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Beam dynamics in $e\bar{e}$ storage rings and a stochastic Schrödinger-like equation

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Abstract

The longitudinal dynamics of electrons in $e\bar{e}$ storage rings has been studied, when the radiation damping and quantum excitation of synchrotron radiation are taken into account. It has been shown that the electron beam propagates according to the law specified by a stochastic Schrödinger-like equation, in which the role of Planck's constant is played by an effective longitudinal thermal beam emittance. © 1997 Published by Elsevier Science B.V.

1. Introduction

The dynamics of high-energy leptons in a storage ring device differs essentially from that of heavy particles. The reason is that the contribution of synchrotron radiation, taking place whenever the trajectory of a charged particle is bent, becomes significant and cannot be neglected in general. Consequently, the electron dynamics is dissipative, provided the radiation friction force is present. Moreover, quantum fluctuations of synchrotron radiation (due to the random acts of photon emission) give rise to a Langevin-type force in the equations of motion. Thus, the analogy with classical Brownian motion of a particle suspended in a viscous medium is complete with the radiation friction force playing the role of the well-known Stokes force. Fluctuations, implied by the existence of many degrees of freedom in the beam, are also present, which adds to the complexity of the problem. We further assume that these fluctuations do not lead to subsequent dissipation (the only source of dissipation being the synchrotron radiation), which is equivalent to considering the beam fluid inviscid. Their implementation in the formalism developed here is achieved in a way similar to that of Ref. [1].

In the present paper we study the motion of an electron coupled to the thermal bath constituting the random photon field. In addition, we take into account the effect produced by the rest of the beam, substituting it by a random inhomogeneous medium [1], where our test particle has been suspended. As a result, the beam

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propagation is governed by a stochastic Schrödinger-like equation in which the role of Planck's constant is played by an effective longitudinal beam emittance. The latter is a random quantity in its turn with an evolution law specified by a corresponding Fokker–Planck equation.

An approach similar to ours has been recently proposed [2], based on analogies between relativistic particle beam motion and non-relativistic quantum mechanics.

2. Hamiltonian description of beam dynamics with synchrotron radiation

Our starting point is the system of stochastic differential equations [3,4]

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{\nabla}_p H, \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s} = -\boldsymbol{\nabla} H - C(s)\boldsymbol{\nabla}_p H, \tag{2.1a}$$

$$-\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\partial H}{\partial \mathcal{H}}, \qquad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}s} = -\frac{\partial H}{\partial(-t)} + c^2 C(s)\frac{\partial H}{\partial \mathcal{H}}, \tag{2.1b}$$

describing the motion of an electron with rest mass m_0 and charge e in a storage ring device, when the synchrotron radiation is taken into account. Here $\boldsymbol{x} = (x_1, x_2)$ is the particle displacement in a plane transverse to the orbit, $\boldsymbol{p} = (p_1, p_2)$ is the conjugate momentum and s is the curve length along the circumference of the accelerator. The Hamiltonian H (which is in fact the longitudinal momentum) reads

$$H = -(1 + \boldsymbol{x}\cdot\boldsymbol{K})\sqrt{(\mathcal{H} - e\varphi)^2/c^2 - m_0^2c^2 - (\boldsymbol{p} - e\boldsymbol{A})^2} - (1 + \boldsymbol{x}\cdot\boldsymbol{K})eA_s, \tag{2.2}$$

where $\boldsymbol{K} = (K_1, K_2)$ is the local curvature of the orbit (for plane orbits $K_2 = 0$), $\mathcal{H}$ is the total energy of the electron and φ, A_s and $\boldsymbol{A} = (A_1, A_2)$ define the electromagnetic field providing beam acceleration and focusing (we exclude beam–beam interaction from consideration throughout the present paper). Furthermore,

$$C(s) = C_1(s) + C_2(s)\xi(s), \tag{2.3}$$

where the coefficients $C_1(s)$ and $C_2(s)$ are defined in terms of the mean and fluctuating parts of the radiated power as follows [4,5],

$$C_1(s) = \frac{p^4 c_1(s)}{c^2}, \qquad C_2(s) = \frac{p^2\sqrt{\gamma^3 c_2(s)}}{c^2}. \tag{2.4}$$

In Eqs. (2.4) p is the total momentum of the electron, γ is the relativistic factor and the quantities $c_1(s)$ and $c_2(s)$ are given by

$$c_1(s) = \frac{2r_e}{3m_0^3c}K_1^2(s), \qquad c_2(s) = \frac{55 r_e \hbar c}{24\sqrt{3}m_0^3}K_1^3(s), \tag{2.5}$$

where r_e is the classical electron radius,

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_0 c^2}. \tag{2.6}$$

The quantity $\xi(s)$ is a white noise variable with formal correlation properties

$$\langle\xi(s)\rangle = 0, \qquad \langle\xi(s)\xi(s_1)\rangle = \delta(s - s_1). \tag{2.7}$$

First of all let us introduce the non-canonical scaling transformation

$$\boldsymbol{x} \;\Rightarrow\; \boldsymbol{p}_1 = \frac{\boldsymbol{p}}{p_s}, \tag{2.8a}$$

$$\tau = -v_s t \quad \Rightarrow \quad h = \frac{\mathcal{H}}{v_s p_s} \tag{2.8b}$$

and specify the new Hamiltonian

$$H_1 = \frac{RH}{p_s}, \tag{2.9}$$

where v_s is the velocity of the synchronous particle, p_s is its momentum and R is the mean machine radius. Then Eqs. (2.1) become

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\theta} = \nabla_{p_1} H_1, \qquad \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}\theta} = -\nabla H_1 - \frac{\mathcal{C}(\theta)}{p_s}\nabla_{p_1} H_1, \tag{2.10a}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}\theta} = \frac{\partial H_1}{\partial h}, \qquad \frac{\mathrm{d}h}{\mathrm{d}\theta} = -\frac{\partial H_1}{\partial \tau} + \frac{\mathcal{C}(\theta)}{\beta_s^2 p_s}\frac{\partial H_1}{\partial h}, \tag{2.10b}$$

where the new independent variable is the azimuth θ. Next we perform a canonical transformation given by the generating function

$$F_2^{(1)}(\boldsymbol{x}, \boldsymbol{p}_2, \tau, \eta; \theta) = \boldsymbol{x} \cdot \boldsymbol{p}_2 + \tau(\eta + 1/\beta_s^2) + R\eta\theta \tag{2.11}$$

and obtain the equations of motion for the new canonical variables

$$\sigma = \tau + R\theta, \qquad \eta = h - 1/\beta_s^2 \tag{2.12}$$

in terms of the new Hamiltonian

$$H_2 = H_1 + R\eta + \frac{R\mathcal{C}(\theta)}{\beta_s^2 p_s}\sigma \tag{2.13}$$

that are quite the same compared to (2.10b). Obviously, the equations describing the evolution of the transversal degrees of freedom in terms of the new Hamiltonian (2.13) become similar to Eqs. (2.10a). Hence

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\theta} = \nabla_{p_1} H_2, \qquad \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}\theta} = -\nabla H_2 - \frac{\mathcal{C}(\theta)}{p_s}\nabla_{p_1} H_2, \tag{2.14a}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{\partial H_2}{\partial \eta}, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}\theta} = -\frac{\partial H_2}{\partial \sigma} + \frac{\mathcal{C}(\theta)}{\beta_s^2 p_s}\frac{\partial H_2}{\partial \eta}. \tag{2.14b}$$

Expanding the first term on the right-hand side of the original Hamiltonian (2.2) in the small quantities $|\boldsymbol{p}_1|$ and η we obtain [6]

$$H_2 = H_2^{(0)} + H_2^{(1)} + H_2^{(2)} + H_2^{(3)} + \ldots, \tag{2.15}$$

$$H_2^{(0)} = \frac{R\eta^2}{2\gamma_s^2} + \frac{R\mathcal{C}(\theta)}{\beta_s^2 p_s}\sigma + \frac{eR\mathcal{E}_0(\theta)}{\omega p_s}\cos\left(\frac{\omega\sigma}{v_s} - k\theta + \phi_0\right), \tag{2.15a}$$

$$H_2^{(1)} = -R\eta(\boldsymbol{x} \cdot \boldsymbol{K}), \tag{2.15b}$$

$$H_2^{(2)} = \frac{R}{2}\boldsymbol{p}_1^2 + \frac{1}{2R}(G_1 x_1^2 + G_2 x_2^2), \tag{2.15c}$$

$$H_2^{(3)} = R(\boldsymbol{x} \cdot \boldsymbol{K})\left(\frac{\boldsymbol{p}_1^2}{2} + \frac{g_0}{2R^2}(x_1^2 - x_2^2)\right) + \frac{\lambda_0}{6R^2}(x_1^3 - 3x_1 x_2^2), \tag{2.15d}$$

where $\mathcal{E}_0(\theta)$ is the accelerating field, ω and ϕ_0 are its frequency and phase respectively and k is the acceleration harmonic. The coefficients $g_0(\theta)$ and $\lambda_0(\theta)$ specify the quadrupolar and the sextupolar strengths of the guiding magnetic field, while $G_1(\theta)$ and $G_2(\theta)$ determine the linear stability of the machine. If terms of higher order in the transverse coordinates are retained, then contributions due to higher multipoles would appear in the Hamiltonian (2.15). Let us now consider a second canonical transformation specified by the generating function

$$F_2^{(2)}(\boldsymbol{x},\hat{\boldsymbol{p}},\sigma,\hat{\eta};\theta) = \sigma\hat{\eta} + \hat{\boldsymbol{p}}\cdot(\boldsymbol{x} - \hat{\eta}\boldsymbol{D}) + \frac{\hat{\eta}}{R}\boldsymbol{x}\cdot\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}\theta} - \frac{\hat{\eta}^2}{2R}\boldsymbol{D}\cdot\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}\theta}, \tag{2.16}$$

where $\boldsymbol{D}(\theta)$ is the dispersion function, satisfying the equation

$$\frac{\mathrm{d}^2\boldsymbol{D}_k(\theta)}{\mathrm{d}\theta^2} + G_k(\theta)\boldsymbol{D}_k(\theta) = R^2K_k(\theta) \qquad (k=1,2). \tag{2.17}$$

The old and the new canonical variables are related via the expressions

$$\boldsymbol{x} = \hat{\boldsymbol{x}} + \hat{\eta}\boldsymbol{D}, \qquad \boldsymbol{p}_1 = \hat{\boldsymbol{p}} + \frac{\hat{\eta}}{R}\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}\theta}, \tag{2.18a}$$

$$\sigma = \hat{\sigma} + \hat{\boldsymbol{p}}\cdot\boldsymbol{D} - \frac{\hat{\boldsymbol{x}}}{R}\cdot\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}\theta}, \qquad \eta = \hat{\eta}. \tag{2.18b}$$

Retaining terms quadratic in the momenta $\hat{\boldsymbol{p}}$, $\hat{\eta}$ the new Hamiltonian can be written as [6]

$$\hat{H} = \hat{H}_0 + \hat{H}_2, \tag{2.19}$$

$$\hat{H}_0 = -\frac{R\mathcal{K}}{2}\hat{\eta}^2 + \frac{RC(\theta)}{\beta_s^2 p_s}\sigma + \frac{\Delta E_0}{2\pi\omega p_s}\cos\left(\frac{\omega\sigma}{v_s} + \varphi_0\right), \tag{2.19a}$$

$$\hat{H}_2 = \frac{R}{2}\hat{\boldsymbol{p}}^2 + \frac{1}{2R}\left(G_1\hat{x}_1^2 + G_2\hat{x}_2^2\right) + \mathcal{V}(\boldsymbol{x};\theta), \tag{2.19b}$$

where $\mathcal{V}(\boldsymbol{x};\theta)$ collects all non-linear terms due to the guiding magnetic field. Here $\Delta E_0 = \langle eR\mathcal{E}_0(\theta)\rangle_\theta$ is the maximum energy gain per turn, $\mathcal{K}$ is the autophasing coefficient (phase slip factor) and φ_0 is the phase of the accelerating voltage,

$$\mathcal{K} = \alpha_M - \frac{1}{\gamma_s^2}, \qquad \varphi_0 = \phi_0 - \Xi_0, \tag{2.20a}$$

$$\Xi_0 = \arctan\left(\frac{\langle eR\mathcal{E}_0(\theta)\sin k\theta\rangle_\theta}{\langle eR\mathcal{E}_0(\theta)\cos k\theta\rangle_\theta}\right). \tag{2.20b}$$

The equations of motion for the new canonical variables given by the canonical transformation (2.18) in terms of the new Hamiltonian (2.19) are written now in the form

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}\theta} = \boldsymbol{\nabla}_{\hat{p}}\hat{H} - \frac{C\boldsymbol{D}}{\beta_s^2 p_s}\frac{\partial H_2}{\partial\eta}, \tag{2.21a}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{p}}}{\mathrm{d}\theta} = -\boldsymbol{\nabla}_{\hat{x}}\hat{H} - \frac{C}{p_s}\boldsymbol{\nabla}_{p_1}H_2 - \frac{C}{R\beta_s^2 p_s}\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}\theta}\frac{\partial H_2}{\partial\eta}, \tag{2.21b}$$

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\theta} = \frac{\partial\hat{H}}{\partial\hat{\eta}} + \frac{C\boldsymbol{D}}{p_s}\cdot\boldsymbol{\nabla}_{p_1}H_2, \qquad \frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\theta} = -\frac{\partial\hat{H}}{\partial\hat{\sigma}} + \frac{C}{\beta_s^2 p_s}\frac{\partial H_2}{\partial\eta}. \tag{2.21c}$$

In what follows we shall consider the synchrotron motion only and neglect the betatron motion and the synchro-betatron coupling, which generally takes place by virtue of the expressions (2.18). It is possible, however, to cast Eqs. (2.21c) into a Hamiltonian form [7-9], if the time-dependent "synchrotron radius" is introduced according to the expression

$$R_s(\theta) = R\exp\left(-\frac{R\mathcal{K}}{\beta_s^2 p_s}\int\limits_0^\theta \mathrm{d}\theta_1\, C(\theta_1)\right). \tag{2.22}$$

Then Eqs. (2.21c) become identical to the Hamilton equations

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\theta} = \frac{\partial \hat{H}_s}{\partial \hat{\eta}}, \qquad \frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\theta} = -\frac{\partial \hat{H}_s}{\partial \hat{\sigma}}, \tag{2.23}$$

where

$$\hat{H}_s = \frac{R_s(\theta)}{2}\hat{\eta}^2 - \frac{R\mathcal{K}}{R_s(\theta)}\left[\frac{RC(\theta)}{\beta_s^2 p_s}\hat{\sigma} + \frac{\Delta E_0}{2\pi\omega p_s}\cos\left(\frac{\omega\hat{\sigma}}{v_s} + \varphi_0\right)\right] \tag{2.24}$$

is the Hamiltonian function, which is not the energy of our system as has been pointed out in Ref. [7].

3. The stochastic Schrödinger-like equation

Following in quite the same manner the strategy outlined in our previous paper [1] we introduce in the Hamiltonian (2.24) the random longitudinal velocity field $z_s(\theta)$ with the formal correlation properties

$$\langle z_s(\theta)\rangle_s = 0, \qquad \langle z_s(\theta) z_s(\theta_1)\rangle_s = \epsilon_{s0} R_s(\theta)\delta(\theta - \theta_1). \tag{3.1}$$

The notation $\langle\ldots\rangle_s$ implies a statistical average over the ensemble of realizations of the process $z_s(\theta)$ and ϵ_{s0} is the longitudinal thermal beam emittance. Note that the average in (3.1) does not affect the process $\xi(\theta)$ responsible for the quantum fluctuations of the synchrotron radiation. This means that the correlation function $\epsilon_{s0}R_s(\theta)$ is in its turn a random function (strictly speaking, a functional of the process $\xi(\theta)$). Taking into account the temporal behavior of the synchrotron radius $R_s(\theta)$ we obtain the set of equations describing the evolution of the Madelung fluid,

$$\frac{\partial\rho}{\partial\theta} + \frac{\partial}{\partial\hat{\sigma}}(v_\sigma\rho) = 0, \tag{3.2a}$$

$$\rho u_\sigma = -\frac{\epsilon_{s0}R_s}{2}\frac{\partial\rho}{\partial\hat{\sigma}}, \tag{3.2b}$$

$$\frac{\partial v_\sigma}{\partial\theta} + v_\sigma\frac{\partial v_\sigma}{\partial\hat{\sigma}} = \frac{v_\sigma}{R_s}\frac{\mathrm{d}R_s}{\mathrm{d}\theta} - R_s\frac{\partial U_\sigma}{\partial\hat{\sigma}} + u_\sigma\frac{\partial u_\sigma}{\partial\hat{\sigma}} - \frac{\epsilon_{s0}R_s}{2}\frac{\partial^2 u_\sigma}{\partial\hat{\sigma}^2}, \tag{3.2c}$$

where

$$U_\sigma(\hat{\sigma};\theta) = -\frac{R\mathcal{K}}{R_s(\theta)}\left[\frac{RC(\theta)}{\beta_s^2 p_s}\hat{\sigma} + \frac{\Delta E_0}{2\pi\omega p_s}\cos\left(\frac{\omega\hat{\sigma}}{v_s} + \varphi_0\right)\right]. \tag{3.3}$$

It can be checked that the system (3.2) is equivalent to the Schrödinger-like equation

$$\mathrm{i}\epsilon_{s0}R_s(\theta)\frac{\partial\Psi}{\partial\theta} = -\frac{\epsilon_{s0}^2 R_s^2(\theta)}{2}\frac{\partial^2\Psi}{\partial\hat{\sigma}^2} + U_s(\hat{\sigma};\theta)\Psi \tag{3.4}$$

through the well-known ansatz

$$\Psi(\hat{\sigma};\theta) = \sqrt{\rho(\hat{\sigma};\theta)}\exp\left[\mathrm{i}S(\hat{\sigma};\theta)\right], \tag{3.5}$$

where

$$v_\sigma(\hat{\sigma};\theta) = \epsilon_{s0} R_s(\theta) \frac{\partial S(\hat{\sigma};\theta)}{\partial \hat{\sigma}}, \qquad U_s(\hat{\sigma};\theta) = R_s(\theta) U_\sigma(\hat{\sigma};\theta). \tag{3.6}$$

In order to simplify the subsequent exposition let us adopt the following notations,

$$\alpha_1 = \frac{R\mathcal{K}}{2\pi\beta_s^2 p_s} \int_0^{2\pi} \mathrm{d}\theta\, C_1(\theta), \qquad \alpha_2 = \frac{\sqrt{R}\mathcal{K}}{2\pi\beta_s^2 p_s} \int_0^{2\pi} \mathrm{d}\theta\, C_2(\theta), \tag{3.7a}$$

$$\Phi_s(\hat{\sigma};\theta) = -\Phi_{s0}(\hat{\sigma};\theta) - \alpha_2 \hat{\sigma} \xi(\theta), \tag{3.7b}$$

$$\Phi_{s0}(\hat{\sigma};\theta) = \alpha_1 \hat{\sigma} + \frac{\mathcal{K}\Delta E_0}{2\pi\omega p_s} \cos\left(\frac{\omega\hat{\sigma}}{v_s} + \varphi_0\right), \tag{3.7c}$$

$$\hat{\epsilon}_s(\theta) = \epsilon_s(\theta) \exp\left(-\alpha_2 \int_0^\theta \mathrm{d}\tau\, \xi(\tau)\right), \qquad \epsilon_s(\theta) = \epsilon_{s0} \mathrm{e}^{-\alpha_1\theta}. \tag{3.7d}$$

Then the stochastic Schrödinger-like equation (3.4) takes its final form,

$$\mathrm{i}\hat{\epsilon}_s(\theta) \frac{\partial \Psi}{\partial \theta} = -\frac{R\hat{\epsilon}_s^2(\theta)}{2} \frac{\partial^2 \Psi}{\partial \hat{\sigma}^2} + \Phi_s(\hat{\sigma};\theta)\Psi. \tag{3.8}$$

The relevant quantity one can obtain from Eq. (3.8) is the beam wave function

$$\psi(\hat{\sigma};\theta) = \langle \Psi(\hat{\sigma};\theta) \rangle_\xi, \tag{3.9}$$

averaged over the realizations of the process $\xi(\theta)$. Performing the above-mentioned statistical average in (3.8) we get the following equation,

$$\mathrm{i}\left\langle \hat{\epsilon}_s(\theta) \frac{\partial \Psi}{\partial \theta} \right\rangle = -\frac{R}{2} \left\langle \hat{\epsilon}_s^2(\theta) \frac{\partial^2 \Psi}{\partial \hat{\sigma}^2} \right\rangle - \Phi_{s0}(\hat{\sigma};\theta)\psi - \alpha_2 \hat{\sigma} \langle \xi(\theta)\Psi \rangle, \tag{3.10}$$

which is unfortunately, not closed with respect to $\psi(\hat{\sigma};\theta)$, for it contains the yet unknown correlators on both sides. Hence, we have to find a way to split correlations of the type $\langle \hat{\epsilon}_s^n[\xi(\theta)]\Psi[\xi(\theta)] \rangle$ and $\langle \xi(\theta)\Psi[\xi(\theta)] \rangle$.

In order to solve this problem we use the method of the characteristic functional, initially proposed by Furutsu and Novikov and later developed by Klyatskin and Tatarsky (see Ref. [10] and references therein). For two given generic functionals $F[\xi(\theta)]$ and $G[\xi(\theta)]$ of the stochastic process $\xi(\theta)$, their correlation can be computed according to the expression

$$\langle F[\xi(\theta)] G[\xi(\tau)] \rangle = \left\langle \Omega_\theta^{(F)} \left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta\xi(\tau)} \right) G[\xi(\tau)] \right\rangle, \tag{3.11}$$

where

$$\Omega_\theta^{(F)}[v(\tau)] = \frac{\langle F[\xi(\theta)] \exp[\mathrm{i}\int_0^\theta \mathrm{d}\tau\, \xi(\tau)v(\tau)] \rangle}{\langle \exp[\mathrm{i}\int_0^\theta \mathrm{d}\tau\, \xi(\tau)v(\tau)] \rangle} \tag{3.12}$$

and $\delta/\delta\xi(\tau)$ denotes the functional derivative with respect to the process $\xi(\tau)$. The statistical properties of the latter are entirely specified by the characteristic functional

$$\Phi_\theta[v] = \left\langle \exp\left(\mathrm{i} \int_0^\theta \mathrm{d}\tau\, \xi(\tau)v(\tau) \right) \right\rangle, \tag{3.13}$$

or alternatively by the generating cumulant functional $\Theta_\theta[v]$ defined as

$$\Phi_\theta[v] = \exp(\Theta_\theta[v]). \tag{3.14}$$

If the process $\xi(\theta)$ is a Gaussian stochastic process the generating cumulant functional becomes

$$\Theta_\theta[v] = -\frac{1}{2}\int_0^\theta d\tau_1 \int_0^\theta d\tau_2\, \langle\xi(\tau_1)\xi(\tau_2)\rangle v(\tau_1)v(\tau_2). \tag{3.15}$$

Noting that

$$\Omega_\theta^{(\xi)}[v] = \frac{1}{\mathrm{i}v(\theta)}\left(\frac{d}{d\theta}\Theta_\theta[v]\right) = \mathrm{i}\int_0^\theta d\tau\, \langle\xi(\theta)\xi(\tau)\rangle v(\tau), \tag{3.16}$$

by virtue of (3.11) we immediately obtain the Furutsu–Novikov formula

$$\langle\xi(\theta)\Psi[\xi]\rangle = \int_0^\theta d\tau\, \langle\xi(\theta)\xi(\tau)\rangle\left\langle\frac{\delta\Psi[\xi]}{\delta\xi(\tau)}\right\rangle, \tag{3.17}$$

which we shall apply later to split the correlation in the last term of Eq. (3.10). As far as the correlator $\langle\hat{\epsilon}_s^n(\theta)\Psi[\xi(\tau)]\rangle$ is concerned, straightforward calculations give

$$\Omega_\theta^{(\hat{\epsilon}_s^n)}[v] = \epsilon_s^n(\theta)\exp(\Theta_\theta[v + \mathrm{i}n\alpha_2] - \Theta_\theta[v]) \tag{3.18}$$

and making use of (3.11) once more we find

$$\langle\hat{\epsilon}_s^n(\theta)\Psi[\xi(\tau)]\rangle = \epsilon_s^n(\theta)\exp\left(\frac{n^2\alpha_2^2}{2}\int_0^\theta d\tau_1\int_0^\theta d\tau_2\, \langle\xi(\tau_1)\xi(\tau_2)\rangle\right)$$
$$\times\left\langle\Psi\left(\xi(\tau) - n\alpha_2\int_0^\theta d\tau_1\, \langle\xi(\tau)\xi(\tau_1)\rangle\right)\right\rangle. \tag{3.19}$$

In writing Eq. (3.19) we have utilized the well-known formula for the functional shift

$$G[\xi(\tau) + \varphi(\tau)] = \exp\left(\int d\tau_1\, \varphi(\tau_1)\frac{\delta}{\delta\xi(\tau_1)}\right)G[\xi(\tau)]. \tag{3.20}$$

What remains now is to compute the functional derivative $\delta\Psi[\xi]/\delta\xi(\theta)$ entering the Furutsu–Novikov formula (3.17). For that purpose we first observe that provided the Schrödinger-like equation (3.8) is of first order with respect to the time θ, its solution at the instant θ will functionally depend on the stochastic process $\xi(\theta_1)$ for $0 \leqslant \theta_1 \leqslant \theta$ and $\Psi[\xi]$ does not change under variations of $\xi(\theta_1)$ for $\theta_1 < 0$, $\theta_1 > \theta$. Therefore, the following causality condition holds,

$$\frac{\delta\Psi[\xi(\theta)]}{\delta\xi(\theta_1)} = 0 \qquad (\theta_1 < 0; \theta_1 > \theta). \tag{3.21}$$

Next varying Eq. (3.8) for $0 \leqslant \theta_1 \leqslant \theta$ we get

$$\mathrm{i}\frac{\partial}{\partial\theta}\left(\frac{\delta\Psi[\xi]}{\delta\xi(\theta_1)}\right) = -\frac{R\hat{\epsilon}_s}{2}\frac{\partial^2}{\partial\hat{\sigma}^2}\left(\frac{\delta\Psi[\xi]}{\delta\xi(\theta_1)}\right) + \frac{\alpha_2 R\hat{\epsilon}_s}{2}\frac{\partial^2\Psi[\xi]}{\partial\hat{\sigma}^2} + \hat{\epsilon}_s^{-1}\Phi_s(\hat{\sigma};\theta)\frac{\delta\Psi[\xi]}{\delta\xi(\theta_1)} + \alpha_2\hat{\epsilon}_s^{-1}\left[\Phi_s(\hat{\sigma};\theta) - \hat{\sigma}\delta(\theta-\theta_1)\right]\Psi[\xi]. \tag{3.22}$$

Formally integrating the last equation in the limit $\theta_1 \to \theta$ with the condition (3.21) in hand we obtain the desired expression for the functional derivative, namely

$$\frac{\delta\Psi[\xi]}{\delta\xi(\theta_1)} = \mathrm{i}\alpha_2\hat{\epsilon}_s^{-1}\hat{\sigma}\Psi[\xi]. \tag{3.23}$$

By virtue of (3.17), (3.19) and (3.23) having started from the averaged Schrödinger-like equation (3.10) we finally arrive at

$$\mathrm{i}\epsilon_s(\theta)\frac{\partial\langle\Psi[\xi-\alpha_2]\rangle}{\partial\theta} = -\frac{R\epsilon_s^2(\theta)\mathrm{e}^{3\alpha_2^2\theta/2}}{2}\frac{\partial^2\langle\Psi[\xi-2\alpha_2]\rangle}{\partial\hat{\sigma}^2} - \mathrm{e}^{-\alpha_2^2\theta/2}[\Phi_{s0}(\hat{\sigma};\theta) - \tfrac{1}{2}\alpha_2^2\hat{\sigma}]\psi - \tfrac{1}{2}\mathrm{i}\alpha_2^2\hat{\sigma}^2\epsilon_s^{-1}(\theta)2\langle\Psi[\xi+\alpha_2]\rangle. \tag{3.24}$$

In the lowest order in the parameter α_2, Eq. (3.24) reduces to

$$\mathrm{i}\epsilon_s(\theta)\frac{\partial\psi}{\partial\theta} = -\frac{R\epsilon_s^2(\theta)\mathrm{e}^{3\alpha_2^2\theta/2}}{2}\frac{\partial^2\psi}{\partial\hat{\sigma}^2} - \mathrm{e}^{-\alpha_2^2\theta/2}[\Phi_{s0}(\hat{\sigma};\theta) - \tfrac{1}{2}\alpha_2^2\hat{\sigma}]\psi - \tfrac{1}{2}\mathrm{i}\alpha_2^2\hat{\sigma}^2\epsilon_s^{-1}(\theta)\psi. \tag{3.25}$$

It may be worth mentioning here that the distribution of the effective longitudinal beam emittance (3.7d) is logarithmically normal,

$$W(\hat{\epsilon}_s;\theta|\epsilon_{s0};0) = \frac{1}{\alpha_2\hat{\epsilon}_s\mathrm{e}^{\alpha_1\theta}\sqrt{2\pi\theta}}\exp\left(-\frac{\left[\alpha_1\theta + \ln\left(\hat{\epsilon}_s/\epsilon_{s0}\right)\right]^2}{2\alpha_2^2\theta}\right) \tag{3.26}$$

and satisfies the Fokker-Planck equation,

$$\frac{\partial W(\hat{\epsilon}_s;\theta|\epsilon_{s0};0)}{\partial\theta} = \left(\alpha_1 + \tfrac{1}{2}\alpha_2^2\hat{\mathcal{L}}_s\right)\hat{\mathcal{L}}_s W(\hat{\epsilon}_s;\theta|\epsilon_{s0};0), \tag{3.27a}$$

$$\hat{\mathcal{L}}_s = \frac{\partial}{\partial\hat{\epsilon}_s}\hat{\epsilon}_s, \tag{3.27b}$$

which is a direct consequence of (3.7d). For the nth moment of the effective longitudinal emittance one readily obtains

$$\langle\hat{\epsilon}_s^n(\theta)\rangle = \epsilon_{s0}^n\exp\left[-n(\alpha_1 - \tfrac{1}{2}n\alpha_2^2)\theta\right]. \tag{3.28}$$

The latter expression indicates exponential growth of higher moments of the distribution (3.26) starting from

$$n > \frac{2\alpha_1}{\alpha_2^2}. \tag{3.29}$$

4. Concluding remarks

In the present paper we have studied the longitudinal dynamics of an electron circulating in an $e\bar{e}$ storage ring when both synchrotron radiation and interaction between particles in the beam are taken into account. It has been shown that the electron beam propagates according to the law specified by a stochastic Schrödinger-like equation. Particle motion is governed by a beam wave function averaged over the realization of the stochastic process, describing the quantum fluctuations of synchrotron radiation. The squared modulus of the beam wave function gives the longitudinal bunch profile, while its phase contains information concerning the energy spread within the bunch.

Let us finally note that the radiating electron beam represents a system being very far from thermodynamic equilibrium. It acts as a constant solicitor in the energy transfer from the power source to the incoherent electromagnetic background. Therefore, macroscopic quantities as for example the effective longitudinal thermal beam emittance, do not exhibit relaxation towards equilibrium values, which is indicated by Eqs. (3.26) and (3.28).

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