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# EQUILIBRIUM FLUCTUATIONS IN AN N-PARTICLE COASTING BEAM: SCHOTTKY NOISE EFFECTS*

G. Bassi, Cockcroft Institute and U. of Liverpool
J. A. Ellison, K. Heinemann†, U. of New Mexico

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*Abstract*

We discuss the longitudinal dynamics of an unbunched beam with a collective effect due to the vacuum chamber and with the discreteness of an N-particle beam (Schottky noise) included. We start with the 2N equations of motion (in angle and energy) with random initial conditions. The 2D phase space density (Klimontovich density) for the N particles is a sum of delta functions and satisfies the Klimontovich equation and the Vlasov equation. An arbitrary function of the energy also satisfies the Vlasov equation and we linearize about a convenient equilibrium density taking the initial conditions to be independent, identically distributed random variables with the equilibrium distribution. The linearized equations can be solved using a Laplace transform in time and a Fourier series in angle. The resultant stochastic process for the phase space density is analyzed and compared with a known result. Work is in progress to study the full nonlinear problem.

## INTRODUCTION

We study the effect of a finite number of particles (Schottky noise) in a case where it is believed the Vlasov equation is a reasonable approximation to the evolution dynamics. Perhaps the simplest context to study this is the $N$ particle 2D coasting beam with collective effects modeled by an impedance and with phase space variables $(\theta,\epsilon)$ where $\theta$ is the azimuthal angle and $\epsilon = (E-E_r)/E_r$ where $E = m\gamma c^2$. We consider a Vlasov equilibrium density, $f_{eq}$, choose $N$ independent and identically distributed random variables from that distribution and study the evolution of the Klimontovich density in a linearized approximation.

## MODEL

Our initial value problem (IVP) for the $N$ particle coasting beam is

$$\dot{\theta}_a = \omega(\epsilon_a) := \omega_r + k\epsilon_a\ , \qquad \theta_a(0) = \theta_{a0}, \quad (1)$$

$$\dot{\epsilon}_a = \sum_{b=1}^{N} W(\theta_a - \theta_b)\ , \qquad \epsilon_a(0) = \epsilon_{a0}\ , \quad (2)$$

where $\omega_r > 0$ is the angular velocity of the reference particle and $k > 0$ is the slip factor. Note that the vector field is divergence free, so the flow is measure preserving. In Appendix I we argue that a reasonable form for $W$ is

$$W(\theta) := -(\frac{q\omega_r}{2\pi})^2 \frac{1}{E_r} \sum_{n\in\mathbb{Z}} Z_n e^{in\theta}\ ,\ Z_0 = 0, \quad (3)$$

where $\beta_r = \omega_r R/c,\ E_r = m\gamma_r c^2$ and $R$ is the machine radius and $Z_n$ is the elementary machine impedance. We assume that the $Z_n$ decay sufficiently fast as $|n| \to \infty$ so that $W$ is a smooth, $2\pi$-periodic function of zero mean, i.e., $\int_0^{2\pi} W(\theta) = 0$.

We abbreviate the IVP (1-2) by $\dot{z} = w(z), z(0) = z_0$, where $z := (\theta_1, \epsilon_1, ..., \theta_N, \epsilon_N)^T$, and consider (1-2) as a random IVP specified by a density $\Psi_0 = \Psi_0(z)$. This density evolves by the Liouville equation

$$\partial_t \Psi + w(z)\cdot\nabla_z \Psi = 0\ , \quad \Psi(z,0) = \Psi_0(z)\ , \quad (4)$$

where $w$ is determined by (1-2). Clearly $\Psi(z,t) = \Psi_0(\varphi(-t,z))$, where $\varphi(t,z_0)$ denotes the solution of (1-2).

The Klimontovich density $F(\theta,\epsilon,t;z_0)$ is

$$F := \frac{1}{N}\sum_{a=1}^{N} \delta_p(\theta-\theta_a(t))\delta(\epsilon-\epsilon_a(t))\ , \quad (5)$$

where $\delta_p$ is the $2\pi$-periodic delta function. In probability theory, $F$ is sometimes called an empirical density. In the following we will suppress the $z_0$ dependence. Calculation of the partial derivatives of $F$ from (5) shows that $F$ satisfies the Klimontovich equation

$$\partial_t F + \omega(\epsilon)\partial_\theta F + \sum_{a=1}^{N} W(\theta-\theta_a(t))\partial_\epsilon F = 0\ , \quad (6)$$

and the Vlasov equation

$$\partial_t F + \omega(\epsilon)\partial_\theta F + N\mathcal{L}(F)\partial_\epsilon F = 0, \quad (7)$$

both with the initial condition

$$F(\theta,\epsilon,t=0) = \frac{1}{N}\sum_{a=1}^{N} \delta_p(\theta-\theta_{a0})\delta(\epsilon-\epsilon_{a0})\ . \quad (8)$$

The operator $\mathcal{L}$ in the Vlasov equation is

$$\mathcal{L}(\chi)(\theta,t) := \int \chi(\theta',\epsilon',t)W(\theta-\theta')d\theta' d\epsilon'\ . \quad (9)$$

The Klimontovich equation (6) and the Vlasov equation (7) are not the same, e.g., a function only of $\epsilon$ satisfies (7) (since $W$ has zero mean) but not (6).

Taking the expected value of (7), with respect to the only random quantity $z_0$, and defining $f := \mathbb{E}F$ leads to

$$\partial_t f + \omega(\epsilon)\partial_\theta f + N\mathcal{L}(f)\partial_\epsilon f = -N\mathbb{E}(\mathcal{L}(\delta F)\partial_\epsilon \delta F)\ , \quad (10)$$

where $\delta F := F - \mathbb{E}F = F - f$ is the fluctuation of $F$. Thus $f$ is an approximate solution of the Vlasov equation if the rhs of (10) is small. Equation (10) is the analogue of the corresponding equation in the BBGKY hierarchy where $f$ is equal to the single particle probability density $f_1$ (see Appendix II).

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† heinemanatmath.unm.edu

 

## ANALYSIS

We now study the deviation of the Klimontovich density $F$ from the Vlasov equilibrium density $f = f_{eq}(\epsilon)$. Thus we take

$$\Psi_0(z) = f_{eq}(\epsilon_1)\cdots f_{eq}(\epsilon_N) \tag{11}$$

and note that $\mathbb{E}F(\theta,\epsilon,t=0) = f_{eq}(\epsilon)$, where $\int f_{eq}(\epsilon)d\theta d\epsilon = 1$. Let $F = f_{eq} + G$ then $G$ satisfies

$$\partial_t G + \omega(\epsilon)\partial_\theta G + N\mathcal{L}(G)(f'_{eq}(\epsilon) + \partial_\epsilon G) = 0. \tag{12}$$

We will study an approximation to $G$ by dropping terms in (12) which are nonlinear in $G$. The linearized IVP is

$$\partial_t G + \omega(\epsilon)\partial_\theta G + N\mathcal{L}(G)f'_{eq}(\epsilon) = 0\ , \tag{13}$$

$$G(\theta,\epsilon,t=0) = -f_{eq}(\epsilon) + \frac{1}{N}\sum_{a=1}^{N}\delta_p(\theta-\theta_{a0})\delta(\epsilon-\epsilon_{a0}). \tag{14}$$

In the following, $G$ will refer to this approximate $G$. Since $\mathbb{E}G(\theta,\epsilon,t=0) = 0$ it follows from (13) that $\mathbb{E}G(\theta,\epsilon,t) = 0$ whence $\delta F = G$.

To solve the IVP (13),(14) we expand $G$ in a Fourier series, giving

$$\partial_t G_n + in\omega(\epsilon)G_n + 2\pi N f'_{eq}(\epsilon)W_n H_n(t) = 0,$$
$$G_n(\epsilon,0) = \frac{1}{2\pi N}\sum_{a=1}^{N}\delta(\epsilon-\epsilon_{a0})e^{-in\theta_{a0}} \tag{15}$$

for $n \neq 0$ where

$$H_n(t) := \int_{-\infty}^{\infty} G_n(\epsilon,t)d\epsilon\ , \tag{16}$$

so (15) is an integro-differential equation. For $n = 0$, $G_n(\epsilon,t) = G_n(\epsilon,0)$ whence we only study $n \neq 0$. We take the Laplace transform in the form $\tilde{G}_n(\epsilon,\Omega) := \int_0^\infty e^{i\Omega t}G_n(\epsilon,t)dt$, giving us

$$\tilde{G}_n(\epsilon,\Omega) = \frac{iG_n(\epsilon,0)}{\Omega - n\omega(\epsilon)} - \frac{i2\pi N W_n f'_{eq}(\epsilon)\tilde{H}_n(\Omega)}{\Omega - n\omega(\epsilon)}, \tag{17}$$

where $\tilde{H}_n$ is the Laplace transform of $H_n$. Note that for functions which have a Laplace transform, the transform is analytic on a set of the form $\{z \in \mathbb{C} : \Im m(z) > c_0\}$, for some real $c_0$.

To obtain $\tilde{H}_n$ we integrate (17) over $\epsilon$ and use the initial condition in (15) yielding

$$D_n(\Omega)\tilde{H}_n(\Omega) =: \tilde{H}^i_n(\Omega), \tag{18}$$

where

$$\tilde{H}^i_n(\Omega) := \frac{i}{2\pi N}\sum_{a=1}^{N}\frac{e^{-in\theta_{a0}}}{\Omega - n\omega(\epsilon_{a0})}, \tag{19}$$

and the dispersion function

$$D_n(\Omega) := 1 + i2\pi N W_n\int_{-\infty}^{\infty}\frac{f'_{eq}(\epsilon)d\epsilon}{\Omega - n\omega(\epsilon)}\ . \tag{20}$$

The dispersion function also arises naturally in the context of (13). Let $G(\theta,\epsilon,t) = B(\epsilon)\exp(i(n\theta - \Omega_0 t))$ then it follows that $B(\epsilon) = -i2\pi N(\Omega_0 - n\omega(\epsilon))^{-1}f'_{eq}(\epsilon)W_n\int Bd\epsilon$. Integrating over $\epsilon$ gives $D_n(\Omega_0)\int Bd\epsilon = 0$ and thus solutions of the given form exist only if $\Omega_0$ is a zero of the dispersion function.

Physically, a natural equilibrium distribution is a Gaussian. However, a Cauchy distribution allows certain integrals to be evaluated analytically [1] and so we take $f_{eq}(\epsilon) := \frac{\alpha}{2\pi^2(\alpha^2+\epsilon^2)}$ with $\alpha > 0$. In this case the dispersion function takes the form

$$D_n(\Omega) = \frac{(\Omega - \Omega_{n1})(\Omega - \Omega_{n2})}{(\Omega - \Omega_p)^2}, \tag{21}$$

where

$$\Omega_{n1} := n\omega_r + a_n sgn(nb_n) - i(|n|k\alpha - |b_n|)\ ,$$
$$\Omega_{n2} := n\omega_r - a_n sgn(nb_n) - i(|n|k\alpha + |b_n|)\ ,$$
$$\Omega_p := n\omega_r - i\alpha nk sgn(n)\ ,$$

and

$$W_n =: |W_n|\exp(i[\theta_n + \pi])\ ,$$
$$a_n := \sqrt{\frac{2\pi N|n||W_n|}{k}}\cos(\frac{\theta_{|n|}}{2} - \frac{\pi}{4})\ ,$$
$$b_n := \sqrt{\frac{2\pi N|n||W_n|}{k}}\sin(\frac{\theta_{|n|}}{2} - \frac{\pi}{4})\ .$$

A partial fraction expansion on (18) leads to

$$\tilde{H}_n(\Omega) = \frac{i}{2\pi N}\sum_{a=1}^{N}e^{-in\theta_{0a}}\Big(\frac{A_1(\epsilon_{0a})}{\Omega - \Omega_{n1}} + \frac{A_2(\epsilon_{0a})}{\Omega - \Omega_{n2}} + \frac{A_3(\epsilon_{0a})}{\Omega - n\omega(\epsilon_{0a})}\Big), \tag{22}$$

where

$$A_3(\epsilon_{0a}) := (D_n(n\omega(\epsilon_{0a})))^{-1}, \tag{23}$$

and, for $(j,k) = (1,2)$ or $(2,1)$

$$A_j(\epsilon_{0a}) := \frac{(\Omega_{nj} - \Omega_p)^2}{(\Omega_{nj} - \Omega_{nk})(\Omega_{nj} - n\omega(\epsilon_{0a}))}. \tag{24}$$

Inverting (22) gives

$$H_n(t) = \frac{1}{2\pi N}\sum_{a=1}^{N}e^{-in\theta_{0a}}\Big(A_1(\epsilon_{0a})e^{-i\Omega_{n1}t} + A_2(\epsilon_{0a})e^{-i\Omega_{n2}t} + A_3(\epsilon_{0a})e^{-in\omega(\epsilon_{0a})t}\Big). \tag{25}$$

The $G_n$ can now be determined from (15) which then gives $G$.



 

We finish by studying $H_n$. We first note that $\mathbb{E}(H_n(t)) = 0$. When $\Im m(\Omega_{nj}) < 0$, i.e., in the case of linear stability, we have for large $t$ the covariance:

$$\begin{aligned}&\mathbb{E}(H_n(t)^* H_n(t+s)) \\ &= \frac{1}{2\pi N}\int_{-\infty}^{\infty} f_{eq}(\epsilon)\frac{e^{-in\omega(\epsilon)s}}{|D_n(n\omega(\epsilon))|^2}d\epsilon \\ &=: C_n(s)\ .\end{aligned} \tag{26}$$

where $*$ denotes complex conjugation. Thus $H_n$ becomes a weakly stationary stochastic process for large $t$. Making the change of the variable $\epsilon = (\lambda - n\omega_r)/nk$ we see that $C_n(s)$ is the Fourier transform of

$$\sigma(\lambda) := \frac{1}{N|n|k|D_n(\lambda)|^2} f_{eq}\left(\frac{\lambda - n\omega_r}{nk}\right), \tag{27}$$

which is therefore the spectral density of the weakly stationary process. Note that $b_n$ is proportional to $\sqrt{N}$, thus the linear stability is lost if $N$ is sufficiently large. Also, linear stability is lost as $\alpha \to 0$, i.e., when $f_{eq}(\epsilon) \to \delta(\epsilon)/2\pi$. Thus the hydrodynamical approximation to the Vlasov equation would not be valid.

We now compare our result with an approach used in [2]. There a ‘noise power spectrum’ is computed by considering the quantity $A(\Delta) := 2\Delta\mathbb{E}(\tilde{H}_n(\lambda+i\Delta)\tilde{H}_n(\lambda+i\Delta)^*)$ in the limit $\Delta \to 0+$. For $\Delta > 0$ one has by (18),(19) that

$$\begin{aligned}&\mathbb{E}(\tilde{H}_n(\lambda+i\Delta)\tilde{H}_n(\lambda+i\Delta)^*) \\ &= \frac{1}{4\pi^2 N^2 |D_n(\lambda+i\Delta)|^2}\ \cdot \\ &\qquad \sum_{a=1}^{N}\mathbb{E}\left(\frac{1}{(\lambda - n\omega(\epsilon_{a0}))^2 + \Delta^2}\right),\end{aligned} \tag{28}$$

since cross terms do not contribute because of statistical independence. Thus $A(\Delta) = \frac{1}{\pi N|D_n(\lambda+i\Delta)|^2}\int_{-\infty}^{\infty}\frac{\Delta f_{eq}(\epsilon)d\epsilon}{(\lambda-n\omega(\epsilon))^2+\Delta^2}$. Using $\delta(x) = (1/\pi)\lim_{\Delta\to 0+}\frac{\Delta}{x^2+\Delta^2}$ we find $\lim_{\Delta\to 0+} A(\Delta) = \sigma(\lambda)$.

Thus the two calculations give the same result. Note however that the second calculation does not use the specific $f_{eq}$, it only uses (18)-(20). Nor does it seem to care about the analyticity properties of $D_n$. We suspect that, the procedure in [2] gives the spectral density of $H_n$ if the latter is weakly stationary. However, $H_n$ generically is not weakly stationary. It seems likely that the procedure in [2] does not make sense if the roots of $D_n$ are in the upper half plane. If they are in the lower half plane, the process $H_n$ is not weakly stationary because of the decaying exponents. We must leave open the question why the two calculations are in agreement for our special case.

## DISCUSSION

We are pursuing the issues raised after (28). In addition, we are interested in what is really measured in an accelerator. Is it the spectral density? What is measured if the process is not weakly stationary?

Work is in progress to study the full nonlinear problem. We will continue with the Klimontovich approach developed here and investigate, in our context, the related *tour de force* nonlinear iteration calculation of Vlaicu [1] as outlined in Chapter 5 of [3]. In addition we will investigate the BBGKY approach, the approach discussed in [4] and the large deviation approach of Donsker and Varadhan, see, e.g., [5, 6].

## APPENDIX I

To see that (1),(2) are not unreasonable we first note that from the Lorentz equation $m\dot{\gamma}c^2 = q\mathbf{E}\cdot\mathbf{v}$ where $\mathbf{E}$ is the electric field and $\mathbf{v}$ is the velocity. For our case of circular motion $\mathbf{E}\cdot\mathbf{v} \approx E_{az}\omega_r R$ where $E_{az}$ is the azimuthal field and $R$ is the radius. The current $I(\theta,t)$ is approximately $qN\rho(\theta,t)\omega_r$ where $N\rho$ is the particle density. Let $V(\theta,t) := -2\pi R E_{az}$, then solving Maxwell’s equations by a Laplace transform in $t$ and a Fourier series in $\theta$ gives $\tilde{V}_n(\Omega) = Z(n,\Omega)\tilde{I}_n(\Omega)$ with $Z(0,\Omega) = 0$. See for example [7], where $Z(n,\Omega)$ is called the complete impedance. Approximating $Z(n,\Omega)$ by $Z_n := Z(n, n\omega_r)$ we obtain $V_n(t) = Z_n I_n(t)$ and thus $V(\theta,t) = qN\omega_r\sum_{n\in Z} Z_n\rho_n(t)e^{in\theta}$. Equations (1),(2) follow if we define $W_n := -Z_n(q\omega_r)^2/(4\pi^2 E_r)$.

## APPENDIX II

We assume that $\Psi_0(z_1,...,z_N)$ is symmetric under permutations of the $z_1,...,z_N$. Thus, by the special form of $W$ (whence of $w$), $\Psi(z_1,...,z_N,t)$ is also symmetric under the permutations so we get $f(\theta,\epsilon,t) \equiv \mathbb{E}F(\theta,\epsilon,t) = f_1(\theta,\epsilon,t)$, where $f_j(\theta_1,\epsilon_1,...,\theta_j,\epsilon_j,t) := \int\ \Psi(\theta_1,\epsilon_1,..,\theta_N,\epsilon_N,t)d\theta_{j+1}d\epsilon_{j+1}\cdots d\theta_N d\epsilon_N$ for $j = 1,2,...,N-1$ and $f_N := \Psi$.

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