \quad (5.8)$$

and that of $n_N(\vec{p})$

$$\int n_N(\vec{p}) d\vec{p} = |\alpha_N|^2. \quad (5.9)$$

5c. Physical Interpretation of Phase Incoherence

The physical interpretation of phase incoherence in $\psi_N(\vec{r})$ can be understood by defining

$$C_m(\vec{p}) = \frac{1}{V_C} \int_{V_m} \psi_N(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \quad (5.10)$$

as the contribution to the Fourier transform of ψ_N from the region of volume $\sim V_C$ centred at \vec{r}_m . Then the

contribution to $n(\vec{p})$ of $\psi_N(\vec{r})$ is

$$n_N(\vec{p}) = \left| \int \psi_N(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) d\vec{r} \right|^2 = \left| \sum_m C_m(\vec{p}) \right|^2 = \sum_m |C_m(\vec{p})|^2 \quad (5.11)$$

$n_N(\vec{p})$ is the sum of independent contributions from regions of volume $\sim V_C$. The sum over cross terms,

$\sum_{m \neq m'} C_m^*(\vec{p}) C_{m'}(\vec{p})$, has an average value of zero due to the random phase of the different regions, when

$|\vec{r}_m - \vec{r}_{m'}| \gg r_C$. ψ_N can be physically interpreted as describing a situation when the atoms are localised. ψ_S

describes delocalised particles in the sense that all regions of space contribute coherently to $n_S(\vec{p})$, which cannot be

divided into the sum of independent contributions from localised regions.⁵⁰

5d. Superfluid and Normal Fluid

The vanishing of cross terms between $\psi_S(\vec{r})$ and $\psi_N(\vec{r})$ implies that the two components behave as separate quantum systems⁵¹. Since, according to equation 5.1, $\psi(\vec{r})$ is a superposition of $\psi_S(\vec{r})$ and $\psi_N(\vec{r})$, each particle in the fluid contributes simultaneously to both components which can be interpreted as interpenetrating fluids, without any mutual interaction. It is clear that these properties suggest the identification of the superfluid with the phase coherent component of $\psi(\vec{r})$ and the normal fluid with the phase incoherent component. Thus $|\alpha_S|^2$ can be identified as the superfluid fraction and $|\alpha_N|^2$ as the normal fluid fraction.

The presence of BEC is a necessary condition for two fluid behaviour. If BEC is not present, the division of $\psi(\vec{r})$ into perfectly coherent and perfectly incoherent components with no interaction cannot be made. As discussed in section 3d, in the absence of BEC $\psi(\vec{r})$ must either be completely phase incoherent or must be a localised function of \vec{r} ³⁰. In the former case $|\alpha_S|^2 = 0$ whereas in the latter case integrations over \vec{r} , of the form in equation 5.4, are restricted to a volume Ω , of atomic dimensions. The cross-terms between the two components in averages over \vec{r} are then $\sim \sqrt{V}/(\Omega N) \sim 1$, rather than $\sim 1/\sqrt{N}$ and not negligible. In either case the absence of BEC means that two fluid behaviour is not possible.

Fluctuations in f are also not negligible if BEC is absent. If BEC is absent, $\psi(\vec{r})$ is completely incoherent and $\bar{\psi}$, defined in equation 2.23 is the sum of $\sim N$ randomly phased contributions. Fluctuations in $\bar{\psi}$ are then comparable to $\bar{\psi}$. If BEC is absent because $\psi(\vec{r})$ is phase coherent, but localised within a volume Ω of atomic dimensions, the size of Ω and hence f , will depend strongly on the arrangement of other particles in the fluid. Again fluctuations in f are not negligible. In the presence of BEC a new quantum thermodynamical variable $\bar{\psi}$ can be defined which has negligible. In the absence of BEC quantum fluctuations in $\bar{\psi}$ are not negligible and $\bar{\psi}$ is not a thermodynamic quantity.

6. Interpretation of Two Fluid Behaviour

6a. The two fluid postulates

The basic postulates of the two fluid model are¹ that superfluid ⁴He behaves as if it were a mixture of a normal fluid, of density ρ_N and a superfluid of density ρ_S .

$$\rho(\vec{r}) = \rho_S(\vec{r}) + \rho_N(\vec{r}) \quad (6.1)$$

where ρ is the ordinary density of liquid ${}^4\text{He}$. The liquid flow is determined by

$$\vec{j} = \rho_S \vec{v}_S + \rho_N \vec{v}_N \quad (6.2)$$

where v_S is the superfluid velocity and v_N the normal fluid velocity. The superfluid has zero entropy and its velocity is assumed to be irrotational.

$$\nabla \times \vec{v}_S = 0 \quad (6.3)$$

The superfluid motion is assumed to be governed by the ‘‘condensate wave function’’⁵⁶ which is a macroscopic function with similar properties to the elementary one particle wave function. This implies that superfluid circulation must be quantised over macroscopic length scales.

6b. The Density

A macroscopic number density can be defined in terms of $\psi(\vec{r})$ as⁵²

$$\rho(\vec{r}) = \frac{N}{V_L} \int_{V_L} |\psi(\vec{r}')|^2 d\vec{r}' \quad (6.4)$$

$\rho(\vec{r})$ is the number of particles/unit volume at \vec{r} . The integration is taken over a volume V_L , centred at \vec{r} . The size of the volume V_L is chosen to satisfy

$$V_C \ll V_L \ll V \quad (6.5)$$

V_L is small on macroscopic scales, but still much greater than the coherence volume V_C ⁵³. Equation 6.1 follows from equations 5.4 and 6.4, with

$$\rho_S(\vec{r}) = \frac{N}{V_L} \int_{V_L} |\alpha_S \psi_S(\vec{r}')|^2 d\vec{r}' \quad (6.6)$$

with a similar expression for $\rho_N(\vec{r})$. From postulate 2, $\rho_0(\vec{r})$ is the density of the fluid in the ground state, at $T = 0$. Hence

$$\rho_S(\vec{r}) = |\alpha_S|^2 \rho_0(\vec{r}) \quad (6.8)$$

6c. The Energy and Entropy

It follows from equation 2.21 and similar arguments to those used in section 5.1 that the energy of the fluid can be written as the sum of the energies of the coherent and incoherent components.

$$E = \int \psi^* \hat{H} \psi dr = \int \psi_S^* \hat{H} \psi_S dr + \int \psi_N^* \hat{H} \psi_N dr = E_S + E_N \quad (6.9)$$

with cross terms $\int \psi_S^* \hat{H} \psi_N dr \sim 1/\sqrt{N}$. It follows⁵⁴ that the partition function factorises

$$Z = Z_S Z_N \quad (6.10)$$

and two components contribute separately to the entropy of the fluid.

$$S = -kT \log Z = -kT \log Z_S + kT \log Z_N = S_S + S_N \quad (6.11)$$

From postulate 2 the entropy S_S of the coherent component is identical to that of the ground state and hence is zero, as assumed in the two fluid model.

6d. Superfluid flow

Using equation 2.20, the macroscopic particle flow can be defined as

$$F(\vec{r}) = \frac{N}{MV_L} \int_{V_L} |\psi(\vec{r})|^2 \nabla \phi(\vec{r}) d\vec{r} \quad (6.12)$$

It follows from similar arguments to those used in section 5a that $F(\vec{r})$ can be separated into two components, with weights $|\alpha_S|^2$ and $|\alpha_N|^2$.

$$F(\vec{r}) = |\alpha_S|^2 F_S(\vec{r}) + |\alpha_N|^2 F_N(\vec{r}) \quad (6.13)$$

where

$$F_S(\vec{r}) = \frac{1}{M} \int |\psi_S(\vec{r})|^2 \nabla \phi_S(\vec{r}) d\vec{r} \quad (6.14)$$

with a similar expression for $F_N(\vec{r})$. If $\nabla \phi_S(\vec{r})$ does not vary significantly over the volume V_L , the macroscopic superfluid flow at \vec{r} can be defined in a similar way to the macroscopic density as

$$F_S(\vec{r}) = \frac{\hbar \nabla \phi_S(\vec{r})}{M} \frac{1}{V_L} \int_{V_L} |\psi_S(\vec{r}')|^2 d\vec{r}' = \rho_S v_S \quad (6.15)$$

where

$$v_S(\vec{r}) = \frac{\hbar \nabla \phi_S(\vec{r})}{M} \quad (6.16)$$

It follows from equation 6.14 that the superfluid flow must be irrotational as assumed in equation 6.3. Note that macroscopic flow of the normal fluid cannot be defined in an analogous way to equation 6.13, since $\nabla \phi_N(\vec{r})$ varies randomly over atomic length scales.

6e. The condensate wave function

The macroscopic condensate wave function $\Psi_C(\vec{r})$ can be defined as

$$\Psi_C(\vec{r}) = \frac{1}{V_L} \int_{V_L} \psi(\vec{r}') d\vec{r}' \quad (6.17)$$

From equation 2.22 $\Psi_C(\vec{r})$ is the local value of the order parameter at \vec{r} . The incoherent component $\psi_N(\vec{r})$ does not contribute to the condensate wave function, as its average over the macroscopic volume V_L is zero⁵⁵. The function $\psi(\vec{r})$ can be regarded as the microscopic form of the condensate wave function⁵⁶, which governs macroscopic superfluid properties.

6f. Flux Quantisation

It follows from the fact that $\psi(\vec{r})$ must be a single valued function that,

$$\oint \nabla \psi(\vec{r}) \cdot d\vec{r} = \alpha_S \oint \nabla \psi_S(\vec{r}) \cdot d\vec{r} + \alpha_N \oint \nabla \psi_N(\vec{r}) \cdot d\vec{r} = 0 \quad (6.18)$$

where the integration is taken over a closed path in the fluid. Due to the random phase variation of $\psi_N(\vec{r})$ on atomic length scales

$$\oint \nabla \psi_N(\vec{r}) \cdot d\vec{r} = 0 \quad (6.19)$$

for any macroscopic path. However the integral over the phase coherent part $\psi_S(\vec{r})$ can also be zero if the condition,

$$\oint \nabla \phi_S(\vec{r}) \cdot d\vec{r} = 2\pi n \quad (6.20)$$

is satisfied over a macroscopic path, where n is an integer. Equations 6.15 and 6.19 imply the possibility of quantised circulation in the superfluid component⁵⁷.

$$\oint \vec{v}_S \cdot d\vec{r} = \frac{nh}{M} \quad (6.21)$$

According to equations 6.6 and 6.14, the magnitude of this quantised superfluid circulation should be proportional to the weight, $|\alpha_S|^2$ of the superfluid component. This behaviour is well known from experiments on superfluid gyroscopes⁵⁸.

6g Temperature dependence of the condensate and superfluid fractions

From equations 2.19 and 2.23, f is determined by the average value of $\psi(\vec{r})$ over the fluid volume. Hence $\psi_N(\vec{r})$ does not contribute to the condensate. Since according to postulate 2, the contribution of $\psi_S(\vec{r})$ to the condensate is identical to that of the ground state it follows that

$$f(T)/f(0) = |\alpha_S(T)|^2 \quad (6.22)$$

where $f(T)$ is the condensate fraction at temperature T . Figure 3 compares experimental measurements of $f(T)/f(0)$ in superfluid ^4He with tabulated values⁶⁷ of the superfluid fraction $|\alpha_S(T)|^2$. It can be seen that equation 6.20 is satisfied within experimental error.

The roles of the condensate and superfluid fractions can be distinguished as follows. The condensate fraction at $T = 0$ is determined essentially by the size of the total volume $\sim fV$, within which $\psi(\vec{r})$ is non zero. The superfluid fraction is determined by the degree of phase coherence in $\psi(\vec{r})$. The proportionality between the condensate and superfluid fractions implies that BEC is necessary for two fluid behaviour, since $f = 0$, implies that the superfluid fraction is also zero. However the absolute magnitude of f is irrelevant. Two fluid behaviour will occur, with a 100% superfluid fraction at $T = 0$, for any finite value of f .

6g Interpretation of Superfluid and Normal Fluid

The superfluid component of the liquid can be interpreted as being due to the residual long range phase coherence in $\psi(\vec{r})$ at finite temperatures. The normal fluid is due to the random noise in the phase of $\psi(\vec{r})$, which is generated by thermal processes. In the IBG this noise is generated by the excitation of particles from the condensate. In superfluid helium the noise is generated by the creation of phonons and rotons in the liquid. This interpretation ties in closely with the standard Landau theory⁵⁹ of the normal fluid. Above the transition temperature, the phase of the many particle wave function $\Psi_i(\vec{r}, \vec{s})$ varies randomly if *any* of the particle co-ordinates is changed by a distance greater than the coherence length r_C .

7. The Dynamic Structure Factor

The dynamic structure factor $S(\vec{q}, \omega)$ is defined in terms of the many particle wave functions as

$$S(\vec{q}, \omega) = \frac{1}{N} \sum_i \eta_i(T) \sum_f \left| \sum_n \int \Psi_i^*(\vec{r}_N) \exp(i\vec{q} \cdot \vec{r}_n) \Psi_f(\vec{r}_N) d\vec{r}_1, d\vec{r}_2 \dots d\vec{r}_n \right|^2 \delta(\omega + E_i - E_f) \quad (7.1)$$

where $\Psi_f(\vec{r}_N)$ is the final state wave function, the sum over n is over all particles and the Dirac δ function expresses conservation of energy. For a system of indistinguishable particles, all n make the same contribution to $S(\vec{q}, \omega)$ and 7.1 can be written in terms of integration over \vec{r} and \vec{s} as²⁸

$$S(\vec{q}, \omega) = N \sum_i \eta_i \sum_f \left| \int \Psi_i^*(\vec{r}, \vec{s}) \exp(i\vec{q} \cdot \vec{r}) \Psi_f(\vec{r}, \vec{s}) d\vec{r} d\vec{s} \right|^2 \delta(\omega + E_i - E_f) \quad (7.2)$$

According to equation 2.6, the many particle wave function $\Psi_i(\vec{r}, \vec{s})$ is proportional to $\psi_i(\vec{r} | \vec{s})$ and therefore can also be divided into parts which are completely coherent and completely incoherent in \vec{r} ⁶⁰.

$$\Psi_i(\vec{r}, \vec{s}) = \alpha_S \Psi_{iS}(\vec{r}, \vec{s}) + \alpha_N \Psi_{iN}(\vec{r}, \vec{s}) \quad (7.3)$$

Thus equation 7.2 can be written as

$$S(\vec{q}, \omega) = N \sum_i \eta_i \sum_f |A_{iS}(\vec{q}) + A_{iN}(\vec{q})|^2 \delta(\omega + E_i - E_f) \quad (7.4)$$

where

$$A_{iN}(\vec{q}) = \alpha_N \int \Psi_{iN}^*(\vec{r}, \vec{s}) \Psi_f(\vec{r}, \vec{s}) \exp(i\vec{q} \cdot \vec{r}) d\vec{r} d\vec{s} \quad (7.5)$$

with a similar expression for $A_{iS}(\vec{q})$. It is assumed that, since $\Psi_{iN}(\vec{r}, \vec{s})$ is completely incoherent in \vec{r} ,

$\Psi_{iN}^*(\vec{r}, \vec{s}) \Psi_f(\vec{r}, \vec{s})$ is also completely incoherent⁶¹ in \vec{r} . Defining

$$A_{iNm}(\vec{s}, \vec{q}) = \frac{\alpha_N}{V_C} \int_{V_m} \Psi_{iN}^*(\vec{r}, \vec{s}) \Psi_f(\vec{r}, \vec{s}) \exp(i\vec{q} \cdot \vec{r}) d\vec{r} \quad (7.6)$$

as the contribution to $A_{iN}(\vec{q})$ from a region centred at \vec{r}_m , of volume $V_m \sim V_C$, $A_{iN}(\vec{q})$ can be written as

$$A_{iN}(\vec{q}) = \int \sum_m A_{iNm}(\vec{s}, \vec{q}) d\vec{s} \quad (7.7)$$

$A_{iNm}, A_{iNm'}$ are randomly phased if $|\vec{r}_m - \vec{r}_{m'}| > r_C$ and $A_{iN}(\vec{q})$ is therefore the superposition of $\sim N$ randomly phased contributions from regions of volume $\sim V_C$. One implication of this is that $A_{iN}(\vec{q})$ will be randomly phased

with respect to $A_{iS}(\vec{q})$. Thus in the sum over occupied states i , cross-terms of the form $\sum_i \eta_i A_{iS}^* A_{iN}$ average to

zero and $S(\vec{q}, \omega)$ in equation 7.4 separates into two terms

$$S(\vec{q}, \omega) = |\alpha_S|^2 S_S(\vec{q}, \omega) + |\alpha_N|^2 S_N(\vec{q}, \omega) \quad (7.8)$$

where

$$S_N(\vec{q}, \omega) = N \sum_i \eta_i \sum_f |A_{iN}(\vec{q})|^2 \delta(\omega + E_i - E_f) \quad (7.9)$$

and similarly for $S_S(\vec{q}, \omega)$. The division of $S(\vec{q}, \omega)$ into a part with weight proportional to the superfluid fraction and a part with weight proportional to the normal fluid fraction, was first suggested by Woods and Svensson¹⁵ on empirical grounds, but has no previous theoretical basis.

The contribution of the two components can be interpreted physically in a similar way to that in section 5c. Due to the random phase of the terms A_{iNm} , the sum $\sum_{m \neq m'} A_{iNm}^* A_{iNm'}$ of cross terms between different regions m, m' averages to zero and⁶²

$$|A_{iN}(\vec{q})|^2 = \left| \alpha_S \int \sum_m A_{iNm}(\vec{s}, \vec{q}) d\vec{s} \right|^2 = |\alpha_S|^2 \int \sum_m |A_{iNm}(\vec{s}, \vec{q})|^2 d\vec{s} \quad (7.10)$$

Equation 7.10 implies that $S_N(\vec{q}, \omega)$ is composed of the sum of independent contributions from regions of volume $\sim V_C$. The contributions $|A_{iNm}(\vec{s}, \vec{q})|^2$ can be interpreted as excitations that are localised within a region of size $\sim 1/r_C$. From equation 7.5 such contributions necessarily have a width in \vec{q} , which is at least $\sim 1/r_C$. Thus the normal fluid cannot contribute to sharp peaks in $S(\vec{q}, \omega)$. On the other hand the contribution to $S(\vec{q}, \omega)$ from the superfluid consists of coherent contributions from the total volume of the fluid and will have a width in \vec{q} which is $\sim 1/\sqrt[3]{V}$. According to postulate 2, the intensity of sharp peaks in $S_S(\vec{q}, \omega)$ should be the same as those in the ground state⁶³ and therefore the total intensity of sharp peaks in $S(\vec{q}, \omega)$ should be exactly proportional to $|\alpha_S|^2$ ⁶⁴.

This prediction agrees closely with experiment^{15,16}. Although the measured intensities of sharp peaks in the most recent neutron scattering measurements of $S(\vec{q}, \omega)$ deviate slightly from exact proportionality to the superfluid fraction, this could be due to the assumptions that were used in the data analysis⁶⁵.

8. The Density and Static Structure Factor

The anomalous temperature dependence of both the density and static structure factor can be simply explained in terms of the changing proportions of the superfluid and normal fluid. It follows from equations 6.1 and 6.6 that the temperature dependence of the density is determined by

$$\rho(T) = |\alpha_S(T)|^2 \rho_0 + |\alpha_N(T)|^2 \rho_N(T) \quad (8.1)$$

We take the crude approximation that $\psi_N(\vec{r})$, like $\psi_S(\vec{r})$ is independent of temperature⁶⁶. This implies that

$$\psi_N = \psi(T_\lambda) \quad (8.2)$$

and that the density of the normal fluid is independent of temperature and equal to its value at the transition temperature,

$$\rho_N(T) = \rho(T_\lambda) \quad (8.3)$$

the solid lines in figure 3 are obtained from equations 8.1, 8.3 and tabulated⁶⁷ values of $|\alpha_S|^2$.

The static structure factor $S(\vec{q})$ is defined as

$$S(\vec{q}) = \int S(\vec{q}, \omega) d\omega \quad (8.4)$$

and from equation 7.8 can also be split into independent contributions from the superfluid and normal fluid. It follows from equation 8.2 that

$$S(\vec{q}, T) = |\alpha_S(T)|^2 S(\vec{q}, 0) + |\alpha_N(T)|^2 S(\vec{q}, T_\lambda) \quad (8.5)$$

where $S(\vec{q}, T)$ is $S(\vec{q})$ at temperature T . The solid line in Figure 3 is calculated from 8.5 and tabulated values⁶⁷ of $|\alpha_S|^2$ and $|\alpha_N|^2$, while the data points⁶⁸ were taken from the experimental work of Sears and Svensson¹⁷. Thus the anomalous decrease in density and spatial order as the temperature is reduced in the superfluid, can be qualitatively explained in terms of the increasing weight of the ground state component of $\psi(\vec{r})$. The decreased density and spatial order of the ground state component is quantitatively consistent^{29,30} with the assumption that the ground state fluid contains more spaces in its structure than the normal fluid. The number of spaces in the superfluid is quantitatively consistent³⁰ with the assumption that $\psi(\vec{r})$ in the ground state can form a connected network between the atoms in the fluid. This would provide a physical connection, which would allow the phase coherence of $\psi_0(\vec{r})$ to be maintained over macroscopic length scales.

8 Summary

It has been shown that the phase coherence properties of the many particle Schrödinger wave function provide a simple explanation of many of the two fluid properties of liquid ⁴He. The conditional wave function $\psi(\vec{r})$ describes the \vec{r} dependence of the N particle Schrödinger wave function $\Psi_i(\vec{r}, \vec{s})$, of a particular many particle state i , at a fixed

configuration \vec{s} of $N - 1$ of the particles. $\psi(\vec{r})$ has a microscopic structure which depends upon i and \vec{s} and in ${}^4\text{He}$ $\psi(\vec{r})$ occupies the spaces between the other $N - 1$ particles. However the average properties of $\psi(\vec{r})$ over macroscopic volumes of space are independent of both \vec{s} and i in the presence of BEC. At $T = 0$ $\psi(\vec{r})$ must be non-zero and completely phase coherent within a macroscopic volume $\geq fV$, where f is the condensate fraction and V the total liquid volume³⁰. It has been shown that $\psi(\vec{r})$ obtained from the ideal Bose gas and correlated basis functions can be written as the superposition of a completely phase coherent part $\psi_s(\vec{r})$ and a completely phase incoherent part $\psi_N(\vec{r})$. The main postulate of the paper is that this is also true in superfluid ${}^4\text{He}$. It has been shown that these two components behave as interpenetrating fluids with negligible interaction and that they satisfy the equations of the standard two fluid model of superfluid ${}^4\text{He}$. The coherent component has been identified with the superfluid and the incoherent component with the normal fluid. The theory is only valid in the presence of BEC and this suggests that BEC is a necessary condition for two fluid behaviour.

The further postulate that $\psi_s(\vec{r})$ is identical to the conditional wave function $\psi_0(\vec{r})$ of the ground state, at all temperatures below the superfluid transition, provides a simple explanation of the anomalous temperature dependence of the liquid density and the dynamic and static structure factors in superfluid ${}^4\text{He}$. The postulates imply that f and the amplitude of sharp peaks in $S(\vec{q}, \omega)$ are both *exactly* proportional to the superfluid fraction, in good agreement with experiment. This second postulate should be a good *approximation* in ${}^4\text{He}$, since the amplitude $|\psi(\vec{r})|$ is determined essentially by the liquid structure, which changes little between the superfluid and normal fluid. However the zero entropy of the superfluid suggests that the coherent component occupies a unique state and that postulate 2 is true more generally.

Appendix. One Dimensional CBF model

The simplest form of CBF wave functions is⁴⁶

$$\Psi(\vec{r}_N) = \left[\prod_{\vec{k}} \Lambda_{\vec{k}}(\vec{r}_N) \right] \Psi_0(\vec{r}_N) \quad (\text{A.1})$$

where $\vec{r}_N = \vec{r}_1, \vec{r}_2, \dots, \vec{r}_{N-1}, \vec{r}_N$, Ψ_0 is the ground state wave function and

$$\Lambda_{\vec{k}}(\vec{r}_N) = \sum_{n=1}^N \exp(i\vec{k} \cdot \vec{r}_n) \quad (\text{A.2})$$

represents an excitation of momentum $\hbar\vec{k}$. The product of Ψ_0 and terms $\Lambda_{\vec{k}}$, with different \vec{k} , represents a wave function containing multiple excitations. $\Lambda_{\vec{k}}$ can be divided into a part which depends upon $\vec{r}_1 (= \vec{r})$ and a part which depends only upon $\vec{s} = \vec{r}_2, \dots, \vec{r}_{N-1}, \vec{r}_N$.

$$\Lambda_{\vec{k}} = \exp(i\vec{k} \cdot \vec{r}_1) + \sum_{n \neq 1} \exp(i\vec{k} \cdot \vec{r}_n) = \exp(i\vec{k} \cdot \vec{r}) + \Lambda'_{\vec{k}}(\vec{s}) \quad (\text{A.3})$$

It follows from equation A.3 and some elementary manipulations that (to within a normalisation factor which depends only on \vec{s}) the functional dependence of $\psi(\vec{r} | \vec{s})$ on \vec{r} is,

$$\psi(\vec{r} | \vec{s}) = [R_S(\vec{s}) + R_N(\vec{r}, \vec{s})] \psi_0(\vec{r} | \vec{s}) \quad (\text{A.4})$$

where $\psi_0(\vec{r} | \vec{s})$ is the conditional wave function of the ground state,

$$R_S(\vec{s}) = \prod_{\vec{k}} \Lambda'_{\vec{k}}(\vec{s}) \quad (\text{A.5})$$

depends only on \vec{s} and

$$R_N(\vec{r}, \vec{s}) = \prod_{\vec{k}} \Lambda_{\vec{k}}(\vec{r}, \vec{s}) - R_S(\vec{s}) \quad (\text{A.6})$$

It follows from equation A.4 that

$$|\psi(\vec{r} | \vec{s})|^2 = \left\{ |R_S(\vec{s})|^2 + |R_N(\vec{r}, \vec{s})|^2 + I_C(\vec{r} | \vec{s}) \right\} |\psi_0(\vec{r} | \vec{s})|^2 \quad (\text{A.7})$$

where

$$I_C(\vec{r} | \vec{s}) = R_S^*(\vec{s}) R_N(\vec{r}, \vec{s}) + R_N^*(\vec{r}, \vec{s}) R_S(\vec{s}) \quad (\text{A.8})$$

Figure 1a shows a configuration \vec{s} of 1500 non-overlapping ‘atoms’ of diameter 16 pixels, positioned randomly on a one-dimensional grid with 65536 pixels. A numerical simulation of the functions R_S and R_N of equation A.5 and A.6 was made for the configuration of Figure 1a, by randomly generating $M = 200$ values of \vec{k} according to a Gaussian distribution and calculating the appropriate products in the definitions of R_S and R_N . In liquid helium at $\sim 2\text{K}$ the thermal excitations are dominated by rotons which have a \vec{k} value centred at the first maximum of the static structure factor and with a spread of values $\Delta k / k \sim 0.1$. In the simulation k values were generated randomly with a distribution $\exp(-(k - k_0)^2 / (2\sigma^2))$. k_0 was chosen to correspond to the first maximum in the static structure factor of the model and $\sigma = 0.1$. In fact the conclusions are insensitive to the values chosen for k_0 and σ .

The real part of the calculated cross-term I_C in equation A.8 is shown in Figure 1c and the product of I_C with $|\psi_0(\vec{r})|^2$ is shown in Figure 1d. The phase of the cross terms varies randomly with \vec{r} . It is found numerically, that $\int I_C |\psi_0|^2 d\vec{r}$ has an average value of zero and that fluctuations from zero, as \vec{s} and \vec{k} are varied, take a Gaussian form, with a second moment which is proportional $1/\sqrt{N}$. Fluctuations are negligible in the limit of a very large number of particles.

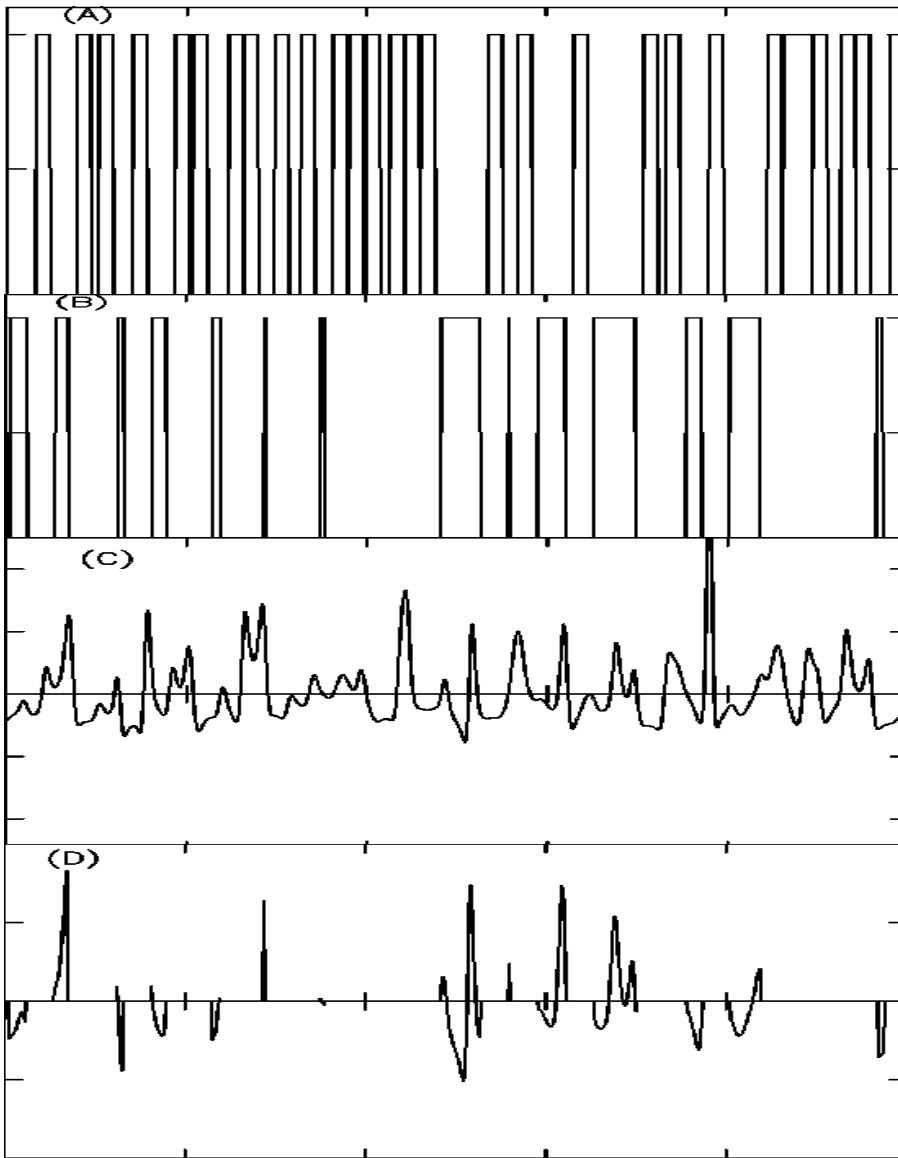


Figure 1 (A). The first 1000 pixels of a 1-D array of 65536 pixels, showing atoms of diameter $a=16$ pixels. The positions of the atoms were generated randomly, with the proviso that no two atoms lie within a distance a , which would correspond to hard core overlap. **(B)** The conditional wave function $\psi_0(\vec{r})$ corresponding to the configuration \vec{s} , of figure 1a. $\psi_0(\vec{r})$ is zero if \vec{r} is within a distance a of any of the atoms at \vec{s} and has uniform amplitude elsewhere. The function $\psi_s(\vec{r})$ in liquid ^4He has a very similar form. **(C)** The calculated cross-term I_C in equation A.8. This also illustrates the form of $\psi_N(\vec{r})$ in the ideal Bose gas. **(D)** The product of I_C with $|\psi_0(\vec{r})|^2$. The figure also illustrates the form of $\psi_N(\vec{r})$ in liquid ^4He . $\psi_N(\vec{r})$ consists of randomly phased regions of non zero amplitude, occupying the spaces between particles.

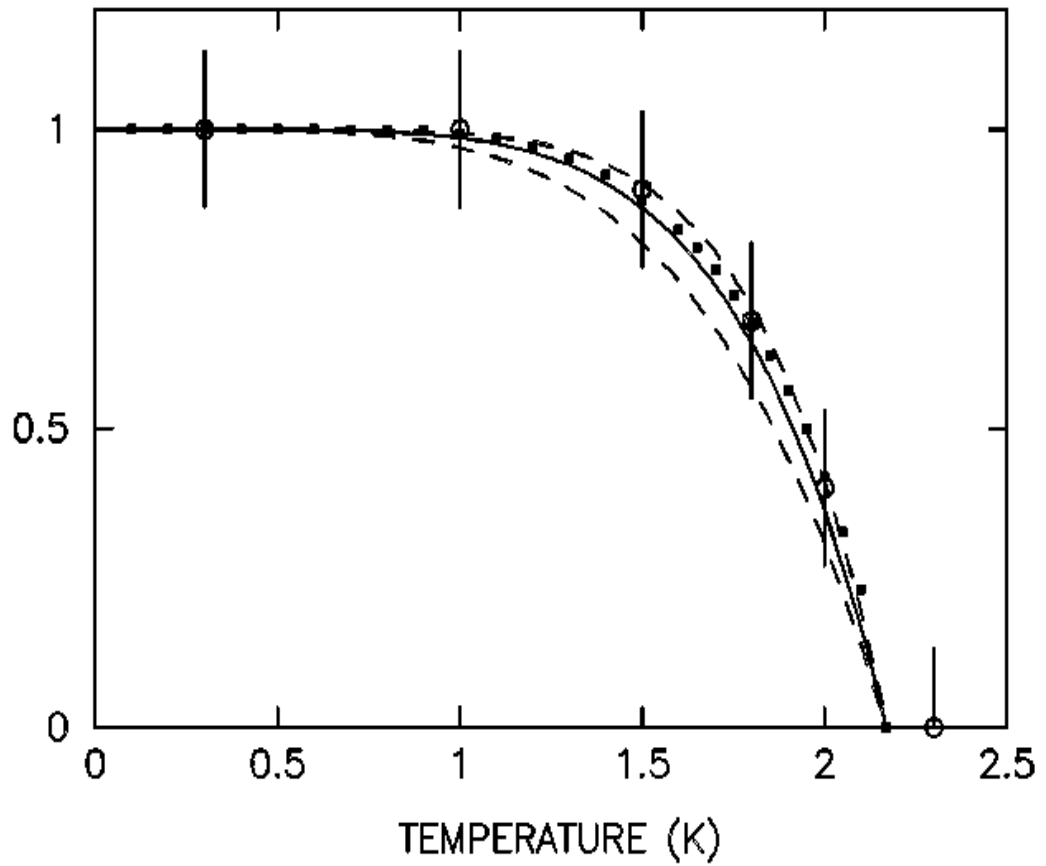


Figure 2. Experimental test of equation 6.21. The points ■ ■ ■ give the superfluid fraction⁶⁷ The solid line is from Glyde et al¹¹ who give $f(T)/f(0) = 1 - (T/T_\lambda)^\gamma$, with $\gamma = 5.5 \pm 1.0$. The dashed lines define the corresponding error. The points o o o are taken from Sosnick et al⁹. The earlier values derived by Sears⁸ are also consistent with equation 6.21, as are the Path Integral Monte-Carlo calculations of Ceperley⁵

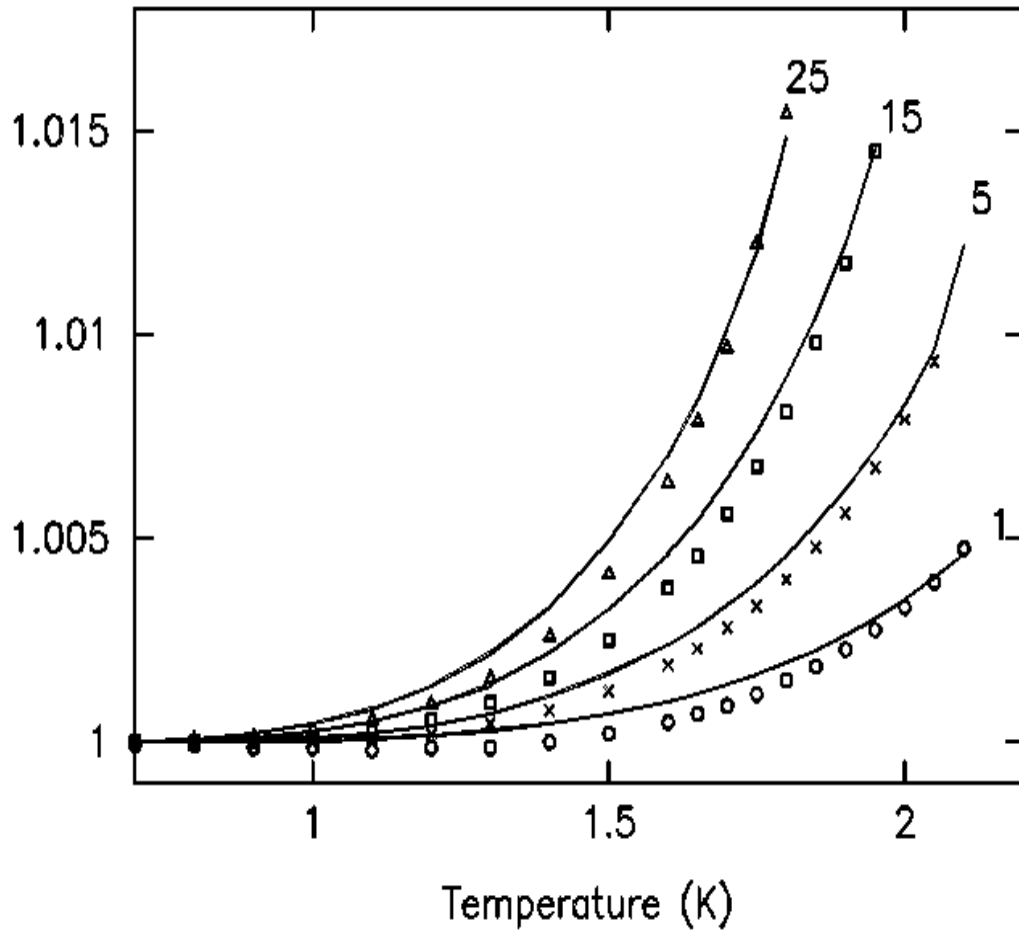


Figure 3 The points are the ratio of the liquid densities $\rho(T)/\rho(0)$ at 1bar, 5 bar, 15 bar and 25 bar. The lines were calculated from equation 6.8 and tabulated values of the weights of the superfluid and normal fluid fractions. The numerical values were taken from reference 67.

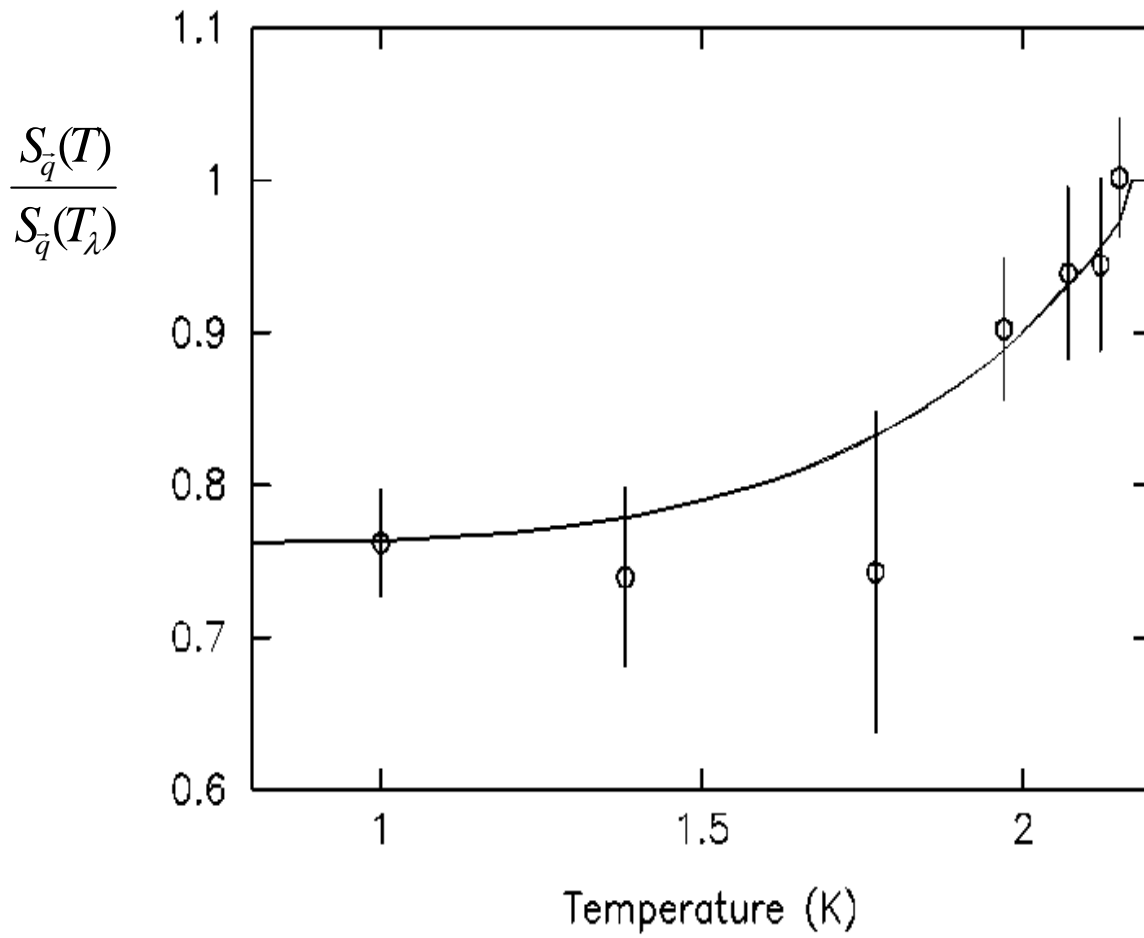


Figure 4. Shows $S_{\bar{q}}(T)/S_{\bar{q}}(T_{\lambda})$ as a function of temperature T . The solid line is calculated from equation 8.5. The points were taken from the experimental work of Sears and Svensson¹⁷.

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31 $|\psi_i(\vec{r} | \vec{s})|^2$ is formally the conditional probability distribution $P_i(\vec{r} | \vec{s})$. See for example D. Sivia, *Data Analysis. A Bayesian Tutorial*, (Clarendon Press, Oxford 1996), Chapter 1.

32. Reference 33, Chapter 13.

33 K. Huang, *Statistical Mechanics*, 2nd Edition, (John Wiley and Sons, New York 1987), Appendix 1.

34 O. Penrose and L. Onsager Phys Rev **104** 576 (1956).

35 It can be shown numerically³⁰ that equation 2.16 is satisfied for the Feynman model.

36 D C. Champeney, *Fourier Transforms and their Physical Applications*, (Academic Press, London, 1988).

37 C N Yang, Rev. Mod. Phys. **34** 694 (1962) .

38 However if the absence of BEC is due to condition 1 not being satisfied, the results and conclusions of the paper are unaffected. Note that at $T = 0$, the phase of $\psi(\vec{r})$ is independent of \vec{r} in any Bose system³³. This implies that if BEC is absent at $T = 0$, condition 1 is not satisfied and that $\psi_0(\vec{r})$ must be a localised function of \vec{r} .

39 S.G. Lipson and H. Lipson, *Optical Physics*, (Cambridge University Press, Cambridge, 1981), Chapter 9.

40 Landau and Lifshitz, *Quantum Mechanics*, (Pergamon Press, New York, 1977), section 61.

41 This is the number of ways N identical particles can be put in different plane wave states, with N_1 in state \vec{k}_1 , N_2 in state \vec{k}_2 etc.

42 The number of terms that contain $\phi_1(\vec{r}_1)$ is $(N-1)! / [(N_1-1)! N_2! \dots]$. This is the number of ways the other $N-1$ particles can be distributed with N_1-1 in state 1, N_2 in state 2 etc.

43 There are $[N! / (N_1! N_2! \dots)] N_m / N$ terms contributing to B_m

44 R .P. Feynman, Phys. Rev. **94** 262 (1954).

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46 C.E. Campbell in *Excitations in two dimensional and three dimensional quantum fluids*, Ed A.F.G. Wyatt and H.J. Lauter (Plenum, New York 1991), section 9.1.

47 Correlations of phase between neighbouring regions have negligible effect in the limit $N, V \rightarrow \infty$.

48 Numerical calculations show that the model discussed in the appendix displays these properties.

49. If $\psi_N(\vec{r})$ is a localised function, then equation 5.4 is still satisfied in the limit $N, V \rightarrow \infty$. Assume $\psi_N(\vec{r})$ is localised within a region Ω . Then $\psi_N(\vec{r}) \sim 1/\sqrt{\Omega}$, $\psi_S(\vec{r}) \sim 1/\sqrt{V}$ and

$$\int \psi_S \psi_N d\vec{r} = \int_{\Omega} \psi_S \psi_N d\vec{r} \sim \sqrt{\Omega/V}, \text{ whereas } \int |\psi_S|^2 d\vec{r} \text{ and } \int |\psi_N|^2 d\vec{r} \text{ are } \sim 1.$$

50 Note that this argument is formally identical to that used in the separation of coherent and incoherent scattering. See for example S W. Lovesey, *Theory of Neutron Scattering from Condensed Matter Vol 1*, (Oxford University Press, New York 1984), Chapter 1.

51 See reference 33, section 8.1

52 This must also be averaged over different \vec{s}, i as in equation 2.19, but for a macroscopic volume V_L the average is independent of \vec{s}, i , to within terms $1/\sqrt{N_L}$.

53 $\rho(\vec{r})$ is independent of the arrangement of particles within the volume V_L , to within terms $\sim \sqrt{1/(N_L)}$, where N_L is the number of atoms in the volume V_L .

54 H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd Edition, (John Wiley and Sons, New York, 1985), Section 16.2.

55 The amplitude of $\Psi_C(\vec{r})$ is also independent of the precise configuration of atoms within the volume V_L , to within terms $\sim \sqrt{1/(N_L)}$.

56 D.R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, (Adam Hilger, Bristol, 1990)

57 W.F. Vinen Proc.Roy.Soc. A 260 218 (1961)

58 J.S.Langer and J.D. Reppy, Progress in Low Temp. Physics, ed C. J. Gorter, Vol VI, ch 1, (Amsterdam, North Holland, 1970).

59 L. Landau and E.M. Lifshitz, *Statistical Physics*, 3rd Edition, (Pergamon Press, Oxford, 1978) p 192-197.

60 This is true for essentially every \vec{s} and i .

61 One could regard this as being true by definition, rather than as an additional postulate. The product of a completely phase incoherent function $\psi_N(\vec{r})$ with any function other than one which is proportional $\psi_N^*(\vec{r})$, must also be a completely phase incoherent function.

62 Equation 7.10 is true if the phase of $A_{iNm}(\vec{s}, \vec{q})$ is independent of \vec{s} and also when the phase varies completely randomly with \vec{s} . It seems reasonable to suppose that it is true when the phase is partially coherent in \vec{s} .

63 The sharp peaks at finite temperature are broader than those at $T = 0$, due to lifetime effects, which are not treated in this paper.

64 . Since f is proportional to ρ_s , this prediction of is in agreement with Glyde-Griffin theory which predicts that the intensity of sharp peaks in $S(\vec{q}, \omega)$ is proportional to f . H R Glyde and A Griffin Phys. Rev. Lett. **65** 1454 (1990), H. R. Glyde Phys Rev Lett **75** 4238 (1995).

65 In the data analysis of ref. 16 the approximation was made that the broad component is identical to the product of $S(\vec{q}, \omega)$ at $T = T_\lambda$, with a temperature dependent Bose occupation factor.

66 This is not true in the IBG for example.

67 J S Brooks and R J Donnelly . *J Phys. Chem. Ref. Data* **6** 51 (1977).

68. Sears and Svensson list values of f with errors, calculated using the assumption that

$[1 - S(\vec{q}, T)] = (1 - f)^2 [1 - S(\vec{q}, T_\lambda)]$. From this expression $S(\vec{q}, T) / S(\vec{q}, T_\lambda)$ and the associated statistical errors can be calculated.