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A FLAVOUR-SYMMETRIC PERSPECTIVE ON NEUTRINO MIXING [a]

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ABSTRACT

A review and consolidation of some of our more recent publications, many with our various collaborators. While we cannot resist mentioning Tribimaximal mixing, our main theme is Flavour Symmetry, in particular Flavour-Symmetric Observables, scalar (or pseudo-scalar) under $S3_l \times S3_\nu$. Our "best guess" for the smallest neutrino mixing angle remains: $\sin\theta_{13} = \sqrt{2\Delta m^2_{\rm sol}/(3\Delta m^2_{\rm atm})} \simeq 0.13$.

1. Introduction

Tri-bimaximal mixing was first put forward in 1999[1] (10 years ago!) as a viable alternative to the original "trimaximal" ansatz[2)3] on which we were then focussed[4)5)6)7]. Of course many authors before us had come close to proposing tribimaximal mixing, some very close [2)8)9)10)11], and we can say it is perhaps only that we realised the need for a distinct, empirically-based name for this specific mixing pattern (reflecting its "bimaximal"[12)13)14] and "trimaximal" character, hence "tri-bimaximal mixing"[15]):

$$P_{\rm TBM} := (|U_{l\nu}|^2)_{\rm TBM} \;\; \simeq \;\; \begin{matrix} \\ e \\ \mu \\ \tau \end{matrix} \begin{matrix} \nu_1 & \nu_2 & \nu_3 \\ \left(\begin{matrix} 2/3 \\ 1/6 \\ 1/6 \end{matrix}\right. & \begin{matrix} 1/3 \\ 1/3 \\ 1/3 \end{matrix} & \left.\begin{matrix} 0 \\ 1/2 \\ 1/2 \end{matrix}\right) \end{matrix}. \tag{1}$$

In his CERN lecture celebrating 50 years of parity violation, T. D. Lee credits us with "a tremendous achievement"[16] (fortunately, elsewhere in his talk, making it clear it is the *experiments* which are in fact tremendous). Actually, T. D. Lee's whole lecture[16] is an inspiration to anyone working on fermion mixing, experimenter or theorist alike, with the CKM and MNS matrices cast as "the cornerstones of particle physics"!

Our original 'derivation' of tri-bimaximal mixing[15] relied on a particular basis, the so-called 'circulant basis' (or 'cyclic basis') where the charged lepton mass matrix (hermitian square) took a cyclically symmetric ($C3$ invariant), 3×3 'circulant' form.

[a]Presented by W. G. Scott at "Neutrino Telescopes", Venice, Italy, 10-13 March 2009.

Assuming the neutrino mass matrix (hermitian square) then takes a $C2$-invariant 2×2 circulant form, the resulting mixing matrix (MNS matrix) $U = U_l^\dagger U_\nu$ is given by:

$$\begin{matrix} e \\ \mu \\ \tau \end{matrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{1}{3}} & \frac{1}{\sqrt{3}} \\ \frac{\omega}{\sqrt{3}} & \sqrt{\frac{1}{3}} & \frac{\bar{\omega}}{\sqrt{3}} \\ \frac{\bar{\omega}}{\sqrt{3}} & \sqrt{\frac{1}{3}} & \frac{\omega}{\sqrt{3}} \end{pmatrix} \overset{\begin{matrix}\nu_1 & \nu_2 & \nu_3\end{matrix}}{\begin{pmatrix} \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} \\ 0 & 1 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \end{pmatrix}} = \begin{matrix} e \\ \mu \\ \tau \end{matrix} \overset{\begin{matrix}\nu_1 & \nu_2 & \nu_3\end{matrix}}{\begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\frac{i}{\sqrt{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \frac{i}{\sqrt{2}} \end{pmatrix}} \qquad (2)$$

where the RHS is the tri-bimaximal form (Eq. 1) in a particular phase convention ($\omega = \exp(i2\pi/3)$ and $\bar{\omega} = \exp(-i2\pi/3)$ are the complex cube roots of unity).

The above factorisation of tribimaximal mixing, into trimaximal and 2×2 maximal contributions, has since been exploited by many authors [17) 18) 19)], although by no means all derivations invoking finite groups depend directly on it, see e.g. the $\Delta(27)$ and T' models[20)21)]. Note that the popular lepton mass matrix M_l of the form[18)]:

$$M_l = \frac{1}{\sqrt{3}} \begin{pmatrix} m_e & m_\mu & m_\tau \\ m_e & m_\mu\omega & m_\tau\bar{\omega} \\ m_e & m_\mu\bar{\omega} & m_\tau\omega \end{pmatrix} \Rightarrow M_l M_l^\dagger = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \quad \begin{matrix} a = \frac{m_e^2}{3} + \frac{m_\mu^2}{3} + \frac{m_\tau^2}{3} \\ b = \frac{m_e^2}{3} + \frac{m_\mu^2\omega}{3} + \frac{m_\tau^2\bar{\omega}}{3} \end{matrix} \qquad (3)$$

has a 'circulant' hermitian square $M_l M_l^\dagger$ (operating between left-handed fields) so that our original derivation included this case. The (charged-lepton) circulant basis would indeed seem to be a useful basis for appreciating possible underlying symmetries.

Of course the usual (charged-lepton) 'flavour basis' is still the most approriate basis for understanding neutrino oscillation phenomenology, matter effects etc. One may ask, then, how the symmetries are manifested in the charged-lepton flavour basis. Consider, in particular, our 2-parameter generalisation[24)25)] of tribimaximal mixing ("ν_2-trimaximal mixing") which implements only the constraint that the ν_2 be trimaximally mixed, interpolating between 'tri-chi-maximal' and 'tri-phi-maxamal' mixing[24)]:

$$U(\chi,\phi) = \begin{pmatrix} \sqrt{2/3}c_\chi c_\phi + i\sqrt{2/3}s_\chi s_\phi & 1/\sqrt{3} & -\sqrt{2/3}c_\chi s_\phi - i\sqrt{2/3}s_\chi c_\phi \\ -\frac{c_\chi c_\phi + i s_\chi s_\phi}{\sqrt{6}} - \frac{c_\chi s_\phi - i s_\chi c_\phi}{\sqrt{2}} & 1/\sqrt{3} & -\frac{c_\chi c_\phi - i s_\chi s_\phi}{\sqrt{2}} + \frac{c_\chi s_\phi + i s_\chi c_\phi}{\sqrt{6}} \\ -\frac{c_\chi c_\phi + i s_\chi s_\phi}{\sqrt{6}} + \frac{c_\chi s_\phi - i s_\chi c_\phi}{\sqrt{2}} & 1/\sqrt{3} & \frac{c_\chi c_\phi - i s_\chi s_\phi}{\sqrt{2}} + \frac{c_\chi s_\phi + i s_\chi c_\phi}{\sqrt{6}} \end{pmatrix}. \qquad (4)$$

Setting $\phi = 0$ gives triχ-maximal mixing and setting $\chi = 0$ gives tri-ϕmaximal mixing (tri-χmaximal respects μ-τ reflection symmetry[26)], while tri-ϕmaximal conserves CP). Clearly a tri-maximal eigenvector (mixing matrix column) can only result (with the phase choice, Eq. 4) if the row sums of the neutrino mass matrix (M_ν) are all equal, whereby its hermitian square ($M_\nu M_\nu^\dagger$) must have all row and column sums equal. [b]

[b]Matrices with all row and column sums equal are clearly expressible as linear combinations of permutation matrices, and as such play a key role in the representation theory of finite groups (within a given group and within a given representation such 'group matrices' [22)] form a representation of the group 'ring' [23)]). The neutrino mass matrix in the flavour basis (at least its hermitian square) is thus empirically an 'S3 group matrix' (in the natural representation of S3) whereby we initially dubbed our ansatz (Eq. 4) 'S3 group mixng' ("ν_2-trimaximal mixing" is simpler and as descriptive).

Thus the neutrino mass matrix in the flavour basis (in the above phase convention) is a simple form of 'magic-square' [27) 28) 29)], as was emphasised also by Lam [30)]. Our paper with Bjorken[31)] re-parametrises the mixing Eq. 4 as a function of the complex mixing elememt U_{e3} (it also introduced the notion of a matrix of unitarity-triangle angles [32)]).

Adding the requirement that the mass matrix be real (no CP violation) the mixing Eq. 4 reduces to tri-ϕ-maximal mixing[24)]. An intriguing construction of this case has been given by R. Friedberg and T. D. Lee [33)] based on a kind of 'translational invariance' in the space of the (neutrino) grassmann fields. However, their generalisation to the complex case seems to involve additional parameters [34)]. In the (charged-lepton) flavour basis, our most economical derivation of Eq. 4, requires only that the neutrino mass matrix (hermitian square) should commute with the 'democratic' (mass) matrix which is our version of 'democracy' symmetry[28)]:

$$[M_\nu M_\nu^\dagger, D] = 0, \qquad \text{where}: \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{5}$$

wherby the mixing Eq. 4 then follows, covering both the real and complex cases. [c]

If we impose instead μ-τ (reflection[26)]) symmetry (in the spirit of Lam [35)] - but please note that it <u>is</u> a μ-τ symmetry [26) 36)] and <u>not</u> a 2-3 symmetry [35)]!) the mixing Eq. 4 reduces to the 'tri-χmaximal' form. It should be emphasised that our 'μ-τ *reflection* symmetry' ('mutativity') [26)] includes a complex conjugation in its definition:

$$(E^T.M_\nu M_\nu^\dagger.E)^* = M_\nu M_\nu^\dagger, \qquad \text{where}: \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{6}$$

and thereby predicts a rather strong form of $\mu-\tau$ universality $|U_{\mu i}| = |U_{\tau i}| \;\; \forall \; i = 1-3$, allowing for non-zero U_{e3} and $U_{\mu 3} \neq 1/\sqrt{2}$. Thus our symmetry [26)] is substantively more general than the original (real) form proposed by Lam[35)] and others[36)].

2. Jarlskog Invariance and Flavour Symmetry

Of course all flavour-oscillation observables must be basis- and phase-convention independent. Jarlskog invariance[37)] (also known as weak-basis invariance[38)]) is readily appreciated working in a weak basis (where by definition the charged-current weak interaction is real, diagonal and universal). We may then as usual diagonalise the charged-lepton mass matrix such that the neutrino mass matrix violates charged-lepton flavour, or, we may (equivalently) choose to diagonlise the neutrino mass-matrix instead, such that the charged-lepton mass matrix violates neutrino "flavour"

[c]Within a given group and within a given representation, class matrices are obtained by summing, with equal weight, all the group elements within a given class.[22)] By definition, class matrices commute with all the group elements and hence with all group matrices.[22)] Thus in the case of the natural representaion of $S3$ (see footnote 1) any $S3$ group matrix commutes with the 'democratic' matrix D, which in this case is the only independent, non-trivial $S3$ class matrix.[25)27)]

(violation of neutrino "flavour" here means violation of neutrino *mass* as in neutrino decay $\nu_3 \longrightarrow \nu_2\gamma$ - we are <u>not</u> talking about old-fashioned "neutrino-flavour-eigenstate" ν_e, ν_μ, ν_τ neutrino flavour!). Jarlskog invariance is then understood as the freedom to 'rotate' continuously in the space spanned by these two extremes with any unitary transformation applied to both mass matrices (the 'circulant basis' above is an example of an 'intermediate' basis, where neither mass matrix is diagonal).

We are also interested in flavour-symmetry, and in the use of flavour-symmetric observables in particular. We will assume here for simplicity that the freedom to redefine the right-handed fields has been used to render the mass matrices themselves hermitian, and at the same time, we will introduce a useful, more-compact, notation:

$$L := M_l \qquad\qquad N := M_\nu \tag{7}$$

We may then readily construct the following flavour-symmetric *mass* observables:

$$L_1 := \mathrm{Tr}\, L = m_e + m_\mu + m_\tau \qquad N_1 := \mathrm{Tr}\, N = m_1 + m_2 + m_3 \tag{8}$$
$$L_2 := \mathrm{Tr}\, L^2 = m_e^2 + m_\mu^2 + m_\tau^2 \qquad N_2 := \mathrm{Tr}\, N^2 = m_1^2 + m_2^2 + m_3^2 \tag{9}$$
$$L_3 := \mathrm{Tr}\, L^3 = m_e^3 + m_\mu^3 + m_\tau^3 \qquad N_3 := \mathrm{Tr}\, N^3 = m_1^3 + m_2^3 + m_3^3 \tag{10}$$

These variables determine the masses through the characteristic equation(s):

$$\lambda_l^3 - (\mathrm{Tr}\, L)\lambda_l^2 + (\mathrm{Pr}\, L)\lambda_l - (\mathrm{Det}\, L) = 0 \tag{11}$$
$$\lambda_\nu^3 - (\mathrm{Tr}\, N)\lambda_\nu^2 + (\mathrm{Pr}\, N)\lambda_\nu - (\mathrm{Det}\, N) = 0 \tag{12}$$

The coefficients in these equations are themselves flavour-symmetric observables:

$$\mathrm{Tr}\, L = m_e + m_\mu + m_\tau = L_1 \qquad \mathrm{Tr}\, N = m_1 + m_2 + m_3 = N_1 \tag{13}$$
$$\begin{aligned}\mathrm{Pr}\, L &= m_e m_\mu + m_\mu m_\tau + m_\tau m_e & \mathrm{Pr}\, N &= m_1 m_2 + m_2 m_3 + m_3 m_2 \\ &= (L_1^2 - L_2)/2 & &= (N_1^2 - N_2)/2\end{aligned} \tag{14}$$
$$\begin{aligned}\mathrm{Det}\, L &= m_e m_\mu m_\tau & \mathrm{Det}\, N &= m_1 m_2 m_3 \\ &= (L_1^3 - 3L_1L_2 + 2L_3)/6 & &= (N_1^3 - 3N_1N_2 + 2N_3)/6\end{aligned} \tag{15}$$

where Pr stands for the sum of the Principal 2×2 minors, invariant under similarity transformations as are the trace and determinant (which are the sum of 1×1 and 3×3 principal minors respectively). Of particular importance are the mass discriminants:

$$\begin{aligned}L_\Delta &:= \sqrt{L_2^3/2 + 3L_1^4L_2/2 + 6L_1L_2L_3 - 7L_1^2L_2^2/2 - 3L_3^2 - 4L_1^3L_3/3 - L_1^6/6} \\ &= (m_e - m_\mu)(m_\mu - m_\tau)(m_\tau - m_e).\end{aligned} \tag{16}$$
$$\begin{aligned}N_\Delta &:= \sqrt{N_2^3/2 + 3N_1^4N_2/2 + 6N_1N_2N_3 - 7N_1^2N_2^2/2 - 3N_3^2 - 4N_1^3N_3/3 - N_1^6/6} \\ &= (m_1 - m_2)(m_2 - m_3)(m_3 - m_1).\end{aligned} \tag{17}$$

which change sign under odd permutations of flavour labels (charged-leptons and neutrinos separately). More precisely, the discriminants have $\bar{1} \times \bar{1}$ symmetry under $S3_l \times S3_\nu$, while all the other flavour-symmetric mass observabes above are 1×1.

The Jarlskog invariant $\mathcal{J}$ is the archetypal flavour-symmetric *mixing* observable

$$\mathcal{J} = \mathrm{Im}\,\Pi_{l\nu} = -i\frac{\mathrm{Det}\,[L,N]}{2L_\Delta N_\Delta} \tag{18}$$

measuring the violation of CP symmetry, with no reference to particular flavour labels (Eq. 42 defines $\Pi_{l\nu}$). That $\mathcal{J}$ has $\bar{1}\times\bar{1}$ symmetry under $S3_l \times S3_\nu$ follows by virtue of the product of mass discriminants $L_\Delta N_\Delta$ appearing in the denominator of Eq. 18.

3. Six New Flavour-Symmetric Mixing Observables

In terms of the matrix T (see Eq. 37) of traces of anti-commutators (c.f Eq. 18) we define:

$$\mathcal{F} := \mathrm{Det}\,P = \frac{\mathrm{Det}\,T}{L_\Delta N_\Delta} = \frac{\mathrm{Det}\,(\mathrm{Tr}\,\{L^m, N^n\})}{2L_\Delta N_\Delta} \tag{19}$$

Somewhat as $\mathcal{J}=0$ protects a neutrino source against matter vs. anti-matter analysis, so $\mathcal{F}=0$ protects against flavour analysis (since the transition-probability matrix $\mathcal{P}$ cannot be inverted if $\mathrm{Det}\,P=0$). $\mathcal{F}$ (like $\mathcal{J}$) is "odd-odd" ($\bar{1}\times\bar{1}$) under $S3_l \times S3_\nu$.

We may define all our new mixing observables[39] as homogeneous polynomials in w, x, y, z, parameterising the P-matrix in terms of deviations from trimaximal mixing:

$$P \;:=\; (|U|^2) =: \begin{pmatrix} 1/3-w-x & 1/3+w & 1/3+x \\ 1/3-y-z & 1/3+y & 1/3+z \\ 1/3+w+x+y+z & 1/3-w-y & 1/3-x-z \end{pmatrix}. \tag{20}$$

The polynomials are then chosen so as to have definite symmetry under $S3_l \times S3_\nu$:

$$\mathcal{G} := (1\times 1)^{(2)} \;=\; 2(w^2+x^2+y^2+z^2+wx+wy+xz+yz)+(wz+xy) \tag{21}$$

$$\mathcal{F} := (\bar{1}\times\bar{1})^{(2)} \;=\; 3(wz-xy) \tag{22}$$

$$\begin{aligned}\mathcal{C} := (1\times 1)^{(3)} \;=\; & 9(xyz+wyz+wxz+wxy) \\ & +9/2(xy(x+y)+wz(w+z))\end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{A} := (\bar{1}\times\bar{1})^{(3)} \;=\; & 2(w^3-x^3-y^3+z^3) \\ & +3(wx(w-x)+wy(w-y)+yz(z-y)+xz(z-x)) \\ & +3(xy(w+z)-wz(x+y)) \\ & +3/2(wz(w+z)-xy(x+y))\end{aligned} \tag{24}$$

$$\begin{aligned}\mathcal{B} := (\bar{1}\times 1)^{(3)} \;=\; & 3\sqrt{3}[(w^2y+wy^2-x^2z-z^2x+wxy+wyz-xyz-wxz) \\ & +1/2(w^2z-wz^2+xy^2-x^2y)]\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{D} := (1\times\bar{1})^{(3)} \;=\; & 3\sqrt{3}[(w^2x+wx^2-y^2z-z^2y+wxy+wxz-xyz-wyz) \\ & +1/2(w^2z-wz^2+yx^2-y^2x)]\end{aligned} \tag{26}$$

Our polynomials are all $C3_l \times C3_\nu$ invariant and form a natural basis of the $C3\times C3$ invariant polynomial ring [40]. They are "plaquette invariant" in that the definitions above are independent of which "plaquette" is chosen to define w, x, y, z in Eq. 20.

With only four parameters needed to fix the mixing magnitudes, not all of our six flavour-symmetric mixing observables (FSMOs) can be independent. We have:

$$\begin{aligned} \mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2+\mathcal{D}^2 &= \mathcal{G}(\mathcal{F}^2+3\mathcal{G}^2)/2 && (27)\\ 2(\mathcal{AC}+\mathcal{BD}) &= \mathcal{F}(\mathcal{G}^2+3\mathcal{F}^2)/2 && (28) \end{aligned}$$

constituting the two constraints needed to reduce six to four. Of course neither can our variables be entirely independent of Jarlskog $\mathcal{J}$ and we have also:

$$\mathcal{J}^2 = 1/108 - \mathcal{G}/18 + 2\mathcal{C}/27 - \mathcal{F}^2/36 \qquad (29)$$

3.1. Expression in terms Mass Matrices

All our variables can of course be readily written in terms of traces:

$$\begin{aligned} 2\mathcal{G}-1 &:= \mathrm{Tr}\,P^T.P = \mathrm{Tr}\,T^T.L_G.T.N_G && (30)\\ 6\mathcal{F} &:= \mathrm{Tr}\,P^T.\epsilon.P.\epsilon^T = \mathrm{Tr}\,T^T.L_F.T.N_F^T && (31)\\ 2\mathcal{C}/3-\mathcal{G}/2-1/6 &:= \mathrm{Tr}\,P^T.K = \mathrm{Tr}\,T^T.L_C.Q.N_C^T && (32)\\ 2\mathcal{A}-2\mathcal{F} &:= \mathrm{Tr}\,P^T.\epsilon.K.\epsilon^T = \mathrm{Tr}\,T^T.L_A.Q.N_A^T && (33)\\ 2\mathcal{B}/\sqrt{3} &:= \mathrm{Tr}\,P^T.\epsilon.K = \mathrm{Tr}\,T^T.L_A.Q.N_C^T && (34)\\ 2\mathcal{D}/\sqrt{3} &:= \mathrm{Tr}\,P^T.K.\epsilon^T = \mathrm{Tr}\,T^T.L_C.Q.N_A^T && (35) \end{aligned}$$

where the K-matrix[41] is the real part of the mixing-matrix "plaquette products" (Eq. 42) with ϵ the totally anti-symmetric 3×3 "epsilon" matrix [27,28] (Eq. 64).

In terms of anti-commutators (A_{mn}) and commutators (C_{mn}) of mass matrices:

$$A_{mn} = \{L^m, N^n\} \qquad C_{mn} = -i[L^m, N^n] \qquad (36)$$

we define the T-matrix[41] and Q-matrix[41] as traces of anti-commutators (i.e. of products) and quadratic (products of) commutators respectively:

$$T = \frac{1}{2}\begin{pmatrix} \mathrm{Tr}\,A_{00} & \mathrm{Tr}\,A_{01} & \mathrm{Tr}\,A_{02}\\ \mathrm{Tr}\,A_{10} & \mathrm{Tr}\,A_{11} & \mathrm{Tr}\,A_{12}\\ \mathrm{Tr}\,A_{20} & \mathrm{Tr}\,A_{2,1} & \mathrm{Tr}\,A_{22} \end{pmatrix}; \quad Q = \frac{1}{2}\begin{pmatrix} \mathrm{Tr}\,C_{11}^2 & \mathrm{Tr}\,C_{11}C_{12} & \mathrm{Tr}\,C_{12}^2\\ \mathrm{Tr}\,C_{11}C_{21} & \mathrm{Tr}\,C_{11}C_{22} & \mathrm{Tr}\,C_{12}C_{22}\\ \mathrm{Tr}\,C_{21}^2 & \mathrm{Tr}\,C_{21}C_{22} & \mathrm{Tr}\,C_{22}^2 \end{pmatrix} \qquad (37)$$

related to the P and K matrices (respectively) by simple mass-moment transforms [41].

The matrices L_G, L_F, L_C, L_A and N_G, N_F, N_C, N_A (Eqs. 30-33) are just functions of our flavour-symmetric mass observables (Eqs. 8-10) for example:

$$L_G = \begin{pmatrix} L_0 & L_1 & L_2\\ L_1 & L_2 & L_3\\ L_2 & L_3 & L_4 \end{pmatrix}^{-1} \qquad L_F = \frac{1}{L_\Delta}\begin{pmatrix} 0 & L_2 & -L_1\\ -L_2 & 0 & L_0\\ L_2 & -L_0 & 0 \end{pmatrix} \qquad (38)$$

$$L_0 = 3; \quad L_4 = L_1^4/6 + L_1L_3/3 + L_2^2/2 - L_1^2L_2; \quad L_\Delta = \mathrm{Det}\,L_G^{-1}.$$

The explicit form of the matrix L_C is somewhat less succinct:

$$L_C = \frac{1}{L_\Delta^2}\begin{pmatrix} L_C(1,1) & (L_1^2 - L_2)/2 & (L_1^2 - L_2)/2 \\ -(L_1^3 - L_3) & 3L_1^2 - L_2 & -2L1 \\ (3L_1^2 - L_2)/2 & -4L_1 & L_0 \end{pmatrix} \tag{39}$$
$$L_0 = 3; \quad L_C(1,1) = L_3 L_1/3 + L_2^2/4 - L_2 L_1^2 + 5L_1^4/12$$

while the L_A matrix is just a little too lengthy to usefully display here. Obviously, identical expressions (with $L \to N$) obtain for the corresponding neutrino matrices.

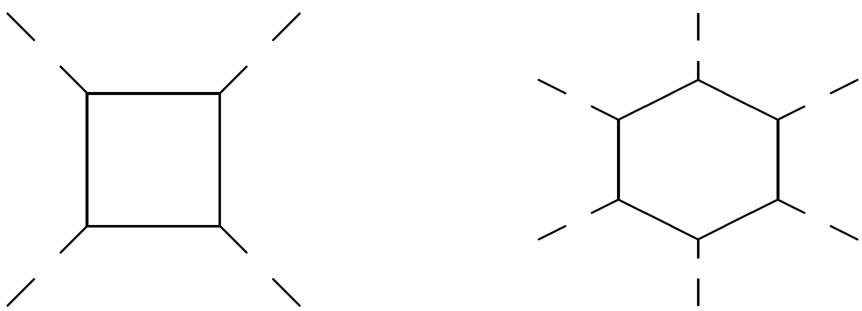
Remarkably our $(\bar{1} \times \bar{1})^{(2)}$ mixing observable $\mathcal{F}$ is not only expressible in terms of the T-matrix but also in terms of the matrix of cubic (products of) commutators[42]:

$$\mathcal{F} = \frac{\mathrm{Det}\, T}{L_\Delta N_\Delta} = \frac{\mathrm{Det}\, C^{(3)}}{L_\Delta N_\Delta \mathrm{Det}^3\, C} \qquad C^{(3)} := \begin{pmatrix} \mathrm{Tr}\, C_{11}^3 & \mathrm{Tr}\, C_{11}^2 C_{12} & \mathrm{Tr}\, C_{11} C_{12}^2 \\ \mathrm{Tr}\, C_{11}^2 C_{21} & \mathrm{Tr}\, C_{11}^2 C_{22} & \mathrm{Tr}\, C_{21} C_{12}^2 \\ \mathrm{Tr}\, C_{11} C_{21}^2 & \mathrm{Tr}\, C_{12} C_{21}^2 & \mathrm{Tr}\, C_{11} C_{22}^2 \end{pmatrix} \tag{40}$$

where $C := C_{11} := -i[L, N]$ is the usual Jarlskog commutator ($\mathrm{Det}\, C = \mathrm{Tr}\, C^3/3$).

3.2. Interpretation and Flavour-Symmetric Mixing Constraints

Our $(1 \times 1)^{(2)}$ variable $\mathcal{G}$ can obviously be related to the flavour-averaged asymptotic [41] survival probability (via $\mathrm{Tr}\, P^T.P$). Interestingly (and perhaps less obviously) both $\mathcal{G}$ and $\mathcal{C}$ can be related to certain flavour-summed loop amplitudes as follows:



$$\sum_{l,\nu} \Pi_{l\nu} = (\mathcal{G} - 1)/2 + 9i\mathcal{J} \qquad\qquad \sum_{l\pm\nu} \Omega_{l\pm\nu} = 2/9\mathcal{C} - 1/3\mathcal{G} + 1/9 \tag{41}$$

(we focus purely on the flavour structure here, in the limit that all masses are zero). The individual plaquettes (4-plaquette) and hexaplaquettes (6-plaquette) contribute:

$$\Pi_{l\nu} \;:=\; U_{l-1,\,\nu-1}U^*_{l-1,\,\nu+1}U_{l+1,\,\nu+1}U^*_{l+1,\,\nu-1}; \qquad K_{l\nu} := \mathrm{Re}\,\Pi_{l\nu} \quad \mathrm{mod}\, 3 \tag{42}$$
$$\Omega_{l\mp\nu} \;:=\; U_{l-1,\,\pm\nu}U^*_{l-1,\,\pm\nu+1}U_{l,\,\pm\nu+1}U^*_{l,\,\pm\nu-1}U_{l+1,\,\pm\nu-1}U^*_{l+1,\,\pm\nu} \quad \mathrm{mod}\, 3 \tag{43}$$

where indices are to be interpreted mod 3, i.e. $e + 1 = \mu$, $\mu + 1 = \tau$, $\tau + 1 = e$ etc.

Most importantly, flavour-symmetric mixing variables can be used to specify (up to permutations) various suggestive and often phenomenologically relevant mixing ansatze with their constraints/symmetries, for example:

$$\text{“Democracy symmetry''} \quad \Leftrightarrow \quad \mathcal{F} = 0 \qquad \mathcal{C} = 0 \tag{44}$$

$$\text{“}\mu - \tau \text{ refl. symmetry''} \quad \Leftrightarrow \quad \mathcal{F} = 0 \qquad \mathcal{A} = 0 \tag{45}$$

$$\text{“Tribimaximal Mixing''} \quad \Leftrightarrow \quad \mathcal{F} = \mathcal{C} = \mathcal{A} = \mathcal{J} = 0 \tag{46}$$

Comparing Eq. 44 and Eq. 45 one notices a kind of “duality” between the democracy and $\mu - \tau$-reflection symmetries, with $\mathcal{C}$ and $\mathcal{A}$ interchanging roles between the two cases. The condition for tribimaximal mixing, Eq. 46, is given in terms of the Jarlskog variable [37] equivalently setting $\mathcal{J} = 0$ for simplicity, in place of $\mathcal{G} = 1/6$.

Some further examples of ansatze and flavour-symmetric constraints are given in Table 1. Our variables are normalised such that their maximum values are unity, corresponding to the case of “no mixing” as in the first line of Table 1.

Mixing ansatz	$\mathcal{F}$	$\mathcal{G}$	$\mathcal{C}$	$\mathcal{A}$	Symmetries	$18\mathcal{J}^2$	$\mathcal{B}$	$\mathcal{D}$
No Mixing	1	1	1	1	CP	0	0	0
Tribimaximal Mix.*	0	$\frac{1}{6}$	0	0	Dem., μ-τ, CP	0	0	$\frac{1}{12\sqrt{3}}$
Trimaximal Mix.	0	0	0	0	Dem., μ-τ	$\frac{1}{6}$	0	0
ν_2-Trimaximal*	0	-	0	-	Democracy	-	0	-
Two-equal P-rows*	0	-	-	0	e.g. μ-τ	-	0	-
Two-equal P-columns	0	-	-	0	e.g. 1-2	-	-	0
Altaerlli-Feruglio*	0	-	$\frac{6G-1}{8}$	0	e.g. μ-τ, CP	0	0	-
Triχmaximal Mix.*	0	-	0	0	Dem., μ-τ	-	0	-
Triϕmaximal Mix.*	0	$\frac{1}{6}$	0	-	Dem., CP	0	0	-
Bimaximal Mix.	0	$\frac{1}{8}$	$-\frac{1}{32}$	0	Dem., CP, 1-2	0	0	0

Table 1: Possible and proposed mixing schemes/constraints expressed in terms of our Flavour-Symmetric Mixing Observacles (FSMOs). Those marked with an asterisk are still phenomenologically viable. Although our four $L \leftrightarrow N$ symmetric variables $\mathcal{F}$, $\mathcal{G}$, $\mathcal{C}$ and $\mathcal{A}$ are sufficient to completely fix the mixing (up to permutations) we include $\mathcal{B}$ and $\mathcal{D}$ (and $\mathcal{J}^2$) for completeness.

Some rather less restrictive flavour-symmetric constraints may also be written:

$$8\mathcal{C}^3 - 27\mathcal{F}^2(\mathcal{C}\mathcal{G} - \mathcal{A}\mathcal{F}) = 0 \quad \Leftrightarrow \quad |U_{\alpha i}|^2 = 1/3 \qquad (\text{any}\,\alpha, \text{any}\, i) \tag{47}$$

$$8\mathcal{B}^3 - 27\mathcal{F}^2(\mathcal{B}\mathcal{G} - \mathcal{D}\mathcal{F}) = 0 \quad \Leftrightarrow \quad |U_{\alpha i}|^2 = |U_{\beta i}|^2 \quad (\alpha \neq \beta, \text{any}\, i) \tag{48}$$

$$8\mathcal{D}^3 - 27\mathcal{F}^2(\mathcal{D}\mathcal{G} - \mathcal{B}\mathcal{F}) = 0 \quad \Leftrightarrow \quad |U_{\alpha i}|^2 = |U_{\alpha j}|^2 \quad (\text{any}\,\alpha, i \neq j) \tag{49}$$

corresponding respectively to one mixing-element with modulus-squared 1/3 (Eq. 47), two elements with equal modulus in the same column (Eq. 48) or, indeed, with equal modulus in the same row (Eq. 49). A single mixing element zero corresponds to:

$$\mathcal{K} = 0 \text{ and } \mathcal{J} = 0 \quad \Leftrightarrow \quad |U_{\alpha u}| = 0 \qquad (\text{any}\,\alpha, \text{any}\, i) \tag{50}$$

where:

$$\mathcal{K} := \mathrm{Det}\, K = \mathcal{A}/27 + \mathcal{F}^3/54 - \mathcal{F}\mathcal{C}/27 - \mathcal{F}/54. \tag{51}$$

An approach to Eq. 50 such that $\mathcal{J} \to 0$ with $\mathcal{K} \equiv 0$ offers an explanation[32)39)43)44)] for one near-right unitarity-triangle angle, e.g. $\alpha \simeq 90^o$ in the quark sector[45)].

One extraordinarily simple/powerful condition (now excluded even for the quarks):

$$\mathcal{B} = \mathcal{D} \ \Leftrightarrow \ (|U|) = (|U^T|) \tag{52}$$

would have guaranteed a completely symmetric mixing (moduli) matrix!

While we have not succeeded to solve Eqs. 21-24 for w, x, y, z in terms of $\mathcal{G}, \mathcal{F}, \mathcal{C}, \mathcal{A}$, we may understand rather simply how each of our observables affects the symmetry of the P-matrix by setting $\mathcal{F} = f$, $\mathcal{C} = c$ and $\mathcal{A} = a$ (where $f, c, a \ll \mathcal{G}, \quad 0 < \mathcal{G} < 1/6$) and then solving to leading order in each of f, c and a:

$$P \sim \begin{pmatrix} \frac{1}{3} + 2\sqrt{\frac{\mathcal{G}}{6}} + \frac{2c}{9\mathcal{G}} & \frac{1}{3} - \frac{4c}{9\mathcal{G}} & \frac{1}{3} - 2\sqrt{\frac{\mathcal{G}}{6}} + \frac{2c}{9\mathcal{G}} \\ \frac{1}{3} - \sqrt{\frac{\mathcal{G}}{6}} - \frac{f}{2\sqrt{6\mathcal{G}}} - \frac{c}{9\mathcal{G}} + \frac{a}{3\mathcal{G}} & \frac{1}{3} + \frac{f}{\sqrt{6\mathcal{G}}} + \frac{2c}{9\mathcal{G}} & \frac{1}{3} + \sqrt{\frac{\mathcal{G}}{6}} - \frac{f}{2\sqrt{6\mathcal{G}}} - \frac{c}{9\mathcal{G}} - \frac{a}{3\mathcal{G}} \\ \frac{1}{3} - \sqrt{\frac{\mathcal{G}}{6}} + \frac{f}{2\sqrt{6\mathcal{G}}} - \frac{c}{9\mathcal{G}} - \frac{a}{3\mathcal{G}} & \frac{1}{3} - \frac{f}{\sqrt{6\mathcal{G}}} + \frac{2c}{9\mathcal{G}} & \frac{1}{3} + \sqrt{\frac{\mathcal{G}}{6}} + \frac{f}{2\sqrt{6\mathcal{G}}} - \frac{c}{9\mathcal{G}} + \frac{a}{3\mathcal{G}} \end{pmatrix} \tag{53}$$

An expansion about tribimaximal mixing is now readily achieved setting $\mathcal{G} = 1/6 - g$ and expanding to first order in g also, with $g \ll 1/6$.

$$P(g, f, c, a) \sim \begin{pmatrix} \frac{2}{3} - g + \frac{4c}{3} & \frac{1}{3} - \frac{8c}{3} & g + \frac{4c}{3} \\ \frac{1}{6} + \frac{g}{2} - \frac{f}{2} - \frac{2c}{3} + 2a & \frac{1}{3} + f + \frac{4c}{3} & \frac{1}{2} - \frac{g}{2} - \frac{f}{2} - \frac{2c}{3} - 2a \\ \frac{1}{6} + \frac{g}{2} + \frac{f}{2} - \frac{2c}{3} - 2a & \frac{1}{3} - f + \frac{4c}{3} & \frac{1}{2} - \frac{g}{2} + \frac{f}{2} - \frac{2c}{3} + 2a \end{pmatrix}, \tag{54}$$

thus forming the basis of an interesting and potentially useful parameterisation of the mixing matrix in terms of small deviations[46)47)48)] from tribimaximal mixing.

4. Extremisation of Flavour-Symmeric "Actions/Potentials"

To the extent that the neutrino mixing is in any sense "maximal" one might say that experiment *suggests* that some kind of "extremisation" may be at work here. Given Eq. 46 above, it is rather obvious that (up to permutations) tribimaximal mixing can be guaranteed by requiring that the flavour-symmetric function:

$$V(\mathcal{G}, \mathcal{F}, \mathcal{C}, \mathcal{A}) = \mathcal{F}^2 + \mathcal{C}^2 + \mathcal{A}^2 + \mathcal{J}^2 \tag{55}$$

be extremal with respect to $\mathcal{F}, \mathcal{C}, \mathcal{A}, \mathcal{J}$, since $(\partial V/\partial \mathcal{F})_{CAJ} = 2\mathcal{F} = 0 \Rightarrow \mathcal{F} = 0$ etc. The choice of extremisation variables is not crucial here, since such functions would generally yield TBM for arbitrary (e.g. PDG) choice of variables (clearly the function Eq. 55 is far from unique, e.g. arbitrary coefficients could obviously be included).

In our 1994 paper[4)], we tentatively floated the idea that extremisation might also account for the observed spectrum of masses. The simplest example we had in mind was very trivial. Take the action/potential to be just the determinant of the mass matrix Det M (we are considering a generic fermion type here, and we are still thinking of $M = L, N, U \dots$ most naturally as a 'hermitised' mass matrix as in Eq. 7 above):

$$V(M) = \text{Det}\, M = xyz(\sqrt{2}\Phi)^3 \tag{56}$$

where x, y, z are now (c.f. Section 3) the usual (diagonalised) Yukawa couplings. Taking the SM higgs field Φ to be essentially fixed by more significant terms elsewhere in the Lagrangian, we extremise with respect to the Yukawa couplings x, y, z themselves:

$$\left.\begin{array}{l} (\partial V/\partial x)_{yz} = yz = 0 \\ (\partial V/\partial y)_{zx} = zx = 0 \\ (\partial V/\partial z)_{xy} = xy = 0 \end{array}\right\} \quad \Rightarrow \text{ e.g.} \left\{\begin{array}{c} x \neq 0 \\ y = 0 \\ z = 0 \end{array}\right. \text{ or } \left\{\begin{array}{c} x = 0 \\ y \neq 0 \\ z = 0 \end{array}\right. \text{ or } \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z \neq 0 \end{array}\right. \tag{57}$$

The solutions require two light (zero mass) fermions and one (potentially) heavy, i.e. non-zero mass fermion, surely a good starting point as regards the observed quark and lepton mass spectra. Notice that the symmetry of our "action" forbids to tell which fermion will turn out to be heavy (indeed there is no distinction between x, y, z a priori). One might say that the choice, e.g. $z \neq 0$, is made spontaneously.

Our 2005 paper[29)] made a somewhat more serious attack on these kinds of issues, especially as regards the mixing, adopting a strictly (Jarlskog) covariant approach. We made particular use of a theorem in matrix calculus[49)]:

$$\partial_X \text{ Tr } XA = A^T \tag{58}$$

which enabled us to extremise directly with respect to the mass matrices (i.e. with respect to the Yukawa couplings, within the SM). As a warm-up exercise, we began by extremising the Jarlskog determinant $\text{Det}\, C = \text{Tr}C^3/3$ (here $C := C_{11} := -i[L, N]$):

$$\begin{array}{l} (\partial_L \text{ Tr } C_{11}^3/3)^T = +i[N, C^2] = 0 \\ (\partial_N \text{ Tr } C_{11}^3/3)^T = -i[L, C^2] = 0 \end{array} \quad \Rightarrow \quad P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \tag{59}$$

which, as expected, yields trimaximal mixing (it should be said that 2×2 maximal mixing in any sector also provides a solution). When extremising for the mixing at fixed masses, the zeros on the RHS of Eqs. 59 get replaced by polynomials in the mass matrices determining Lagrange multipliers (having almost no effect in practice).

Similarly, extremising the sum of the Principal minors ($\text{Pr}\, C = C^2/2$):

$$\begin{array}{l} (-\partial_L \text{Tr}\, C^2/2)^T = +i[N, C] = 0 \\ (-\partial_N \text{Tr}\, C^2/2)^T = -i[L, C] = 0 \end{array} \quad \Rightarrow \text{ e.g. } P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \tag{60}$$

yields, on the same basis, uniquely 2×2 maximal mixing, as one might expect (we are considering solutions which apply independently of the values of the masses -

it turns out that there is also a unique mass-dependent solution[29]), which although not entirely without interest, certainly does not agree with experiment). We remark that we see our extremisation equations, Eqs. 60, as analogous to the Yang-Mills [50]) equations, cf. $[\nabla_\mu, F^{\mu\nu}] = 0$, derivable from the quadratic Lagrangian $\mathcal{L} = -\mathrm{Tr}\ F^2/2$. Yukawa couplings are thus seen as dynamical variables, analogous to gauge fields.

4.1. Extremisation of an Arbitrary Function of the Jarlskog Commutator

We can consider extremising some function of both Tr C^2 and Tr C^3 *together* to obtain potentially more realistic predictions (note that $\mathrm{Tr}\,C = 0$ identically for a commutator). If the function to be extremised is denoted A then, at the extremum:

$$r\ \partial_L \mathrm{Tr} C^2 + \partial_L \mathrm{Tr} C^3 = 0 \tag{61}$$

$$r\ \partial_N \mathrm{Tr} C^2 + \partial_N \mathrm{Tr} C^3 = 0 \tag{62}$$

is the most general matrix extremisation condition (traces of higher powers than third are always reducible to cubic or less via the characteristic equation). In Eqs. 61-62 the dimensionful scalar parameter r is just the ratio of partial derivatives:

$$r = \frac{\partial A \,/\, \partial \mathrm{Tr} C^2}{\partial A \,/\, \partial \mathrm{Tr} C^3} \tag{63}$$

likewise understood as being evaluated at the extremum. Essentially, the resulting constraint equations are simply a linear combination of the constraints Eqs. 59-60.

Seeking solutions, we may work in any basis we choose, and we choose to work in the usual charged-lepton flavour basis where the charged-lepton mass matrix is diagonal, in a phase convention where all the imaginary parts of the off-diagonal elements in the neutrino mass-matrix are equal up to signs (so that the matrix of imaginary parts is proportional to the epsilon matrix ϵ, hence 'the epsilon phase convention' [28])). The neutrino mass-matrix is then (redefining x, y, z yet again!):

$$N = M_\nu = \begin{matrix} \\ e \\ \mu \\ \tau \end{matrix} \begin{matrix} \quad e \quad\quad\quad \mu \quad\quad\quad \tau \\ \begin{pmatrix} a & z+id & y-id \\ z-id & b & x+id \\ y+id & x-id & c \end{pmatrix} \end{matrix} \qquad \epsilon = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{64}$$

The seven variables a, b, c, x, y, z, d then determine the three neutrino masses and the four mixing parameters (with unphysical phase dependence neatly eliminated!).[d]

In solving for fixed masses, there is also a determinantal consistency condition:

$$\begin{vmatrix} 1 & a & a^2+y^2+z^2+2d^2 \\ 1 & b & x^2+b^2+z^2+2d^2 \\ 1 & c & x^2+y^2+c^2+2d^2 \end{vmatrix} = 0. \tag{65}$$

[d]In this section we continue to consider hermitian mass matrices, in the first instance (rather than, say, hermitian squares), as being the more fundamental and natural for extremisation.

formed from the cefficients of the relevant Lagrange multipliers. Looking initially for solutions with 'democracy' symmetry (as solves Eq. 65) we have:

$$\left.\begin{array}{l} a-b=x-y \\ b-c=y-z \\ c-a=z-x \end{array}\right\} \quad \Rightarrow \text{ `democracy'} \left\{\begin{array}{l} a=x+\sigma \\ b=y+\sigma \\ c=z+\sigma \end{array}\right. \tag{66}$$

where σ is a constant mass offset.

Since L is diagonal, the off-diagonal elements in Eq. 61 vanish. The real parts give:

$$\begin{aligned} 2\,(m_\mu+m_\tau-2m_e)\,(d^2-y\,z) \;+\; 2\,(m_\mu-m_\tau)\,(y-z)\,x \;&=\; 0 \\ 2\,(m_\tau+m_e-2m_\mu)\,(d^2-z\,x) \;+\; 2\,(m_\tau-m_e)\,(z-x)\,y \;&=\; 0 \\ 2\,(m_e+m_\mu-2m_\tau)\,(d^2-x\,y) \;+\; 2\,(m_e-m_\mu)\,(x-y)\,z \;&=\; 0 \end{aligned} \tag{67}$$

explicitly cyclically symmetric, and the imaginary parts give:

$$\begin{aligned} &2r\,d\,(m_\mu+m_\tau-2m_e)(y\;+\;z)\;+\;2r\,d\,(m_\mu-m_\tau)\,(y-z) \\ &\qquad\qquad =\; 3\,(m_e-m_\mu)\,(m_\tau-m_e)\;(y-z)\,(d^2-xy-yz-zx) \\ &2r\,d\,(m_\tau+m_e-2m_\mu)(z\;+\;x)\;+\;2r\,d\,(m_\tau-m_e)\,(z-x) \\ &\qquad\qquad =\; 3\,(m_\mu-m_\tau)\,(m_e-m_\mu)\;(z-x)\,(d^2-xy-yz-zx) \\ &2r\,d\,(m_e+m_\mu-2m_\tau)(x\;+\;y)\;+\;2r\,d\,(m_e-m_\mu)(x-y) \\ &\qquad\qquad =\; 3\,(m_\tau-m_e)\,(m_\mu-m_\tau)\;(x-y)\,(d^2-xy-yz-zx) \end{aligned} \tag{68}$$

also explicitly cyclic symmetric. A solution to Eq. 67-68 is:

$$\begin{array}{ll} x=\pm\dfrac{\sqrt{XYZ}}{X} & X=d^2-\dfrac{2dr(m_\mu-m_\tau)}{3(m_e-m_\mu)(m_\tau-m_e)} \\ y=\pm\dfrac{\sqrt{XYZ}}{Y} & Y=d^2-\dfrac{2dr(m_\tau-m_e)}{3(m_\mu-m_\tau)(m_e-m\mu)} \\ z=\pm\dfrac{\sqrt{XYZ}}{Z} & Z=d^2-\dfrac{2dr(m_e-m_\mu)}{3(m_\tau-m_e)(m_\mu-m_\tau)} \end{array} \tag{69}$$

where the cyclic symmetry is clearly respected in the solution.

The operative parameter is now r/d, fixing the mixing and (if we are prepared to assume, eg. $m_1 << m_2 << m_3$) also the neutrino mass hierarchy. Clearly, from Eq. 69 the mixing approaches the so-called 'simplest'[28] form ($|y|, |z| << |x|$) as the denominator factor X goes to zero ($X \to 0$ corresponds to $r/d \to 0.168$ GeV). In the extreme limit, the mixing matrix tends finally to the familiar tri-bimaximal [15] form (with an exact degeneracy $\Delta m^2_{12} \to 0$ occuring at the pole). This qualitative tendency to the 'simplest' anatz[28] seen here (as the approach to tribimaximal mixing) is the most phenomenologically encouraging result we have found so far from this part of our extremisation programme. In quantitative terms, however, even this tendency brings little satisfaction in practice.

Constraining r/d to the observed mass hierarchy (assuming $m_1 << m_2 << m_3$) we have $\Delta m^2_{12}/\Delta m^2_{23} \simeq m^2_2/m^2_3 \simeq 0.035$ for $r/d = 0.245$ GeV, with a mixing matrix:

$$(|U_{l\nu}|^2) \;=\; \begin{matrix} \\ e \\ \mu \\ \tau \end{matrix} \begin{matrix} \nu_1 & \nu_2 & \nu_3 \\ \end{matrix} \!\!\! \begin{pmatrix} .48072 & .33333 & .18595 \\ ..40735 & .33333 & .25932 \\ .11194 & .33333 & .55473 \end{pmatrix} \not\simeq \begin{matrix} e \\ \mu \\ \tau \end{matrix} \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \quad (70)$$

We see (disappointingly) that in practice we are too far from the pole for the 'simplest' [28] approximation to apply, whereby, we are left with too large a value for $|U_{e3}|$ above. Note that the poles at $Y \to 0$ ($r/d \to 0.148$ GeV) and $Z \to 0$ ($r/d \to 42.4$ GeV) correspond to 'permuted' forms of tribimaximal mixing having a $\tau - e$ symmetry (with $|U_{\mu 3}| \to 0$) and an $e - \mu$ symmetry (with $|U_{\tau 3}| \to 0$) respectively, and do not therefore seem to be phenomenologically relevant.

Although democracy (Eq. 66) is an elegant and promising way to implement the determinant condition (Eq. 65), it will be clear that it is not forced on us here. We therefore explore the trajectory of solutions generated dropping this constraint, as we move away from the solution Eq. 70 keeping the neutrino mass hierachy constant (ie. taking $\Delta m^2_{12}/\Delta m^2_{23} \simeq m^2_2/m^2_3 \simeq 0.035$). We find that we can reduce the value of $|U_{e3}|$, but that $|U_{e2}|$ also decreases (and furthermore decreases faster than $|U_{e3}|$):

$$(|U_{l\nu}|^2) \;=\; \begin{matrix} e \\ \mu \\ \tau \end{matrix} \overset{\begin{matrix} \nu_1 & \qquad \nu_2 & \qquad \nu_3 \end{matrix}}{\begin{pmatrix} .64243 & .21957 & .13800 \\ .30883 & .34235 & .34882 \\ .04874 & .43808 & .51318 \end{pmatrix}} \not\simeq \begin{matrix} e \\ \mu \\ \tau \end{matrix} \overset{\begin{matrix} \nu_1 & \nu_2 & \nu_3 \end{matrix}}{\begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}} \quad (71)$$

Eq. 71 is simply a representative 'compromise' solution which (even so) is still very far from viable. The mass matrix parameters are $(x/d, y/d, z/d) = (-3.270, -1.883, 2.240)$; $(a/d, b/d, c/d) = (1.725, 3.979, 5.735)$ and $r/d = 1.086$ GeV. We have not suceeded to obtain analytical solutions when dropping the democracy constraint.

5. Our Best Guess So Far: The 'Simplest' Ansatz

We have seen that extremisation of a general flavour-symmetric scalar function of the Jarlskog commutator C, may be said to *point to* the 'simplest' ansatz. The 'simplest' ansatz was put forward on aesthetic grounds in 2004[28)], having been previously proposed as "an amusing specialisation of tri-χ-maximal mixng" in 2002[24)].

To appreciate the 'simplest' ansatz[28)], one had best focus again on hermitian-squares of mass-matrices. In the usual charged-lepton flavour basis (keeping the epsilon phase convention), one simply takes the neutrino mass matrix (hermitian-square) to be a linear combination of the 3×3 identity matrix, the $\mu - \tau$-exchange operator E (Eq. 6)(with a negative coefficient) and the epsilon matrix ϵ (Eq. 64):

$$M_\nu M_\nu^\dagger = \sigma I + xE + id\epsilon \quad (72)$$

$$= \begin{pmatrix} \sigma + x & id & -id \\ -id & \sigma & x + id \\ id & x - id & \sigma \end{pmatrix} \quad (73)$$

with the two mass-squared differences fixing the parameters x and d ($x < 0$):

$$\Delta m^2_{\rm atm} = m_3^2 - m_1^2 = 2\sqrt{x^2 + 3d^2} \quad (74)$$

$$\Delta m^2_{\rm sol} = m_2^2 - m_1^2 = \sqrt{x^2 + 3d^2} + x. \quad (75)$$

The parameter σ in Eq. 73 represents an overall offset on the neutrino mass-squared spectrum, precisely determined only when the masses of all three neutrinos (i.e. including the lightest neutrino) are known. The offset σ cannot influence the mixing.

Clearly, the mass matrix (Eq. 73) commutes with the democracy operator (Eq. 5) and with the $\mu - \tau$-reflection operator (Eq. 6), so that the 'simplest' ansatz builds-in the 'democracy' and 'mutativity' symmetries. The resulting mixing matrix is simply a reparametrisation of triχmaximal mixing[24)], but with the mixing angle determined:

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} c_\chi & \sqrt{\frac{1}{3}} & -i\sqrt{\frac{2}{3}} s_\chi \\ -\frac{c_\chi}{\sqrt{6}} + i\frac{s_\chi}{\sqrt{2}} & \sqrt{\frac{1}{3}} & -\frac{c_\chi}{\sqrt{2}} + i\frac{s_\chi}{\sqrt{6}} \\ -\frac{c_\chi}{\sqrt{6}} - i\frac{s_\chi}{\sqrt{2}} & \sqrt{\frac{1}{3}} & \frac{c_\chi}{\sqrt{2}} + i\frac{s_\chi}{\sqrt{6}} \end{pmatrix} \qquad \tan 2\chi = \sqrt{3} d/x. \quad (76)$$

This leads us finally to a simple prediction for the reactor neutrino mixing angle in terms of measured mass-squared differences:

$$\sin\theta_{13} = \sqrt{2\Delta m^2_{\rm atm}/3\Delta m^2_{\rm sol}} \simeq 0.13 \pm 0.03. \quad (77)$$

This long-standing prediction[24)28)] lies in the peak of Carl Albright's distribution![51)]

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