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A FEW MATHEMATICAL NOTES ON CONSTANT PERIMETER LAYERS IN COIL WINDINGS

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ABSTRACT

The definitions for ruled and developable surfaces given in the literature are restated and interpreted in this note, and then follows a description of constant perimeter layers in terms of geodesics and surface friction effects are mentioned. It is shown how a constant perimeter surface can be generated from a space curve, the space curve being the first or any turn of the layer.

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INTRODUCTION

Recently, constant perimeter surfaces have been used for magnets with saddle shaped coils (ref 3,4). These surfaces enable the turns of wire of each layer to be wound on top of the previous layer, with a reversal in the direction of filling, as happens in a cotton reel. It will be observed that for the cylindrical surface of the cotton reel all the turns of a layer have the same length. Surfaces without this constant perimeter property are not suitable for layers which are to be filled in either direction. For example, it is only possible to wind a layer on a cone when the layer turns are filled moving away from the cone apex; if filled towards the apex, the winding tension will constantly pull the wires out of place. The general principles which determine whether shapes are constant perimeter or not are given in this report.

SPACE CURVES

A brief summary of the fundamentals of curves in space (see ref 1,2) is given here as a preliminary to the discussion on surfaces. The general space curve equation in terms of a parameter u is

$$\underline{r} = \underline{r}_0(u) \qquad \text{or} \qquad x = x_0(u)\ ,\ y = y_0(u)\ ,\ z = z_0(u)$$

At each point of the curve there is a trihedron of 3 mutually perpendicular unit vectors (see fig 1):

1. Tangent $\underline{t} = \underline{r}' = \dot{\underline{r}}/\dot{s}$

2. Principal normal $\underline{n} = \underline{r}''/k = (\ddot{\underline{r}}\,\dot{s} - \dot{\underline{r}}\,\ddot{s})/(k\dot{s}^3)$

3. Binormal $\underline{b} = \underline{t} \times \underline{n} = \dot{\underline{r}} \times \ddot{\underline{r}}/|\dot{\underline{r}} \times \ddot{\underline{r}}| = \dot{\underline{r}} \times \ddot{\underline{r}}/(k\dot{s}^3)$

where k = curvature $= \frac{1}{R} = \sqrt{x''^2 + y''^2 + z''^2} = |\dot{\underline{r}} \times \ddot{\underline{r}}|/\dot{s}^3$

s = arc length $\quad \dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = |\dot{\underline{r}}| \qquad \ddot{s} = \dfrac{\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}}{\dot{s}} = \dfrac{\dot{\underline{r}}.\ddot{\underline{r}}}{\dot{s}}$

$\underline{r}' = \dfrac{d\underline{r}}{ds}\ ,\quad \dot{\underline{r}} = \dfrac{d\underline{r}}{du}$ etc

Fig 1 indicates that the curve, together with its centre of curvature, are momentarily in the $\underline{t}, \underline{n}$ plane. Moving along the space curve this plane rotates, and the rate defines the torsion:

$$\tau = \frac{d\theta}{ds}$$

and it is positive if the rotation is right handed.

The rate of change of trihedron vectors as they move along the curve are given by the following formulae, usually attributed to Frenet,

$$\underline{t}' = \dot{\underline{t}}/\dot{s} = k\underline{n} \qquad \underline{n}' = \dot{\underline{n}}/\dot{s} = \tau\underline{b} - k\underline{t} \qquad \underline{b}' = \dot{\underline{b}}/\dot{s} = -\tau\underline{n}$$

These equations are consistent with the view that the trihedron is rotating about the axis $\underline{a} = k\underline{b} + \tau\underline{t}$. The last Frenet formula enables the torsion to be found by dividing $\underline{b}'$ by $\underline{n}$.

It is well known that if $\underline{r}$ is the position of a particle and u is the time co-ordinate, the total acceleration $\ddot{\underline{r}}$ is resolvable into $\underline{t}$ and $\underline{n}$ components (accelerations along and perpendicular to the curve). Similarly in the general case $\ddot{\underline{r}}$ is always in the $\underline{t},\underline{n}$ plane, and if $\dot{s}$ is a constant it is directed along $\underline{n}$.

RULED SURFACES

A ruled surface is a surface which can be generated by moving a straight line continuously in space. If a ruling direction $\underline{p}$ is specified at each point on a space curve, the general equation of a ruled surface is obtained:

$$\underline{r}(u,v) = \underline{r}_o(u) + v\underline{p}(u)$$

where $\underline{r}_o(u)$ is the generator curve,
$\underline{p}(u)$ is the ruling at the point u ,
$\underline{r}(u,v)$ is the surface in terms of the parameters u and v .
Lines of constant u and constant v form a curvilinear mesh on the surface.

DEVELOPABLE RULED SURFACES

A developable ruled surface is a surface that can be rolled on a plane, touching along the entire ruling as it rolls. Cylinders and cones are two simple examples and in fact more complicated cases can be regarded as made from infinitesimal sections of these examples. Developable surfaces can always be made out of a sheet of paper by bending without stretching, and consequently they can be unrolled flat with no distortion of the paper to obtain the development (ref 2).

The feature which distinguishes developable surfaces from other ruled surfaces is the fixed tangent plane for the whole length of each ruling. A twisted strip is an example of an undevelopable surface because the tangent plane rotates with the ruling as axis (see fig 2). If an attempt is made to flatten out the twisted strip as if it were developable, the lengths of arc between pairs of points on the surface will change, whereas developable surfaces have identical arc lengths on the original and on the development.

It is best to regard developable surfaces as ending at the point where neighbouring ruled lines intersect e.g. the apex of a cone.

MATHEMATICAL CONDITIONS FOR A DEVELOPABLE SURFACE

Consider the space curve which lies on the general ruled surface and has $v = v_1$ = constant. This surface curve crosses the rulings at a distance $v_1 p$ from the generator curve $\underline{r}_0$.

v_1 surface curve: $\underline{r} = \underline{r}_0(u) + v_1 \underline{p}(u)$

tangent vector: $\dot{\underline{r}} = \dot{\underline{r}}_0(u) + v_1 \dot{\underline{p}}(u)$

$$= \dot{s}_0 \underline{t}_0 + v_1 \dot{\underline{p}}$$

If this tangent, which is known to be in the tangent plane of the surface, is always in the plane of $\underline{t}_0$ and $\underline{p}$ as v_1 is varied, we can conclude that the tangent plane is fixed and the surface is developable. We thus require $\dot{\underline{r}}$ to be coplanar with $\underline{t}_0$ and $\underline{p}$ and this will be so if $\dot{\underline{p}}$, $\underline{t}_0$ and $\underline{p}$ are coplanar.

GEODESICS

Geodesic lines on a surface are those surface curves which have the shortest arc length between surface points. Of course any surface curve is also a curve in space and it will have a principal normal; if the principal normal coincides with the normal to the surface at all points the curve is a geodesic line (ref 1,2). This is almost self-evident; consider the local projection of the curve on the surface tangent plane, then if the normals coincide the projections are locally straight instead of curved.

We find the properties of geodesics on developable surfaces by considering any straight line on the development. When the development is rolled up to form the original, the curvature plane and the principal normal of the line will become perpendicular to the surface. Therefore every straight line on a development corresponds to a geodesic on the surface, and conversely all geodesics develop to straight lines. Alternatively we could have used shortest distance arguments to show that straight lines are the geodesics of a plane, and then stated that developing does not change arc lengths. It also follows that every geodesic on a developable surface is accompanied by an infinity of parallel geodesics extending over the whole surface.

CONSTANT PERIMETER LAYERS

For a surface to be suitable for coil winding in either direction it is necessary that the surface closes on itself, and it must have a set of geodesics which are parallel, uniformly spaced, and are continuous loops around the surface. The turns of wire can then be laid on the geodesics without any tendency to slip sideways. Usually concave geodesic lines are not allowed because of the complication needed to prevent the wire lifting over the concave portion. The path of the parallel geodesics on the curved surface must be known and indicated to the coil winder, usually by making a boundary of the surface along one of them. The above statements may need slight modification because in practice wires have to cross to neighbouring geodesics once every turn, but these crossings are ignored in this report.

The conditions can only be met on a developable ruled surface (ref 2). Very many sets of parallel lines may be drawn on the plane development of a developable surface, and each set will give a set of parallel and uniformly spaced geodesics. To obtain a usable constant perimeter surface it is necessary that a set of parallel geodesics are identified which form closed loops on the surface.

A set of parallel geodesics loops on a developable surface all have the same length and hence the name - Constant Perimeter Layer.

APPROXIMATE CONSTANT PERIMETER SURFACES

It is clear that friction will enable turns to be laid on paths which depart from geodesic lines, and in these cases the $\underline{r}''$ $(= k\underline{n})$ vector for the path of the turn will not be normal to the surface. Fig 3 shows that the winding tension in the wire is transmitted to the surface as a force F along $\underline{r}''$, and this must be balanced by the normal reaction R and the frictional force μR from the surface. Therefore if $\underline{r}''$ is inclined to the surface normal by more than the angle $\tan^{-1}\mu$, where μ is the coefficient of friction, the wire will slip sideways. We can thus define an Approximate Constant Perimeter

Layer as one where there is no turn which has $\underline{r}''$ inclined at an angle more than $\tan^{-1}\mu$. Such a surface will serve as a coil winding "either direction" surface but it will be important to make adequate allowance for the unreliability of the experimental μ values.

GENERATION OF A CONSTANT PERIMETER LAYER FROM AN ARBITRARY SPACE CURVE

When designing the end geometry for a magnet coil it may happen that one of the turns, usually the first, is fixed by geometrical or other considerations. In these circumstances there will be a necessity to find the developable surface which has the given turn as a geodesic, because this will be the constant perimeter surface for the rest of the layer. Whilst tackling the difficult problem of the twisted elastica end for a saddle coil Rosten suggests (ref 5) that the surface ruled out by the binormal vector of a curve is a constant perimeter surface. This is an approximation, true for surfaces which have narrow widths measured along the binormal. However an exactly constant perimeter surface of unrestricted width is obtained if the trihedron rotation axis of the given curve is used instead of the binormal. The equation of this ruled surface is

$$\underline{r}(u,v) = \underline{r}_0(u) + v\,\underline{p}(u) \qquad \underline{p} = \underline{b}_0 + K\underline{t}_0$$

where $\underline{r}_0(u)$ is the given space curve

$\underline{t}_0, \underline{b}_0, \underline{n}_0$ are the trihedron vectors of $\underline{r}_0$ curve at the point u

$K = \tau/k$ = ratio of torsion and curvature for $\underline{r}_0$ curve.

Since the ruling $\underline{p}$ is in the $\underline{b}_0, \underline{t}_0$ plane, the surface generated is normal to $\underline{n}_0$ and the original curve $\underline{r}_0$ will be a geodesic as is required. Since the ruling is along the axis of rotation of the trihedron, we expect $\dot{\underline{p}}$ to have no component in the $\underline{n}_0$ direction, which is confirmed algebraically by differentiating the expression for $\underline{p}$, and using the Frenet formulae:

$$\dot{\underline{p}} = \dot{\underline{b}}_0 + K\dot{\underline{t}}_0 + \dot{K}\underline{t}_0 = -\tau\dot{s}_0\underline{n}_0 + K\dot{s}_0 k\underline{n}_0 + \dot{K}\underline{t}_0 = \dot{K}\underline{t}_0$$

Thus $\underline{p}$, $\dot{\underline{p}}$ and $\underline{t}_0$ are all in the same plane, the $\underline{t}_0, \underline{b}_0$ plane, and the surface is developable. Therefore the surface is the required constant perimeter surface. Although there is an infinity of developable surfaces which pass through the given turn, there is only one constant perimeter surface which has the original turn as a geodesic line.

We can also show that the v = constant lines are the set of parallel geodesics for the layer turns. The perpendicular separation of the line $v = v_1$ from the original curve $\underline{T}_0$, which is the $v = 0$ line, is $v_1 p \sin\varphi$ where φ is the angle between $\underline{t}_0$ and $\underline{p}$.

$$v_1 p \sin\varphi = v_1 |\underline{p} \times \underline{t}_0| = v_1 |(\underline{b}_0 + \kappa \underline{t}_0) \times \underline{t}_0| = v_1 |\underline{b}_0 \times \underline{t}_0| = v_1$$

Hence the separation is fixed and the v = constant lines are all parallel, and they will form a set of constant perimeter winding lines for this surface.

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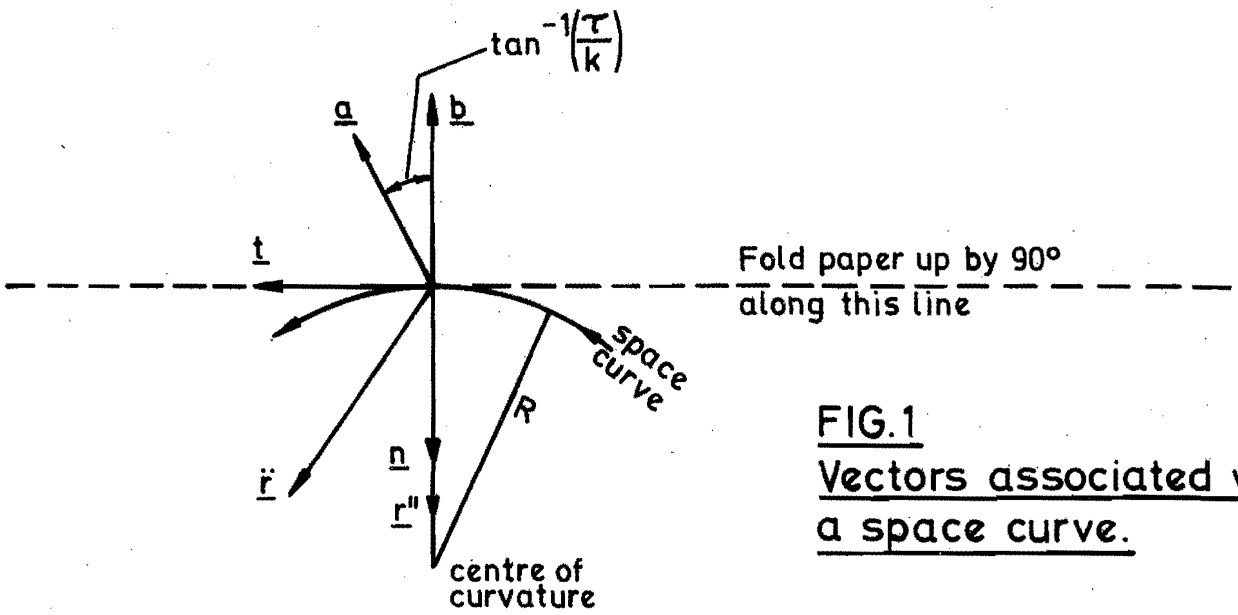
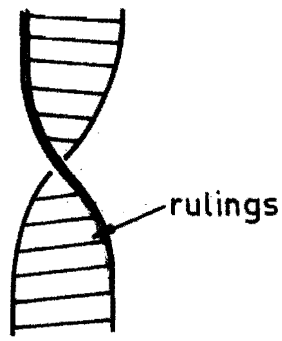


FIG. 1
Vectors associated with
a space curve.

FIG. 2
Twisted strip

An undevelopable ruled
surface.



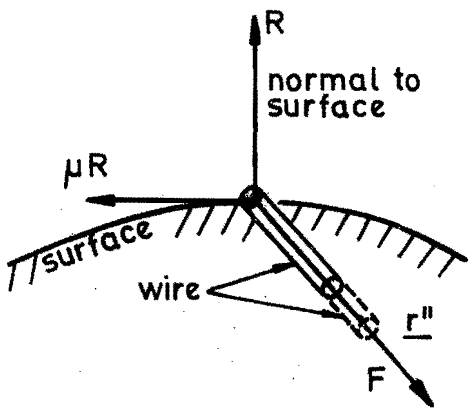


FIG. 3
Wire on non-geodesic path

F and r'' are in the same direction
and coplanar with R and μR;
the wire passes perpendicularly
through this plane.