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ABSTRACT

In this paper we will show explicitly that the Witten's interaction 3-vertex is a solution to the comma overlap equations; hence establishing the equivalence between the conventional and the "comma" formulation of interacting string theory at the level of vertices.

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1 Introduction

In the comma approach to string field theory [1], the Witten's theory of open-bosonic-string [2] is formulated as a generalized Yang-Mills theory. In the comma approach instead of choosing the bosonic coordinates X_0 of the centre of mass and the coordinates X_n for all other points relative to it as variables to describe the string, one chooses as an alternative the coordinates

$$\chi^r(\sigma,\tau) = \begin{cases} X(\sigma,\tau), & if\ r=1\,, \\ X(\pi-\sigma,\tau), & if\ r=2\,, \\ \sigma\in[0,\pi/2)\,, \end{cases} \tag{1.1}$$

where $r = 1, 2$ refers to the left (L), right (R) parts of the string and $\chi^{1,2}$ are restricted by

$$\lim_{\sigma\to\pi/2}\chi^1(\sigma,\tau) = \lim_{\sigma\to\pi/2}\chi^2(\sigma,\tau) \tag{1.2}$$

Now the midpoint[2] varies freely while the variation of the neighboring points on the string is restricted.

In [3, 4, 5] a Fock space representation of the comma theory was developed. There one introduces comma operators, b_k^L, b_k^R (corresponding to each half of the string) satisfying the commutation relations[3]

$$[b_k^{r,\mu}, b_l^{s,\nu}] = \delta^{rs}\delta_{n+m,0}\eta^{\mu,\nu}, \tag{1.3}$$

analogous to the commutation relations of a_n^μ of the full string. The Fock space corresponding, to each half (comma) of the string was introduced

[2]Another approach in which the midpoint plays a central role has been discussed in ref. [6], although in a different context from refs. [3, 4, 5].

[3]The space-time indices will be suppressed for the rest of the paper.

in [3, 4, 5] leading to a basis for the Fock space of our string in which the overall motion of the string is unrestricted but neighboring points on the string are not permitted to wander far from each other. The change of basis from the full string to the comma theory is not a simple one . The explicit transformation relating both of them has been developed in [3, 4, 5]. The commonly used Witten's string field theory (WSFT) has of course the merit of being simple for constructing solutions of the free-string equation of motion. The comma theory has the advantage of being more transparent in questions connected with gauge invariance. In [5] we have addressed the question of equivalence between the conventional and the comma formulations of interacting string field theory. In [5] we examined the relationship of the interacting vertex in both theories. In fact we were able to show directly that the identity vertex ($\int$) , the $2 - Vertex$ and the $4 - Vertex$ in WSFT constructed in [7, 8, 9, 10, 11] are solutions to the overlap equations in the comma theory, hence establishing the equivalence (at least) at the level of the vertex for $N = 1, 2, 4$. However, the assumption that the *cubic* interaction vertex (which of central importance) also solves the comma overlaps was verified subjected to the assumption, that $V_3 = I\, V_4$ (true in Witten's theory) continue to hold in the comma theory. This serious shortcoming is now removed by giving a direct proof that the *cubic* vertex in WSFT is indeed a solution of the comma overlaps.

The results presented here are obtained in the same was as in [5]; the details, however, are rather more involved, as one might have anticipated. Although several computational details are relegated to appendices, this paper is not meant to be self-contained in the sense that we relay on [5] for notation and indeed for many other details only alluded to below. For readers interested in the formulation of the comma theory, a cursory reading of refs. [1] should suffice. Other subtleties related to the anomaly cancellation for the identity vertex, the two vertex, the $3 - vertex$ and the $4 - vertex$ are discussed in [5].

2 Comma Vertices

In the comma formulation of string field theory the elements of the theory are defined by δ-function type overlaps. The N interaction vertex is given by

$$V[\chi_1^r, \chi_2^r, \ldots, \chi_N^r, \varphi^r] = e^{iQ^{\varphi}(\phi/2)} \prod_{i=1}^{N} \prod_{\sigma=0}^{\pi/2} \delta(\chi_i^1(\sigma) - \chi_{i-1}^2(\sigma))\delta(\varphi(\sigma)) . \quad (2.1)$$

The index i refers to the ith string (it is understood that $i = 0$ and N are identified) and 1 , 2 refer to the left and right parts of the string respectively[4]. The ghost δ-function has the same structure as the coordinates one and Q^{φ} is the ghost number insertion. The formulation of the elements of the comma theory in the oscillator Hilbert space is given in ref. [3, 4, 5] where they are represented as exponentials of quadratic forms. The *cubic* interaction vertex is given by

$$|V_3^C >= e^{-\sum_{i=1}^{3} b^{L\dagger}_{i\,n} b^{R\dagger}_{i-1\,n}} \prod_{i=1}^{3} |0>_i^L \, |0>_i^R . \quad (2.2)$$

This vertex has the advantage of being more transparent and it is trivial to generalize to an arbitrary number of strings which is not true in the case of the standard formulation of string field theory [7, 8, 9, 10, 11]. In fact the generalization of the above vertex is simply

$$|V_N^C >= e^{-\sum_{i=1}^{N} b^{L\dagger}_{i\,n} b^{R\dagger}_{i-1\,n}} \prod_{i=1}^{N} |0>_i^L \, |0>_i^R . \quad (2.3)$$

In the oscillator Hilbert space of the comma theory [5], the δ functions, for the coordinates[5], translate into operator overlap equations, namely

$$\left[\chi_i^L(\sigma) - \chi_{i-1}^R(\sigma)\right] |V_N >= 0 \, , \sigma \in [0, \pi/2) \, , \quad (2.4)$$

[4]To avoid confusion we may use the alternative label $L(R)$ to refer to the *left* (*right*) parts of the string respectively.

[5]The ghost degrees of freedom, in the bosonized representation, have the same structure apart from some mid-point insertions which will be addressed later.

and $i = 1, 2, .., N$. In addition, conservation of momentum requires

$$\left[\wp_j^L(\sigma) + \wp_{j-1}^R(\sigma)\right] |V_N >= 0 \,. \tag{2.5}$$

These are now the overlaps defining equations for the comma vertices. To simplify the calculation we introduce a new set of coordinates. Following Gross and Jevicki [7] we define

$$\mathcal{Q}_k^r(\sigma) = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \chi_l^r(\sigma) e^{\frac{2\pi i l k}{N}}, \; r = 1, 2, \tag{2.6}$$

and similar ones for the momenta. The corresponding creation and annihilation operators introduced in [5] are defined similarly and (for $N \geq 3$) satisfy the commutation relations

$$\left[\mathcal{B}_n^r, \overline{\mathcal{B}^s}_{-m}\right] = \delta^{rs}\delta_{nm}, \qquad \left[\mathcal{B}_n^r, \mathcal{B}_{-m}^s\right] = 0 \,. \tag{2.7}$$

The advantage of this new set of variables is that it leads to the separation of degrees of freedom in the overlap equations. In the complex coordinates, the overlap conditions $\chi_j^L(\sigma) = \chi_{j-1}^R(\sigma)$ for the three string vertex, read

$$\mathcal{Q}^L(\sigma) = e^{2\pi i/3} \mathcal{Q}^R(\sigma) \;\; , \sigma \in [0, \pi/2) \,, \tag{2.8}$$

$$\mathcal{Q}_3^L(\sigma) = \mathcal{Q}_3^R(\sigma) \;\; , \sigma \in [0, \pi/2) \,, \tag{2.9}$$

where $\mathcal{Q}^r(\sigma) \equiv \mathcal{Q}_1^r(\sigma) = (\overline{\mathcal{Q}_2^r(\sigma)})$. For the complex momenta the overlap conditions $\wp_j^L(\sigma) = -\wp_{j-1}^R(\sigma)$ translate into

$$\mathcal{P}^L(\sigma) = -e^{2\pi i/3} \mathcal{P}^R(\sigma) \;\; , \sigma \in [0, \pi/2) \,, \tag{2.10}$$

$$\mathcal{P}_3^L(\sigma) = -\mathcal{P}_3^R(\sigma) \;\; , \sigma \in [0, \pi/2) \,, \tag{2.11}$$

where $\mathcal{P}^r(\sigma) \equiv \mathcal{P}_1^r(\sigma) = (\overline{\mathcal{P}_2^r(\sigma)})$. The overlap conditions on $\mathcal{Q}^r(\sigma)$ and $\mathcal{P}^r(\sigma)$ determine the form of the comma $3-Vertex$. The interaction vertex, in the comma theory is

$$|V_3^C> = exp\left(-\frac{1}{2}(\mathcal{B}_3^\dagger|\mathcal{I}|\mathcal{B}_3^\dagger) - (\mathcal{B}^\dagger|\mathcal{H}|\overline{\mathcal{B}^\dagger})\right)\prod_{i=1}^{3}|0>_i^1\,|0>_i^2\,, \qquad (2.12)$$

with concrete matrices $\mathcal{I}$ and $\mathcal{H}$ (see ref. [5]).

Now our task is to show that Witten's *cubic* vertex constructed in [7, 8, 9, 10, 11] is a solution of the above overlap equations. At this point it is worth mentioning that it is not a trivial matter to show this; one encounters various double infinite sums (the second coming from integrating σ over the range $[0, \pi/2)$ in formulating the comma theory). Here the double infinite sums may not converge absolutely and the convergence may depend on the order of the sums. In fact when performing these sums using the theory of contour integration one end up with many divergent pieces of different orders. The amazing thing is that all these infinities of different magnitudes conspire among them selves to leave us with a finite number! The case of the full string [7, 8, 9, 10, 11] is different, the expression for the vertices involve absolutely convergent sums. This ambiguity is not an accident, we have seen in [3, 4] that Witten's theory can be viewed as an infinite dimensional local matrix algebra where the star product "$*$" becomes matrix multiplication over infinite dimensional matrices that does not conserve associativity. In the standard formulation of string field theory, the Witten's *cubic* vertex in oscillator basis (see ref. [7]) is given by

$$|V_3^W> = exp\left(-\frac{1}{2}(A_3^\dagger|C|A_3^\dagger) - (A^\dagger|U|\overline{A^\dagger})\right)\prod_{i=1}^{3}|0>_i\,, \qquad (2.13)$$

where $C_{nm} = (-)^n\delta_{nm}$ and U_{nm} is an infinite dimensional matrix constructed in [7]. The explicit form of the matrix U is given in appendix A. With the help of the change of representation formulas (see ref. [5]) we are able to show that the comma overlaps are satisfied by the cubic vertex[6].

[6]By this we mean to show that the comma overlaps continue to hold in the Oscillator Hilbert space of the Witten's cubic vertex.

3 Proving the Overlap Equations

Coordinate Overlaps. We first note that the second equation of the coordinates overlaps, (2.9), is the same as the overlap equation for the identity vertex and therefore the proof follows from the form of the vertex[7]. Hence we are only left with (2.8) to verify. The overlap conditions on $Q^r(\sigma)$, eq. (2.8), imply that their Fourier components satisfy

$$\left[Q^L_{2n} - e^{2\pi i/3}Q^R_{2n}\right]|V_3>= 0.n \geq 0 \tag{3.1}$$

Using the change of representation (see ref. [5]) the above equation, in the full string oscillator Hilbert space reads

$$\left[-i\sqrt{3}Q_0 + \frac{2\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}Q_{2n+1}\right]|V_3^W>= 0\,, \tag{3.2}$$

$$\left[-i\sqrt{3}Q_{2n} - 2\sum_{m=0}^{\infty}B_{2n\,2m+1}Q_{2m+1}\right]|V_3^W>= 0. \tag{3.3}$$

Commuting the annihilation operators in (3.2) through the creation operators in $|V_3^W>$ yields a sum of creation operators acting on $|V_3^W>$, hence

$$-\sum_{m=0}^{\infty}[.........]\,A^{\dagger}_m|V_3^W>\,, \tag{3.4}$$

where the expression in the square bracket is given by

$$\frac{\sqrt{3}}{2}(U_{m\,0}+\delta_{m\,0}) + \frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)^{3/2}}(U_{m\;2n+1}+\delta_{m\;2n+1})\,. \tag{3.5}$$

Since the states $A^{\dagger}_m|V_3^W>$ are linearly independent, the expression in (3.5) must vanish for all values of m. Now there are three cases to consider $m = 0$, $2k \geq 2$, $2k+1 \geq 1$. For $m = 0$, equation (3.5) reduces to

[7] See ref. [5].

$$\left[\frac{\sqrt{3}}{2}(U_{0\,0}+1)+\frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-)^k}{(2n+1)^{3/2}}U_{0\,2n+1}\right]. \qquad (3.6)$$

Using the expression for U_{nm} in ref. [7] (see also appendix A), eq. (3.6) becomes

$$\left[\frac{\sqrt{3}}{2}(U_{0\,0}+1)-\frac{2}{\pi}(1-U_{0\,0})\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)^2}\right], \qquad (3.7)$$

where a_n are the coefficients obtained in the expansion of the function $((1+x)/(1-x))^{1/3}$. The sum in the above expression can be found explicitly using the contour integral representation

$$a_n=\frac{1}{2\pi i}\oint_0 dz\frac{1}{z^{n+1}}\left(\frac{1+z}{1-z}\right)^{1/3}. \qquad (3.8)$$

Hence one obtains the following value[8]

$$\sum_{n=0}^{\infty}\frac{a_{2n+1}}{(2n+1)^2}=\frac{\sqrt{3}}{4}\pi(3\,ln3-4\,ln2)=\frac{\sqrt{3}\pi}{4}\frac{1+U_{00}}{1-U_{00}}. \qquad (3.9)$$

This proves that equation (3.7) is identically zero for $m=0$. The next case to consider is $m=2k$. Now eq. (3.5) becomes

$$\left[\frac{\sqrt{3}}{2}U_{2k\,0}+\frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)^{3/2}}U_{2k\,2n+1}\right]. \qquad (3.10)$$

Using the explicit values of the matrix U, the above expression reduces to

$$-\frac{\sqrt{3}}{2}(1-U_{00})\frac{A_{2k}}{\sqrt{2k}}+\frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)^{3/2}}\left[\frac{-i}{2}(2k)^{1/2}(2n+1)^{1/2}\right.$$

[8]To perform the sum we notice that in the complex plane we have the cuts along the lines $(1,\infty)$ and $(-1,\infty)$. Deforming the contours and picking the integrals along the cuts; then commuting the sum and the integral one sees that the series inside the integral is easily converted to another integral over a logarithmic function in two integration variables (say x,y). Performing the change of variable $t=\frac{x+1}{x-1}$ followed by the change of variables $x=1/t$ the whole expression reduces to various forms of special functions which are easily read from standard tables of integrals.

$$\left\{ \frac{A_{2k}B_{2n+1} - B_{2k}A_{2n+1}}{(2k) + (2n+1)} + \frac{A_{2k}B_{2n+1} + B_{2k}A_{2n+1}}{(2k) - (2n+1)} \right\} +$$

$$i(1 - U_{00}) \frac{A_{2k}A_{2n+1}}{(2k)^{1/2}(2n+1)^{1/2}} \Bigg] , \qquad (3.11)$$

where A_n are the coefficients given by

$$\left(\frac{1+ix}{1-ix} \right)^{1/3} = \sum_{n=0}^{\infty} A_{2n} x^{2n} + i \sum_{n=0}^{\infty} A_{2n+1} x^{2n+1}, \qquad (3.12)$$

and the corresponding coefficients B_n, are obtained in the expansion of $((1+ix)/(1-ix))^{2/3}$. It is worth mentioning that there is only a sign deference between them and the analogous coefficients a_n and b_n in the expansion of the functions $((1+x)/(1-x))^{1/3}$ and $((1+x)/(1-x))^{2/3}$

$$A_n = (-)^{n/2} a_n \, , \quad n = 2k, \qquad (3.13)$$

$$A_n = (-)^{(n-/2)} a_n \, , \quad n = 2k+1, \qquad (3.14)$$

and likewise for the B's. All the sums needed in the above expression are worked out at the end of the paper. Putting the explicit values for all the sums we see that (3.11) is identically zero[9]. The last case to consider is $m = 2k+1 \geq 1$. In this case eq. (3.5) reduces to

$$\left[\frac{\sqrt{3}}{2} U_{2k+1\,0} + \frac{2i}{\pi} \sum_{k=0}^{\infty} \frac{(-)^n}{(2k+1)^{3/2}} (U_{2k+1\;2n+1} + \delta_{kn}) \right] , \qquad (3.15)$$

where

$$U_{nm} = (-)^{n+m} U_{mn}, \qquad (3.16)$$

and

[9]The reason we are skipping details so far is that some of the sums needed to finish the proof at later stages is more complicated and these which we shall give in detail.

$$U_{2k+1\,2n+1} = \frac{1}{2}\left(\frac{2k+1}{2}\right)^{1/2}\left(\frac{2n+1}{2}\right)^{1/2}(E(U'+\overline{U}')E)_{2k+1\;2n+1} - \frac{U_{2k+1\;0}U_{0\;2n+1}}{1-U_{0\,0}}. \tag{3.17}$$

In this case it is a tedious job to perform the sums in (3.15) due to the complicated form of the matrix $(E(U'+\overline{U}')E)$. The complication arises when $n = k$ since in this case the expression for the matrix $(E(U' + \overline{U}')E)$ becomes ill defined and one must consider a limiting procedure ($n \to k$). The explicit expression of the matrix $(E(U' + \overline{U}')E)$ for the diagonal and off diagonal elements is give in ref. [7]; however we shall replace the two expressions given in ref. [7] by a single expression valid for both the diagonal and off diagonal elements (see appendix A). First lets consider the sum over the matrix U in (3.15). It is clear from (3.17) that this sum consists of two terms , one over the nonzero elements and one over the zero elements. We call these two terms Σ_1 and Σ_2 respectively. Using the explicit value of $(E(U' + \overline{U}')E)_{2k+1\,2n+1}$, Σ_1 reduces to

$$\Sigma_1 = \frac{1}{\sqrt{2}}\left(\frac{2k+1}{2}\right)^{1/2}\frac{(-)^k}{(2k+1)}\left[2a_{2k+1}\Sigma_0^b - \frac{2}{(2k+1)}\right.$$
$$\left. + \frac{1}{(2k+1)}\sum_{n=0}^{\infty}\frac{b_1 a_{2n+1} - a_1 b_{2n+1}}{2n}\right] + \frac{1}{2}\frac{(-)^k}{(2k+1)^{3/2}}, \tag{3.18}$$

where we have made use of the identities[10]

$$\sum_{n=0}^{\infty}\frac{a_{2k+1}b_{2n+1} + b_{2k+1}a_{2n+1}}{(2k+1)+(2n+1)} = \frac{2}{2k+1}, \tag{3.19}$$

$$\sum_{n=0}^{\infty}\frac{a_{2k+1}b_{2n+1} - b_{2k+1}a_{2n+1}}{(2k+1)-(2n+1)} = \frac{1}{2k+1}\sum_{n=0}^{\infty}\frac{b_1 a_{2n+1} - a_1 b_{2n+1}}{(2n+1)-1}. \tag{3.20}$$

The sum in (3.18) is potentially divergent and must be treated with care. The term for $n = 0$ should be understood as the limit of $n \to 0$ which follow from our definition of the diagonal elements for the matrix $(E(U' + \overline{U}')E)$ (see appendix A). Hence one writes

[10]For the derivation of these identities see appendix B.

$$\sum_{n=0}^{\infty} \frac{b_1 a_{2n+1} - a_1 b_{2n+1}}{2n} = lim_{n\to 0} \frac{b_1 a_{2n+1} - a_1 b_{2n+1}}{2n} + \sum_{n=1}^{\infty} \frac{b_1 a_{2n+1} - a_1 b_{2n+1}}{2n}. \tag{3.21}$$

The numerical values of the limit and the sum in the above expression are $-1/3$ and $-2/3$ respectively (see appendices A and B). Using this identity and the fact that $\Sigma_0^b = \frac{1}{2}\pi\sqrt{3}$, Σ_1 reduces to

$$\Sigma_1 = \frac{1}{2}\pi\sqrt{3}\frac{(-)^k}{(2k+1)^{1/2}} a_{2k+1} - \frac{(-)^k}{(2k+1)^{3/2}}. \tag{3.22}$$

Using the values of the zero elements for the matrix U, Σ_2 reduces to

$$\Sigma_2 = -\frac{\pi}{4}\sqrt{3}(1+U_{00})\frac{(-)^k}{\sqrt{2k+1}} a_{2k+1}. \tag{3.23}$$

Combining Σ_1 and Σ_2 we obtain

$$\sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)^{3/2}} U_{2k+1\,2n+1} = \frac{1}{2}\pi\sqrt{3}\frac{(-)^k}{(2k+1)^{1/2}} a_{2k+1} - \frac{(-)^k}{(2k+1)^{3/2}}$$

$$-\frac{\pi}{4}\sqrt{3}U_{00}\frac{(-)^k}{\sqrt{2k+1}} a_{2k+1} \tag{3.24}$$

Now using the explicit value of $U_{2k+1\,0}$ and (3.24) we see that the expression in the square bracket in (3.15) vanish. This completes the proof of equation (3.2). It remains to verify equation (3.3). Again commuting the annihilation operators through the creation part of $|V_3^W>$ we end up with

$$-\sum_{m=0}^{\infty} [.........]\, A_m^\dagger |V_3^W>, \tag{3.25}$$

where now the expression in the squared bracket is given by

$$\frac{\sqrt{3}}{2}\left(\frac{2}{2n}\right)^{1/2}(U_{m\,2n}+\delta_{m\,2n}) - i\sum_{k=0}^{\infty} B_{2n\,2k+1}\left(\frac{2}{2k+1}\right)^{1/2}(U_{m\,2k+1}+\delta_{m\,2k+1}). \tag{3.26}$$

Now we have to show that this expression vanish for all values of $m \geq 0$. The case for $m = 0$ is a straight forward to demonstrate and there is no need to give it here. The other two cases (i.e., $m = 2l \geq 2 \;\; 2l+1 \geq 1$), however, are much harder. For $m = 2l$ the above expression reduces to

$$\frac{\sqrt{3}}{2}\left(\frac{2}{2n}\right)^{1/2}(U_{2l\,2n}+\delta_{l\,n}) - i\sum_{k=0}^{\infty} B_{2n\,2k+1}\left(\frac{2}{2k+1}\right)^{1/2} U_{2l\,2k+1}. \quad (3.27)$$

Substituting the explicit values for the matrices B and U and performing some of the sums with the help of the following identity[11]

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\frac{A(B)_{2m+1}}{2m+1\pm 2l} = \frac{(-)^n}{\pi}\left[\frac{1}{2n\mp 2l}\right.$$

$$\left.\left(\Sigma^{a(b)}_{2n} - \Sigma^{a(b)}_{\pm 2l}\right) + \frac{1}{2n\pm 2l}\left(\pm\Sigma^{a(b)}_{\pm 2l} - \Sigma^{a(b)}_{-2n}\right)\right], \;\; l \neq n, \quad (3.28)$$

and using the result in the appendices, the above expression reduces (for $l \neq n$) to

$$\frac{\sqrt{3}}{2}\left(\frac{2}{2n}\right)^{1/2}(U_{2l\,2n}+\delta_{l\,n}) + \frac{(-)^n}{\pi}\left(\frac{2l}{2}\right)^{1/2} A_{2l}$$

$$\left[\frac{1}{2l-2n}\left(\frac{3}{2}\Sigma^b_{2n} - \frac{3}{2}\Sigma^b_{2l}\right) + \frac{1}{2l+2n}\left(\frac{3}{2}\Sigma^b_{2n} - \frac{3}{2}\Sigma^b_{2l}\right)\right] - \frac{(-)^n}{\pi}\left(\frac{2l}{2}\right)^{1/2} B_{2l}$$

$$\left[\frac{1}{2l-2n}\left(\frac{3}{2}\Sigma^a_{2n} - \frac{3}{2}\Sigma^a_{2l}\right) - \frac{1}{2l+2n}\left(\frac{3}{2}\Sigma^a_{2n} + \frac{3}{2}\Sigma^a_{2l}\right)\right] + \sqrt{\frac{3}{2}}(1-U_{0\,0})\frac{A_{2n}}{2n}\frac{A_{2l}}{\sqrt{2l}}, \quad (3.29)$$

where

$$\Sigma^{a(b)}_{\pm 2n} = \sum_{m=0}^{\infty}\frac{a(b)_{2m+1}}{\pm 2n+2m+1}, \quad (3.30)$$

[11] This identity can be verified using the results in appendix A.

and we have used the identities

$$\sum_{k=0}^{\infty} B_{2n\ 2k+1}\frac{A_{2k+1}}{2k+1} = \frac{\sqrt{3}}{2}\frac{A_{2n}}{2n}, \tag{3.31}$$

$$\Sigma^a_{-2n} = -\frac{1}{2}\Sigma^a_{2n}\ , \Sigma^a_{2n} = +\frac{1}{2}\Sigma^b_{2n}, \tag{3.32}$$

which can be verified using the integral representation of the coefficients $a's$ and $b's$. All other sums involved above are given in ref. [7] and appendices A and B. Recalling that

$$U_{2l\ 2n} = U'_{2l\ 2n} - \frac{U_{2l\ 0}U_{0\ 2n}}{1-U_{0\ 0}}, \tag{3.33}$$

where

$$U'_{2l\ 2n} = \frac{1}{2}\sqrt{\frac{2l}{2}}\sqrt{\frac{2n}{2}}(E(U'+\overline{U}')E)_{2l\ 2n}, \tag{3.34}$$

we see after some algebra that the expression in (3.29) vanish. To complete this case we still need to consider $n = l$. For this case the expression in (3.5) reduces to

$$\frac{\sqrt{3}}{2}\left(\frac{2}{2n}\right)^{1/2}(U_{2n\ 2n}+1) - i\sum_{k=0}^{\infty} B_{2n\ 2k+1}\left(\frac{2}{2k+1}\right)^{1/2}U_{2n\ 2k+1}. \tag{3.35}$$

This expression involves many sums of the form $\Sigma^{a(b)}_{\pm 2n}$ and $\tilde{\Sigma}^{a(b)}_{\pm 2n}$. $\Sigma^{a(b)}_{\pm 2n}$ has been considered previously and

$$\tilde{\Sigma}^{a(b)}_{\pm 2n} = \sum_{m=0}^{\infty}\frac{a(b)_{2m+1}}{(\pm 2n+2m+1)^2}. \tag{3.36}$$

These sums can be found explicitly (see refs. [7]) however, in this case we need not evaluate them since the ones coming from the expansion of the second term over U in the above expression cancel against the ones coming from the $U_{2n\ 2n}$ in the first term. Thus if one expands the second term and substitute the explicit values for the U matrix (these are given in appendix A); then with the help of the identity[12]

$$\sum_{m=0}^{\infty} B_{2n\,2m+1} \frac{A(B)_{2m+1}}{\pm 2n + (2m+1)} = \frac{(-)^n}{\pi} \left[\frac{1}{2(2n)} \left(\Sigma_{2n}^{a(b)} - \Sigma_{-2n}^{a(b)} \right) - \tilde{\Sigma}_{\pm 2n}^{a(b)} \right], \tag{3.37}$$

the above expression reduces to

$$-\frac{\sqrt{3}}{4} \left(\frac{2}{2n} \right)^{1/2} a_{2n} b_{2n} + \frac{1}{2\pi} \left(\frac{2}{2n} \right)^{1/2} b_{2n} \left(\Sigma_{2n}^{a} - \Sigma_{-2n}^{a} \right), \tag{3.38}$$

which is identically zero, since

$$\Sigma_{-2n}^{a} = -\frac{1}{2} \Sigma_{2n}^{a} = -\frac{1}{2} \sqrt{\frac{1}{3}} \pi a_{2n}, \tag{3.39}$$

as can easily bee seen. Hence (3.27) vanish for all values of $m = 2l \geq 2$. It remains to consider $m = 2l+1 \geq 1$. For $m = 2l+1 \geq 1$, (3.26) reduces to

$$\frac{\sqrt{3}}{2} \left(\frac{2}{2n} \right)^{1/2} U_{2l+1\ 2n} - i \sum_{k=0}^{\infty} B_{2n\ 2k+1} \left(\frac{2}{2k+1} \right)^{1/2} \left(U_{2l+1\ 2k+1} + \delta_{l\ k} \right). \tag{3.40}$$

The sum involving the U matrix is very delicate when k takes the value l. We will see how to handle it below. Using the explicit values of U, we get

$$\sum_{k=0}^{\infty} B_{2n\ 2k+1} \left(\frac{2}{2k+1} \right)^{1/2} U_{2l+1\ 2k+1} = \sum_{k=0}^{\infty} B_{2n\ 2k+1} \left(\frac{2}{2k+1} \right)^{1/2}$$

$$\left[\frac{1}{2} \left(\frac{2l+1}{2} \right)^{1/2} \left(\frac{2k+1}{2} \right)^{1/2} (E(U' + \overline{U}')E)_{2l+1\ 2k+1} - \frac{U_{2l+1\ 0} U_{0\ 2k+1}}{1 - U_{0\ 0}} \right]$$

$$= T_1 + T_2. \tag{3.41}$$

[12]This identity can be verified using the results of ref. [7].

To evaluate the above sums we make use of the following identity[13]

$$\sum_{m=0}^{\infty} B_{2n\,2m+1} \frac{A(B)_{2m+1}}{(2m+1) \pm (2k+1)} = \frac{(-)^n}{\pi}$$

$$\left[\frac{1}{2n \mp (2k+1)} \Sigma_{2n}^{a(b)} - \frac{1}{2n \pm (2k+1)} \Sigma_{-2n}^{a(b)} - \left(\frac{1}{2n \mp (2k+1)} - \frac{1}{2n \pm (2k+1)}\right) S_{\pm(2k+1)}^{a(b)}\right], \tag{3.42}$$

where

$$S_{\pm(2k+1)}^{a(b)} = \sum_{n=0}^{\infty} \frac{a(b)_{2n+1}}{(2n+1) \pm (2k+1)}, \tag{3.43}$$

and identities given in (3.19), (3.20) and (3.21). Hence

$$T_1 = \left(\frac{2l+1}{2}\right)^{1/2} \left[\frac{\sqrt{3}}{2}\pi \left(a_{2l+1} b_{2n} + \frac{2n}{2l+1} b_{2l+1} a_{2n}\right) - \frac{3}{2l+1}\right] B_{2n\,2l+1}, \tag{3.44}$$

$$T_2 = -\left(\frac{3}{2}\right)^{1/2} \frac{(-)^{n+l}}{\pi} (1 - U_{00}) \frac{a_{2l+1} a_{2n}}{(2l+1)^{1/2}(2n)}. \tag{3.45}$$

The values of the matrix $U_{2l+1\,2n}$ is given in appendix A. Putting everything in (3.40) we see that it is identically zero. This completes the proof for the coordinate overlaps.

Momentum Overlaps

The momentum overlaps in the comma theory are given by (2.10) and (2.11). The second equation is the same as the overlap equation of the identity vertex and therefore the proof follows from the form of the vertex. Now the overlap condition on $\mathcal{P}^r(\sigma)$, given by eq. (2.10), implies that its Fourier components satisfy

[13]The derivation of this identity is quite complicated; it makes use of appendix A and the identities already derived in appendix B. The complication arises when m takes the value k, since for this term the expression becomes ill defined. See appendix A for handling such terms.

$$\left[P^L_{2n} + e^{2i\pi/3} P^R_{2n}\right] |V_3 >= 0 \quad n \geq 0, \tag{3.46}$$

as well as their complex conjugate. The proof of the case for $n = 0$ does not involve new identities other than those used to prove the coordinate part and therefore there is no need to give it here although we have checked that the overlap equation for the zero mode is indeed satisfied. However, for $n \geq 1$ the proof requires deriving new identities . Using the change of representation formulas (see ref. [5]), eq. (3.46) reduces for $n \geq 1$ to

$$\frac{1}{2} e^{i\pi/3} \sum_{k=0}^{\infty} [\ldots\ldots\ldots\ldots] \, A^\dagger_3 |V^W_3 >, \tag{3.47}$$

where the expression in the square bracket is

$$\left(\frac{2n}{2}\right)^{1/2} (-U_{k\,2n} + \delta_{k\,2n}) - 2i\sqrt{3} \sum_{m=0}^{\infty} B_{2n\,2m+1} \left(\frac{2m+1}{2}\right)^{1/2}$$

$$\times \left(U_{k\,2m+1} - \delta_{k\,2m+1}\right) . \tag{3.48}$$

In order for (3.47) to vanish, (3.48) must vanish for all values of k since the states $A^\dagger_k |V^W_3 >$ are all linearly independent. Therefore to prove the overlap equation (3.46), it is sufficient to show that (3.48) is identically zero for all values of k. For $k = 0$, (3.48) is zero as can be easily seen by explicit substitution (no new identities are needed here). The other two case, i.e., $k = even \geq 2$ and $k = odd \geq 1$ need careful treatment since they involve quantities which are potentially divergent. Setting $k = 2l$ in (3.48) we obtain

$$-\left(\frac{2n}{2}\right)^{1/2} (U_{2l\,2n} - \delta_{l\,n}) - 2i\sqrt{3} \sum_{m=0}^{\infty} B_{2n\,2m+1} \left(\frac{2m+1}{2}\right)^{1/2} U_{2l\,2m+1}. \tag{3.49}$$

First let us consider $l \neq n$ since it is the easier of the two case to prove. Then if one substitute the explicit values of the matrix U for $l \neq n$ and make use of the following identity[14]

[14]This identity can easily be derive using the fact $m/(n \pm m) = \pm(1 - n/(n \pm m))$ and the results of appendix A.

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\frac{(2m+1)}{2l\pm(2m+1)}A(B)_{2m+1}=\frac{(-)^n}{\pi}\left[\mp 2\Sigma^{a(b)}_{\pm 2l}+\frac{2n}{2l-2n}\right.$$

$$\left.\left(\pm\Sigma^{a(b)}_{\pm 2n}\mp\Sigma^{a(b)}_{\pm 2l}\right)+\frac{2n}{2l+2n}\left(\pm\Sigma^{a(b)}_{\pm 2l}-\Sigma^{a(b)}_{\mp 2n}\right)\right],\quad l\neq n, \tag{3.50}$$

(after a lengthy, otherwise a straight forward exercise) we see that (3.49) is indeed zero. The case $l=n$ contains potentially divergent terms and requires proving new identities involving the sums of the Taylor modes. Substituting the explicit values of the matrix U in (3.49) and making use of the identity[15],

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\frac{(2m+1)}{2n\pm(2m+1)}A(B)_{2m+1}=\frac{(-)^n}{\pi}\left[(2n)\tilde{\Sigma}^{a(b)}_{\pm 2n}\mp\frac{3}{2}\Sigma^{a(b)}_{\pm 2n}\mp\Sigma^{a(b)}_{\mp 2n}\right], \tag{3.51}$$

we get

$$2\left(\frac{2n}{2}\right)^{1/2}\left[1+\frac{\sqrt{3}}{\pi}\frac{2n}{2}\left(\frac{2}{3}b_{2n}\tilde{\Sigma}^a_{2n}-\frac{4}{3}b_{2n}\tilde{\Sigma}^a_{-2n}-\frac{2}{3}a_{2n}\tilde{\Sigma}^b_{2n}-\frac{4}{3}a_{2n}\tilde{\Sigma}^b_{-2n}\right)\right], \tag{3.52}$$

for $l=n$. The values of $\tilde{\Sigma}$ for $n\leq 0$ are related to the values of $\tilde{\Sigma}$ for $n\geq 0$ through the following identities (the proof is given in appendix B)

$$\tilde{\Sigma}^a_{-n}-\frac{1}{2}\tilde{\Sigma}^a_n=\frac{3}{2}\Sigma^a_0 S^a_n, \tag{3.53}$$

$$\tilde{\Sigma}^b_{-n}+\frac{1}{2}\tilde{\Sigma}^b_n=\frac{1}{2}\Sigma^b_0 S^b_n. \tag{3.54}$$

Hence (3.52) reduces to

$$2\left(\frac{2n}{2}\right)^{1/2}\left[1-\frac{2n}{2}\left(b_{2n}S^a_{2n}+a_{2n}S^b_{2n}\right)\right]. \tag{3.55}$$

[15] See previous footnote.

Now using the identity (see appendix B for derivation)

$$a_{2n} S^b_{2n} + b_{2n} S^a_{2n} = \frac{2}{2n}, \tag{3.56}$$

we see that (3.55) is identically zero. This completes the proof for $k = 2l \geq 2$. Next we consider $k = odd = 2l + 1 \geq 1$. For this case (3.48) reduces to

$$-\left(\frac{2n}{2}\right)^{1/2} U_{2l+1\,2n} - 2i\sqrt{3}\sum_{m=0}^{\infty} B_{2n\,2m+1}\left(\frac{2m+1}{2}\right)^{1/2}(U_{2l+1\,2m+1} - \delta_{l\,m})\,. \tag{3.57}$$

The explicit value of the matrix element $U_{2l+1\,2n}$ is given in appendix A. The sum over U has not been considered before and therefore need to be evaluated. However, in performing the sum over U one has to be extra careful when m takes the value l, since this term is potentially divergent. Consider the sum over the matrix U in (3.57) , i.e.,

$$\sum_{m=0}^{\infty} B_{2n\,2m+1}\left(\frac{2m+1}{2}\right)^{1/2} U_{2l+1\,2m+1} = \frac{1}{4}\left(\frac{2l+1}{2}\right)^{1/2}\Sigma_I$$

$$+\frac{1}{2}\left(\frac{2l+1}{2}\right)^{1/2} B_{2n\,2l+1} - \left(\frac{1}{2}\right)^{1/2}\frac{U_{2l+1\,0}}{1-U_{00}}\Sigma_{II}, \tag{3.58}$$

where

$$\Sigma_I = \sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)(E(U' + \overline{U}')E)_{2l+1\,2m+1}, \tag{3.59}$$

and

$$\Sigma_{II} = \sum_{m=0}^{\infty} B_{2n\,2m+1}(2m+1)^{1/2} U_{0\,2m+1}, \tag{3.60}$$

Σ_{II} can be easily evaluated using the identity

$$\sum_{m=0}^{\infty} B_{2n\,2m+1} A_{2m+1} = -\frac{1}{2\sqrt{3}} A_{2n}, \tag{3.61}$$

which can be checked by converting the sum into integral form and then evaluating it using the theory of special functions (we will not do it here since this is relatively easy to do). Hence

$$\left(\frac{1}{2}\right)^{1/2} \frac{U_{2l+1\,0}}{1-U_{00}} \Sigma_{II} = -\frac{1}{4\sqrt{3}}(1-U_{00}) \left(\frac{2}{2l+1}\right)^{1/2} A_{2l+1} A_{2n}. \qquad (3.62)$$

Next we consider Σ_I. Substituting the explicit values of the matrices B and U in (3.59) and then making use of the identities (3.19), (3.20) and (3.61) we get (skipping a rather tedious algebra)

$$\frac{1}{4}\left(\frac{2l+1}{2}\right)^{1/2} \Sigma_I = \frac{1}{2}\left(\frac{2l+1}{2}\right)^{1/2} \left\{ \frac{1}{\sqrt{3}} \left[(-)^{n+l+1} b_{2l+1} a_{2n} \right.\right.$$

$$\left.\left. +\pi \left(\frac{2l+1}{2}\right) a_{2l+1} b_{2n} B_{2l+1\,2n} \right] - \left[\frac{\pi}{\sqrt{3}} \left(\frac{2l+1}{2}\right) b_{2l+1} a_{2n} - 1 \right] B_{2n\,2l+1} \right\}. \qquad (3.63)$$

Putting everything together and making use of the fact

$$n\, B_{nm} = -m\, B_{mn}, \qquad (3.64)$$

we see that (3.57) is indeed zero. This completes the proof of the P overlaps.

Ghost Overlaps

To complete the proof we have to see if the ghost part of the Witten's' *cubic* vertex satisfies (Violates) the comma overlaps in exactly the same way as in the standard formulation. The ghost part of the Witten's vertex is given by

$$|V_3^{\phi}> = e^{\frac{3}{2}i\phi(\pi/2)} |V_3^{\phi,0}>, \qquad (3.65)$$

where $|V_3^{\phi,0}>$ has the same form as the orbital part of the vertex. The ghost factor $e^{\frac{3}{2}i\phi(\pi/2)}$ corresponding to ghost number 3/2 is the right ghost number [2]. Expanding the phase factor and commuting the annihilation operator through the creation part of the vertex results in doubling the creation part of the insertion. Thus one has

$$|V_3^{\phi}>= exp\left(\sqrt{3}\sum_{n=1}^{\infty}\frac{(-)^n}{\sqrt{2n}}A^{\dagger}_{3\ 2n}\right)|V_3^{\phi,0}> . \tag{3.66}$$

The quadratic part of the vertex $|V_3^{\phi,0}>$ satisfies the comma overlap equations since it has the same structure as the orbital part which solves the comma overlaps as we have already seen. However, when one includes the ghost insertion this is no longer the case. To see this one first observes that the comma overlaps for V_3 are blind to the phase factor[16] (insertion) apart from

$$\mathcal{Q}^L_{3\ 2n} = \mathcal{Q}^R_{3\ 2n}\,, \quad n \geq 0, \tag{3.67}$$

$$\mathcal{P}^L_{3\ 2n} = -\mathcal{P}^R_{3\ 2n}\,, \quad n \geq 0. \tag{3.68}$$

In fact the first of these equations is also blind to the insertion factor, since it contains only odd modes in the annihilation-creation operator A_3 which clearly commute with the even modes in the phase factor. On the other hand, the second equation contains even modes of the operator A_3 and therefore is not satisfied by the vertex due to the insertion. To see this notice that

$$\mathcal{P}^r_{3\ 2n} exp\left(\sqrt{3}\sum_{n=1}^{\infty}\frac{(-)^n}{\sqrt{2n}}A^{\dagger}_{3\ 2n}\right) = exp\left(\sqrt{3}\sum_{n=1}^{\infty}\frac{(-)^n}{\sqrt{2n}}A^{\dagger}_{3\ 2n}\right)\left[-\frac{\sqrt{3}}{2}\frac{(-)^n}{\sqrt{2n}} + \mathcal{P}^r_{3\ 2n}\right], \tag{3.69}$$

where $r = 1, 2$ refers to the left and right parts of the string respectively. Thus commuting the overlaps through the insertion factor and collecting terms we obtain

$$exp\left(\sqrt{3}\sum_{n=1}^{\infty}\frac{(-)^n}{\sqrt{2n}}A^{\dagger}_{3\ 2n}\right)\left[-\sqrt{3}\frac{(-)^n}{\sqrt{2n}} + +\mathcal{P}^L_{3\ 2n} = -\mathcal{P}^R_{3\ 2n}\right]. \tag{3.70}$$

Now it is clear that the overlaps in the square bracket are not satisfied by the quadratic part of the ghost vertex because of the presence of a

[16]The reason for this is that the other overlaps describe different strings in the complex space as we have already seen.

$c - number$. This is the same violation seen in the operator formulation of Witten's string field theory (see ref. [7]). Therefore the comma overlaps are satisfied (violated) by the Witten's *cubic* vertex in exactly the same way as in the case of the standard formulation[8, 9, 7, 10, 11]. This completes the proof that the Witten's *cubic* vertex is a solution to the comma overlaps.

4 Conclusion

We have demonstrated that the *cubic* vertex in Witten's string field theory, making use of the operator formalism as developed in ref. [8, 9, 7, 10, 11], solves the comma overlaps. However, as we have seen in [5], there are still few subtleties which must be understood if one is to get greater confidence in the comma approach to string field theory. For example, as pointed out in ref. [5], while in the full string formulation, the K and the $BRST$ invariances require some specific ghost insertions at the midpoint of the string for consistency, it does not seem to be the case in the comma formalism. In the comma theory both the orbital and the ghost parts of the *cubic* vertex are invariant separately! Also in the same ref. We have seen that the associativity anomaly in the star algebra of the standard formulation of string field theory disappears in the comma theory. The Q or gauge invariance in the comma representation have only been mentioned in ref. [5]. However, for a complete discussion, it is useful to use a fermionic representation of the comma ghost. In the fermionic representation it is possible to construct the analogous comma (ghost) *cubic* vertex and examine other properties of standard string theory in the comma representation, For consistency, we still have to show that the *cubic* ghost vertex in the fermionic representation of Witten string field theory still solves the comma overlaps of the ghost in the fermionic representation. This and Q will be given in a separate publication [12].

Appendix A

The coefficients a_n and b_n. In the first part of this appendix we give the properties of the coefficients in the Taylor series expansion of the functions

$$\left(\frac{1+x}{1-x}\right)^{1/3} = \sum_{n=0}^{\infty} a_n x^n, \tag{A.1}$$

$$\left(\frac{1+x}{1-x}\right)^{2/3} = \sum_{n=0}^{\infty} b_n x^n. \tag{A.2}$$

They are instrumental in the proof of the overlap equation. Most of the results listed here are derived in ref. [7], or follow from results given there. The integral form of the coefficients a_n is given by

$$a_n = \frac{1}{2\pi i}\oint_0 dz \frac{1}{z^{n+1}} \left(\frac{1+z}{1-z}\right)^{1/3}, \tag{A.3}$$

and likewise for the b_n with $1/3 \to 2/3$. This form can be utilized to derive the various recursion relations satisfied by the Taylor modes. Integrating (A.3) by parts leads to

$$(n+1)a_{n+1} - \frac{2}{3}a_n - (n-1)a_{n-1} = 0, \tag{A.4}$$

likewise we get for b_n

$$(n+1)b_{n+1} - \frac{4}{3}b_n - (n-1)b_{n-1} = 0. \tag{A.5}$$

Making use of the same integral representation, one could derive the cross-recursion relations

$$\frac{4}{3}a_n = (-)^n \left[(n+1)b_{n+1} - 2nb_n + (n-1)b_{n-1}\right], \tag{A.6}$$

$$\frac{2}{3}b_n = (-)^n \left[(n+1)a_{n+1} - 2na_n + (n-1)a_{n-1}\right]. \tag{A.7}$$

In the text we meet various sums involving these coefficients. The primary ones being

$$\Sigma_n^a \equiv \sum_{n+m=odd} \frac{a_n}{n+m}, \tag{A.8}$$

$$S_n^a \equiv \sum_{n+m=even} \frac{a_n}{n+m}, \tag{A.9}$$

and analogously for $a \to b$. All these sums are absolutely convergent. All the sums given above have been evaluated in ref. [7]. Here we merely quote the results. For the sums labeled by Σ the results are

$$\Sigma_0^a = \frac{1}{2}\sqrt{\frac{1}{3}}\pi, \quad \Sigma_n^a = \sqrt{\frac{1}{3}}\pi a_n, \quad \Sigma_0^a = -\frac{1}{2}\Sigma_n^a, \tag{A.10}$$

$$\Sigma_0^b = \frac{1}{2}\pi\sqrt{3}, \quad \Sigma_n^b = \sqrt{\frac{1}{3}}\pi b_n, \quad \Sigma_0^b = \frac{1}{2}\Sigma_n^b, \tag{A.11}$$

where n is a positive integer. The results for the sums labeled by S are are given by

$$S_n^a = \left(\frac{3}{2} - ln2)\right) a_n + \frac{3}{2}\sum_{k=0}^{n-1}(-)^k \frac{a_n a_{n-k-1}}{k+1}, \tag{A.12}$$

$$S_n^b = \left(\frac{3}{4} + ln2)\right) b_n + \frac{3}{4}\sum_{k=0}^{n-1}(-)^k \frac{b_n b_{n-k-1}}{k+1}, \tag{A.13}$$

for $n > 1$. For $n = 1$, the results are given by the same expressions without the summations over k. The S_n^a and S_n^b satisfy the same recursion relations as the a_n and b_n respectively

$$(n+1)S_{n+1}^a - \frac{2}{3}S_n^a - (n-1)S_{n-1}^a = 0, \tag{A.14}$$

$$(n+1)S_{n+1}^b - \frac{4}{3}S_n^b - (n-1)S_{n-1}^b = 0, \tag{A.15}$$

for $n > 1$. Another sum involving the Taylor modes which appears in the text is

$$\tilde{\Sigma}^a \equiv \sum_{n+m=even} \frac{a_n}{(n+m)^2}, \tag{A.16}$$

and like wise for b with $(a \to b)$. The values of these sums are given in ref. [7] and we shall not reproduce them here. However, it is important to notice that these sums satisfy the following recursion relations

$$(n+1)\tilde{\Sigma}^a_{n+1} = \frac{2}{3}\tilde{\Sigma}^a_n + (n-1)\tilde{\Sigma}^a_{n-1} + \Sigma^a_{n+1} - \Sigma^a_{n-1}, \tag{A.17}$$

$$(n+1)\tilde{\Sigma}^b_{n+1} = \frac{4}{3}\tilde{\Sigma}^b_n + (n-1)\tilde{\Sigma}^b_{n-1} + \Sigma^b_{n+1} - \Sigma^b_{n-1}. \tag{A.18}$$

The matrix elements U_{nm}. We shall first give the explicit values of the matrix elements $U_{n\,m}$ appearing in Witten's vertex. They have been derived explicitly in ref. [7]. The matrices U and U' are related by

$$U_{n\,m} = U'_{n\,m} - \frac{U_{n\,0}U_{0\,m}}{1-U_{0\,0}}, \tag{A.19}$$

where

$$U'_{n\,m} = \frac{1}{2}\sqrt{\frac{n}{2}}\sqrt{\frac{m}{2}}(E(U'+\overline{U}')E)_{n\,m}, \tag{A.20}$$

and

$$(E(U'+\overline{U}')E)_{n\,m} = 2(-)^{n+1}\left(\frac{A_nB_m + B_nA_m}{n+m} + \frac{A_nB_m - B_nA_m}{n-m}\right), \tag{A.21}$$

for $n+m = even$ and $n, m \geq 1$,

$$(E(U'-\overline{U}')E)_{n\,m} = -2i\left(\frac{A_nB_m - B_nA_m}{n+m} + \frac{A_nB_m + B_nA_m}{n-m}\right), \tag{A.22}$$

for $n+m = odd$ and $n, m \geq 1$. For the zero mode components

$$(E(U'+\overline{U}')E)_{0\,m} = -2\sqrt{2}\left(\frac{A_m}{m}\right), \quad m = 2k, \tag{A.23}$$

$$(E(U'-\overline{U}')E)_{0\,m} = i2\sqrt{2}\left(\frac{A_m}{m}\right), \quad m = 2k+1. \tag{A.24}$$

The diagonal elements, $(E(U'-\overline{U}')E)_{n\,n}=0$ while $(E(U'+\overline{U}')E)_{n\,n}$ are given by

$$(E(U'+\overline{U}')E)_{n\,n}=-2\left[(-)^n\frac{A_nB_n+1}{n}+\Delta_n\right], \qquad (A.25)$$

where

$$\Delta_{n=2k}=\frac{1}{n}-\frac{2}{\sqrt{3}}\frac{(-)^{n/2}}{\pi}\left[\left(\tilde{\Sigma}^a_n+\tilde{\Sigma}^a_{-n}\right)B_n-\left(\tilde{\Sigma}^b_n-\tilde{\Sigma}^b_{-n}\right)A_n\right], \qquad (A.26)$$

and

$$\Delta_{n=2k+1}=\frac{3}{n}-2\sqrt{3}\frac{(-)^{(n-1)/2}}{\pi}\left[\left(\tilde{\Sigma}^b_n+\tilde{\Sigma}^b_{-n}\right)A_n-\left(\tilde{\Sigma}^a_n-\tilde{\Sigma}^a_{-n}\right)B_n\right]. \qquad (A.27)$$

The symbol $\tilde{\Sigma}^{a(b)}_n$ has been introduced before. For completeness we also give the matrix elements of E. They are

$$(E^{-1})_{n\,m}=\sqrt{\frac{n}{2}}\delta_{n\,m}+\delta_{n\,0}\delta_{m\,0}. \qquad (A.28)$$

It is worth observing that expressions for the matrix elements of U for the off diagonal elements defined by (A.21) for $n=odd$, $m=odd$ and the diagonal elements defined by (A.25) for $n=odd$ can be combined into a single expression. This observation is instrumental in making the evaluation of the sums much easier. To derive a single expression for U we need the value of Δ_1. Setting $n=1$ in (A.27) gives

$$\Delta_1=3-\frac{2\sqrt{3}}{\pi}\left[\left(\tilde{\Sigma}^b_1+\tilde{\Sigma}^b_{-1}\right)A_1-\left(\tilde{\Sigma}^a_1-\tilde{\Sigma}^a_{-1}\right)B_1\right]. \qquad (A.29)$$

The sums in the above expression are easily converted into integrals using the integral representation of the coefficients a_n and b_n. For example

$$\sum_{m=0}^{\infty}\frac{a_{2m}}{(2m+1)^2}=1+\sum_{m=1}^{\infty}\frac{a_{2m}}{(2m+1)^2}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} \frac{1}{2\pi i} \oint_0 dz \frac{1}{z^{2m+1}} \left(\frac{1+z}{1-z}\right)^{1/3}$$

$$= 1 + \frac{\sqrt{3}}{2\pi} \int_1^{\infty} dx \int_0^1 \frac{dy}{y} \left[\frac{1}{2} \ln\left(\frac{x+y}{x-y}\right) - \frac{y}{x}\right]$$

$$\left[\left(\frac{x+1}{x-1}\right)^{1/3} - \left(\frac{x+1}{x-1}\right)^{-1/3}\right], \qquad (A.30)$$

making the change of variables $t = (x+1)/(x-1)$ followed by the change variables $x = 1/t$ the above expression reduces to

$$1 + \frac{\sqrt{3}}{\pi} \int_0^1 dx \left(x^{1/3} - x^{-1/3}\right) \left\{\frac{1}{(1-x)(1+x)} - \frac{1}{2} \frac{1}{(1-x)^2}\right.$$

$$\left.\int_0^1 \frac{dy}{y} \ln\left(\frac{\alpha(y) + x\gamma(y)}{\gamma(y) + x\alpha(y)}\right) (x^{1/3} - x^{-1/3})\right\}, \qquad (A.31)$$

where $\alpha(y) = 1 + y$ and $\gamma(y) = 1 - y$. Similarly one rewrite all the sums in integral forms. Converting all the sums in the expression for Δ_1 to integrals and making use of the fact,

$$\frac{1}{(a+bx)^2} = -\frac{1}{b}\left(\frac{1}{a+bx}\right)', \qquad (A.32)$$

the expression for Δ_1, after performing several integrations (by parts[17]), reduces to

$$\Delta_1 = 3 - \frac{4}{\pi^2}\left[\psi'(4/3) - \beta'(4/3) - \beta'(2/3) + \psi'(2/3)\right] = \frac{1}{3}. \qquad (A.33)$$

Now the value of Δ_1 is used to calculate the value of $(E(U' + \overline{U}')E)_{11}$ which is needed as a boundary condition for $(E(U' + \overline{U}')E)_{n\; n=odd}$. Skipping the details, we get

[17]When integrating by parts it is crucial that one takes the limits at the end of the calculation to preserve all the divergent pieces. The divergent pieces will cancel among them selves rendering the whole thing finite.

$$(E(U'+\overline{U}')E)_{n\,m} = 2(-)^{n+1}\left(\frac{A_nB_m + B_nA_m}{n+m} + \frac{A_nB_m - B_nA_m}{n-m}\right)$$

$$+\left(\frac{2}{n}\right)^{1/2}\left(\frac{2}{m}\right)^{1/2}\delta_{nm}, \tag{A.34}$$

for $n = odd$, $m = odd$ which is the case of interest[18]. It is clear that for the off diagonal elements the above expression reduces to that in (A.21). For the diagonal elements it is understood that one must consider a limiting procedure ($m \to n$). If one takes the limit $(m \to n)$ one recovers the expression in (A.25) for $n = odd$. Below we shall give the relevant limit needed to recover (A.25) for $n = odd$. Consider

$$\lim_{n\to m}\frac{a_{2n-1}b_{2m-1} - b_{2n-1}a_{2m-1}}{(2n-1)-(2m-1)}. \tag{A.35}$$

Writing the above expression in integral representation we get

$$\lim_{\epsilon\to 0}\left(\frac{sin\pi/3}{\pi}\right)^2 \frac{1}{2\epsilon}\int_1^\infty\int_1^\infty dxdy\left(\frac{1}{x^{2m+2\epsilon}}\frac{1}{y^{2m}} - \frac{1}{x^{2m}}\frac{1}{y^{2m+2\epsilon}}\right)$$

$$\left[\left(\frac{x+1}{x-1}\right)^{1/3} + \left(\frac{x+1}{x-1}\right)^{-1/3}\right]\left[\left(\frac{y+1}{x-1}\right)^{2/3} + \left(\frac{y+1}{y-1}\right)^{-2/3}\right]. \tag{A.36}$$

Using the fact

$$\lim_{\epsilon\to 0}\frac{x^{-2\epsilon} - y^{-2\epsilon}}{2\epsilon} = ln\left(\frac{y}{x}\right), \tag{A.37}$$

the above expression reduced to

$$-\frac{\sqrt{3}}{2\pi}b_{2m-1}\int_1^\infty \frac{dx}{x^{2m}}lnx\left[\left(\frac{x+1}{x-1}\right)^{1/3} + \left(\frac{x+1}{x-1}\right)^{-1/3}\right]$$

[18]For $n = even$, $m = even$ it is more convenient to use the expressions given by (A.21) and (A.25).

$$+\frac{\sqrt{3}}{2\pi}a_{2m-1}\int_1^\infty \frac{dx}{x^{2m}}lnx\left[\left(\frac{x+1}{x-1}\right)^{2/3}+\left(\frac{x+1}{x-1}\right)^{-2/3}\right]$$

$$=-\frac{\sqrt{3}}{\pi}b_{2m-1}\sum_{n=0}^{\infty}a_{2n}\int_1^\infty \frac{dx}{x^{2(m+n)}}lnx+\frac{\sqrt{3}}{\pi}a_{2m-1}\sum_{n=0}^{\infty}b_{2n}\int_1^\infty \frac{dx}{x^{2(m+n)}}lnx$$

$$=\frac{\sqrt{3}}{\pi}\left[a_{2m-1}\tilde{\Sigma}^b_{2m-1}-b_{2m-1}\tilde{\Sigma}^a_{2m-1}\right]. \qquad \text{(A.38)}$$

A particular case of interest is the value of the limit when $m=1$. In this case, eq. (A.38) gives

$$\lim_{n\to 0}\frac{b_1a_{2n-1}-a_1b_{2n-1}}{2n}=\frac{\sqrt{3}}{\pi}\left[a_1\tilde{\Sigma}^b_1-b_1\tilde{\Sigma}^a_1\right]=-\frac{1}{3}. \qquad \text{(A.39)}$$

To see this, let us first consider $\tilde{\Sigma}^a_1$. Writing $\tilde{\Sigma}^a_1$ in integral form we obtain

$$\tilde{\Sigma}^a_1=-\frac{1}{2}\frac{d}{dk}I(k,q)|_{k=0,q=1/3}, \qquad \text{(A.40)}$$

where

$$I(k,q)=\int_0^1 dtt^k\left(\frac{1+t}{1-t}\right)^q+\int_0^1 dtt^k\left(\frac{1+t}{1-t}\right)^{-q}. \qquad \text{(A.41)}$$

This integral is easily evaluated using the *hypergeometric* function

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dtt^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}. \qquad \text{(A.42)}$$

Using (A.42), and after some algebraic manipulation of the the gamma functions, eq. (A.40) gives

$$\tilde{\Sigma}^a_1=\frac{\pi}{\sqrt{3}}\left(ln\frac{3}{2}+\frac{1}{6}\right). \qquad \text{(A.43)}$$

Similar steps give

$$\tilde{\Sigma}^b_1=\frac{2\pi}{\sqrt{3}}\left(ln\frac{3}{2}-\frac{1}{12}\right). \qquad \text{(A.44)}$$

Substituting (A.43) and (A.44) in the LHS of (A.39) we obtain the desired result.

Appendix B

Sums involving a_n and b_n continue . A particular combination of sums which appears in the text is

$$a_1 \sum_{n=1}^{\infty} \frac{b_{2n+1}}{1-(2n+1)} - b_1 \sum_{n=1}^{\infty} \frac{a_{2n+1}}{1-(2n+1)}, \tag{B.1}$$

which when written in integral form becomes

$$-\frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{2n} \frac{1}{2\pi i} \oint_0 \frac{dz}{z^{2n+2}} \left[\left(\frac{1+z}{1-z}\right)^{2/3} - 2\left(\frac{1+z}{1-z}\right)^{1/3} \right]. \tag{B.2}$$

Deforming the contour and picking up the integrals along the cuts we obtain

$$\frac{1}{\pi\sqrt{3}} \int_1^{\infty} \frac{dx}{x^2} \ln\left(\frac{x^2}{x^2-1}\right) \left[\left(\frac{x+1}{x-1}\right)^{2/3} + \left(\frac{x+1}{x-1}\right)^{-2/3} - 2\left(\frac{x+1}{x-1}\right)^{1/3} - 2\left(\frac{x+1}{x-1}\right)^{-1/3} \right]. \tag{B.3}$$

Employing the change of variables $t = (x+1)/(x-1)$ followed by the change of variables $x = 1/t$ the whole expression simplifies

$$-\frac{1}{\pi\sqrt{3}} \int_0^1 \frac{dx}{(x+1)^2} \ln\left[\frac{(x+1)^2}{4x}\right] \left(x^{2/3} + x^{-2/3} - 2x^{1/3} - 2x^{-1/3}\right). \tag{B.4}$$

This expression can be further reduced by integration by parts[19]. Doing all that, we end up with

$$-\frac{1}{\pi\sqrt{3}} \left[\frac{9}{2} - \beta(1/3) + 2\beta(2/3) - 2\beta(4/3) + \beta(5/3)\right] = -\frac{2}{3}. \tag{B.5}$$

[19]When integrating by parts one has to be careful since integration by parts give rise to many divergent terms as one approaches the branch points. These divergent terms have different magnitudes and one must keep track of all of them as they conspire at the end to give a finite number.

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Deriving various identities used in the text. First lets derive the identity in (3.19). Consider

$$W_{mn} = W_{nm} \equiv \frac{a_m b_n + b_m a_n}{m+n}. \tag{B.6}$$

Now it is not hard to see (by direct substitution) that

$$(m+1)W_{m+1\,n} - (m-1)W_{m-1\,n} + (n+1)W_{m\,n+1} - (n-1)W_{m\,n-1} = 0. \tag{B.7}$$

for $m+n =$ odd integer. Letting $m \to 2m-1$, $n \to 2n$ and suming over $m \geq 1$ we get

$$(2n+1)W_{2n+1} = (2n-1)W_{2n-1}, \tag{B.8}$$

where

$$W_{n=odd} = \sum_{m=1}^{\infty} \frac{a_n b_{2m-1} + b_n a_{2m-1}}{(2m-1)+n}. \tag{B.9}$$

From the recursion relation (B.8) it follows that

$$W_{2n-1} = \frac{W_1}{2n-1}, \tag{B.10}$$

where

$$W_1 = \sum_{m=1}^{\infty} \frac{a_1 b_{2m-1} + b_1 a_{2m-1}}{2m}. \tag{B.11}$$

The value of W_1 can be evaluated using the integral representation of the Taylor coefficients. The steps involved in calculating W_1 are similar to the ones leading to (B.5). Doing so we get $W_1 = 2$ and identity (3.19) follows. Next consider the identity given in (3.20). If we define

$$T_{nm} = T_{mn} \equiv \frac{a_m b_n - b_m a_n}{m-n}, \tag{B.12}$$

then it follows that

$$(n+1)T_{n+1\,m}-(n-1)T_{n-1\,m}+(m+1)T_{n\,m+1}-(m-1)T_{n\,m-1}=0, \quad \text{(B.13)}$$

for $m+n=$ odd integer. Now letting $m \to 2m-1 \geq 1$, $n \to 2n \geq 2$ and summing over m we arrive at

$$(2n+1)\sum_{m=1}^{\infty} T_{2n+1\,2m-1} = (2n-1)\sum_{m=1}^{\infty} T_{2n-1\,2m-1}. \quad \text{(B.14)}$$

from which (3.20) follows.

To derive (3.56) let $n \rightleftharpoons m$ in (B.7) and then letting $n \to 2n-1 \geq 1$, $m \to 2m \geq 2$ and summing over m we get

$$2n\left(a_{2n}S^b_{2n}+b_{2n}S^a_{2n}\right)-(2n-2)\left(a_{2n-2}S^b_{2n-2}+b_{2n-2}S^a_{2n-2}\right)$$

$$=\frac{2}{2n}\left[a_{2n-1}+b_{2n-1}-a_{2n-2}b_{2n-2}\right]. \quad \text{(B.15)}$$

The expression inside the square bracket is most easily evaluated using the integral representation of the Taylor coefficients. The resulting integrals are easily evaluated and yield zero. Hence, (B.15) reduces to .

$$2n\left(a_{2n}S^b_{2n}+b_{2n}S^a_{2n}\right)-(2n-2)\left(a_{2n-2}S^b_{2n-2}+b_{2n-2}S^a_{2n-2}\right)=0. \quad \text{(B.16)}$$

Solving (B.16) we get

$$C_{2n} \equiv a_{2n}S^b_{2n}+b_{2n}S^a_{2n}=\frac{2}{2n}C_2. \quad \text{(B.17)}$$

This solution depends on the value of C_2 . The value of C_2 is given by

$$C_2 = a_2S^b_2+b_2S^a_2=1, \quad \text{(B.18)}$$

which can be easily checked using the fact

$$S^{a(b)}_2=\frac{1}{2}a(b)S^{a(b)}_1+\frac{1}{2}, \quad \text{(B.19)}$$

and (3.19). Substituting the value of C_2 back into (B.17) we arrive at (3.56).

We conclude this appendix by deriving two more identities which are needed in the proof of the *momentum* overlaps. Consider the recursion relation given in (A.17)

$$(n+1)\tilde{\Sigma}^a_{n+1} = \frac{2}{3}\tilde{\Sigma}^a_n + (n-1)\tilde{\Sigma}^a_{n-1} + \Sigma^a_{n+1} - \Sigma^a_{n-1}. \tag{B.20}$$

Letting $n \to -n$ in the above expression and subtracting the result from $\frac{1}{2}\times$ eq. (B.20) and remembering that $\Sigma^a_{-n} = -\frac{1}{2}\Sigma^a_n$, we get

$$(n+1)\left[\frac{1}{2}\tilde{\Sigma}^a_{n+1} - \tilde{\Sigma}^a_{-(n+1)}\right] - (n-1)\left[\frac{1}{2}\tilde{\Sigma}^a_{n-1} - \tilde{\Sigma}^a_{-(n-1)}\right]$$

$$-\frac{2}{3}\left[\frac{1}{2}\tilde{\Sigma}^a_n - \tilde{\Sigma}^a_{-n}\right] = 0. \tag{B.21}$$

The above equation is the same as (A.14) and therefore has a solution proportional to S^a_n. Hence[20]

$$\frac{1}{2}\tilde{\Sigma}^a_n - \tilde{\Sigma}^a_{-n} = \kappa S^a_n. \tag{B.22}$$

This solution depends on one constant, κ. Setting $n = 1$ in (B.22) it follows that

$$\kappa = \frac{1}{S^a_1}\left(\frac{1}{2}\tilde{\Sigma}^a_1 - \tilde{\Sigma}^a_{-1}\right). \tag{B.23}$$

Now direct substitution yields

$$\kappa = -\frac{3}{2}\Sigma^a_0. \tag{B.24}$$

Substituting this into (B.22) and rearranging terms we obtain

$$\tilde{\Sigma}^a_{-n} = \frac{1}{2}\tilde{\Sigma}^a_n + \frac{3}{2}\Sigma^a_0 S^a_n. \tag{B.25}$$

[20]From the integral representation of S_n, it follows that when $n \to 0$, $S_n \to c/n$ as required.

Similarly staring with (A.18) we can easily obtain $\tilde{\Sigma}_n^b$ for negative values of n

$$\tilde{\Sigma}_{-n}^b = -\frac{1}{2}\tilde{\Sigma}_n^b + \frac{1}{2}\Sigma_0^b S_n^b \tag{B.26}$$

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