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SCHOTTKY NOISE EFFECTS

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Proceedings of PAC07, Albuquerque, New Mexico, USA
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Abstract

We discuss the longitudinal dynamics of an unbunched beam with a collective effect due to the vacuum chamber and with the discreteness of an N-particle beam (Schottky noise) included. We start with the 2N equations of motion (in angle and energy) with random initial conditions. The 2D phase space density (Klimontovich density) for the N particles is a sum of delta functions and satisfies the Klimontovich equation and the Vlasov equation. An arbitrary function of the energy also satisfies the Vlasov equation and we linearize about a convenient equilibrium density taking the initial conditions to be independent, identically distributed random variables with the equilibrium distribution. The linearized equations can be solved using a Laplace transform in time and a Fourier series in angle. The resultant stochastic process for the phase space density is analyzed and compared with a known result. Work is in progress to study the full nonlinear problem.

INTRODUCTION

We study the effect of a finite number of particles (Schottky noise) in a case where it is believed the Vlasov equation is a reasonable approximation to the evolution dynamics. Perhaps the simplest context to study this is the N-particle 2D coasting beam, collective effects modeled by an impedance and with phase space variables (θ, ε) where θ is the azimuthal angle and ε = (E - E_r)/E_r where E = mγc^2. We consider a Vlasov equilibrium density, f_0, with the initial condition (1-2) as a random IVP specified by a density Ψ_0 = Ψ(ζ). The Klimontovich density f(θ, t, z) satisfies

\[ \frac{\partial}{\partial t} f(\theta, t, z) + \frac{\partial}{\partial \theta} \left( \omega(\epsilon) f(\theta, t, z) \right) = 0, \]

where \( \omega(\epsilon) = \frac{\beta_r}{m} \left( E - E_r \right) \) and \( \beta_r = \omega_r, R/c \). Here, \( E_r = m\gamma c^2 \) and \( R \) is the machine radius and \( Z_n \) is the elementary machine impedance. We assume that \( Z_n \) decay sufficiently fast as \(|n| \rightarrow \infty\) so that \( W \) is a smooth, 2π-periodic function of zero mean, i.e.,

\[ f_0^{2\pi} W(\theta) = 0. \]

We abbreviate the IVP (1-2) by \( \dot{z} = w(z), z(0) = z_0 \), where \( z := (\theta_1, \epsilon_1, ..., \theta_N, \epsilon_N) \), and consider (1-2) as a random IVP specified by a density \( \Psi_0 = \Psi(z) \). This density evolves by the Liouville equation

\[ \partial_t \Psi + w(z) \cdot \nabla_z \Psi = 0, \quad \Psi(z, 0) = \Psi_0(z), \]

where \( w \) is determined by (1-2). Clearly \( \Psi(z, t) = \Psi_0(\varphi(-t, z)) \), where \( \varphi(t, z) \) denotes the solution of (1-2). The Klimontovich density \( F(\theta, t, z; \epsilon, \theta_0) \) is

\[ F := \frac{1}{N} \sum_{a=1}^{N} \delta(\theta - \theta_a(\epsilon)) \delta(\epsilon - \epsilon_a(\epsilon)) , \]

where \( \delta \) is the 2π-periodic delta function. In probability theory, \( F \) is sometimes called an empirical density. In the following we will suppress the \( z_0 \) dependence. Calculation of the partial derivatives of \( F \) from (5) shows that \( F \) satisfies the Klimontovich equation

\[ \partial_t F + \omega(\epsilon) \partial_\theta F + \sum_{a=1}^{N} W(\theta - \theta_a(\epsilon)) \partial_\epsilon F = 0, \]

and the Vlasov equation

\[ \partial_t F + \omega(\epsilon) \partial_\theta F + N(L(F) \partial_\epsilon F = 0, \]

both with the initial condition

\[ F(\theta, t, 0) = \frac{1}{N} \sum_{a=1}^{N} \delta(\theta - \theta_a) \delta(\epsilon - \epsilon_a) . \]

The operator \( L \) in the Vlasov equation is

\[ L(\chi)(\theta, t) := \int \chi(\theta', \epsilon', t) \delta(\theta - \theta') d\theta' d\epsilon' . \]

The Klimontovich equation (6) and the Vlasov equation (7) are not the same, e.g., a function only of \( \epsilon \) satisfies (7) (since \( W \) has zero mean) but not (6).

Taking the expected value of (7), with respect to the only random quantity \( z_0 \), and defining \( f := EF \) leads to

\[ \partial_t f + \omega(\epsilon) \partial_\theta f + N(L(f) \partial_\epsilon f = -N\epsilon E(L(\delta F) \partial_\epsilon \delta F), \]

where \( \delta F := F - EF = f \) is the fluctuation of \( F \). Thus \( f \) is an approximate solution of the Vlasov equation if the rhs of (10) is small. Equation (10) is the analogue of the corresponding equation in the BBGKY hierarchy where \( f \) is equal to the single particle probability density \( f_1 \) (see Appendix II).

05 Beam Dynamics and Electromagnetic Fields

\* Work supported by DOE grant DE-FG02-99ER41104
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4318
**ANALYSIS**

We now study the deviation of the Klimontovich density $F$ from the Vlasov equilibrium density $f = f_{eq}(\epsilon)$. Thus we take

$$G(\theta, \epsilon, t = 0) = f_{eq}(\epsilon),$$

and note that $\mathbb{E}F(\theta, \epsilon, t = 0) = f_{eq}(\epsilon)$, where $\int f_{eq}(\epsilon) \, d\epsilon = 1$. Let $F = f_{eq} + G$ then $G$ satisfies

$$\partial_t G + \omega(\epsilon) \partial_\theta G + N \mathcal{L}(G)(f_{eq}(\epsilon) + \partial_\epsilon G) = 0. \quad (12)$$

We will study an approximation to $G$ by dropping terms in (12) which are nonlinear in $G$. The linearized IVP is

$$\partial_t G + \omega(\epsilon) \partial_\theta G + N \mathcal{L}(G)f_{eq}(\epsilon) = 0, \quad (13)$$

where $G(\theta, \epsilon, t = 0) = 0$. It follows from (13) that $\mathbb{E}G(\theta, \epsilon, t = 0) = 0$, whence $\mathbb{E} F = G$.

To solve the IVP (13), we expand $G$ in a Fourier series, giving

$$\partial_t G_n + i \omega_n(\theta) G_n + 2 \pi N f_{eq}(\epsilon) W_n H_n(t) = 0,$$

$$G_n(\epsilon, 0) = \frac{1}{2 \pi N} \sum_{a=1}^N \delta(\epsilon - \epsilon_0) e^{-in\theta_0}, \quad (14)$$

for $n \neq 0$ where

$$H_n(t) := \int_{-\infty}^{\infty} G_n(\epsilon, t) \, d\epsilon, \quad (16)$$

so (15) is an integro-differential equation. For $n = 0$, $G_0(\epsilon, t) = G_n(\epsilon, 0)$ whence we only study $n \neq 0$. We take the Laplace transform in the form

$$G_n(\epsilon, \Omega) := \int_0^\infty e^{it \Omega} G_n(\epsilon, t) \, dt,$$

then it follows from (13) that

$$\mathbb{E}G_n(\epsilon, \Omega) = \frac{iG_n(\epsilon, 0)}{\Omega - n\omega(\epsilon)} - \frac{i2 \pi N W_n f_{eq}(\epsilon) \tilde{H}_n(\Omega)}{\Omega - n\omega(\epsilon)},$$

(17)

where $\tilde{H}_n$ is the Laplace transform of $H_n$. Note that for functions which have a Laplace transform, the transform is analytic on a set of the form $\{z \in \mathbb{C} : \Re m(z) > c_0\}$, for some real $c_0$.

To obtain $\tilde{H}_n$, we integrate (17) over $\epsilon$ and use the initial condition in (15) yielding

$$D_n(\Omega) \tilde{H}_n(\Omega) = \tilde{H}_n^i(\Omega), \quad (18)$$

where

$$\tilde{H}_n^i(\Omega) := \frac{i}{2 \pi N} \sum_{a=1}^N \frac{e^{-in\theta_0}}{\Omega - n\omega(\epsilon_0)},$$

and the dispersion function

$$D_n(\Omega) := 1 + i 2 \pi N W_n \int_{-\infty}^{\infty} \frac{f_{eq}(\epsilon) \, d\epsilon}{\Omega - n\omega(\epsilon)}. \quad (20)$$

The dispersion function also arises naturally in the context of (13). Let $G(\theta, \epsilon, t) = B(\epsilon) \exp(i nt \theta - \Omega_0 t)$ then it follows that $B(\epsilon) = -i 2 \pi N (\Omega_0 - n\omega(\epsilon))^{-1} f_{eq}(\epsilon) W_n \int B \, d\epsilon$. Integrating over $\epsilon$ gives $D_n(\Omega_0) \int B \, d\epsilon = 0$ and thus solutions of the given form exist only if $\Omega_0$ is a zero of the dispersion function.

Physically, a natural equilibrium distribution is a Gaussian. However, a Cauchy distribution allows certain integrals to be evaluated analytically [1] and so we take $f_{eq}(\epsilon) := \frac{\alpha}{2\pi(x^2 + \alpha^2)}$ with $\alpha > 0$. In this case the dispersion function takes the form

$$D_n(\Omega) = \frac{(\Omega - \Omega_{n1})(\Omega - \Omega_{n2})}{(\Omega - \Omega_p)^2},$$

(21)

where

$$\Omega_{n1} := n \omega_r + a_n \text{sgn}(nb_n) - i[|n|ka_1 - |b_n|],$$

$$\Omega_{n2} := n \omega_r - a_n \text{sgn}(nb_n) - i[|n|ka_1 + |b_n|],$$

$$\Omega_p := n \omega_r - i \alpha \text{sgn}(n),$$

and

$$W_n := |W_n| \exp(i[\theta_n + \pi]),$$

$$a_n := \sqrt{\frac{2 \pi |n| |W_n|}{k} \cos(\frac{\theta_n |n|}{2} - \frac{\pi}{4})},$$

$$b_n := \sqrt{\frac{2 \pi |n| |W_n|}{k} \sin(\frac{\theta_n |n|}{2} - \frac{\pi}{4})}.$$

A partial fraction expansion on (18) leads to

$$\tilde{H}_n(\Omega) = \frac{i}{2 \pi N} \sum_{a=1}^N e^{-in\theta_0} \left( \frac{A_1(\epsilon_0)}{\Omega - \Omega_{n1}} + \frac{A_2(\epsilon_0)}{\Omega - \Omega_{n2}} + \frac{A_3(\epsilon_0)}{\Omega - n\omega(\epsilon_0)} \right),$$

(22)

where

$$A_3(\epsilon_0) := (D_n(n\omega(\epsilon_0)))^{-1},$$

and, for $(j, k) = (1, 2)$ or $(2, 1)$

$$A_j(\epsilon_0) := \frac{(\Omega_{nj} - \Omega_p)^2}{(\Omega_{nj} - \Omega_{nk})(\Omega_{nj} - n\omega(\epsilon_0))}. \quad (24)$$

Inverting (22) gives

$$H_n(t) = \frac{1}{2 \pi N} \sum_{a=1}^N e^{-in\theta_0} \left( A_1(\epsilon_0) e^{-i\Omega_{n1}t} + A_2(\epsilon_0) e^{-i\Omega_{n2}t} + A_3(\epsilon_0) e^{-i\Omega_{n3}t} \right).$$

(25)
We finish by studying $H_n$. We first note that $\mathbb{E}(H_n(t)) = 0$. When $\Im n(\Omega_{n}) < 0$, i.e., in the case of linear stability, we have for large $t$ the covariance:

$$\mathbb{E}(H_n(t)\ast H_n(t+s)) = \frac{1}{2\pi N} \int_{-\infty}^{\infty} f_{eq}(\epsilon) d\epsilon =: C_n(s),$$

(26)

where $\ast$ denotes complex conjugation. Thus $H_n$ becomes a weakly stationary stochastic process for large $t$. Making the change of the variable $\epsilon = (\lambda - n\omega_r)/nk$ we see that $C_n(s)$ is the Fourier transform of

$$\sigma(\lambda) := \frac{1}{N|n|k} D_n(\lambda)^2 \mathbb{E}(\frac{\lambda - n\omega_r}{nk}),$$

(27)

which is therefore the spectral density of the weakly stationary process. Note that $b_n$ is proportional to $\sqrt{N}$, thus the linear stability is lost if $N$ is sufficiently large. Also, linear stability is lost as $\alpha \rightarrow 0$, i.e., when $f_{eq}(\epsilon) \rightarrow \delta(\epsilon)/2\pi$. Thus the hydrodynamical approximation to the Vlasov equation would not be valid.

We now compare our result with an approach used in [2]. There a ‘noise power spectrum’ is computed by considering the quantity $A(\Delta) := 2\Delta \mathbb{E}(\tilde{H}_n(\lambda + i\Delta)\tilde{H}_n(\lambda + i\Delta)^*)$ in the limit $\Delta \rightarrow 0^+$. For $\Delta > 0$ one has by (18),(19) that

$$\mathbb{E}(\tilde{H}_n(\lambda + i\Delta)\tilde{H}_n(\lambda + i\Delta)^*) = \frac{1}{4\pi^2 N^2 |D_n(\lambda + i\Delta)|^2} \sum_{a=1}^{N} \mathbb{E}(\frac{\lambda - n\omega_r(\epsilon_a)}{nk})^2,$$

(28)

since cross terms do not contribute because of statistical independence. Thus $A(\Delta) = \frac{1}{\pi |D_n(\lambda + i\Delta)|^2} \int_{-\infty}^{\infty} \frac{\Delta f_{eq}(\epsilon) d\epsilon}{(\lambda - n\omega_r(\epsilon_a))^2 + \Delta^2}$.

Thus the two calculations give the same result. Note however that the second calculation does not use the specific $f_{eq}$, it only uses (18)-(20). Nor does it seem to care about the analyticity properties of $D_n$. We suspect that, the procedure in [2] gives the spectral density of $H_n$ if the latter is weakly stationary. However, $H_n$ generically is not weakly stationary. It seems likely that the procedure in [2] does not make sense if the roots of $D_n$ are in the upper half plane. If they are in the lower half plane, the process $H_n$ is not weakly stationary because of the decaying exponents. We must leave open the question why the two calculations are in agreement for our special case.

**DISCUSSION**

We are pursuing the issues raised after (28). In addition, we are interested in what is really measured in an accelerator. Is it the spectral density? What is measured if the process is not weakly stationary?

**APPENDIX I**

To see that (1),(2) are not unreasonable we first note that from the Lorentz equation $m^2c^2 = qE \cdot v$ where $E$ is the electric field and $v$ is the velocity. For our case of circular motion $\mathbf{E} \cdot \mathbf{v} \approx E_{az} \omega_r R$ where $E_{az}$ is the azimuthal field and $R$ is the radius. The current $I(\theta, t)$ is approximately $qN\rho(\theta, t)\omega_r$ where $N\rho$ is the particle density. Let $V(\theta, t) := -2\pi RE_{az}$, then solving Maxwell’s equations by a Laplace transform in $t$ and a Fourier series in $\theta$ gives $V_n(\Omega) = Z(n,\Omega)I_n(\Omega)$ with $Z(0,\Omega) = 0$. See for example [7], where $Z(n,\Omega)$ is called the complete impedance. Approximating $Z(n,\Omega)$ by $Z_n := Z(n, n\omega_r)$ we obtain $V_n(t) = Z_n I_n(t)$ and thus $V(\theta, t) = qN\omega_r \sum_{n \in \mathbb{Z}} Z_n \rho_n(t)e^{i\omega t}$. Equations (1),(2) follow if we define $W_n := -Z_n(q\omega_r)^2/(4\pi^2 E_r)$.

**APPENDIX II**

We assume that $\Psi_0(z_1, \ldots, z_N)$ is symmetric under permutations of the $z_1, \ldots, z_N$. Thus, by the special form of $W$ (whence of $w$), $\Psi(z_1, \ldots, z_N, t)$ is also symmetric under the permutations so we get $f(\theta, \epsilon, t) \equiv \mathbb{E}F(\theta, \epsilon, t) = f_1(\theta, \epsilon, t)$, where $f_j(\theta_1, \epsilon_1, \ldots, \theta_j, \epsilon_j, t) := \int \Psi(\theta_1, \epsilon_1, \ldots, \theta_N, \epsilon_N, t)d\theta_{j+1}d\epsilon_{j+1}\cdots d\theta_N d\epsilon_N$ for $j = 1, 2, \ldots, N-1$ and $f_N := \Psi$.

**ACKNOWLEDGEMENTS**

We are grateful to P. Colestock, R. Kashuba, and I. Vlaicu for introducing us to this problem. Discussions with Y. Elskens in Senigallia are greatly acknowledged.

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