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Saturation and Cross Field Coupling of Beat-Driven 3-D Plasma Waves

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Saturation and Cross Field Coupling of
Beat-Wave Driven 3-D Plasma Waves

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Using 3-D relativistic fluid equations the dynamics for the plasma beat wave generated by two circularly polarized electromagnetic waves with Gaussian profiles is obtained. Unlike the linear theory we find there is a strong cross field coupling between the longitudinal and radial electric field components of the beat wave resulting in a nonlinear phase change and energy transfer between these two components. The resulting beat wave structure being controlled more by this cross-field coupling than by the other nonlinear terms such as the nonlinear coupling to the laser beams with the result that the accelerating region of the plasma beat wave is much more restrictive than previously calculated.

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Introduction

Recently there has been considerable interest in the nonlinear excitation of large amplitude plasma waves by beating two laser beams in an underdense plasma [1,2] as a particle accelerator and known as the beat-wave accelerator. The scheme depends on the generation of a large amplitude plasma wave with a plasma velocity close to the velocity of light. Such a wave can be produced by beating two co-linear laser beams, with frequencies and wavenumber (ω_1, k_1) and (ω_2, k_2) in a plasma where the frequency and wavenumber of the plasma wave satisfy the following renonance conditions
$\omega_1 - \omega_2 = \omega_p$, $k_1 - k_2 = k_p$ where ω_p, k_p are the frequency and wavenumber of the plasma wave resulting from Raman forward scattering. The laser beams exert a periodic force (the ponderomotive force) on the electrons to produce charge separation and hence plasma oscillations at the resonant frequency $\omega_p = (4\pi e^2 n_o/m_e)^{\frac{1}{2}}$. If $\omega_p \ll \omega_{1,2}$ then the phase velocity of the plasma wave $v_{ph} = \omega_p/k_p$ is equal to the group velocity of the laser $v_g = c(1 - \omega_p^2/\omega_{1,2}^2)^{\frac{1}{2}}$ which is almost equal to the speed of light c in an underdense plasma.

Previous studies on the nonlinear behaviour of the large amplitude plasma wave have concentrated mostly on the one-dimensional treatment and considered only the longitudinal component. However, there is also a ponderomotive force associated with the transverse spatial variation of the pump profile. Most laser beams have a Gaussian transverse profile. This transverse ponderomotive force produces a transverse variation of the plasma beat wave giving rise to a periodic radial electric field. The importance of these radial electric fields as a possible focussing mechanism has been noted by several authors (3-6). Ruth and Chao (3) first pointed out that the radial electric field in quadrature with the longitudinal field, produces a strong defocusing force over the first half of the accelerating phase range and a strong focusing force over the second half. This results in the acceptable accelerating phase range of only $5\pi/16$. It has also been pointed out (5) that the radial field could be used in place of quadrapole magnets to provide the final focusing of an accelerated beam. The radial fields can also set up betatron oscillations (6) for the injected particles which could limit the accelerating or focusing schemes.

The theory for the generation of radial fields in the beat-wave accelerator for Gaussian pump profiles has been previously considered (6) but only for the linear case where the longitudinal field and radial field of the plasma are treated as independent variables being coupled only to the driving pump fields and not to each other. In this paper account is taken of the cross field coupling between the longitudinal and transverse components of the Langmuir wave which is excited by the beating between two laser beams with Gaussian cross sections. It is shown that the effect of the cross field coupling between the longitudinal and radial fields of the plasma wave allows for energy transfer between these components as well as introducing a nonlinear phase. The two field components are found to be only in phase for a particular radial position. These new effects combine to severely limit the region over which focusing or accelerating of the charged particles can take place. Another important concept introduced in the paper is a radial dependent plasma frequency and a frequency mismatch which is also radially dependent. This has the effect of causing the centre of the wave to saturate at a value lower than the wings. This effect has already been seen in simulations carried out by Mori (7).

The nonlinear equations describing the longitudinal and radial electric fields of the plasma beat-wave are derived in cylindrical co-ordinates and solved numerically assuming no pump depletion and compared with the linear independent solutions.

2. Model and derivations of the nonlinear equations

We shall consider a uniform unmagnetised infinite plasma in which two Gaussian profile laser beams propagate colinearly with frequencies much greater than the plasma frequency. The plasma model we use to analyse the problem is the relativistic fluid equations for electrons together with Maxwell's equations in cylindrical coordinates, the ions are assumed to be immobile and provide overall charge neutrality. The use of the relativistic treatment is necessary when it comes to examine the plasma beat wave which is driven to very large amplitudes such that the electron quiver velocity in both longitudional and radial directions approaches the velocity of light. Previous papers (3,4,6) concerning the three dimensional aspects of the problem have computed the longitudional and radial field components for the non-relativistic situation. In this paper we will consider the weakly relativistic limit where $\gamma = (1 - \underline{v}^2/c^2)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\underline{v}^2/c^2$, where $\underline{v}$ is the electron quiver velocity. We will also ignore Raman cascading and pump depletion.

Starting from the equations:

$$(\frac{\partial}{\partial t} + \underline{v}.\underline{\nabla})\ \underline{P} = -\ e\ (\underline{E} + \frac{1}{c}\ \underline{v} \times \underline{B}) \tag{1}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \tag{2}$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\ \underline{J} \tag{3}$$

$$\underline{\nabla} \cdot \underline{E} = 4\pi\rho \tag{4}$$

where $\underline{P} = \gamma m_o \underline{v}$, $\underline{J} = -\ en\underline{v}$, $\rho = -e(n-n_o)$, n is the electron density and m_o is the electron rest mass.

Introducing the scalar ϕ and vector $\underline{A}$ potentials such that

$\underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$ and $\underline{B} = \underline{\nabla} \times \underline{A}$

equation (1) can be written as

$$\frac{\partial}{\partial t}(\gamma\underline{v} - \frac{e}{m_o c}\ \underline{A}) = -\ \underline{\nabla}(c^2\gamma - \frac{e}{m_o}\Phi) + \underline{v} \times (\underline{\nabla} \times (\gamma\underline{v} - \frac{e}{m_o c}\ \underline{A})) \tag{5}$$

Taking the curl of equ. (5) results in

$$\frac{\partial \underline{\Gamma}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{\Gamma}) \ , \tag{6}$$

with $\underline{\Gamma} = \underline{\nabla} \times (\gamma\underline{v} - \dfrac{e}{m_o c}\ \underline{A})$

From the initial condition $\Gamma(t = o) = o$ and equation (6) it is concluded that $\Gamma(t) = o$ and equation (5) reduces to the following equation of motion

$$\frac{\partial \underline{P}}{\partial t} = -\ e\underline{E} - m_o\underline{\nabla}(c^2\gamma) \tag{7}$$

Using a multiple time scale analysis where the fast and slow timescales are the inverse laser and electron plasma frequencies respectively and using the expressions $\underline{P}=m_o\gamma\underline{v}$, $\gamma^2=1+P_\perp{}^2/m_o{}^2c^2+P^2/m_o{}^2c^2$, equation (7) is reduced to the following differential equation

$$m_o\frac{\partial}{\partial t}\left(\langle\gamma_\perp\rangle\left[1+\tfrac{1}{2}\left(\frac{1}{4\pi e n_o c}\frac{\partial \underline{E}_p}{\partial t}\right)^2\right]\frac{1}{4\pi e n_o}\frac{\partial \underline{E}_p}{\partial t}\right) = -e\underline{E}_p + \underline{F}_{NL} \qquad (8)$$

where $\langle\gamma_\perp\rangle = (1 + P_{o\perp}{}^2/m_o{}^2c^2)^{\frac{1}{2}}$, $\underline{P}_{o\perp} = \frac{e}{m_o c}\underline{A}_o$

$\underline{E}_p$ is the low frequency plasma electric field $\underline{F}_{NL}$ is the ponderomotive force due to the high frequency laser beams and is given by

$$\underline{F}_{NL} = -m_o c^2 \underline{\nabla}\langle\gamma_\perp\rangle \qquad (9)$$

and < > represents averaging over the fast laser period time scale.

In deriving equation (8) we have assumed $\frac{\partial \underline{E}_p}{\partial t} \simeq 4\pi e n_o \underline{v}$, neglecting magnetic field effects and second harmonic terms associated with the plasma and electromagnetic waves. The equation was also derived in the weakly relativistic limit such that $v^2/c^2 << 1$. The ponderomotive force $\underline{F}_{NL}$ is calculated assuming circularly polarized pump waves with Gaussian transverse profile. For the total pump wave amplitude we assume in terms of the vector potential $\underline{A}_o$

$$\begin{aligned}\underline{A}_o &= \underline{A}_1 + \underline{A}_2 \\ &= A_o \exp(-r^2/w_o{}^2) \sum_{i=1}^{2} (\hat{x}\cos(k_i z - \omega_i t) + \hat{y}\sin(k_i z - \omega_i t))\end{aligned} \qquad -(10)$$

where w_o represents the beam width of the two pumps.

Equation (10) can be written as

$$\underline{A}_o = 2A_o \exp(-r^2/w_o{}^2)\,[\hat{x}\cos(k_f z/2 - \omega_f t/2) + \hat{y}\sin(k_f z/2 - \omega_f t/2)]\times \cos(k_s z/2 - \omega_s t/2) \qquad (11)$$

$$\text{where } k_f = k_1 + k_2, \qquad \omega_f = \omega_1 + \omega_2$$
$$k_s = k_1 - k_2 \equiv k_p, \qquad \omega_s = \omega_1 - \omega_2 \equiv \Delta\omega \qquad (12)$$

Using equation (11) $P_{o\perp}$ and $\langle\gamma_\perp\rangle$ are given by

$$P_{o\perp}^2/m_o^2c^2 = \frac{4e^2}{m_o^4c^4} A_o^2 \exp(-2r^2/w_o^2)\cos^2(k_s z/2 - \omega_s t/2) \qquad -(13)$$

and

$$\langle\gamma_\perp\rangle = 1 + 2a_o^2\exp(-2r^2/w_o^2)[1 + \cos(k_p z - \Delta\omega t)] \qquad -(14)$$

resulting in the ponderomotive force being given by

$$\underline{F}_{NL} = -a_o^2 m_o c^2 \underline{\nabla}\{\exp(-2r^2/w_o^2)[1 + \cos\,(k_p z - \Delta\omega t)]\} \qquad -(15)$$

where $a_o^2 = e^2A_o^2/m_o^2c^2 << 1$

Using equation (15) to represent $\underline{F}_{NL}$ and the following normalized variables

$$\tau = \Delta\omega t$$

$$\underline{E} = (ek_p/m_o\omega_p^2)\underline{E}_p \qquad (16)$$

$$R \equiv k_p r, \quad Z = k_p z$$

equation (8) becomes

$$\frac{\partial}{\partial\tau}[\langle\gamma_\perp\rangle \{1 + \tfrac{1}{2}(\tfrac{\partial\underline{E}}{\partial\tau})^2\}\tfrac{\partial\underline{E}}{\partial\tau}] + (\omega_p/\Delta\omega)^2\underline{E} = -\, a_o^2\underline{\nabla}[\exp(-2R^2/(k_p w_o)^2)\mathrm{x}$$

$$(1 + \cos\theta)] \qquad -(17)$$

where θ=Z$-\tau$ and the gradient can be written in dimensionless variables as

$$\underline{\nabla} = \hat{r}\frac{\partial}{\partial R} + \hat{z}\frac{\partial}{\partial Z}$$

Writing $\langle\gamma_\perp\rangle$ in the form $\langle\gamma_\perp\rangle = [1 + a_o^2 \exp-(R\chi)^2][1 + \nu^2(R)\cos\theta]$

(18)

where $\chi = \sqrt{2}/k_p w_o$ and $\nu^2(R) = a_o^2 \exp(-R^2\chi^2)/(1 + a_o^2 \exp(-R^2\chi^2))$

results in the following expression for the electric field $\underline{E}$ of the plasma wave·

$$\frac{\partial}{\partial\tau}[1+\nu^2\cos\theta][1 + \tfrac{1}{2}(\tfrac{\partial\underline{E}}{\partial\tau})^2]\frac{\partial\underline{E}}{\partial\tau} + f^2(R)\underline{E} =$$

$$-\nu^2\exp(R\chi)^2\nabla\{\exp-(R\chi)^2[1 + \cos\theta]\} \qquad (19)$$

We have introduced into equation (19) a "radial dependent" plasma frequency and frequency mismatch also dependent on radial position given by

$\omega_s(R) = \omega_p^2/(1 + a_o^2 \exp-(R\chi)^2)$ and $f(R) = \omega_s(R)/\Delta\omega$

respectively.

Separating the electric field $\underline{E}$ into its components in cylindrical geometry such that

$$\underline{E}\,(R,Z,\tau) = \hat{r}E_r(R,Z,\tau) + \hat{z}E_z(R,Z,\tau) \qquad -(20)$$

And substituting this expression for $\underline{E}$ into equation (19) and carrying out the gradient operation yields to leading order the following equations for E_r and E_z

$$\begin{aligned} &\ddot{E}_r + f^2(1 - \tfrac{3}{2}\dot{E}_r^{\,2} - \tfrac{1}{2}\dot{E}_z^{\,2})E_r - f^2\dot{E}_r\dot{E}_zE_z = 2\nu^2R\chi^2[1+\cos\theta] \\ &\ddot{E}_z + f^2(1 - \tfrac{3}{2}\dot{E}_z^{\,2} - \tfrac{1}{2}\dot{E}_r^{\,2})E_z - f^2\dot{E}_r\dot{E}_zE_r = \nu^2\sin\Theta \end{aligned} \qquad (21)$$

where E_z and E_r satisfy the initial conditions

$$\begin{aligned} &E_r(\tau=0) = 0, \quad \dot{E}_r/_{\tau=0} = 0 \\ &E_z(\tau=0) = 0, \quad \dot{E}_z/_{\tau=0} = 0 \end{aligned} \qquad (22)$$

and the dot represents time derivative

In order to solve the two coupled equations (21) we set $E_r = \kappa v$ and $E_z = \kappa u$ and use the following scaling $\kappa^2 = \varepsilon$, $\nu^2/\kappa = \varepsilon$ where κ is proportional to the pump strength ($\kappa = \nu^{2/3}$) and ε is the small expansion parameter ($\varepsilon = \nu^{4/3}$). This results in the following two coupled equations

$$\ddot{v} + (1 + \varepsilon\sigma)[1 - \tfrac{3}{2}\varepsilon\dot{v}^2 - \tfrac{1}{2}\varepsilon\dot{u}^2]v - (1 + \varepsilon\sigma)\varepsilon\dot{u}\dot{v}u = \varepsilon\alpha^2(1 + \cos\theta) \qquad (23a)$$

$$\ddot{u} + (1 + \varepsilon\sigma)[1 - \tfrac{3}{2}\varepsilon\dot{u}^2 - \tfrac{1}{2}\varepsilon\dot{v}^2]v - (1 + \varepsilon\sigma)\varepsilon\dot{u}\dot{v}v = \varepsilon\sin\theta \qquad (23b)$$

where σ is a detuning parameter such that $f^2(R) = 1 + \epsilon\sigma(R)$ and $\alpha^2 = 2\chi^2 R$. Equation (23) represents the desired coupled set of equations that have been solved numerically, with the results displayed in figures 1(a-c) and 2(a,b), for functions of the form:

$$u(Z,R,\tau) = a_1\ (\tau,R)\ \cos[Z-\tau + \Phi_1(\tau)] \tag{24a}$$

$$v(Z,R,\tau) = a_2\ (\tau,R)\ \sin[Z-\tau + \Phi_2(\tau)] \tag{24b}$$

with the radial variables R entering as a parmeter through the ponderomotive force (α^2).

These equations represent a strongly coupled system. In the absence of the last term on the r.h.s of equation (23) solutions can be obtained analytically in terms of Jacobi elliptic funtions which yield periodic behaviour. In the presence of this term the two components of the logitudinal wave exchange energy, it also introduces a nonlinear phase difference. Figure 1 represents the solution for perfect frequency matching with the detuning factor $\sigma(R) = 0$. All solutions are obtained with the same pump intensity $a_o^2 = 0.01$ but at different radial positions. The solid line represents the longitudinal electric field amplitude $a_1(\tau)$ while the dashed line represents the transverse electric field amplitude $a_2(\tau)$ for radial distances corresponding to $R = 0.1$ (1a), $R = 1.5$ (1b) and $R = 3$ (1c). Close to the axis ($R<1$) the longitudinal field is initially the dominant field, this field component grows and saturates well before the transverse component, however due to the coupling term in equation (23) the transverse component continues to grown and eventually reaches almost the same value. For larger radial values i.e. R = 1.5, and R = 3 both fields grow together reaching similar values in the same time with the transverse field component being sometimes greater than the longitudinal field component.

Fig 2 a-b represent solutions for the same pump intensity $a_o^2 = 0.01$ and beam width $k_p w_o = 3$ and with radius R = 1.5 and R = 3 respectively but with the detuning parameter $\sigma = 1$. Here we observe even more remarkable changes in the two electric field components, especially the double peaked behaviour of the radial component. Notice also that the amplitudes in the wings attain larger values than those close to the axis. A comparison of the maximum values of both field components at different radial positions show that the saturation is not strongly dependent on ϵ on time scales of order ϵ^{-1}.

The phases of the waves also show remarkable behaviour: very rapid phase changes take place for all solutions with the phases changing independently with no well defined phase difference, unlike the solutions of the two uncoupled equations [6].

Conclusions

We have extended the original calculation on the generation of longitudinal plasma waves by a Gaussian profile pump in a cylindrical coordinate system to include the nonlinear coupling between the longitudinal and transverse components. This nonlinear coupling is a consequence of the relativistic mass variation of the electron. In order to make the problem tractable we have neglected magnetic field generation i.e. we have assumed E to be irrotational, which was also assumed in the uncoupled system [6]. Unlike the uncoupled system where the solution for E was found to be irrotational the coupling makes the present solution non-irrotational so that magnetic field effects should be taken into account for consistency. However, using a multiple time scale analysis on the full problem it is found that the zero order term is still irrotational [8].

We also considered the weakly relativistic limit allowing us to expand the Lorentz factor in terms of the smallness parameter v^2/c^2 and neglected instabilities such as Raman cascade and modulational instabilities. The solutions of the two coupled equations describing the wave show radically different behaviour compared to the uncoupled case. There is now no preferred phase between the two components of the wave which will result in no well defined region for focusing or accelerating coherently. The amplitudes of the wave are also quite different with the radial field sometimes exceeding the longitudinal field. We have also noted a radial frequency dependence which allows the wings to grow to larger values than the centre which is in agreement with simulation results [7]. Using the simplified model described in the paper we have been able to investigate more details arising with the generation of beat waves, which have been observed in simulations. A more detailed study including the magnetic field generation and a discussion of instabilities arising from the nonlinear coupling between the two components is at present underway and will be reported in a much fuller version.

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Figure Captions

Fig. 1(a-c): The full line represents the amplitude a_1 and phase Φ_1 for the longitudinal electric field component, and the broken line represents the amplitude a_2 for the radial field component and phase difference $\Phi_1 - \Phi_2$. The laser intensities are assumed equal with $a_o^2 = 0.01$ and laser beam width $k_p w_o = 3$ with no detuning ($\sigma = 0$) but at different radial positions (a) R = 0.1, (b) R = 1.5, (c) R = 3. The time is in units of ω_{pe}^{-1}.

Fig. 2(a-b). Same as figures 1 (b-c) but including detuning $\sigma = 1$.

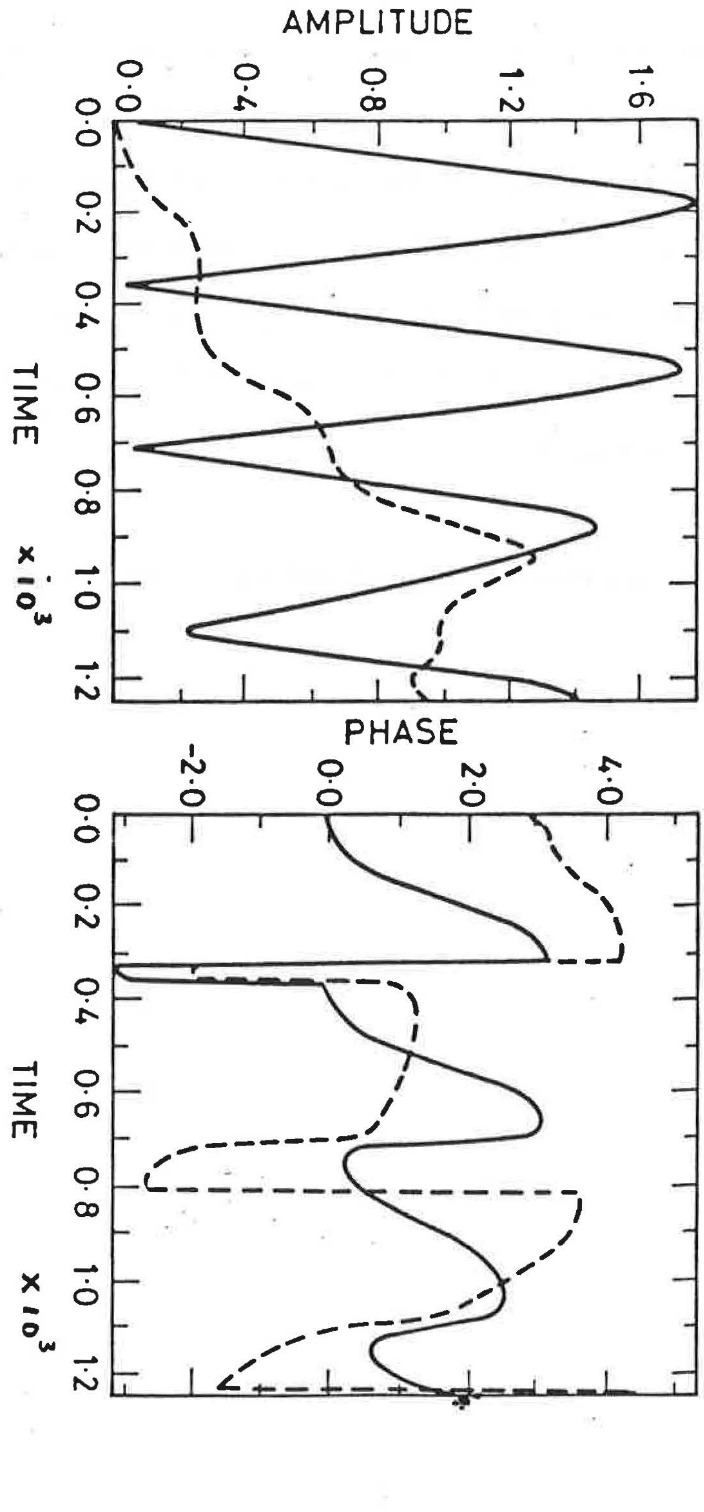
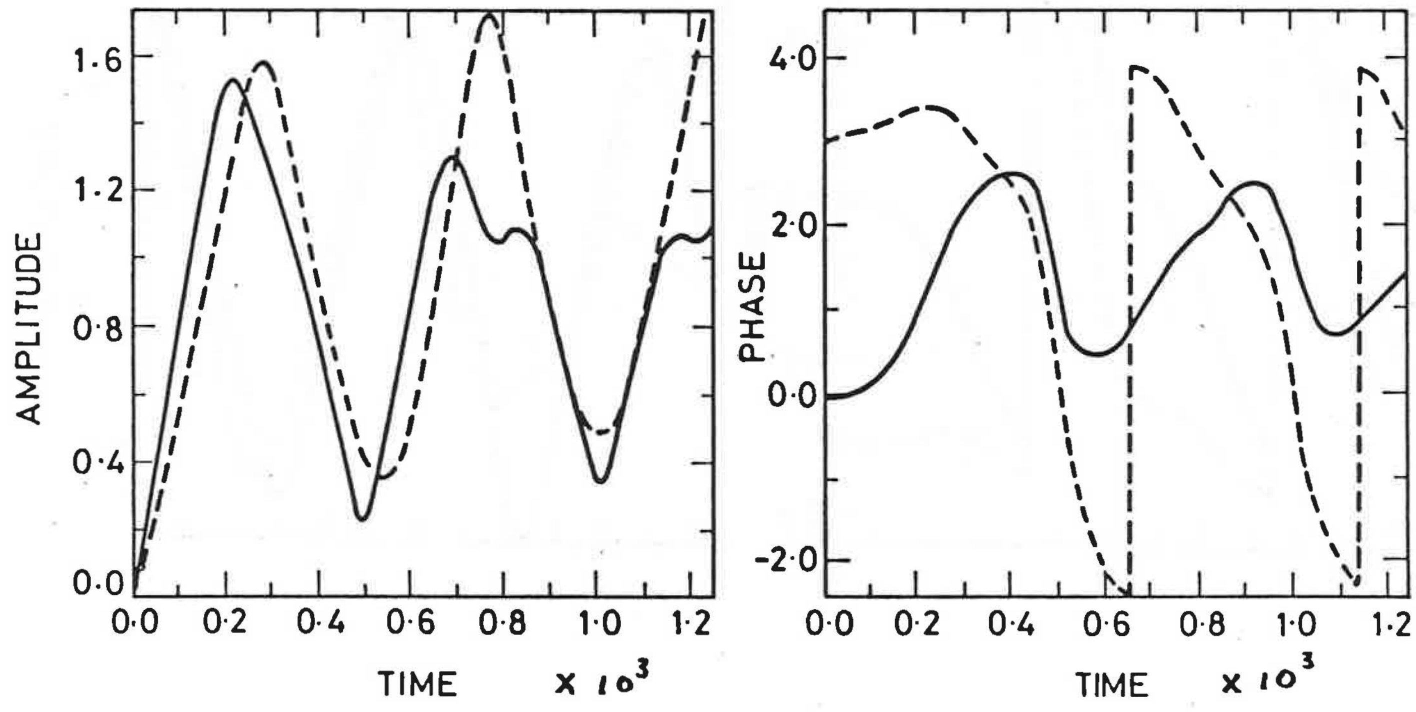


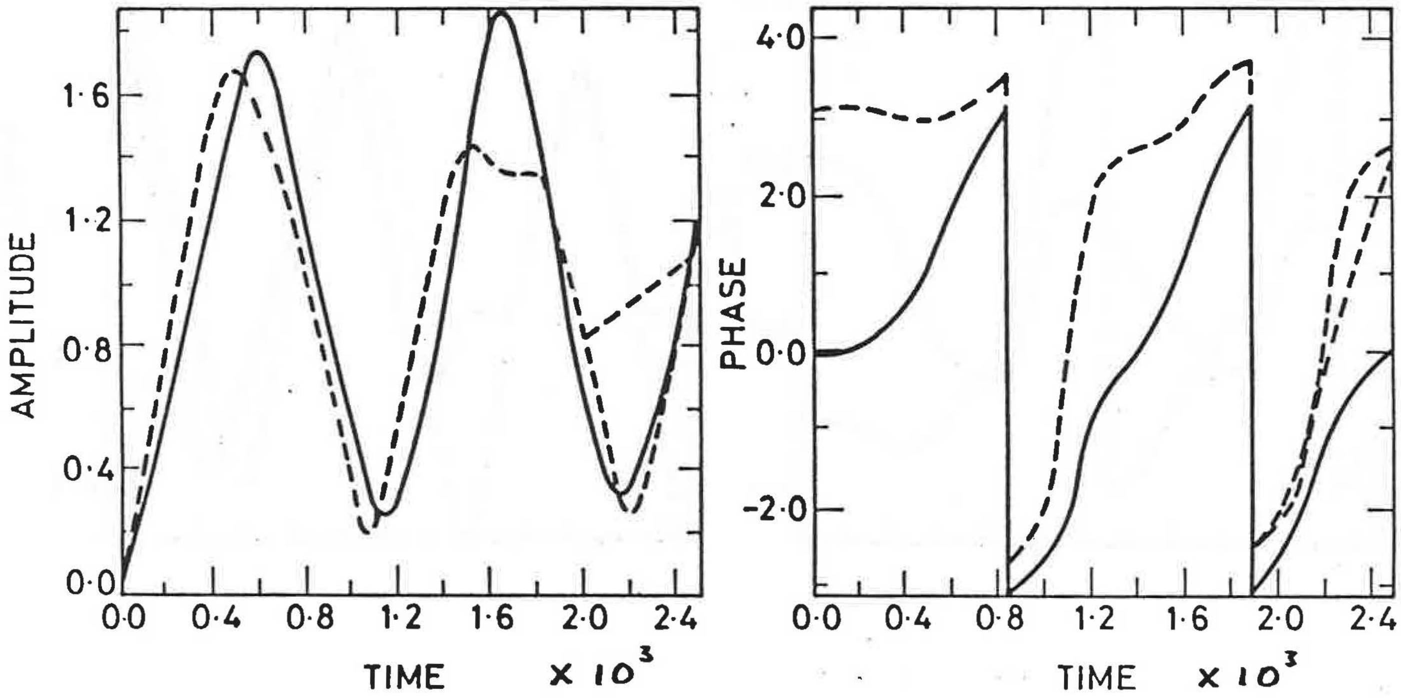
FIG. 1a

FIG. 1 b



AMPLITUDE
1·6
1·2
0·8
0·4
0·0
0·0 0·2 0·4 0·6 0·8 1·0 1·2
TIME x 10³
PHASE
4·0
2·0
0·0
-2·0
0·0 0·2 0·4 0·6 0·8 1·0 1·2
TIME x 10³

FiG. 1 c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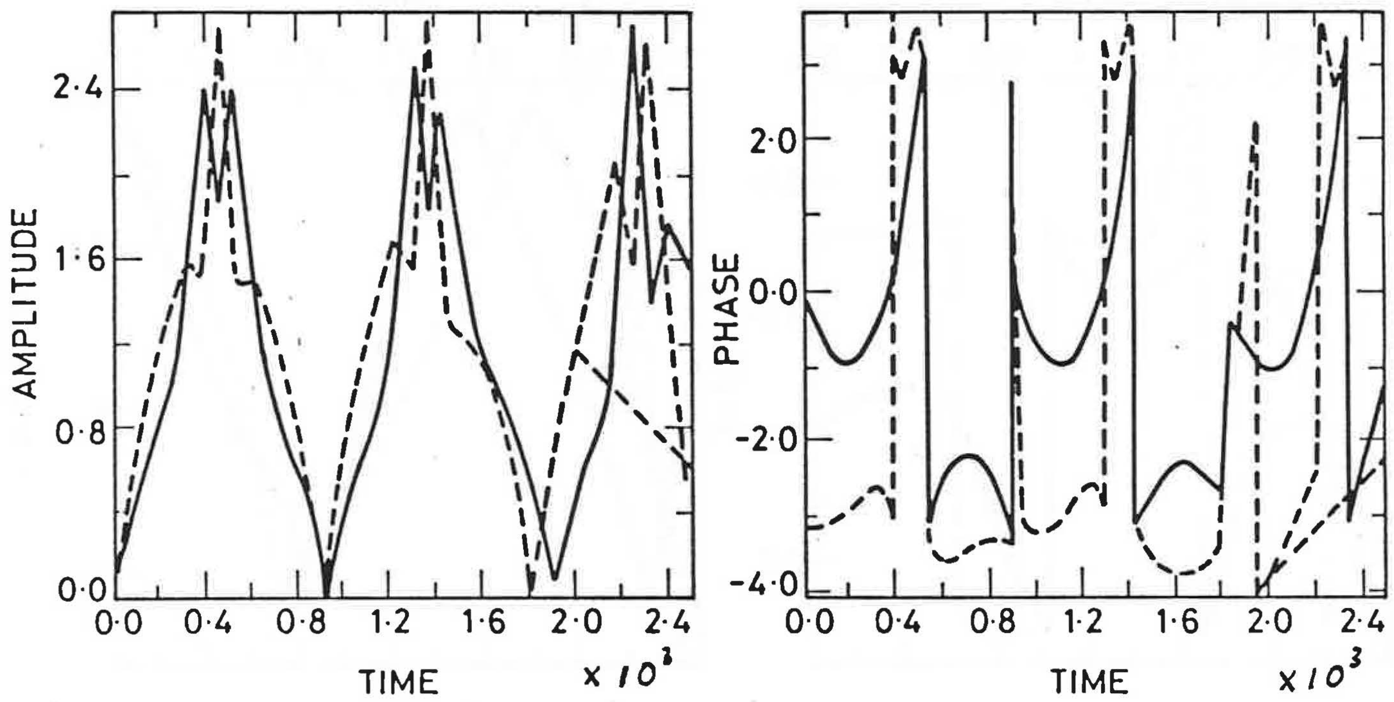
0·4
0·8
1·2
1·6
0·0
0·4
0·8
1·2
1·6
2·0
2·4
TIME
x 10³
PHASE
4·0
2·0
0·0
-2·0
0·0
0·4
0·8
1·2
1·6
2·0
2·4
TIME
x 10³

FIG. 2a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1·2
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2·0
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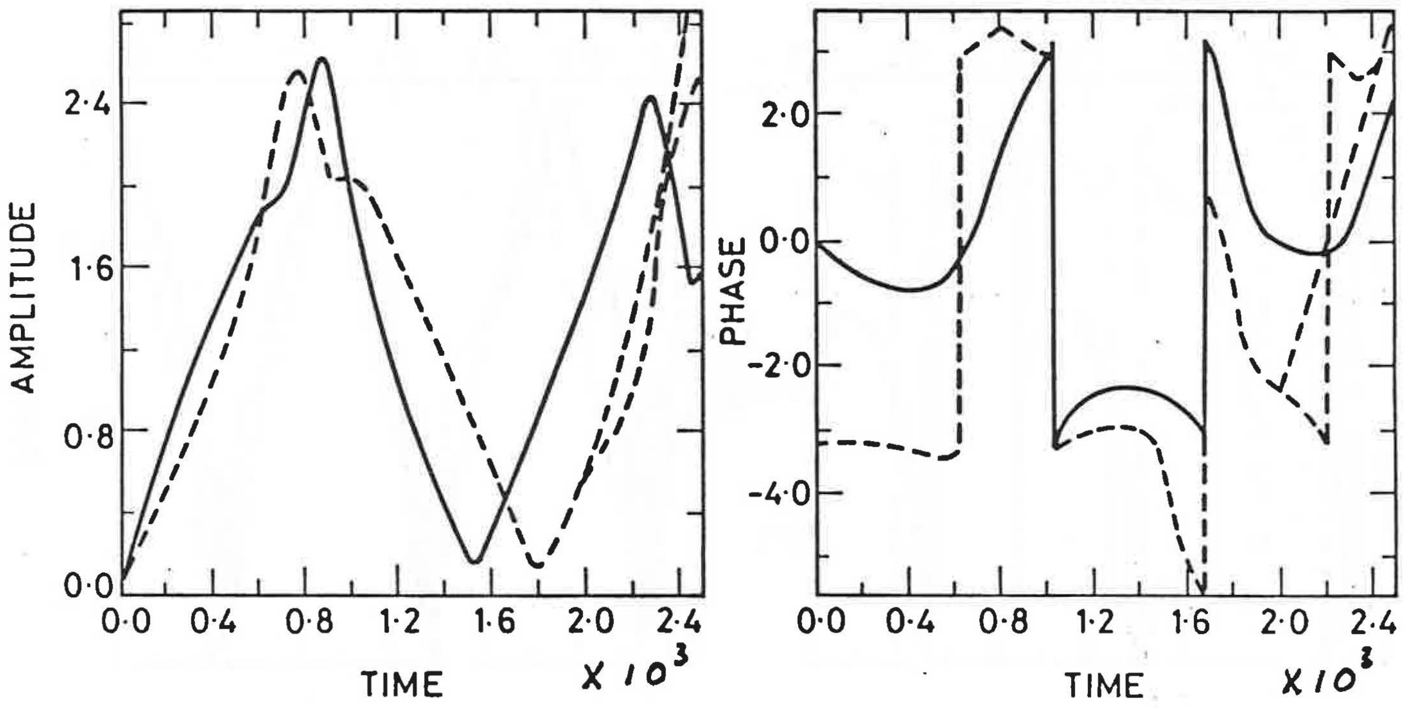
TIME
x 10^3
PHASE
2·0
0·0
-2·0
-4·0
0·0
0·4
0·8
1·2
1·6
2·0
2·4
TIME
x 10^3

FIG. 2b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TIME
X 10^3
PHASE
2·0
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-2·0
-4·0
0·0 0·4 0·8 1·2 1·6 2·0 2·4
TIME
X 10^3