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FREQUENCY MAP ANALYSIS WITH DIFFUSION

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Abstract

As particle amplitudes approach the chaotic boundary of the dynamic aperture limit, their oscillation frequencies - although not incommensurate – diffuse through the tune diagram with an increasing rate which is a good indicator of the onset of long-term instability. The accurate determination of these frequencies using the Interpolated Fourier-Hanning algorithm enables these diffusion rates to be measured in a reasonably small number of samples from a tracking calculation. This in turn allows a frequency map to be constructed from a set of initial starting conditions in a realistic length of time. This frequency map analysis has been implemented using Mathematica and the MAD code, and applied to the EPAC 2000 DIAMOND storage ring lattice.

1. FREQUENCIES OF A NEAR-INTEGRABLE SYSTEM

1.1 KAM Invariant Tori

According to the Kolmogorov-Arnold-Moser theory [1,2,3], a near-integrable conservative system will contain many invariant tori (each of which corresponds to a particular action value), upon which particle trajectories will execute quasi-periodic motion with a fixed frequency. Even in systems arbitrarily close to being integrable, these tori are interleaved with regions of action space in which chaotic motion occurs, and in general the invariant tori for a Cantor set over the space of possible frequency vectors. Diffusion can therefore take place from one action to another due to this fractal connection between chaotic orbits; this is known as Arnold diffusion [2].

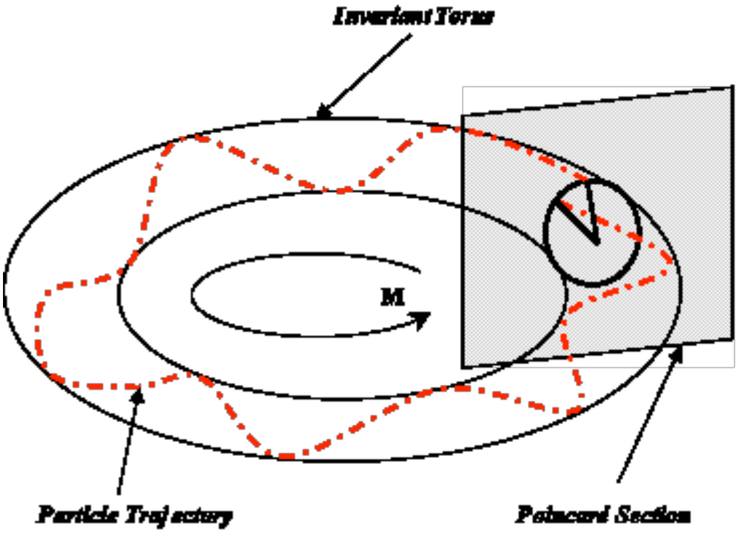


Figure 1. A particular action (i.e. normalised particle amplitude) in a system with 2 degrees of freedom which lies on a KAM torus gives rise to a trajectory with a well-defined frequency. This frequency may be probed by sampling the trajectory using a Poincaré section of the torus. Interleaved between the stable tori are chaotic regions of space for which no frequency can be determined in the long term. However, over a short time period the measured trajectory of a particle in a chaotic region interpolates between the frequencies of the stable tori on either side.

The diffusion rate from one action to another is an indication of the level of non-linearity inherent in the system, and therefore a useful indicator that complete breakdown of the stable tori is about to occur [4]. On the KAM tori the calculated frequency vector from a finite time series at a Poincaré section of the torus will be a good approximation to the actual frequencies, whilst this vector provides a natural interpolation between these frequencies in the chaotic regions, even though in these regions a frequency vector cannot strictly speaking be defined [5].

1.2 Diffusion Rates

The time evolution of the numerically-determined frequencies of a system can be used as a measure of the real diffusion of a trajectory [6.7]. In a system with 2 degrees of freedom (for example, considering only the horizontal motion of a particle in a storage ring) the frequency space will be a line with KAM tori being points on this line which block diffusion from one chaotic region to another. In practical systems, noise – for instance in the magnet power supplies – allows particles to jump across these points. In systems with more degrees of freedom (i.e. in any storage ring that has coupling) the stable tori are still points in action space, but now there is a fractal connection between the chaotic trajectories which are then free to diffuse into each other.

However, it is generally supposed that the diffusion rate is smaller in the vicinity of KAM tori – i.e. that they have some effective width corresponding to a particular time frame; therefore, regions which are densely filled with KAM tori will act as effective barriers to diffusion [8,9]. The analysis of frequencies and diffusion rates, over the action space of interest in a system being studied, can therefore give a good indication of the boundaries for stable motion, and the perturbations which give rise to limited particle stability.

1.3 Long-Term Stability and Electron Rings

The issue of long-term stability is of course crucially important in systems which do not exhibit damping of the trajectory actions, such as proton storage rings (in which there is no significant radiation emission), but electron storage rings have benefited from a frequency map analysis of their particle trajectories. The most studied light source ring to date has been the ALS, and preliminary analyses have been performed of the fundamental frequencies for the DIAMOND storage ring [10]. However, a calculation of the diffusion rates has not been performed, and this is the subject of the following sections.

1.4 Tune and Diffusion Rate Determination

The Interpolated Fourier-Hanning (IFH) algorithm has been previously described [11], and can be shown to give an accuracy of the determined tune which scales as

$$\left|\varepsilon_{IFHan}\right| \le \frac{C_{FHan}}{N^4}, \qquad (1)$$

where N is the number of turns for which the trajectory has been sampled, giving a best estimate of the tune as

$$\nu_{IFHan} = \frac{k}{N} + \frac{1}{2\pi}\arcsin\left[A\left(\left|\phi(\nu_k)\right|, \left|\phi(\nu_{k+1})\right|, \cos\frac{2\pi}{N}\right)\sin\frac{2\pi}{N}\right], \qquad (2)$$

where

$$A(a,b,c) = \frac{-(a+bc)(a-b) + b\sqrt{c^2(a+b)^2 - 2ab\left(2c^2 - c - 1\right)}}{a^2 + b^2 + 2abc} \qquad (3)$$

and the original time series data has had a Hanning filter applied as

$$\phi(\nu_j) = \frac{1}{N}\sum_{n=1}^{N} z(n)\chi(n)e^{-2ni\pi\nu_j}, \quad \chi(n) = 2\sin^2\left(\frac{\pi n}{N}\right). \qquad (4)$$

The IFH algorithm is formally the same as Laskar's numerical analysis of fundamental frequencies (NAFF) method [12,13].

The diffusion rate is calculated in a simple-minded, but effective, way by dividing the available time series for a particular starting action into two equal halves, and calculating the normalised rate of frequency change with time

$$\frac{\partial \nu_{IFHan}}{\partial N} = \frac{\nu_2 - \nu_1}{N}. \qquad (5)$$

2. IMPLEMENTATION

The Interpolated Fourier/Hanning (IFH) method and Diffusion rate calculation has been implemented within Mathematica [14] and incorporated into the Mathematica Interface to MAD (MIM) [15-18]. The plots shown below can therefore be readily calculated for an arbitrary lattice, and example code is available at:

http://www.srd.dl.ac.uk/srs/ap/Software/Mathematica/MadtoMMA/Examples/

Code used to generate the plots above is available with this note.

3. RESULTS FOR DIAMOND

The EPAC 2000 lattice in its non-zero dispersion operating mode was used, since FMA calculations have been previously carried out for this lattice [10]. The optical functions are shown in Figure 2 and Figure 3, with a schematic layout shown in Figure 4. Horizontal and vertical diffusion rates were calculated using equation (5) and 512 turns (using the Lie3 mapping method in MAD). Particles were tracked over a grid of points over the range [1,15] mm horizontally and [1,10] mm vertically, in 0.1 mm steps. The logs of each of the diffusion rates for this starting grid are plotted in Figure 5, and shows that the horizontal and vertical diffusion rates are broadly similar to each other. Many resonances perturb particles close to zero amplitude which are not visible without the diffusion calculation; this indicates that this particular optics would need further work to both improve the nonlinear correction (narrowing the width and strength of these resonances). Similar calculations will be performed with the increased ~540 m circumference lattice (now being worked on) to guide the nonlinear correction and selection of its working points.

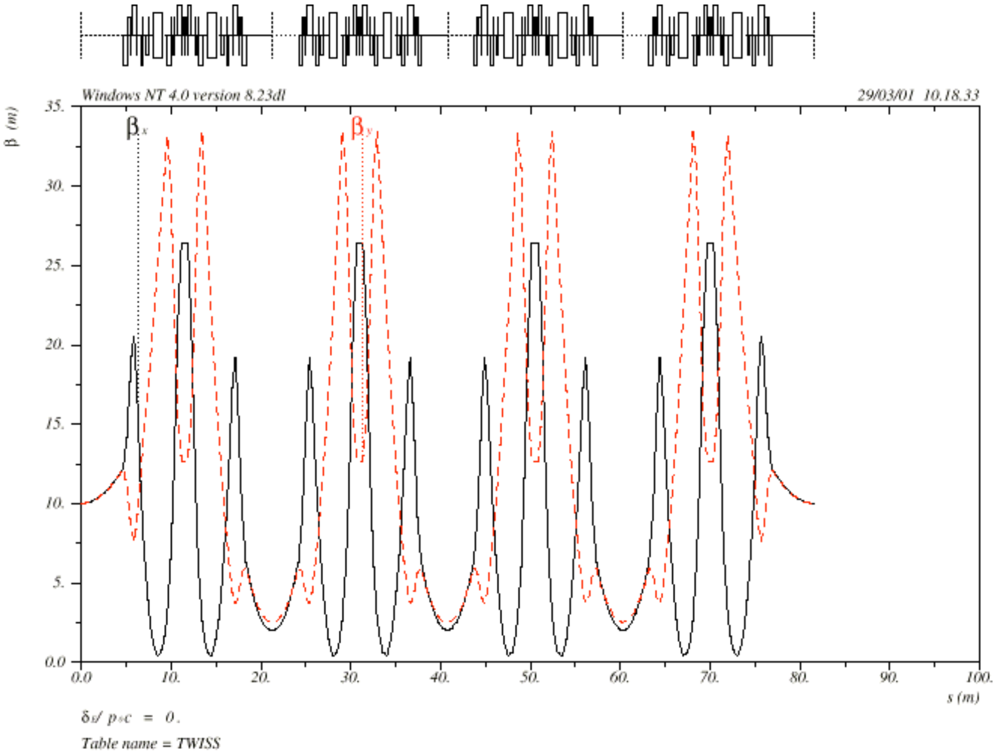


Figure 2. Beta functions of the EPAC 2000 lattice used in these calculations, operating in the non-zero dispersion mode.

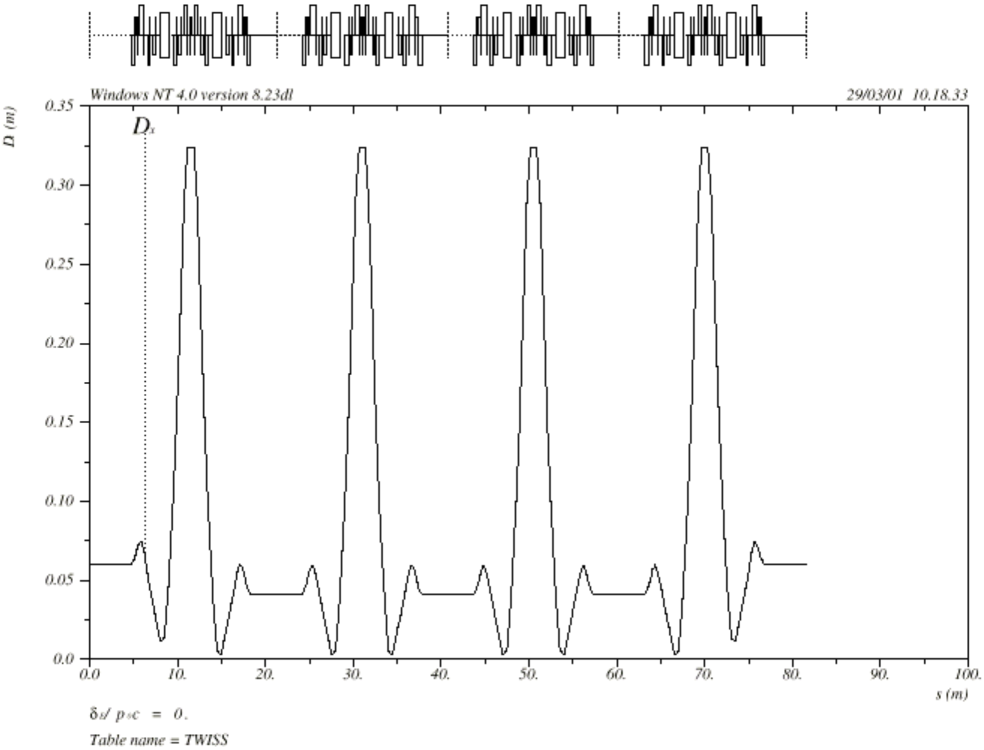
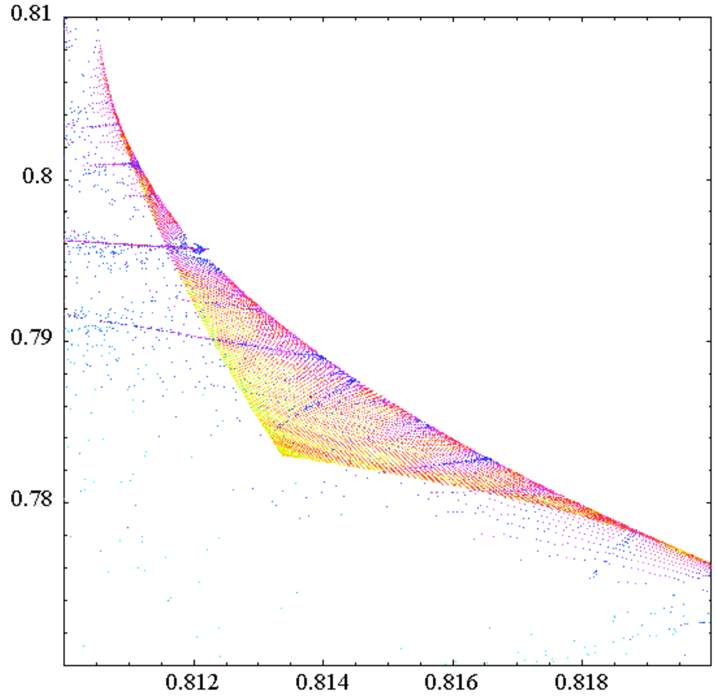
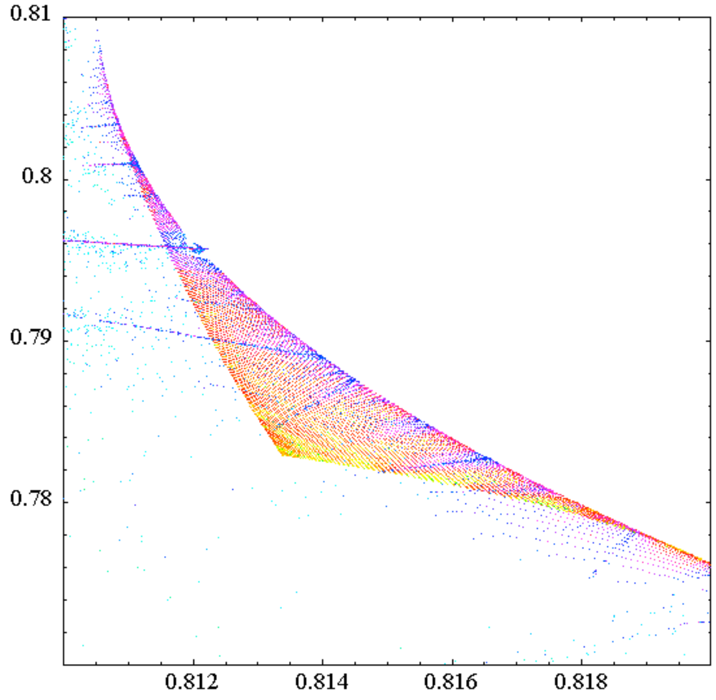


Figure 3. Horizontal dispersion function in the EPAC 2000 lattice used in these calculations, operating in the non-zero dispersion mode.

Figure 4. Layout of 1/6 of the EPAC 2000 lattice.



(a)



(b)

Figure 5. Frequency map analysis of the DIAMOND EPAC 200 lattice, with diffusion rates shown logarithmically in colour. Small diffusion rates are shown in red whilst large diffusion rates are shown in blue. Trapping of particles onto unstable resonances is clearly apparent, where they then suffer greater diffusion, indicating long-term instability. The horizontal and vertical diffusion rates are broadly similar.

4. REFERENCES

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[17] The NLC code home pages is at http://www-project.slac.stanford.edu/lc/local/AccelPhysics/Codes/nlc_simulation_codes.htm

[18] The MadtoMMA Homepage is at http://wwwslap.cern.ch/~jowett/Madtomma/.