CALCULATION OF THE TOUSCHEK LIFETIME IN ELECTRON STORAGE RINGS

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Abstract: Various formulae for calculating the Touschek lifetime of a ribbon beam of electrons are examined. It is shown that two commonly used approximations can give inaccurate results in certain circumstances. A method is suggested for calculating the lifetime accurately and efficiently using a combination of formulae.

Introduction

In electron storge rings electrons within the same bunch can scatter elastically off each other due to their betatron motion and after collision may be lost if the change introduced in the longitudinal momentum is larger than can be accommodated by the r.f. acceleration system. The effect, first observed on the small storage ring AdA [1], was explained by B. Touschek, and so is most often referred to as Touschek scattering. The loss rate due to this process is given by

$$\frac{dN_b}{dt} = -\alpha N_b^2 \; ; \qquad N_b = \frac{N_0}{1 + \alpha N_0 t}$$

where N_b is the number of electrons per bunch. The half-life, τ, known as the Touschek lifetime, is then given by:

$$\frac{1}{\tau} = \alpha N_0$$

The loss rate depends on the beam current, r.f. acceptance, the distribution of transverse momentum and bunch volume. It is very energy dependent and is often the limiting factor in determining the minimum injection energy of a storage ring. However, even at relatively large energies with the very small beam emittances being proposed for the next generation of synchrotron radiation sources the Touschek lifetime can make a significant contribution to the overall beam lifetime. Loss of the particles may also occur if the transverse motion after collision exceeds the physical or dynamic aperture. This process is often neglected but may be important in certain cases and can be accounted for by introducing an effective r.f. acceptance related to the aperture [2].

Less violent collisions also take place which do not lead to loss but which give rise to a growth in the energy spread and emittance of the particles [3]. The effect called multiple Coulomb scattering, or more commonly intra-beam scattering, will not be discussed further here. In this report the various formulae which have been derived to calculate the Touschek lifetime for an equilibrium distribution of particles, whether this refers to the "natural" values or those including the effects of intra-beam scattering, will be examined and their range of validity explored.

Touschek Lifetime Calculations

Various formulae to compute the Touschek lifetime have been derived [4-7] using different approximations; however, the basic method was the same in all cases. Firstly, the Møller scattering cross-section [8,9] in the centre-of-mass frame of the two electrons involved in the collision is integrated over the angular region for which sufficient momentum can be transferred into the longitudinal direction to result in the loss of both particles. Then the scattering rate is evaluated by integrating the total cross-section over the distribution of transverse momentum in the bunch. It is usual to assume two simplified assumptions:

i) All particles have the same energy, E_0. This is a good approximation since the energy spread in most storage rings is small compared to the r.f. acceptance to ensure adequate quantum lifetime.
ii) The vertical component of transverse momentum can be neglected, or equivalently, that the vertical beam divergence is small compared to the horizontal: $\sigma'_y < \sigma'_x$. In many cases this approximation of a "ribbon electron beam" is reasonably good, since averaged around the circumference typically $\sigma'_x/\sigma'_y \approx$ 5-10 in most machines. However, there are situations where it may be desired to induce a high degree of coupling between horizontal and vertical betatron motion either to produce "round beams" for synchrotron radiation sources, or possibly in order to increase the beam volume and hence increase the Touschek lifetime. In such cases a more complicated analysis is necessary and this topic has received little attention until recently when Miyahara [10] put forward a simplified expression for a perfectly round beam ($\sigma'_x = \sigma'_y$) in the non-relativistic regime, which is in agreement with measurements made on the SOR-RING.

The analysis used to explain the results obtained with AdA [1] and which appears in the book by Bruck [4] was based on the further approximation, that the transverse momentum is non-relativistic. The condition for validity of this approach is that the transverse momentum, p_x, in units of m_0c, is less than unity. For an electron of momentum $\gamma m_0 c$ travelling at an angle θ to the reference orbit this condition reduces to: $\gamma\theta < 1$. In many electron storage rings the value of $\gamma\sigma'_x$, averaged around the circumference, is in the range 0.1 - 0.5 at full energy; however, it can be larger. For example, the SRS (before the recent upgrade [11]) had a typical divergence σ'_x of 0.4 mrad at 2 GeV giving $\gamma\sigma'_x \simeq 1.5$. It is clear therefore that a non-relativistic approximation is not valid in all cases; the errors involved will be considered later.

Bruck's result was as follows:

$$\frac{1}{\tau} = \frac{\sqrt{\pi}\, r_0^2\, c\, N_b\, C(\varepsilon)}{\delta q\, \gamma^2\, \eta^2\, V} \tag{1}$$

where r_0 = classical electron radius
$V = 8\pi^{3/2}\, \sigma_x\, \sigma_y\, \sigma_z$, the bunch volume
δq = rms transverse momentum in units of m_0c, $= \gamma\sigma'_x$
$\eta = \left(\frac{\Delta E}{E_0}\right)_{max}$, the maximum relative energy deviation accepted by the r.f. system
$\varepsilon = (\eta/\delta q)^2$

The function $C(\varepsilon)$ is given by:

$$C(\varepsilon) = -\frac{3}{2}e^{-\varepsilon} + \frac{\varepsilon}{2}\int_\varepsilon^\infty \frac{\ln u}{u}e^{-u}\,du + \frac{(3\varepsilon - \varepsilon \ln\varepsilon + 2)}{2}\int_\varepsilon^\infty \frac{e^{-u}}{u}\,du \tag{2}$$

and is shown in Fig.1. For $\varepsilon \ll 1$ Bruck gives the approximation:

$$C(\varepsilon) = -\ln(\varepsilon) - 2.077 \tag{3}$$

The approximation, shown dotted in Fig.1, is good to 2% for $\varepsilon \lesssim 5.10^{-3}$.

Gitelman and Ritson [5] used a small angle approximation for the cross-section and derived expressions for the Touschek lifetime in the limiting cases

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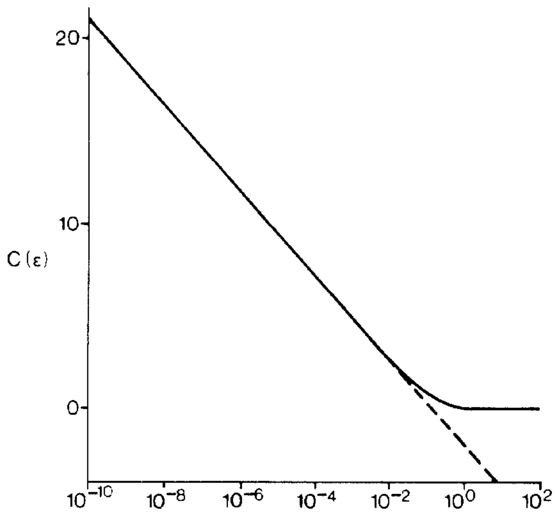

Fig. 1. Non-relativistic approximation for the Touschek lifetime, eqn.(2). The dotted line shows the limiting solution for $\varepsilon \ll 1$ (eqn.(3)).

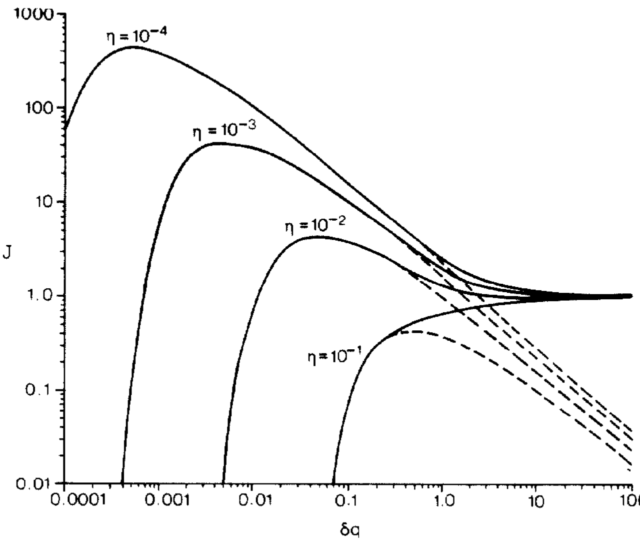

Fig. 2. General formula for the Touschek lifetime, eqn.(5). The dotted lines show the non-relativistic approximation (eqns.(2) and (6)).

of non-relativistic ($p_x \ll 1$) and extreme relativistic ($p_x \gg 1$) transverse momentum. Völkel [6] showed that the small angle approximation was equivalent to taking the limit $\eta \to 0$, and indeed their result for the non-relativistic case is equivalent to eqn.(3) above.

Völkel [6] extended the analysis of ref.[5] using the exact expression for the scattering cross-section. His result was as follows:

$$\frac{1}{\tau} = \frac{4\pi\, r_0^2\, c\, N_b}{\gamma^2\, \eta^2\, V}\, J(\eta, \delta q) \tag{4}$$

The general expression for J, for a Gaussian distribution of momentum, is:

$$J = \int_\eta^\infty dx\, \frac{e^{-\left(\frac{x}{\delta q}\right)^2}}{2\sqrt{\pi}\,\delta q}\, \frac{\sqrt{1+x^2}}{x} \left[\left(1 + \frac{x^2}{1+x^2}\right)^2 \left(1 - \eta^2\, \frac{1+x^2}{x^2}\right)\right. \tag{5}$$

$$\left. + \eta^2\, \frac{x^2}{1+x^2} \left(1 - \eta\, \frac{\sqrt{1+x^2}}{x}\right) + \eta^2\, \frac{4x^2+1}{x^2(1+x^2)}\, \ln\left(\eta\, \frac{\sqrt{1+x^2}}{x}\right)\right]$$

where the symbol x has been used to denote the transverse momentum, p_x. Figure 2 shows the results of numerically integrating eqn.(5) for a wide range of values of η and δq : $10^{-4} < \eta < 10^{-1}$, $10^{-4} < \delta q < 10^{2}$. It can be seen that the lifetime is worse (larger J) the smaller the r.f. acceptance (η), as expected. Generally as the transverse momentum decreases (reducing beam divergence or energy) the lifetime also gets worse, however a minimum is reached at $\eta/\delta q \simeq 0.2$. A minimum is expected since for an individual collision if $\eta/p_x \geqslant 1$ the particles cannot transfer sufficient momentum into the longitudinal plane to cause loss. Of course, as mentioned earlier, the influence of intra-beam scattering in this regime is likely to be strong. At extremely relativistic transverse momenta it can be seen that J approaches unity, independent of r.f. acceptance, as predicted by Gittelman and Ritson [5].

Wiedemann [7] examined the non-relativistic limit ($x \ll 1$) of eqn.(5). In fact, the result obtained was identical to eqn.(1) with:

$$J = \frac{C(\varepsilon)}{4\sqrt{\pi}\,\delta q} \qquad (p_x \ll 1)\ . \tag{6}$$

Results obtained using the non-relativistic approximation are shown in fig.2 by the dotted lines. It can be seen that the approximation is valid for $\delta q \lesssim 0.1$ and becomes increasingly inaccurate for larger δq. Figure 3 shows the error in using the non-relativistic approximation more clearly, and it can be seen that for better than 2% accuracy $\delta q < 0.1 - 0.3$ depending on the value of η. For $\delta q = 1$ errors of between 10% and 40% can result, depending on η.

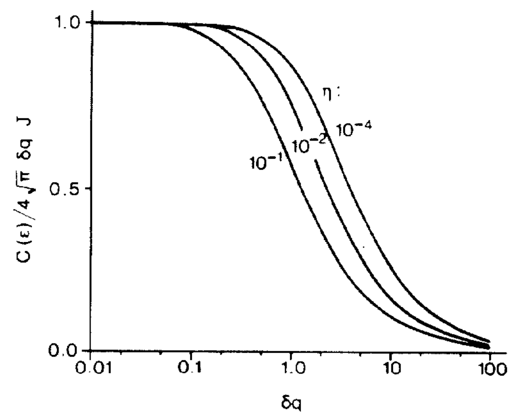

Fig. 3. Ratio of the non-relativistic approximation for the Touschek lifetime to the exact result.

Völkel gave several useful approximations of eqn.(5) for the limiting case $\eta \to 0$. With a Gaussian momentum distribution numerical integration is still required, however with the simplification of a rectangular momentum distribution the following expression was derived:

$$J_{approx} = \frac{(1+\delta p^2)^{1/2}}{\delta p} + \frac{1}{2\delta p}\left\{\ln\left(\frac{2}{\eta}\right) - \frac{23}{4}\right. \tag{7}$$

$$\left. + \frac{1}{2}\ln\left[\frac{(1+\delta p^2)^{1/2} - 1}{(1+\delta p^2)^{1/2} + 1}\right] + \frac{2}{\delta p}\ln\left(\delta p + (1+\delta p^2)^{1/2}\right)\right\}$$

where the limits of the momentum distribution are $\pm\delta p$, related to the rms width by $\delta q = \delta p/\sqrt{3}$. Figure 4 shows that the approximation is accurate for $\delta q/\eta \gtrsim 100$. However, outside this range the approximation becomes very poor and will even predict negative values of J [12], and hence of the lifetime, for $\delta q/\eta \lesssim 3$. The formula

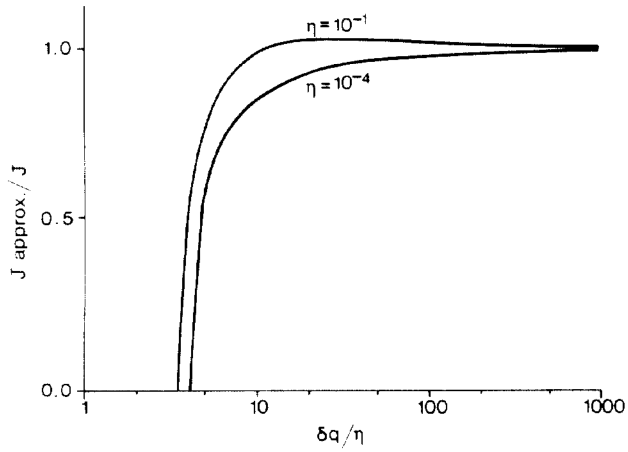

Fig.4. Ratio between a simplified expression for the Touschek integral, eqn.(7), and the exact result, eqn.(5).

eqn.(7) appears in the CERN BEAMPARAM program [13] which should therefore be used with some caution for calculating the Touschek lifetime.

Taking the limits $\eta \to 0$ and $\delta q \ll 1$ Völkel also derived the equivalent result to eqn.(3), namely:

$$J_{approx} = \frac{1}{2\sqrt{\pi}\ \delta q}\left[\ell n\left(\frac{\delta q}{\eta}\right) - 1.039\right] \qquad (8)$$

The range of validity of this equation can be inferred from the results given previously for small $\eta/\delta q$ and the non-relativistic approximation. This formula therefore allows a very simple calculation of Touschek lifetime but for a restricted range of situations.

In applying the formulae to a real machine the scattering rate should be averaged around the storage ring circumference [14] since both the transverse momentum distribution (beam divergence) and beam size vary around the ring. In one case it was shown that errors of up to a factor of two could be produced using average values for the lattice functions compared to averaging the scattering rate [15].

Conclusion

It has been shown that one commonly used formula for computing the Touschek lifetime, eqn.(1), is valid only for non-relativistic transverse momentum and that at high energy with large beam emittance significant errors compared to the more general formula can result. Another approximation, eqn.(7), can break down to the extent that negative lifetimes are indicated, particularly in the case of low energy and low emittance beams with a large r.f. acceptance.

For preference therefore one should use in the general case the result of Völkel, eqns.(4) and (5). However, in order to reduce the amount of computer time used, particularly as the calculation must be performed many times in order to take into account the varying lattice functions around the ring circumference, it is useful to incorporate approximations wherever they are appropriate. Taking as a criterion a requirement for 2% accuracy the two approximations, eqns.(7) and (8), have been examined in detail to determine the ranges of parameter space over which they can be applied. The result is shown in Fig.5. The most useful is eqn.(7) whose range of validity is defined by the constraints $\delta q \geqslant 1$ and $\delta q/\eta \geqslant 130$. The region over which eqn.(8) may be used is given by: $\eta \leqslant 5.10^{-3}$, $\delta q \leqslant 0.2$ and $\delta q/\eta \geqslant 16$. It can be seen that there is a region of parameter space where it may be applied when eqn.(7)

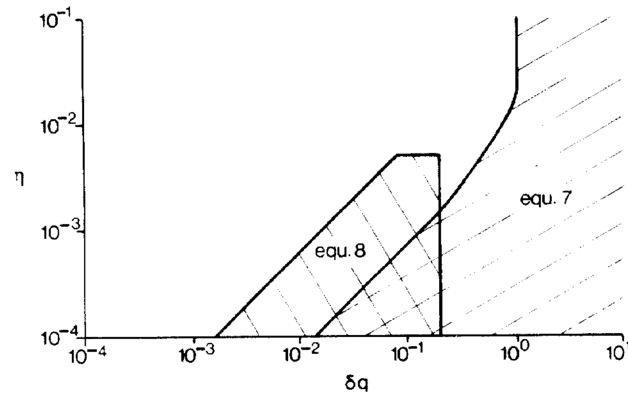

Fig.5. Ranges of validity of two approximations to the Touschek integral for 2% accuracy.

does not give sufficient accuracy. In the region of overlap eqn.(8) would also be preferred as being somewhat more efficient than eqn.(7).

References

[1] C. Bernardini et al., Phys. Rev. Lett., vol.10, 1963, p.407.

[2] ZAP User's Manual, M.S. Zisman et al., LBL-21270, 1986

[3] H. Bruck and J. Le Duff, Proc. 5th Int. Conf. High Energy Accelerators, Frascati, 1965, p.284.

[4] H. Bruck, Accelerateurs Circulaires de Particules, PUF, Paris, 1966. English translation: Los Alamos National Laboratory Report, LA-TR-72-10 Rev.

[5] B. Gittelman and D.M. Ritson, HEPL-291, 1963, Stanford University.

[6] U. Völkel, DESY 67/5, 1967.

[7] H. Wiedemann, PEP Note-27 1973, SLAC.

[8] C. Møller, Ann Phys. vol.14, 1932, p.531.

[9] W. Heitler, The Quantum Theory of Radiation, III Ed., Oxford, 1960, p.239.

[10] Y. Miyahara, Jap. J. Appl. Phys., vol.24, 1985, p.L742.

[11] M.W. Poole et al., these proceedings.

[12] F.H. Wang, INFN Frascati Machine Division Internal Memorandum, T-113.

[13] M. Hanney and E. Keil, CERN-ISR/TM/79-29.

[14] L. Smith, Proc. 1977 NSLS Summer Theory Workshop, BNL 23695, 1977, ch.M.

[15] P. Backer, ESRF Machine Group Note, ESRF-MG-30, 1978.