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February 9, 2009

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ISSN 1358-6254

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A second derivative SQP method: global convergence¹

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ABSTRACT

Sequential quadratic programming (SQP) methods form a class of highly efficient algorithms for solving nonlinearly constrained optimization problems. Although second derivative information may often be calculated, there is little practical theory that justifies exact-Hessian SQP methods. In particular, the resulting quadratic programming (QP) subproblems are often nonconvex, and thus finding their global solutions may be computationally nonviable. This paper presents a second-derivative SQP method based on quadratic subproblems that are either convex, and thus may be solved efficiently, or need not be solved globally. Additionally, an explicit descent-constraint is imposed on certain QP subproblems, which “guides” the iterates through areas in which nonconvexity is a concern. Global convergence of the resulting algorithm is established.

Keywords: Nonlinear programming, nonlinear inequality constraints, sequential quadratic programming, ℓ_1 -penalty function, nonsmooth optimization

AMS classification: 49J52, 49M37, 65F22, 65K05, 90C26, 90C30, 90C55.

¹ This work was supported by the EPSRC grants EP/E053351/1 and EP/F005369/1.

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1. Introduction

In this paper we present a sequential quadratic programming (SQP) method for solving the problem

$$(\ell_1\text{-}\sigma) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \phi(x) = f(x) + \sigma \| [c(x)]^- \|_1,$$

where the constraint vector $c(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the objective function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ are assumed to be twice continuously differentiable, σ is a positive scalar known as the penalty parameter, and we have used the notation $[v]^- = \min(0, v)$ for a generic vector v (the minimum is understood to be component-wise). Our motivation for solving this problem is that solutions of problem $(\ell_1\text{-}\sigma)$ correspond (under certain assumptions) to solutions of the problem

$$(\text{NP}) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad c(x) \geq 0.$$

See [10, 19] for more details on exactly how problems $(\ell_1\text{-}\sigma)$ and (NP) are related.

The precise set of properties that characterize an SQP method is often author dependent. In fact, as the immense volume of literature on SQP methods continues to increase, the properties that define these methods become increasingly blurred. One may argue, however, that the backbone of every SQP method consists of “step generation” and “step acceptance/rejection”. We describe these concepts in turn.

All SQP methods generate a sequence of trial steps, which are computed as solutions of cleverly chosen quadratic or quadratic-related subproblems. Typically, the QP subproblems are closely related to the optimality conditions of the underlying problem and thus give the potential for fast Newton-like convergence. More precisely, the trial steps “approximately” minimize (locally) a quadratic approximation to a Lagrangian function subject to a linearization of all or a subset of the constraint functions. Two major concerns associated with this QP subproblem are incompatible linearized constraints and unbounded solutions. There are essentially two approaches that have been used for handling unbounded solutions. The first approach is to use a positive definite approximation to the Hessian in the quadratic subproblem. The resultant *strictly convex* QP is bounded with a unique minimizer. The second approach allows for a *nonconvex* QP by explicitly bounding the solution via a trust-region constraint. Both techniques have been effective in practice. The issue of incompatible subproblems is more delicate. We first note that the QP subproblem may be “naturally” incompatible – i.e., the set of feasible points is empty. However, even if the linearized constraints are compatible, the feasible region may still be empty if a trust-region constraint is imposed; the trust-region may “cut-off” all solutions to the linear system. Different techniques, such as constraint shifting [25], a special “elastic” mode [16], and a “feasibility restoration” phase [13], have been used to deal with incompatible subproblems.

Strategies for accepting or rejecting trial steps are sometimes referred to as “globalization techniques” since they are the instrument for guaranteeing global convergence. The earliest methods used so-called merit functions to measure the quality of a trial step. A

merit function is a single function that carefully balances the (usually) conflicting aims of reducing the objective function and satisfying the constraints. The basic idea is that a step is accepted if it gives sufficient decrease in the merit function; otherwise, the step is rejected, parameters updated, and a new trial step is computed. More recently, filter methods [13] have become an attractive alternative to a merit function. Filter methods view problem (NP) as a bi-objective optimization problem – minimizing the objective function $f(x)$ and minimizing the constraint violation $\|[c(x)]^-\|$. These methods use the idea of a “filter”, which is essentially a collection of pairs $(\|[c(x)]^-\|, f(x))$ such that no pair dominates another – we say that a pair $(\|[c(x_1)]^-\|, f(x_1))$ dominates a pair $(\|[c(x_2)]^-\|, f(x_2))$ if $f(x_1) < f(x_2)$ and $\|[c(x_1)]^-\| < \|[c(x_2)]^-\|$. Although the use of a merit function and a filter are conceptually quite different, Curtis and Nocedal [11] have shown that a “flexible” penalty approach partially bridges this gap. Their method may be viewed as a continuum of methods with traditional merit function and filter methods as the extrema.

The previous two paragraphs described two properties of all SQP methods – step computation and step acceptance or rejection – and these properties alone may differentiate one SQP method from another. In the context of problem (NP), a further fundamental distinction between SQP methods can be found in how the inequality constraints are used in the QP subproblems. This distinction has spawned a rivalry between essentially two classes of methods, which are commonly known as SEQP and SIQP methods.

Sequential equality-constrained quadratic programming (SEQP) methods solve problem (NP) by solving an *equality* constrained QP during each iterate. The linearized equality constraints that are included may be interpreted as an approximation to the optimal active constraint set. Determining which constraints to include in each subproblem is a delicate task. The approach used by Coleman and Conn [8] includes those constraints that are nearly active at the current point. Then they solve an equality constrained QP in which a second-order approximation to the locally differentiable part of an exact penalty function is minimized subject to keeping the “nearly” active constraints fixed. An alternative approach is to use the solution of a “simpler” auxiliary subproblem as a prediction of the optimal active constraints. Often, the simpler subproblem only uses first-order information and results in a linear program. Merit function based variants of this type have been studied by Fletcher and Sainz de la Maza [14], Byrd et al. [4, 5], while filter based variants have been studied by Chin and Fletcher [7].

Sequential inequality-constrained quadratic programming (SIQP) methods solve problem (NP) by solving a sequence of inequality constrained quadratic subproblems. Unlike the SEQP philosophy, SIQP methods utilize *every* constraint in each subproblem and, therefore, avoid the precarious task of choosing which constraints to include. These methods also have the potential for fast convergence; under standard assumptions, methods of this type correctly identify the optimal active-set in a finite number of iterations and thereafter rapid convergence is guaranteed by the famous result due to Robinson [22, Theorem 3.1]. Probably the greatest disadvantage of SIQP methods is their potential cost; to solve the inequality constrained QP subproblem, both active-set and interior-point algorithms

may require the solution of many equality constrained quadratic programs. However, in the case of moderate-sized problems, there is much empirical evidence that indicates that the additional cost per iteration is often off-set by substantially fewer function evaluations (similar evidence has yet to surface for large-sized problems). SIQP methods that utilize exact second-derivatives must also deal with nonconvexity. To our knowledge, all previous second-order SIQP methods assume that global minimizers of nonconvex subproblems are computed, which is not a realistic assumption in most cases. For these methods, the computation of a local minimizer is unsatisfactory because it may yield an ascent direction. Line-search, trust-region, and filter variants of SIQP methods have been proposed. The line-search method by Gill et al. [16] avoids unbounded and non-unique QP solutions by maintaining a quasi-Newton (sometimes limited-memory quasi-Newton) approximation to the Hessian of the Lagrangian. The SIQP approaches by Boggs, Kearsley and Tolle [1, 2] modify the exact second derivatives to ensure that the reduced Hessian is sufficiently positive definite. Finally, the filter SIQP approach by Fletcher and Leyffer [13] deals with infeasibility by entering a special restoration-phase to recover from bad steps.

The algorithm we propose may be considered an SIQP/SEQP hybrid that is most similar to the $S\ell_1$ QP method proposed by Fletcher [12], which is a second-order method designed for finding first-order critical points of problem $(\ell_1-\sigma)$. The QP subproblem studied by Fletcher is to minimize a second-order approximation to the ℓ_1 -penalty function subject to a trust-region constraint. More precisely, the QP subproblem is obtained by approximating $f(x)$ and $c(x)$ in the ℓ_1 -penalty function by a second- and first-order Taylor approximation, respectively. Unfortunately, the theoretical results of Fletcher's method requires the computation of the *global* minimizer of this (generally) nonconvex subproblem, which is known to be an NP-hard problem. The method we propose is also a second-derivative method that is globalized via the ℓ_1 -merit function, but we do not require the global minimizer of any nonconvex QP. To achieve this goal, our procedure for computing a trial step is necessarily more complicated than that used by Fletcher. Given an estimate x_k of a solution to problem (NP), a search direction is generated from a combination of three steps: a *predictor* step s_k^p is defined as a solution to a *strictly convex* QP subproblem; a *Cauchy* step s_k^{cp} drives convergence of the algorithm and is computed from a special univariate *global* minimization problem; and an (optional) *SQP* step s_k^s is computed from a local solution of a special nonconvex QP subproblem. Since there is considerable flexibility in how we compute the SQP step, we may adaptively choose the SQP subproblem to reflect the likely success of an SIQP or SEQP approach. This observation justifies our claim that our method is an SIQP/SEQP hybrid.

The rest of the paper is organized as follows. This section proceeds to introduce requisite notation and to catalog various model functions used throughout the paper. Section 2 gives a complete description of how we generate the predictor, Cauchy and SQP steps. The algorithm for computing a first-order solution to problem $(\ell_1-\sigma)$ is given in Section 3 and the global convergence of this algorithm is considered in Section 4. Finally, Section 5 gives conclusions and future work.

1.1. Notation

Most of our notation is standard. We let e denote the vector of all ones whose dimension is determined by the context. A local solution of $(\ell_1\text{-}\sigma)$ is denoted by x^* ; $g(x)$ is the gradient of $f(x)$, and $H(x)$ its (symmetric) Hessian; the matrix $H_j(x)$ is the Hessian of $c_j(x)$; $J(x)$ is the $m \times n$ Jacobian matrix of the constraints with i th row $\nabla c_i(x)^T$. For a general vector v , the notation $[v]^- = \min(0, v)$ is used, where the minimum is understood to be component-wise. The Lagrangian function associated with (NP) is $\mathcal{L}(x, y) = f(x) - y^T c(x)$. The Hessian of the Lagrangian with respect to x is $\nabla_{xx}^2 \mathcal{L}(x, y) = H(x) - \sum_{j=1}^m y_j H_j(x)$.

We often consider problem functions evaluated at a specific point x_k . To simplify notation we define the following: $f_k = f(x_k)$, $c_k = c(x_k)$, $g_k = g(x_k)$ and $J_k = J(x_k)$. Given a pair of values (x_k, y_k) , we let H_k and B_k denote symmetric approximations to $H(x_k, y_k)$ in which B_k is required additionally to be positive-definite. In practice H_k may be chosen to be $H(x_k, y_k)$, but this is not necessary in what follows.

We define the indexing sets $\mathcal{V}_k \stackrel{\text{def}}{=} \mathcal{V}(x_k; s_k^{\text{CP}}) = \{i : [c_k + J_k s_k^{\text{CP}}]_i < 0\}$ and $\mathcal{S}_k \stackrel{\text{def}}{=} \mathcal{S}(x_k; s_k^{\text{CP}}) = \{i : [c_k + J_k s_k^{\text{CP}}]_i \geq 0\}$, which contain the indices of the linearized constraints that are *violated* and *satisfied* at the Cauchy step, respectively. Given a generic vector v or matrix V , the notation $[v]_{\mathcal{V}_k}$ and $[V]_{\mathcal{V}_k}$ will denote the rows of v and V , respectively, that correspond to the indices in \mathcal{V}_k . Analogous notation applies to the indexing set \mathcal{S}_k .

1.2. Model functions

We define the following models of $\phi(x)$ for a given estimate x_k of a solution to problem $(\ell_1\text{-}\sigma)$.

- The *linear* model of the merit function:

$$M_k^{\text{L}}(s) \stackrel{\text{def}}{=} M_k^{\text{L}}(s; x_k) = f_k + g_k^T s + \sigma \|[c_k + J_k s]^- \|_1.$$

- The *convex* model of the merit function:

$$M_k^{\text{B}}(s) \stackrel{\text{def}}{=} M_k^{\text{B}}(s; x_k) = f_k + g_k^T s + \frac{1}{2} s^T B_k s + \sigma \|[c_k + J_k s]^- \|_1.$$

- The *faithful* model of the merit function:

$$M_k^{\text{H}}(s) \stackrel{\text{def}}{=} M_k^{\text{H}}(s; x_k) = f_k + g_k^T s + \frac{1}{2} s^T H_k s + \sigma \|[c_k + J_k s]^- \|_1.$$

- The *SQP* model:

$$M_k^{\text{S}}(s) \stackrel{\text{def}}{=} M_k^{\text{S}}(s; x_k, s_k^{\text{CP}}) = \bar{f}_k + (g_k + H_k s_k^{\text{CP}})^T s + \frac{1}{2} s^T H_k s + \sigma \|[c_k + J_k (s_k^{\text{CP}} + s)]^- \|_{\mathcal{V}_k},$$

where s_k^{CP} is the Cauchy step (see Section 2.2) and $\bar{f}_k = f_k + g_k^T s_k^{\text{CP}} + \frac{1}{2} s_k^{\text{CP}T} H_k s_k^{\text{CP}}$.

- The change in the linear model:

$$\Delta M_k^{\text{L}}(s) \stackrel{\text{def}}{=} \Delta M_k^{\text{L}}(s; x_k) = M_k^{\text{L}}(0; x_k) - M_k^{\text{L}}(s; x_k).$$

- The change in the convex model:

$$\Delta M_k^{\text{B}}(s) \stackrel{\text{def}}{=} \Delta M_k^{\text{B}}(s; x_k) = M_k^{\text{B}}(0; x_k) - M_k^{\text{B}}(s; x_k).$$

- The change in the faithful model:

$$\Delta M_k^{\text{H}}(s) \stackrel{\text{def}}{=} \Delta M_k^{\text{H}}(s; x_k) = M_k^{\text{H}}(0; x_k) - M_k^{\text{H}}(s; x_k).$$

- The change in the SQP model:

$$\Delta M_k^{\text{S}}(s) \stackrel{\text{def}}{=} \Delta M_k^{\text{S}}(s; x_k, s_k^{\text{CP}}) = M_k^{\text{S}}(0; x_k, s_k^{\text{CP}}) - M_k^{\text{S}}(s; x_k, s_k^{\text{CP}}).$$

- For a given trust-region radius $\Delta \geq 0$, primal variable x , and penalty parameter σ , we denote the maximum decrease in the *linear* model to be

$$\Delta_{\text{max}}^{\text{L}}(x, \Delta) = M_k^{\text{L}}(0; x) - \min_{\|s\|_{\infty} \leq \Delta} M_k^{\text{L}}(s; x). \quad (1.1)$$

Useful properties of the function $\Delta_{\text{max}}^{\text{L}}$ are given in the next lemma. See Borwein and Lewis [3], Rockafellar [23], and Yuan [26] for more details.

Lemma 1.1. *Consider the definition of $\Delta_{\text{max}}^{\text{L}}$ as given by equation (1.1). Then the following properties hold:*

- (i) $\Delta_{\text{max}}^{\text{L}}(x, \Delta) \geq 0$ for all x and all $\Delta \geq 0$;
- (ii) $\Delta_{\text{max}}^{\text{L}}(x, \cdot)$ is a non-decreasing function;
- (iii) $\Delta_{\text{max}}^{\text{L}}(x, \cdot)$ is a concave function;
- (iv) $\Delta_{\text{max}}^{\text{L}}(\cdot, \Delta)$ is continuous;
- (v) For any fixed $\Delta > 0$, $\Delta_{\text{max}}^{\text{L}}(x, \Delta) = 0$ if and only if x is a stationary point for problem $(\ell_1\text{-}\sigma)$.

Properties (ii) and (iii) allow us to relate the maximum decrease in the linear model for an *arbitrary* radius to the maximum decrease in the linear model for a constant radius. For convenience, we have chosen that constant to be one. The following corollary makes this precise.

Corollary 1.1. *Let x be fixed. Then for all $\Delta \geq 0$*

$$\Delta_{\text{max}}^{\text{L}}(x, \Delta) \geq \min(\Delta, 1) \Delta_{\text{max}}^{\text{L}}(x, 1). \quad (1.2)$$

Proof. See Lemma 3.1 in [5]. □

2. Step Computation

During each iterate of our proposed method we compute a trial step s_k that is calculated from three steps: a predictor step s_k^P , a Cauchy step s_k^{CP} , and an SQP step s_k^S . The predictor step is defined as the solution of a *strictly convex* model for which the global minimum is unique and computable in polynomial time. The Cauchy step is then computed as the *global* minimizer of a specialized one-dimensional optimization problem involving the faithful model M_k^H and is also computable in polynomial time. It will be shown that the Cauchy step alone is enough for proving convergence, but we allow the option for computing an additional SQP step. The SQP step is the solution of a special QP subproblem that utilizes a less restrictive approximation H_k to $H(x_k, y_k)$ and is intended to improve the efficiency of the method by encouraging fast convergence. We begin by discussing the predictor step.

2.1. The predictor step s_k^P

In our algorithm, the predictor step s_k^P plays a role analogous to the role played by the direction of steepest descent in unconstrained trust-region methods. During each iterate of a traditional unconstrained trust-region method, a quadratic model of the objective function is minimized in the direction of steepest descent. The resulting step, known as the Cauchy step, gives a decrease in the quadratic model that is sufficient for proving convergence (see Conn et al. [9]). In our setting, a vector that is directly analogous to the direction of steepest descent is the vector that minimizes the linearization of the ℓ_1 -merit function within a trust-region constraint. However, since we want to incorporate second-order information, we define the predictor step to be the solution to

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \quad M_k^B(s) \quad \text{subject to} \quad \|s\|_\infty \leq \Delta_k^P, \quad (2.1)$$

where B_k is any symmetric positive definite approximation to the Hessian, and $\Delta_k^P > 0$ is the predictor trust-region radius; since B_k is positive definite, problem (2.1) is *strictly convex* and the minimizer is unique. We note that

$$\Delta M_k^B(s_k^P) \geq 0, \quad (2.2)$$

since $M_k^B(s_k^P) \leq M_k^B(0)$ and that problem (2.1) is a non-smooth minimization problem. In fact, it is not differentiable at any point for which the constraint linearization is zero. In practice, we solve the equivalent ‘‘elastic’’ problem [16] defined as

$$\begin{aligned} \underset{s \in \mathbb{R}^n, v \in \mathbb{R}^m}{\text{minimize}} \quad & f_k + g_k^T s + \frac{1}{2} s^T B_k s + \sigma_k e^T v \\ \text{subject to} \quad & c_k + J_k s + v \geq 0, \quad v \geq 0, \quad \|s\|_\infty \leq \Delta_k^P. \end{aligned} \quad (2.3)$$

Problem (2.3) is a *smooth* linearly-constrained convex quadratic program that may be solved using a number of software packages such as LOQO [24] and QPOPT [15], as well as the QP solvers QPA, QPB, and QPC that are part of the GALAHAD [17] library. In addition, if

B_k is chosen to be diagonal, then the GALAHAD package LSQP may be used since problem (2.3) is then a separable convex quadratic program.

The following estimate is [26, Lemma 2.2] transcribed into our notation.

Lemma 2.1. *For a given x_k and σ the following inequality holds:*

$$\Delta M_k^B(s_k^P) \geq \frac{1}{2} \Delta_{max}^L(x_k, \Delta_k^P) \min \left(1, \frac{\Delta_{max}^L(x_k, \Delta_k^P)}{\|B_k\|_2 (\Delta_k^P)^2} \right). \quad (2.4)$$

We note that the proof by Yuan requires the *global* minimum of the associated QP. For a general QP this requirement is not practical since finding the global minimum of a nonconvex QP is NP-hard. This is likely the greatest drawback of any previous method that utilized both *exact* second derivatives and the ℓ_1 -penalty function. In our situation, however, the matrix B_k is positive definite by construction and the global minimum may be found efficiently.

We may further bound $\Delta M_k^B(s_k^P)$ by applying Corollary 1.1.

Corollary 2.1.

$$\Delta M_k^B(s_k^P) \geq \frac{1}{2} \Delta_{max}^L(x_k, 1) \min \left(1, \Delta_k^P, \frac{\Delta_{max}^L(x_k, 1)}{\|B_k\|_2}, \frac{\Delta_{max}^L(x_k, 1)}{\|B_k\|_2 (\Delta_k^P)^2} \right). \quad (2.5)$$

Proof. Follows directly from Corollary 1.1 and Lemma 2.1. \square

The previous corollary bounds the change in the convex model at the predictor step in terms of the maximum change in the linear model within a unit trust-region. Since we wish to drive convergence by using the faithful model, we must derive a useful bound on the change in the *faithful* model. This essential bound is derived from the Cauchy point and is the topic of the next section.

2.2. The Cauchy step s_k^{CP}

In the beginning of Section 2 we stated that the Cauchy step induces global convergence of our proposed method. However, it is also true that the predictor step may be used to drive convergence for a slightly different method; this modified algorithm may crudely be described as follows. During the computation of each iterate, the ratio of actual versus predicted decrease in the merit function is computed, where the predicted decrease is given by the change in the convex model $M_k^B(s)$ at s_k^P . Based on this ratio, the trust-region radius and iterate x_k may be updated using standard trust-region techniques. Using this idea and assuming standard conditions on the iterates generated by this procedure, one may prove convergence to a first-order solution of problem $(\ell_1\text{-}\sigma)$. This may be derived from Fletcher's work [12] by allowing a positive-definite approximation to $H(x_k, y_k)$. However, our intention is to stay as *faithful* to the problem functions as possible. Therefore, in computing the ratio of actual versus predicted decrease in the merit function, we use the decrease in the *faithful* model $M_k^H(s)$ instead of the *strictly convex* model $M_k^B(s)$.

Unfortunately, since the predictor step is computed using the approximate Hessian B_k , the point s_k^P is not directly appropriate as a means for ensuring convergence. In fact, it is possible that $M_k^H(s_k^P) > M_k^H(0)$ so that the predictor step gives an increase in the faithful model. However, a reasonable point is close-at-hand and is what we call the Cauchy step. The basic idea behind the Cauchy step is to make improvement in the *faithful* model in the direction s_k^P by finding the *global* minimizer of $M_k^H(\alpha s_k^P)$ for $0 \leq \alpha \leq 1$. We will see that the Cauchy step allows us to prove convergence by using the quantity $\Delta M_k^H(s_k^{CP})$ as a prediction of the decrease in the merit function.

To be more precise, the Cauchy step is defined as $s_k^{CP} = \alpha_k s_k^P$ where α_k is the solution to

$$\underset{0 \leq \alpha \leq 1}{\text{minimize}} \quad M_k^H(\alpha s_k^P). \quad (2.6)$$

The function $M_k^H(\alpha s_k^P)$ is a piecewise-continuous quadratic function of α for which the exact *global* minimizer may be found efficiently. Before discussing the properties of the Cauchy step, we give the following simple lemma.

Lemma 2.2. *Let $c \in \mathbb{R}^m$, $J \in \mathbb{R}^{m \times n}$, and $s \in \mathbb{R}^n$. Then the following inequality holds for all $0 \leq \alpha \leq 1$:*

$$\|[c + \alpha Js]^- \|_1 \leq \alpha \|[c + Js]^- \|_1 + (1 - \alpha) \|[c]^- \|_1. \quad (2.7)$$

Proof. From the convexity of $\|[\cdot]^- \|_1$ it follows that

$$\|[c + \alpha Js]^- \|_1 = \|[\alpha(c + Js) + (1 - \alpha)c]^- \|_1 \leq \alpha \|[c + Js]^- \|_1 + (1 - \alpha) \|[c]^- \|_1,$$

which is equation (2.7). \square

We now give a precise lower bound for the change in the faithful model obtained from the Cauchy step.

Lemma 2.3. *Let s_k^P and s_k^{CP} be defined as previously. Then*

$$\Delta M_k^H(s_k^{CP}) \geq \frac{1}{2} \Delta M_k^B(s_k^P) \min \left(1, \frac{\Delta M_k^B(s_k^P)}{n \|B_k - H_k\|_2 (\Delta_k^P)^2} \right). \quad (2.8)$$

Proof. For all $0 \leq \alpha \leq 1$, we have

$$\Delta M_k^H(s_k^{CP}) \geq \Delta M_k^H(\alpha s_k^P) \quad (2.9)$$

$$= \sigma \left(\|[c_k]^- \|_1 - \|[c_k + \alpha J_k s_k^P]^- \|_1 \right) - \alpha g_k^T s_k^P - \frac{\alpha^2}{2} s_k^{PT} H_k s_k^P \quad (2.10)$$

$$= \sigma \left(\|[c_k]^- \|_1 - \|[c_k + \alpha J_k s_k^P]^- \|_1 \right) - \alpha g_k^T s_k^P - \frac{\alpha^2}{2} s_k^{PT} B_k s_k^P + \frac{\alpha^2}{2} s_k^{PT} (B_k - H_k) s_k^P. \quad (2.11)$$

Equation (2.9) follows since s_k^{CP} minimizes $M_k^H(\alpha s_k^P)$ for $0 \leq \alpha \leq 1$. Equations (2.10) and (2.11) follow from the definition of M_k^H and from simple algebra. Continuing to bound the

change in the faithful model, we have

$$\begin{aligned} \Delta M_k^H(s_k^{\text{CP}}) &\geq \sigma(\|[c_k]^- \|_1 - \alpha\|[c_k + J_k s_k^{\text{P}}]^- \|_1 - (1 - \alpha)\|[c_k]^- \|_1) \\ &\quad - \alpha g_k^T s_k^{\text{P}} - \frac{\alpha}{2} s_k^{\text{PT}} B_k s_k^{\text{P}} + \frac{\alpha^2}{2} s_k^{\text{PT}} (B_k - H_k) s_k^{\text{P}} \end{aligned} \quad (2.12)$$

$$\begin{aligned} &= \alpha \sigma(\|[c_k]^- \|_1 - \|[c_k + J_k s_k^{\text{P}}]^- \|_1) \\ &\quad - \alpha g_k^T s_k^{\text{P}} - \frac{\alpha}{2} s_k^{\text{PT}} B_k s_k^{\text{P}} + \frac{\alpha^2}{2} s_k^{\text{PT}} (B_k - H_k) s_k^{\text{P}} \end{aligned} \quad (2.13)$$

$$= \alpha \Delta M_k^B(s_k^{\text{P}}) + \frac{\alpha^2}{2} s_k^{\text{PT}} (B_k - H_k) s_k^{\text{P}}. \quad (2.14)$$

Equation (2.12) follows from equation (2.11), Lemma 2.2 and the inequality $\alpha^2 \leq \alpha$, which holds since $0 \leq \alpha \leq 1$. Finally, equations (2.13) and (2.14) follow from simplification of equation (2.12) and from the definition of $\Delta M_k^B(s_k^{\text{P}})$.

The previous string of inequalities holds for *all* $0 \leq \alpha \leq 1$, so it must hold for the value of α that maximizes the right-hand-side. As a function of α , the right-hand-side may be written as $q(\alpha) = a\alpha^2 + b\alpha$ where

$$a = \frac{1}{2} s_k^{\text{PT}} (B_k - H_k) s_k^{\text{P}} \quad \text{and} \quad b = \Delta M_k^B(s_k^{\text{P}}) \geq 0.$$

There are three cases to consider.

Case 1 : $a \geq 0$

In this case the quadratic function $q(\alpha)$ is convex and the maximizer on the interval $[0, 1]$ must occur at $\alpha = 1$. Thus, the maximum of q on the interval $[0, 1]$ is $q(1)$ and may be bounded by

$$q(1) = a + b \geq b \geq \frac{1}{2} b = \frac{1}{2} \Delta M_k^B(s_k^{\text{P}})$$

since $b \geq 0$ and $a \geq 0$.

Case 2 : $a < 0$ and $-b/2a \leq 1$

In this case the maximizer on the interval $[0, 1]$ must occur at $\alpha = -b/2a$. Therefore, the maximum of q on the interval $[0, 1]$ is given by

$$q(-b/2a) = a \frac{b^2}{4a^2} + b \frac{-b}{2a} = -\frac{b^2}{4a}.$$

Substituting for a and b , using the Cauchy-Schwarz inequality, and applying norm inequalities shows

$$q(-b/2a) = \frac{(\Delta M_k^B(s_k^{\text{P}}))^2}{2|s_k^{\text{PT}}(B_k - H_k)s_k^{\text{P}}|} \geq \frac{(\Delta M_k^B(s_k^{\text{P}}))^2}{2\|B_k - H_k\|_2 \|s_k^{\text{P}}\|_2^2} \geq \frac{(\Delta M_k^B(s_k^{\text{P}}))^2}{2n\|B_k - H_k\|_2 \|s_k^{\text{P}}\|_\infty^2}.$$

Finally, since $\|s_k^{\text{P}}\|_\infty \leq \Delta_k^{\text{P}}$, we have

$$q(-b/2a) \geq \frac{(\Delta M_k^B(s_k^{\text{P}}))^2}{2n\|B_k - H_k\|_2 (\Delta_k^{\text{P}})^2}.$$

Case 3 : $a < 0$ and $-b/2a > 1$

In this case the maximizer of q on the interval $[0, 1]$ is given by $\alpha = 1$. Therefore, the maximum of q on the interval $[0, 1]$ is given by $q(1)$ and is bounded by

$$q(1) = a + b > -\frac{1}{2}b + b = \frac{1}{2}b = \frac{1}{2}\Delta M_k^B(s_k^P)$$

since the inequality $-b/2a > 1$ implies $a > -b/2$.

If we denote the maximizer of $q(\alpha)$ on the interval $[0, 1]$ by α^* , then consideration of all three cases shows that

$$q(\alpha^*) \geq \frac{1}{2}\Delta M_k^B(s_k^P) \min\left(1, \frac{\Delta M_k^B(s_k^P)}{n\|B_k - H_k\|_2(\Delta_k^P)^2}\right). \quad (2.15)$$

Returning to equation (2.14), we have

$$\Delta M_k^H(s_k^{CP}) \geq q(\alpha^*) \geq \frac{1}{2}\Delta M_k^B(s_k^P) \min\left(1, \frac{\Delta M_k^B(s_k^P)}{n\|B_k - H_k\|_2(\Delta_k^P)^2}\right),$$

which completes the proof. \square

We note that in the special case $B_k = H_k$, the term $\Delta M_k^B(s_k^P)/(n\|B_k - H_k\|_2(\Delta_k^P)^2)$ should be interpreted as infinity, and then Lemma 2.3 reduces to

$$\Delta M_k^H(s_k^{CP}) \geq \frac{1}{2}\Delta M_k^B(s_k^P), \quad (2.16)$$

which trivially holds since $B_k = H_k$ and $s_k^{CP} = s_k^P$.

We may further bound the change in the faithful model by using Corollary 2.1

Corollary 2.2. *Let s_k^P and s_k^{CP} be defined as previously. Then*

$$\Delta M_k^H(s_k^{CP}) \geq \frac{1}{4}\Delta_{max}^L(x_k, 1) \min(\mathcal{S}_k)$$

where

$$\mathcal{S}_k = \left\{ 1, \Delta_k^P, \frac{\Delta_{max}^L(x_k, 1)}{\|B_k\|_2}, \frac{\Delta_{max}^L(x_k, 1)}{\|B_k\|_2(\Delta_k^P)^2}, \frac{\Delta_{max}^L(x_k, 1)}{2n\|B_k - H_k\|_2}, \frac{\Delta_{max}^L(x_k, 1)}{2n\|B_k - H_k\|_2(\Delta_k^P)^2}, \frac{(\Delta_{max}^L(x_k, 1))^3}{2n\|B_k - H_k\|_2\|B_k\|_2^2(\Delta_k^P)^2}, \frac{(\Delta_{max}^L(x_k, 1))^3}{2n\|B_k - H_k\|_2\|B_k\|_2^2(\Delta_k^P)^6} \right\}.$$

Proof. The bound follows from Corollary 2.1 and Lemma 2.3. \square

Corollary 2.2 provides a bound that is sufficient for proving convergence of our proposed algorithm, but we note that an *approximate* Cauchy point s_k^{ACP} satisfying

$$\Delta M_k^H(s_k^{ACP}) \geq \eta_{ACP}\Delta M_k^H(s_k^{CP}) \text{ for some } 0 < \eta_{ACP} < 1 \quad (2.17)$$

is also sufficient for proving convergence. An approximate Cauchy point may be obtained, for example, by backtracking from the predictor step s_k^P to the Cauchy point s_k^{CP} until

equation (2.17) is satisfied. Since the theory for an *approximate* Cauchy point is identical to the theory for the Cauchy point (modulo a constant factor in the appropriate estimates), we focus primarily on the latter.

The derivation of the bound supplied by Corollary 2.2 relied on minimizing the faithful model along the single direction s_k^p . If the predictor step is a bad search direction for the faithful model (most likely because B_k is, in some sense, a poor approximate to H_k), then convergence is likely to be slow. In order to improve efficiency we may need to make “better” use of the faithful model; the SQP step serves this purpose and more.

2.3. The SQP step s_k^s and the full step s_k

We begin by discussing three primary motivations for an SQP step s_k^s ; we use the word “an” instead of the word “the” since we propose several reasonable alternatives. The first motivation of the SQP step is to improve the rate-of-convergence. The predictor step s_k^p uses a positive-definite approximation B_k to the true Hessian $H(x_k, y_k)$, while the Cauchy step s_k^{cp} is computed as a minimization problem in the direction s_k^p . Therefore, the quality of both the predictor step and the Cauchy step is constrained by how well B_k approximates $H(x_k, y_k)$ (possibly when restricted to the null-space of the Jacobian of the active constraints). The simplest and cheapest choice is $B_k = I$, but this would likely result in a linear convergence rate. In general, if B_k is chosen to more closely approximate $H(x_k, y_k)$ then the predictor step s_k^p becomes more costly to compute, but would likely lead to faster convergence. Of course as B_k is required to be positive definite and since $H(x_k, y_k)$ is usually indefinite, this is typically not even possible. To promote efficiency, therefore, we may compute an SQP step s_k^s from various SQP subproblems that are formed from either the predictor step or the Cauchy step. Once the “quality” of the step s_k^s has been determined, we define the full step accordingly; Sections 2.3.1, 2.3.2 and 2.3.3 make this statement precise.

The previous paragraph may do the Cauchy step injustice; not only does the Cauchy step guarantee convergence of the algorithm, but it may happen that the Cauchy step is an excellent direction. In fact, if we are allowed the choice $B_k = H(x_k, y_k)$ and choose σ sufficiently large, then provided the trust-region radius Δ_k^p is inactive, the resulting Cauchy step $s_k^{cp}(= s_k^p)$ is the classical SQP step for problem (NP). This means that the Cauchy step may be the “ideal” step. As previously stated, however, the choice $B_k = H(x_k, y_k)$ will generally not be permissible. We summarize by saying that the quality of the Cauchy step is strongly dependent on how well B_k “approximates” $H(x_k, y_k)$.

Unfortunately, even if the Cauchy step is an “excellent” direction, it may still suffer from the Maratos effect [9, 18]. The Maratos effect occurs when the linear approximation to the constraint function does not adequately capture the nonlinear behavior of the constraints. As a result, although the unit step may make excellent progress towards finding a solution of problem (NP), it is in fact rejected by the merit function and subsequently the trust-region radius is reduced; this inhibits the natural convergence of Newton’s Method. Avoiding the Maratos effect is the second motivation for the SQP step.

The third motivation for the SQP step is to improve the performance of our method; the quadratic model used in computing the SQP step may use an *indefinite* approximation H_k to the Hessian of the Lagrangian and is, therefore, considered a more faithful model of the merit function.

Since the Cauchy step is good enough to guarantee convergence, we require that the SQP step and the full step be defined so that

$$\Delta M_k^H(s_k) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) \quad (2.18)$$

for some $0 < \eta < 1$. One way to satisfy this condition is to set $s_k = s_k^{\text{CP}}$ regardless of the merit of the k th SQP step (this includes the case when an SQP step is not computed). When this strategy is used for a given iterate, we say that condition (2.18) has been satisfied in a *trivial-way*; otherwise, we say that it has been satisfied in a *non-trivial* way.

2.3.1. An explicitly-constrained SIQP subproblem

This section discusses an SQP step that is the solution of an *explicitly*-constrained SIQP subproblem. The terminology “explicitly-constrained” is used to emphasize that the subproblem has a subset of the linearized constraints as explicit constraints, while the term SIQP indicates that *all* the explicit constraints are imposed as *inequalities*. The subproblem is given by

$$\begin{aligned} \text{(SIQP-E)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} && M_k^S(s) \\ & \text{subject to} && [c_k + J_k(s_k^{\text{CP}} + s)]_{\mathcal{S}_k} \geq 0, \\ & && (g_k + H_k s_k^{\text{CP}} + \sigma J_k^T w_k)^T s \leq 0, \\ & && \|s\|_\infty \leq \Delta_k^S, \end{aligned}$$

where $w_k \in \mathbb{R}^m$ is defined component-wise as

$$[w_k]_i = \begin{cases} -1 & \text{if } i \in \mathcal{V}_k, \\ 0 & \text{if } i \in \mathcal{S}_k, \end{cases} \quad (2.19)$$

and $\Delta_k^S > 0$ is the SQP trust-region radius. The artificial constraint $(g_k + H_k s_k^{\text{CP}} + \sigma J_k^T w_k)^T s \leq 0$ (henceforth referred to as the “descent-constraint”) guarantees that the directional derivative is non-positive. We will soon see that this condition ensures that a useful SQP-Cauchy step s_k^{CS} may easily be defined from *any* local solution to problem (SIQP-E); without the descent-constraint, it is clear that a local minimizer may be an ascent direction for which M_k^H increases. If problem (SIQP-E) was smooth, then the descent-constraint would guarantee that any solution would not cause the model $M_k^S(s)$ to increase. However, since (SIQP-E) is not smooth, it is possible that the model may still increase at a local solution even though it must initially decrease in that direction! Consider problem (SIQP-E) with data $\sigma = 2$, $s_k^{\text{CP}} = 0$, $\Delta_k^S = 2$,

$$c_k = \begin{pmatrix} -4 \\ -11 \\ -11 \end{pmatrix}, \quad J_k = \begin{pmatrix} 16 & 0 \\ 6 & 10 \\ 6 & -10 \end{pmatrix}, \quad g_k = \begin{pmatrix} 34 \\ 4 \end{pmatrix}, \quad \text{and} \quad H_k = \begin{pmatrix} -13 & -24/11 \\ -24/11 & 1 \end{pmatrix}.$$

It can be verified that $s_k^s = (11/6, 0)$ is a local solution and that $M_k^s(s_k^s) > M_k^s(0)$; the model has increased. However, if we define $\alpha_0 = \min(\alpha_B, 1) > 0$, where α_B is defined as

$$\alpha_B = \min \left\{ \left\{ \frac{-[c_k + J_k s_k^{CP}]_i}{[J_k s_k^s]_i} : i \in \mathcal{V}_k \text{ and } [J_k s_k^s]_i > 0 \right\} \cup \{\infty\} \right\}, \quad (2.20)$$

i.e., α_B is the distance in the direction s_k^s to the first point of non-differentiability, then it follows from Lemma 2.4 that the SQP model does not increase for the step $s_k^{CS} = \alpha_s s_k^s$, where α_s is the minimizer of $M_k^s(\alpha s_k^s)$ for $0 \leq \alpha \leq \alpha_0$ (see Figure 2.1 on page 15).

Lemma 2.4. *Let s_k^s be any local solution for problem (SIQP-E) and define $\alpha_0 = \min(\alpha_B, 1) > 0$, where α_B is defined by equation (2.20). It follows that the vector $s_k^{CS} = \alpha_s s_k^s$, where α_s is the minimizer of $M_k^s(\alpha s_k^s)$ for $0 \leq \alpha \leq \alpha_0$, satisfies*

$$\Delta M_k^s(s_k^{CS}) \geq -\frac{1}{2}(g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s \cdot \min \left(\alpha_s, -\frac{(g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s}{n \|H_k\|_2 (\Delta_k^s)^2} \right) \geq 0. \quad (2.21)$$

Moreover;

$$\text{if } s_k^{sT} H_k s_k^s > 0 \text{ then } (g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s < 0. \quad (2.22)$$

Proof. We first prove equation (2.22). Suppose that $s_k^{sT} H_k s_k^s > 0$ and that $(g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s \geq 0$. These conditions and the fact that s_k^s was a local solution to problem (SIQP-E) imply that $s_k^s = 0$. This contradicts $s_k^{sT} H_k s_k^s > 0$ and therefore equation (2.22) holds.

To establish (2.21) we consider the quadratic function $q(\alpha) \stackrel{\text{def}}{=} M_k^s(\alpha s_k^s) = a\alpha^2 + b\alpha + c$ defined on the interval $[0, \alpha_0]$, where

$$a = \frac{1}{2} s_k^{sT} H_k s_k^s, \quad b = (g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s \leq 0, \quad \text{and} \quad c = \bar{f} + \sigma(c_k + J_k s_k^{CP})^T w_k. \quad (2.23)$$

We consider three cases.

Case 1. : $a > 0$ and $-b/2a > \alpha_0$.

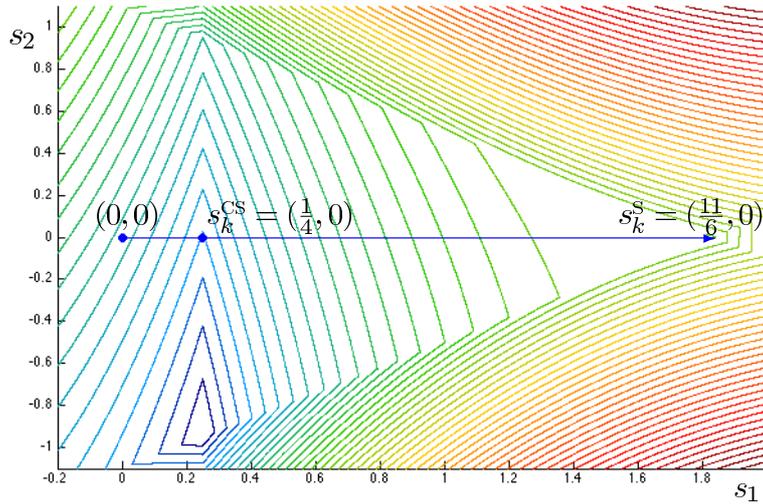
In this case we must have $\alpha_s = \alpha_0$. Using this fact, the definitions of $\Delta M_k^s(s)$ and of $q(\alpha)$, and the inequalities $-b/2a > \alpha_0 = \alpha_s$ and $a > 0$, we have

$$\Delta M_k^s(\alpha_s s_k^s) = q(0) - q(\alpha_s) = -a\alpha_s^2 - b\alpha_s \geq -\frac{1}{2}b\alpha_s = -\frac{\alpha_s}{2}(g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s. \quad (2.24)$$

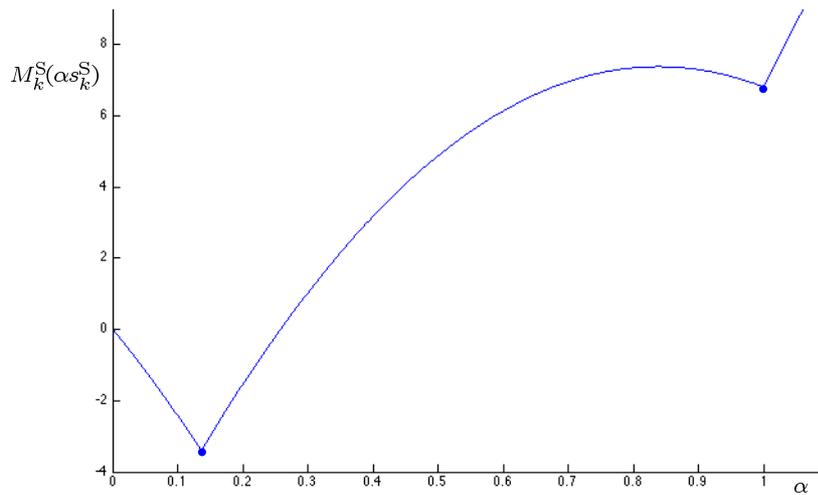
Case 2. : $a > 0$ and $-b/2a \leq \alpha_0$.

In this case we must have $\alpha_s = -b/2a$. Using this fact, the definitions of $\Delta M_k^s(s)$ and of $q(\alpha)$, norm inequalities, and the inequality $\|s_k^s\|_\infty \leq \Delta_k^s$, we have

$$\begin{aligned} \Delta M_k^s(\alpha_s s_k^s) &= q(0) - q(\alpha_s) = -a\alpha_s^2 - b\alpha_s \\ &= \frac{b^2}{4a} = \frac{((g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s)^2}{2 s_k^{sT} H_k s_k^s} \geq \frac{((g_k + H_k s_k^{CP} + \sigma J_k^T w_k)^T s_k^s)^2}{2n \|H_k\|_2 (\Delta_k^s)^2}. \end{aligned} \quad (2.25)$$



(a) Contour graph of the function $M_k^S(s)$.



(b) Graph of $M_k^S(\alpha s_k^S)$ for $0 \leq \alpha \leq 12/11$.

Figure 2.1: (a) Contour graph of $M_k^S(s)$ with local solution $s_k^S = (11/6, 0)$. The vector $s_k^{CS} = (1/4, 0)$ is a local minimizer when constrained to the direction s_k^S . (b) Graph of $M_k^S(\alpha s_k^S)$ for $0 \leq \alpha \leq 12/11$. Notice that $\alpha_s = \alpha_B = 6/44$ and that $M_k^S(s_k^S) > M_k^S(0) > M_k^S(\alpha_s s_k^S) = M_k^S(s_k^{CS})$.

Case 3. : $a \leq 0$.

In this case we must $\alpha_s = \alpha_0$. Using this fact, the definitions of $\Delta M_k^S(s)$ and of $q(\alpha)$, and the inequalities $a \leq 0$ and $b \leq 0$, we have

$$\begin{aligned} \Delta M_k^S(\alpha_s s_k^S) &= q(0) - q(\alpha_s) = -a\alpha_s^2 - b\alpha_s \\ &\geq \max(-a\alpha_s^2, -b\alpha_s) \geq -b\alpha_s \geq -\frac{1}{2}b\alpha_s = -\frac{\alpha_s}{2}(g_k + H_k s_k^{\text{CP}} + \sigma J_k^T w_k)^T s_k^S. \end{aligned} \quad (2.26)$$

The desired bound on $\Delta M_k^S(\alpha_s s_k^S)$ follows from equations (2.24), (2.25), and (2.26). \square

This result suggests that we define the full step as (see Figure 2.2)

$$s_k = \begin{cases} s_k^{\text{CP}} + s_k^S & \text{if } \Delta M_k^S(s_k^S) \geq (\eta - 1)\Delta M_k^H(s_k^{\text{CP}}); \\ s_k^{\text{CP}} + s_k^{\text{CS}} & \text{otherwise.} \end{cases} \quad (2.27)$$

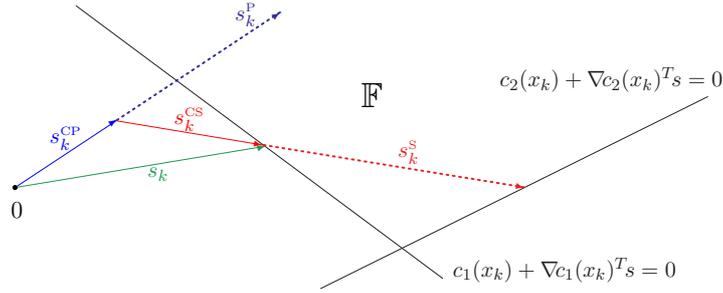


Figure 2.2: The relevant steps for subproblem (SIQP-E) with linearized constraints $c_1(x_k) + \nabla c_1(x_k)^T s = 0$ and $c_2(x_k) + \nabla c_2(x_k)^T s = 0$ and feasible region \mathbb{F} . We have illustrated the case $\Delta M_k^S(s_k^S) < (\eta - 1)\Delta M_k^H(s_k^{\text{CP}})$ so that the full step s_k is defined from the SQP-Cauchy step as $s_k = s_k^{\text{CP}} + s_k^{\text{CS}}$. If the condition $\Delta M_k^S(s_k^S) \geq (\eta - 1)\Delta M_k^H(s_k^{\text{CP}})$ had been satisfied, then the full step would have been defined as $s_k = s_k^{\text{CP}} + s_k^S$. Note that this subproblem allows for the active set at s_k to be *different* than the active set at s_k^{P} and s_k^{CP} .

The next result shows that this definition guarantees that condition (2.18) is satisfied in a non-trivial way.

Lemma 2.5. *If the SQP step and full step are defined by equation (2.27), then they satisfy*

$$\Delta M_k^H(s_k) = \Delta M_k^H(s_k^{\text{CP}}) + \Delta M_k^S(s_k) \geq \eta \Delta M_k^H(s_k^{\text{CP}}).$$

Proof. Using the definitions of M_k^H , ΔM_k^H , M_k^S , and ΔM_k^S , it follows that

$$\Delta M_k^H(s_k) = M_k^H(0) - M_k^H(s_k) = M_k^H(0) - M_k^H(s_k^{\text{CP}}) + M_k^H(s_k^{\text{CP}}) - M_k^H(s_k) = \Delta M_k^H(s_k^{\text{CP}}) + \Delta M_k^S(s_k - s_k^{\text{CP}}).$$

We may then conclude

$$\Delta M_k^H(s_k) \geq \begin{cases} \Delta M_k^H(s_k^{\text{CP}}) + \Delta M_k^S(s_k) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) & \text{if } \Delta M_k^S(s_k) \geq (\eta - 1)\Delta M_k^H(s_k^{\text{CP}}); \\ \Delta M_k^H(s_k^{\text{CP}}) + \Delta M_k^S(s_k^{\text{CS}}) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) & \text{otherwise,} \end{cases}$$

by using Lemma 2.4, equation (2.27), and the fact that $0 < \eta < 1$. \square

This result shows that if subproblem (SIQP-E) is used and the full step is defined from equation (2.27), then the full step is at least as good as the Cauchy step, which is itself good enough to guarantee convergence. We stress that Lemma 2.5 is true as a direct result of the descent-constraint; it is generally not true if the full step is defined from the traditional SQP subproblem.

2.3.2. An SEQP subproblem

The terminology SEQP indicates that this particular SQP subproblem includes a subset of the linearized constraints as *equality* constraints. The choice of which to include is based on a prediction of those constraints that will be active at a solution to problem (NP). Our prediction is based on those constraints whose linearization is *not* strictly satisfied by the predictor step s_k^P . If we define the set $\mathcal{A}(s_k^P) = \{i : [c_k + J_k s_k^P]_i \leq 0\}$, then the subproblem takes the form

$$\begin{aligned} \text{(SEQP)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} && \bar{f}_k + (g_k + H_k s_k^P)^T s + \frac{1}{2} s^T H_k s \\ & \text{subject to} && [J_k s]_{\mathcal{A}(s_k^P)} = 0, \quad \|s\|_2 \leq \Delta_k^S, \end{aligned}$$

where $\bar{f}_k = f_k + g_k^T s_k^{\text{CP}} + \frac{1}{2} s_k^{\text{CP}T} H_k s_k^{\text{CP}}$. Note that we have used the two-norm for the trust-region constraint and that this does not change any of the theoretical results since $\|s\|_\infty \leq \|s\|_2 \leq \Delta_k^S$ for all feasible vectors s . If *a priori* we knew the optimal active set, then we could compute a solution to problem (NP) by solving a sequence of equality constrained QP subproblems; this is equivalent to solving a sequence of subproblems of the form given by (SEQP) if $\mathcal{A}(s_k^P)$ agrees with the optimal active set. The fact that $\mathcal{A}(s_k^P)$ does eventually agree with the optimal active set (under certain assumptions) may be deduced from Robinson [22, Theorem 3.1]. His result implies that if x^* is a solution to problem (NP) that satisfies the strong second-order sufficient conditions for optimality, then there exists a neighborhood of x^* such that if x_k is in this neighborhood then the predictor step s_k^P will correctly identify the optimal active set provided the trust-region constraint is inactive.

Let s_k^S denote the solution to subproblem (SEQP). In an attempt to satisfy condition (2.18) in a non-trivial way, we define the full step s_k as (see Figure 2.3)

$$s_k = \begin{cases} s_k^P + s_k^S & \text{if } \Delta M_k^H(s_k^P + s_k^S) \geq \eta \Delta M_k^H(s_k^{\text{CP}}); \\ s_k^{\text{CP}} & \text{otherwise.} \end{cases} \quad (2.28)$$

We note that this update strategy guarantees that the full step s_k satisfies condition (2.18), but perhaps in a trivial way.

Generally speaking, computing a solution to subproblem (SEQP) is less expensive than computing a solution to subproblem (SIQP); this is certainly an advantage. However, problem (SEQP) has at least two disadvantages. First, its success depends on correctly

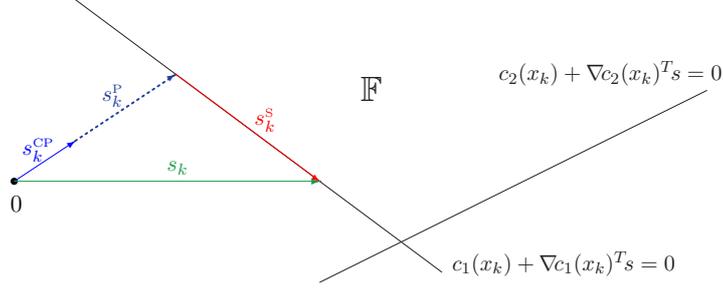


Figure 2.3: The relevant steps for subproblem (SEQP) with linearized constraints $c_1(x_k) + \nabla c_1(x_k)^T s = 0$ and $c_2(x_k) + \nabla c_2(x_k)^T s = 0$ and feasible region \mathbb{F} . The illustration depicts the case $\Delta M_k^H(s_k^P + s_k^S) \geq \eta \Delta M_k^H(s_k^{\text{CP}})$ so that the full step is defined as $s_k = s_k^P + s_k^S$. By construction, the SQP step s_k^S is contained in the null space of the constraints that are not strictly feasible at the predictor step s_k^P , i.e., the constraints whose indices are in the set $\mathcal{A}(s_k^P) = \{1\}$. Compare this with Figure 2.2.

identifying the optimal active set, which may not always occur. In fact, if any of the strong second-order sufficient conditions do not hold at a minimizer, then the predictor step is not guaranteed to correctly identify the optimal active set. Second, the strategy for defining s_k may resort to satisfying condition (2.18) in a trivial way. When this occurs, the cost in computing s_k^S has essentially been wasted. To avoid this, we consider the alternative subproblem

$$\begin{aligned}
 (\text{SEQP-ACP}) \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} && \bar{f} + (g_k + H_k s_k^{\text{ACP}})^T s + \frac{1}{2} s^T H_k s \\
 & \text{subject to} && [J_k s]_{\mathcal{A}(s_k^{\text{ACP}})} = 0, \quad \|s\|_2 \leq \Delta_k^S,
 \end{aligned}$$

where s_k^{ACP} is an approximate Cauchy point as given by equation (2.17), $\eta_{\text{ACP}} \geq \eta$, and $\mathcal{A}(s_k^{\text{ACP}}) = \{i : [c_k + J_k s_k^{\text{ACP}}]_i \leq 0\}$ (see Figure 2.4). It follows that if we define

$$s_k = s_k^{\text{ACP}} + s_k^S, \quad (2.29)$$

then

$$\Delta M_k^H(s_k) \geq \Delta M_k^H(s_k^{\text{ACP}}) \geq \eta_{\text{ACP}} \Delta M_k^H(s_k^{\text{CP}}) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) \quad (2.30)$$

since the solution s_k^S to problem (SEQP-ACP) will decrease the objective function. Therefore, s_k is guaranteed to satisfy condition (2.18) in a nontrivial way and computation is never wasted. We conclude this section by mentioning that the GALAHAD library contains the software package EQP, which may be used to solve subproblems (SEQP) and (SEQP-ACP).

2.3.3. An implicitly-constrained SIQP correction subproblem

This section discusses an SQP step that is computed from a class of *implicitly*-constrained subproblems. We use the terminology implicitly-constrained because we attempt to satisfy

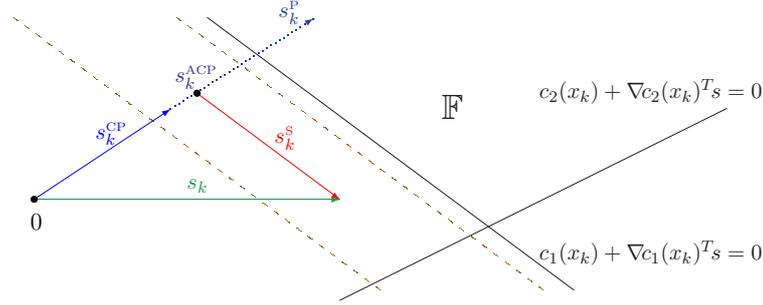


Figure 2.4: The relevant steps for subproblem (SEQP-ACP) with linearized constraints $c_1(x_k) + \nabla c_1(x_k)^T s = 0$ and $c_2(x_k) + \nabla c_2(x_k)^T s = 0$ and feasible region \mathbb{F} . All points that lie along the direction s_k^P and between the two dashed-lines are acceptable approximate Cauchy points as defined by equation (2.17). By construction, the SQP step s_k^S is contained in the null space of the constraints that are not strictly feasible at s_k^{ACP} , i.e., the constraints whose indices are in the set $\mathcal{A}(s_k^{ACP}) = \{1\}$. Compare this with Figure 2.3.

an appropriate constraint function *implicitly* by penalizing the violation of that constraint, while we use the term *correction* to emphasize that their principal use is to avoid the Maratos effect. In fact, the primary advantage of these subproblems over those discussed in Sections 2.3.1 and 2.3.2 is their direct connection to standard subproblems used for avoiding the Maratos effect. Their main disadvantage is that we are no longer guaranteed that the resultant step will provide sufficient decrease in the faithful model M_k^H ; for certain iterations, satisfying condition (2.18) in a trivial way may be required. However, since these subproblems are intended to avoid the Maratos effect, they are likely to be used asymptotically and this is precisely the situation for which we expect them to give sufficient decrease.

The class of implicitly-constrained SIQP correction subproblems is given by

$$\begin{aligned} \text{(SIQP-I)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} \quad \bar{f}_k + (g_k + H_k s_k^{CP})^T s + \frac{1}{2} s^T H_k s + \sigma \|[\chi(s)]^-\|_1 \\ & \text{subject to} \quad \|s\|_\infty \leq \Delta_k^S \end{aligned}$$

for *any* continuous vector-valued function $\chi(s)$ defined for all $\|s\|_\infty \leq \Delta_k^S$, and where $\bar{f}_k = f_k + g_k^T s_k^{CP} + \frac{1}{2} s_k^{CP T} H_k s_k^{CP}$. Using the notation $c_k^C \stackrel{\text{def}}{=} c(x_k + s_k^{CP})$ and $J_k^C \stackrel{\text{def}}{=} J(x_k + s_k^{CP})$, we define the two specific vector-valued functions $\chi(s) = c_k^C + J_k s$ and $\chi(s) = c_k^C + J_k^C s$. The resulting two subproblems are

$$\begin{aligned} \text{(SIQP-I}_1\text{)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} \quad \bar{f}_k + (g_k + H_k s_k^{CP})^T s + \frac{1}{2} s^T H_k s + \sigma \|[c_k^C + J_k s]^-\|_1 \\ & \text{subject to} \quad \|s\|_\infty \leq \Delta_k^S \end{aligned}$$

and

$$\begin{aligned} \text{(SIQP-I}_2\text{)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} \quad \bar{f}_k + (g_k + H_k s_k^{CP})^T s + \frac{1}{2} s^T H_k s + \sigma \|[c_k^C + J_k^C s]^-\|_1 \\ & \text{subject to} \quad \|s\|_\infty \leq \Delta_k^S, \end{aligned}$$

respectively.

Let s_k^S denote the solution to either subproblem (SIQP-I₁) or subproblem (SIQP-I₂). In an attempt to satisfy condition (2.18) in a non-trivial way, we define the full step s_k as (see Figure 2.5)

$$s_k = \begin{cases} s_k^{\text{CP}} + s_k^S & \text{if } \Delta M_k^H(s_k^{\text{CP}} + s_k^S) \geq \eta \Delta M_k^H(s_k^{\text{CP}}); \\ s_k^{\text{CP}} & \text{otherwise.} \end{cases} \quad (2.31)$$

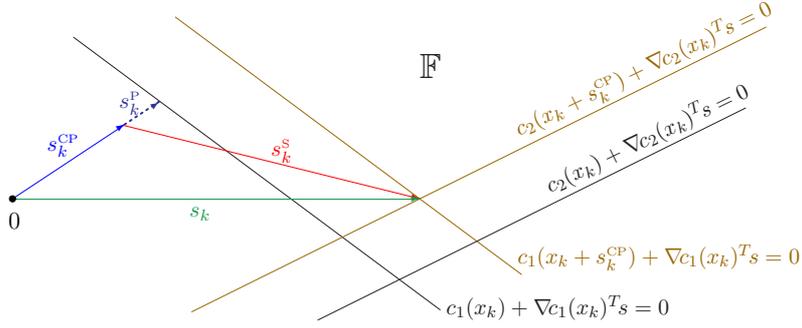


Figure 2.5: The relevant steps for subproblem (SIQP-I₁) with the *shifted* linearized constraints $c_1(x_k + s_k^{\text{CP}}) + \nabla c_1(x_k)^T s = 0$ and $c_2(x_k + s_k^{\text{CP}}) + \nabla c_2(x_k)^T s = 0$ and feasible region \mathbb{F} . We have illustrated the case $\Delta M_k^H(s_k^{\text{CP}} + s_k^S) \geq \eta \Delta M_k^H(s_k^{\text{CP}})$ so that the full step s_k is defined as $s_k = s_k^{\text{CP}} + s_k^S$.

We note that this update strategy guarantees that the full step s_k satisfies condition (2.18), but perhaps in a trivial way.

3. The Algorithm

In this section we describe Algorithm 3.1 – an SQP algorithm for computing a first-order critical point for problem $(\ell_1\text{-}\sigma)$. First, the user supplies an initial estimate (x_0, y_0) of a solution, and then “success” parameters $0 < \eta_s \leq \eta_{vs} < 1$, a maximum allowed predictor trust-region radius Δ_u , expansion and contraction factors $0 < \eta_c < 1 < \eta_e$, sufficient model decrease and approximate Cauchy point tolerances $0 < \eta \leq \eta_{\text{ACP}} < 1$, and SQP trust-region radius factor τ_f are defined. With parameters set, the main “do-while” loop begins. First, the problem functions are evaluated at the current point (x_k, y_k) . Next, a symmetric positive-definite matrix B_k is defined and the predictor step s_k^P is computed as a solution to problem (2.1). Simple choices for B_k would be the identity matrix or perhaps a scaled diagonal matrix that attempts to model the “essential properties” of the matrix $H(x_k, y_k)$. However, computing B_k via a limited-memory quasi Newton update is an attractive option. We leave further discussion of the matrix B_k to a separate paper. Next, we solve problem (2.6) for the Cauchy step s_k^{CP} , calculate the decrease in the model M_k^H as given by $\Delta M_k^H(s_k^{\text{CP}})$, and compute an SQP step if we believe that it will be advantageous. The step computation is completed by defining a full step s_k that satisfies condition (2.18).

This may be done in two ways. First, the SQP step may be skipped and then we simply define $s_k = s_k^{\text{CP}}$; this satisfies condition (2.18) in a trivial way. Second, we may solve any of the subproblems discussed in Sections 2.3.1, 2.3.2, and 2.3.3 and then define s_k according to equations (2.27), (2.28), (2.29), and (2.31), respectively. In either case we are guaranteed that s_k satisfies condition (2.18). Next, we compute $\phi(x_k + s_k)$ and $\Delta M_k^{\text{H}}(s_k)$, and then calculate the ratio r_k of actual versus predicted decrease in the merit function.

Our strategy for updating the predictor trust-region radius and for accepting or rejecting candidate steps is identical to that used by Fletcher [12] and is determined by the ratio r_k . More precisely, if the ratio r_k is larger than η_{VS} , then we believe that the model is a very accurate representation of the true merit function within the current trust-region; therefore, we increase the predictor trust-region radius with the belief that the current trust-region radius may be overly restrictive. If the ratio is greater than η_{S} , then we believe the model is sufficiently accurate and we keep the predictor trust-region radius fixed. Otherwise, the ratio indicates that there is poor agreement between the model M_k^{H} and the merit function and, therefore, we decrease the predictor trust-region radius with the hope that the model will accurately approximate the merit function over the smaller trust-region. As for step acceptance or rejection, we accept any iterate for which r_k is positive, since this indicates that the merit function has decreased. We note that the precise update used for the dual vector y_{k+1} is not important for proving global convergence, so we do not specify any particular update. However, the specific update used is essential when considering the rate-of-convergence and the most obvious update is the multiplier vector from the SQP subproblem. If the SQP step is not computed, then the most obvious update becomes the multiplier vector from the predictor subproblem. We also note that a least-squares multiplier update is possible, but would require solving a specialized inequality-constrained linear program.

Finally, we have the additional responsibility of updating the SQP trust-region radius. In Algorithm 3.1 we set the SQP trust-region radius to a constant multiple of the predictor trust-region radius although the condition $\Delta_{k+1}^{\text{S}} \leq \tau_f \cdot \Delta_{k+1}^{\text{P}}$ for some constant τ_f is also sufficient. Although this update is simple and may be viewed as “obvious”, we believe that it deserves extra discussion. If the predictor trust-region radius is *not* converging to zero on any subsequence, then the algorithm must be making good progress in reducing the merit function. However, a delicate situation arises when the trust-region radius does converge to zero on some subsequence. Since the predictor step must also be converging to zero, it seems natural to require that the full step s_k also converge to zero. Therefore it seems intuitive to require that if $\{x_{k_j}\}_{j \geq 0}$ is any subsequence such that $\lim_{j \rightarrow \infty} \|s_{k_j}^{\text{P}}\|_{\infty} = 0$, then the sequence

$$\{\Delta_{k_j}^{\text{S}} / \|s_{k_j}^{\text{P}}\|_{\infty}\}_{j \geq 0} \text{ remains bounded.} \quad (3.1)$$

A simple way to ensure this condition is by defining the SQP trust-region radius as $\Delta_{k+1}^{\text{S}} \leftarrow \tau_f \cdot \|s_k^{\text{P}}\|_{\infty}$, i.e., set the SQP trust-region radius to be a constant multiple of the size of the predictor *step*. This condition is sufficient for proving convergence, but we prefer the alternate update $\Delta_{k+1}^{\text{S}} \leftarrow \tau_f \cdot \Delta_{k+1}^{\text{P}}$, i.e., set the SQP trust-region radius to be a constant

multiple of the size of predictor *radius*. Corollary 4.1 shows that, in fact, they are equivalent asymptotically, but the update $\Delta_{k+1}^S \leftarrow \tau_f \cdot \Delta_{k+1}^P$ allows for larger value of Δ_k^S globally and has been observed to perform better during initial testing.

Algorithm 3.1. Minimizing the ℓ_1 -penalty function

Input: (x_0, y_0)

Set parameters $0 < \eta_s \leq \eta_{vs} < 1$, $\Delta_u > 0$, $0 < \eta \leq \eta_{ACP} < 1$, and $\tau_f > 0$.

Set expansion and contraction factors $0 < \eta_c < 1 < \eta_e$.

$k \leftarrow 0$

do

Evaluate f_k, g_k, c_k, J_k , and then compute ϕ_k .

Define B_k and H_k to be symmetric approximations to $H(x_k, y_k)$ with B_k positive definite.

Solve problem (2.1) for s_k^P .

Solve problem (2.6) for s_k^{CP} and compute $\Delta M_k^H(s_k^{CP})$.

Compute an SQP step s_k^S (optional).

Define a full step s_k that satisfies condition (2.18).

Evaluate $\phi(x_k + s_k)$ and $\Delta M_k^H(s_k)$.

Compute $r_k = (\phi_k - \phi(x_k + s_k)) / \Delta M_k^H(s_k)$.

if $r_k \geq \eta_{vs}$ [very successful]

$\Delta_{k+1}^P \leftarrow \min(\eta_e \cdot \Delta_k^P, \Delta_u)$ [increase predictor radius]

else if $r_k \geq \eta_s$ [successful]

$\Delta_{k+1}^P \leftarrow \Delta_k^P$ [keep predictor radius]

else [unsuccessful]

$\Delta_{k+1}^P \leftarrow \eta_c \cdot \Delta_k^P$ [decrease predictor radius]

end

if $r_k > 0$ [accept step]

$x_{k+1} \leftarrow x_k + s_k$

$y_{k+1} \leftarrow$ arbitrary

else [reject step]

$x_{k+1} \leftarrow x_k$

$y_{k+1} \leftarrow y_k$

end

$\Delta_{k+1}^S \leftarrow \tau_f \cdot \Delta_{k+1}^P$ [update SQP radius]

$k \leftarrow k + 1$

end do

4. Convergence

This section shows that Algorithm 3.1 is globally convergent. Our main result is that under certain assumptions, there exists a subsequence of the iterates generated by Algorithm 3.1 that converges to a first-order solution of problem $(\ell_1\text{-}\sigma)$. The proof requires two preliminary results as well as two estimates. First, since $f(x)$ and $c(x)$ are continuously differentiable by assumption, there exists a positive constant M such that

$$\left\| \begin{pmatrix} g(x)^T \\ J(x) \end{pmatrix} \right\|_{\infty} \leq M \text{ for all } x \in \mathcal{B}, \quad (4.1)$$

where \mathcal{B} is a closed and bounded subset of \mathbb{R}^n . Second, since the function $h(f, c) = f + \sigma \| [c]^- \|_1$ is convex, there exists a positive constant L such that

$$|h(f_1, c_1) - h(f_2, c_2)| \leq L \left\| \begin{pmatrix} f_1 - f_2 \\ c_1 - c_2 \end{pmatrix} \right\|_{\infty} \quad (4.2)$$

for all (f_1, c_1) and $(f_2, c_2) \in (f(\mathcal{B}), c(\mathcal{B}))$ [23, Theorem 10.4]. Using these bounds we may now state the following lemma, which provides a lower bound on the size of the predictor step. This is essentially [26, Lemma 3.2] except for the use of the infinity norm.

Lemma 4.1. *Let $x_k \in \mathcal{B}$ so that equations (4.1) and (4.2) hold. Then, if $\|s_k^p\|_{\infty} < \Delta_k^p$ then*

$$\|s_k^p\|_{\infty} \geq \frac{1}{2} \Delta_{\max}^L(x_k, 1) \min \left(\frac{1}{LM}, \frac{1}{n(1 + \Delta_u) \|B_k\|_2} \right). \quad (4.3)$$

Corollary 4.1. *Suppose that $\{x_k\}_{k \geq 0} \subset \mathcal{B}$ so that equations (4.1) and (4.2) hold, and that K is a subsequence of the integers such that the following hold:*

- (i) *there exists a number δ such that $\Delta_{\max}^L(x_k, 1) \geq \delta > 0$ for all $k \in K$;*
- (ii) *there exists a positive constant b_B such that $\|B_k\|_2 \leq b_B$ for all $k \in K$;*
- (iii) $\lim_{k \in K} \Delta_k^p = 0$.

Then

$$\|s_k^p\|_{\infty} = \Delta_k^p \text{ for all } k \in K \text{ sufficiently large.} \quad (4.4)$$

Proof. If equation (4.3) holds, then (i), and (ii) imply that $\|s_k^p\|_{\infty}$ is *strictly* bounded away from zero for all $k \in K$. However, this contradicts assumption (iii) for $k \in K$ sufficiently large since $\|s_k^p\|_{\infty} \leq \Delta_k^p$. Therefore, equation (4.3) must not be true and Lemma 4.1 implies that $\|s_k^p\|_{\infty} = \Delta_k^p$ for all $k \in K$ sufficiently large. \square

We may now state our main result. The organization of the proof is based on Fletcher [12, Theorem 14.5.1] and the proof of case 1 is nearly identical.

Theorem 4.1. *Let f and c be twice continuously differentiable functions, and let $\{x_k\}$, $\{H_k\}$, $\{B_k\}$, $\{\Delta_k^p\}$, and $\{\Delta_k^s\}$, be sequences generated by Algorithm 3.1. Assume that the following conditions hold:*

1. $\{x_k\}_{k \geq 0} \subset \mathcal{B} \subset \mathbb{R}^n$, where \mathcal{B} is a closed and bounded set; and
2. there exists positive constants b_B and b_H such that $\|B_k\|_2 \leq b_B$ and $\|H_k\|_2 \leq b_H$ for all $k \geq 0$.

Then, either x_K is a first-order critical point for problem $(\ell_1\text{-}\sigma)$ for some $K \geq 0$, or there exists a subsequence of $\{x_k\}$ that converges to a first-order solution of problem $(\ell_1\text{-}\sigma)$.

Proof. If x_K is a first-order point for problem $(\ell_1\text{-}\sigma)$ for some $K \geq 0$ then we are done. Therefore, we assume that x_k is not a first-order solution to problem $(\ell_1\text{-}\sigma)$ for all k . We consider two cases.

Case 1 : there exists a subsequence of $\{\Delta_k^p\}$ that converges to zero.

Examination of the algorithm shows that this implies the existence of a subsequence S of the integers such that

$$\lim_{k \in S} x_k = x_*, \quad (4.5)$$

$$\lim_{k \in S} \Delta_k^p = 0, \quad (4.6)$$

$$\lim_{k \in S} \|s_k^p\|_\infty = 0, \quad \text{and} \quad (4.7)$$

$$r_k < \eta_s \quad \text{for all } k \in S. \quad (4.8)$$

For a proof by contradiction, we suppose that x_* is not a first-order critical point. This implies that there exists a direction s and a scalar $\rho > 0$ such that $\|s\|_\infty = 1$ and

$$\max_{y \in \partial \|[c_*]^- \|_1} s^T (g_* + \sigma J_*^T y) = -\rho, \quad (4.9)$$

where $\partial \|[c_*]^- \|_1$ is the sub-differential of $\|[\cdot]^- \|_1$ at the point c_* (see [12, Section 14.3] for more details). A Taylor expansion of f at x_k in a general direction v gives

$$f(x_k + \varepsilon v) = f_k + \varepsilon g_k^T v + o(\varepsilon) = f_k + \varepsilon g_k^T v + \frac{\varepsilon^2}{2} v^T H_k v + o(\varepsilon) \quad (4.10)$$

since $\{H_k\}$ is bounded by assumption, while a Taylor expansion of c at x_k gives

$$c(x_k + \varepsilon v) = c_k + \varepsilon J_k v + o(\varepsilon). \quad (4.11)$$

Combining these two equations gives

$$\begin{aligned} \phi(x_k + \varepsilon v) &= f_k + \varepsilon g_k^T v + \frac{\varepsilon^2}{2} v^T H_k v + o(\varepsilon) + \sigma \|[c_k + \varepsilon J_k v + o(\varepsilon)]^- \|_1 \\ &= f_k + \varepsilon g_k^T v + \frac{\varepsilon^2}{2} v^T H_k v + \sigma \|[c_k + \varepsilon J_k v]^- \|_1 + o(\varepsilon) \\ &= M_k^H(\varepsilon v) + o(\varepsilon), \end{aligned} \quad (4.12)$$

where the first equality follows from the definition of ϕ and the Taylor expansions, the second equality follows from the boundedness of $\partial \|\cdot\|_1$, and the last equality follows from the definition of $M_k^H(\varepsilon v)$. The same argument using B_k in place of H_k gives the estimate

$$\phi(x_k + \varepsilon v) = M_k^B(\varepsilon v) + o(\varepsilon). \quad (4.13)$$

Choosing $v = s_k / \|s_k\|_\infty$ and $\varepsilon = \|s_k\|_\infty$ in equation (4.12), and $v = s$ and $\varepsilon = \varepsilon_k$ (we have not yet defined ε_k) in equation (4.13), yields

$$\phi(x_k + s_k) = M_k^H(s_k) + o(\|s_k\|_\infty) \quad \text{and} \quad (4.14)$$

$$\phi(x_k + \varepsilon_k s) = M_k^B(\varepsilon_k s) + o(\varepsilon_k). \quad (4.15)$$

Equation (4.14) then implies the equation

$$r_k = \frac{\phi_k - \phi(x_k + s_k)}{\Delta M_k^H(s_k)} = \frac{\Delta M_k^H(s_k) + o(\|s_k\|_\infty)}{\Delta M_k^H(s_k)} = 1 + \frac{o(\|s_k\|_\infty)}{\Delta M_k^H(s_k)}. \quad (4.16)$$

We now proceed to bound $\Delta M_k^H(s_k)$. For all $k \in S$ we have

$$\Delta M_k^H(s_k) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) \quad (4.17)$$

$$\geq \eta \Delta M_k^H(s_k^{\text{P}}) \quad (4.18)$$

$$= \eta (M_k^H(0) - M_k^H(s_k^{\text{P}})) \quad (4.19)$$

$$= \eta (M_k^B(0) - M_k^B(s_k^{\text{P}}) - \frac{1}{2} s_k^{\text{PT}} (H_k - B_k) s_k^{\text{P}}) \quad (4.20)$$

$$= \eta \Delta M_k^B(s_k^{\text{P}}) - \frac{\eta}{2} s_k^{\text{PT}} (H_k - B_k) s_k^{\text{P}} \quad (4.21)$$

$$= \eta \Delta M_k^B(s_k^{\text{P}}) + o(\|s_k^{\text{P}}\|_\infty) \quad (4.22)$$

Inequalities (4.17) and (4.18) follow from assumption (2.18) and since the Cauchy step maximizes $\Delta M_k^H(s)$ in the direction s_k^{P} . Equations (4.19) - (4.21) follow from the definitions of ΔM_k^H and ΔM_k^B , and by introducing B_k . Finally, equation (4.22) follows since $\{B_k\}$ and $\{H_k\}$ are bounded by assumption.

We now define the scalar-valued sequence $\{\varepsilon_k\}_{k \geq 0}$ such that $\varepsilon_k = \|s_k^{\text{P}}\|_\infty$. It follows that $\|\varepsilon_k s\|_\infty = \|s_k^{\text{P}}\|_\infty$ and, therefore, the vector $\varepsilon_k s$ is feasible for the k th predictor subproblem. It follows that for all $k \in S$ sufficiently large we have

$$\Delta M_k^H(s_k) \geq \eta \Delta M_k^B(\varepsilon_k s) + o(\|s_k^{\text{P}}\|_\infty) \quad (4.23)$$

$$= \eta (\phi_k - \phi(x_k + \varepsilon_k s)) + o(\|s_k^{\text{P}}\|_\infty) \quad (4.24)$$

$$\geq \eta \varepsilon_k (\rho + o(1)) + o(\|s_k^{\text{P}}\|_\infty) \quad (4.25)$$

$$= \eta \rho \varepsilon_k + o(\varepsilon_k) + o(\|s_k^{\text{P}}\|_\infty) \quad (4.26)$$

$$= \eta \rho \|s_k^{\text{P}}\|_\infty + o(\|s_k^{\text{P}}\|_\infty), \quad (4.27)$$

where we have used the convention $\zeta(\varepsilon_k) = o(1)$ to mean that $\zeta(\varepsilon_k) \rightarrow 0$ as $\varepsilon_k \rightarrow 0$. Inequality (4.23) follows from equation (4.22) and since s_k^{P} is a global minimizer for the k th predictor subproblem. Equation (4.24) follows from equation (4.15), while inequality

(4.25) follows from [12, corollary to Lemma 14.5.1]. Finally, equations (4.26) and (4.27) follow from algebra and definition of ε_k .

Equation (4.27) implies the existence of a positive sequence $\{z_k\}$ such that for $k \in S$ sufficiently large

$$\left| \frac{o(\|s_k\|_\infty)}{\Delta M_k^H(s_k)} \right| \leq \left| \frac{o(\|s_k\|_\infty)}{\eta\rho\|s_k^P\|_\infty + o(\|s_k^P\|_\infty)} \right| \quad (4.28)$$

$$\leq \frac{z_k\|s_k\|_\infty}{\frac{1}{2}\eta\rho\|s_k^P\|_\infty} \quad (4.29)$$

$$\leq \frac{2z_k(\|s_k^{CP}\|_\infty + \|s_k^S\|_\infty)}{\eta\rho\|s_k^P\|_\infty} \quad (4.30)$$

$$\leq \frac{2z_k(\|s_k^P\|_\infty + \|s_k^S\|_\infty)}{\eta\rho\|s_k^P\|_\infty} \quad (4.31)$$

$$= \frac{2z_k}{\eta\rho} \left(1 + \frac{\|s_k^S\|_\infty}{\|s_k^P\|_\infty} \right) \quad (4.32)$$

and where $\{z_k\}_S$ is a subsequence that converges to zero as $k \rightarrow \infty$. Inequality (4.28) follows from inequality (4.27), while inequality (4.29) follows from the definition of “little-oh”. Inequality (4.30) follows from the triangle-inequality, and inequalities (4.31) and (4.32) follow from how the Cauchy point s_k^{CP} is computed and simplification.

We now show that the assumptions in Corollary 4.1 are satisfied. Since x_* is not first-order optimal by assumption, it follows from part (v) of Lemma 1.1 that $\Delta_{\max}^L(x_*, 1) \neq 0$. From part (iv) of Lemma 1.1 it follows that $\Delta_{\max}^L(x_k, 1)$ is strictly bounded away from zero for $k \in S$ sufficiently large; this is assumption (i) of Corollary 4.1. Assumptions (ii) and (iii) follow directly from the assumptions in this theorem and the case we are considering.

Equation (4.32), Corollary 4.1, and the SQP trust-region radius update used in Algorithm 3.1 imply

$$\left| \frac{o(\|s_k\|_\infty)}{\Delta M_k^H(s_k)} \right| \leq \frac{2z_k}{\eta\rho} \left(1 + \frac{\|s_k^S\|_\infty}{\Delta_k^P} \right) \leq \frac{2(1 + \tau_f)z_k}{\eta\rho}. \quad (4.33)$$

Finally, inequalities (4.16) and (4.33) show that

$$r_k = 1 + o(1) \text{ for } k \in S. \quad (4.34)$$

This is a contradiction since this implies that for $k \in S$ sufficiently large the identity $r_k > \eta_s$ holds, which violates equation (4.8). Thus, x^* is a first-order critical point if Case 1 occurs.

Case 2: there does not exist a subsequence of $\{\Delta_k^P\}$ that converges to zero.

Examination of the algorithm shows that this implies the existence of a positive number δ and of an infinite subsequence S of the integers such that

$$\lim_{k \in S} x_k = x_*, \quad (4.35)$$

$$\Delta_k^P \geq \delta > 0, \text{ for all } k \quad (4.36)$$

$$r_k \geq \eta_s \text{ for all } k \in S. \quad (4.37)$$

Equation (2.18) and the fact that each $k \in S$ is a successful iterate imply

$$\phi_k - \phi(x_k + s_k) \geq \eta_s \Delta M_k^H(s_k) \geq \eta \eta_s \Delta M_k^H(s_k^{\text{CP}}). \quad (4.38)$$

Corollary 2.2, equation (4.36), the bounds b_B and b_H on B_k and H_k , and the bound $\Delta_k^P \leq \Delta_u$ imply

$$\phi_k - \phi(x_k + s_k) \geq \frac{\eta \eta_s}{4} \Delta_{\max}^L(x_k, 1) \min(\mathcal{S}_k) \quad (4.39)$$

where

$$\mathcal{S}_k = \left\{ 1, \delta, \frac{\Delta_{\max}^L(x_k, 1)}{b_B}, \frac{\Delta_{\max}^L(x_k, 1)}{b_B \Delta_u^2}, \frac{\Delta_{\max}^L(x_k, 1)}{2n(b_B + b_H)}, \frac{\Delta_{\max}^L(x_k, 1)}{2n(b_B + b_H) \Delta_u^2}, \frac{(\Delta_{\max}^L(x_k, 1))^3}{2n(b_B + b_H) b_B^2 \Delta_u^2}, \frac{(\Delta_{\max}^L(x_k, 1))^3}{2n(b_B + b_H) b_B^2 \Delta_u^6} \right\}.$$

Summing over all $k \in S$ yields

$$\sum_{k \in S} \phi_k - \phi(x_k + s_k) \geq \sum_{k \in S} \frac{\eta \eta_s}{4} \Delta_{\max}^L(x_k, 1) \min(\mathcal{S}_k). \quad (4.40)$$

Next, using the monotonicity of $\{\phi(x_k)\}_{k \geq 0}$ it follows that

$$\sum_{k \in S} \phi_k - \phi(x_k + s_k) = \sum_{k \in S} \phi_k - \phi(x_{k+1}) \leq \phi(x_0) - \phi(x_*). \quad (4.41)$$

Combining the two previous inequalities gives

$$\phi(x_0) - \phi(x_*) \geq \sum_{k \in S} \frac{\eta \eta_s}{4} \Delta_{\max}^L(x_k, 1) \min(\mathcal{S}_k), \quad (4.42)$$

which implies

$$\lim_{k \in S} \Delta_{\max}^L(x_k, 1) = 0 \quad (4.43)$$

since the series on the right-hand-side is convergent. Parts (iv) and (v) of Lemma 1.1 then imply that $\Delta_{\max}^L(x_*, 1) = 0$ and that x_* is a first-order critical point.

In both cases we have shown that there exists a limit point x_* that is a first-order critical point. We are done since one of these cases must occur. \square

As stated previously, the proof of case 1 is nearly identical to that given by Fletcher. However, his proof for case 2 does not carry over to our setting because he essentially requires the global minimizer of M_k^H over the trust-region defined by radius Δ_k^P , while we only compute the global minimizer of M_k^H in the *single* direction s_k^P .

The remainder of this section shows that the assumption $\|H_k\|_2 \leq b_H$ for all $k \geq 0$ may be weakened; similar generalizations to Fletcher's algorithm [12] have been given by Yuan [26] and our proofs are similar. These results are of theoretical and practical importance when H_k is defined from a quasi-Newton update to the matrix H_{k-1} , which is the natural choice when second derivatives are not available or may be too expensive

to compute. It was shown by Powell [20] that many quasi-Newton updating schemes only guarantee positive constants c_1 and c_2 such that

$$\|H_k\|_2 \leq c_1 + c_2 \sum_{i=1}^k \|s_k\|_\infty \quad \text{or that} \quad \|H_k\|_2 \leq c_1 + c_2 k \quad \text{for all } k \geq 0. \quad (4.44)$$

As a result, if a quasi-Newton approach is used to update H_k then Theorem 4.1 does not necessarily apply since the matrices H_k may not be bounded and, therefore, Algorithm 3.1 is not guaranteed to converge. The rest of this section shows that global convergence is still guaranteed when either of the assumptions in equation (4.44) hold.

The following lemma and proof is based on [26, Lemma 3.3]. We use the bound

$$\|s_k\|_\infty \leq \Delta_k^p + \tau_f \Delta_k^p = (1 + \tau_f) \Delta_k^p, \quad (4.45)$$

which follows from how Algorithm 3.1 defines the SQP radius, equations (2.27), (2.28), (2.29), (2.31), and the definitions of the predictor step, the Cauchy step, and the various SQP subproblems considered in Section 2.

Lemma 4.2. *Let $\{x_k\} \subset \mathcal{B}$ and suppose that there exists numbers $\delta > 0$ and $b_B \geq 1$ such that $\Delta_{\max}^L(x_k, 1) \geq \delta > 0$ and $\|B_k\|_2 \leq b_B$ for all k . Then there exists a positive constant c_3 such that*

$$\Delta_k^p \geq \frac{c_3}{M_k}, \quad (4.46)$$

where $M_k = b_B + \max_{i \leq k} \{\|H_i\|_2\}$.

Proof. Let \mathcal{S}_k be defined as in Corollary 2.2, and note that Corollary 4.1 implies that there exists a number $\Delta_* > 0$ such that if $\Delta_k^p < \Delta_*$, then $\|s_k^p\|_\infty = \Delta_k^p$. Using the bounds $\|B_k\|_2 \leq b_B$, $\Delta_k^p \leq \Delta_u$, $\Delta_{\max}^L(x_k, 1) \geq \delta > 0$, the definition of M_k , and by defining

$$\kappa_1 = \frac{1}{\Delta_u} \cdot \min \left(1, \frac{\delta}{b_B}, \frac{\delta}{b_B \Delta_u^2}, \frac{\delta}{2n}, \frac{\delta}{2n \Delta_u^2}, \frac{\delta^3}{2n b_B^2 \Delta_u^2}, \frac{\delta^3}{2n b_B^2 \Delta_u^6}, \Delta_u \right) > 0,$$

it follows that

$$\min(S_k) \geq \kappa_1 \min \left(\Delta_k^p, \frac{1}{b_B + \|H_k\|_2} \right) \geq \kappa_1 \min \left(\Delta_k^p, \frac{1}{M_k} \right). \quad (4.47)$$

Since f and c are assumed to be twice continuously differentiable, there exists a constant $0 < \Delta \leq \Delta_*$ such that

$$\left\| \begin{pmatrix} f(x) - f(\bar{x}) - g(\bar{x})^T(x - \bar{x}) \\ c(x) - c(\bar{x}) - J(\bar{x})(x - \bar{x}) \end{pmatrix} \right\|_\infty \leq \frac{(1 - \eta_s) \kappa_2}{2L} \|x - \bar{x}\|_\infty \quad (4.48)$$

for all $\|x - \bar{x}\|_\infty \leq \Delta$, where $\kappa_2 = (\kappa_1 \eta \delta)/4$, η is defined in equation (2.18), L is defined in equation (4.2), and η_s is defined in Algorithm 3.1. We claim the result is true for

$$c_3 = \min \left(\Delta_0^p M_0, \frac{\Delta \eta_c M_0}{1 + \tau_f}, \frac{\delta M_0}{2LM}, \frac{\delta}{2n(1 + \Delta_u)}, \eta_c, \frac{\eta_c \kappa_2 (1 - \eta_s) (1 - \tau_f)}{n(1 + \tau_f)^2} \right).$$

The proof is by induction on k . The bound $\Delta_0^P \geq c_3/M_0$ follows from definition of c_3 . We now assume that the result is true for k , i.e., that

$$\Delta_k^P \geq \frac{c_3}{M_k} \quad (4.49)$$

and proceed to prove it is true for $k+1$. We consider two cases.

Case 1: $\|s_k^P\|_\infty \geq \Delta/(1+\tau_f)$

By construction of the algorithm, definition of c_3 , and monotonicity of the sequence $\{M_k\}$, we have

$$\Delta_{k+1}^P \geq \eta_c \Delta_k^P \geq \eta_c \|s_k^P\|_\infty \geq \eta_c \Delta/(1+\tau_f) \geq c_3/M_0 \geq c_3/M_{k+1}, \quad (4.50)$$

so that the result is true for $k+1$.

Case 2: $\|s_k^P\|_\infty < \Delta/(1+\tau_f)$

We consider three sub-cases. First, suppose that iterate k is successful/very successful and that $\|s_k^P\|_\infty = \Delta_k^P$. Then by the inductive hypothesis and the monotonicity of $\{M_k\}$ we have

$$\Delta_{k+1}^P \geq \|s_k^P\|_\infty = \Delta_k^P \geq c_3/M_k \geq c_3/M_{k+1}$$

and the result is true for $k+1$. Second, suppose that iteration k is successful/very successful, but that $\|s_k^P\|_\infty < \Delta_k^P$. Lemma 4.1, the assumptions of this lemma, definition and monotonicity of the sequence $\{M_k\}$, definition of c_3 , and the bound $\Delta_k^P \leq \Delta_u$, imply

$$\begin{aligned} \Delta_{k+1}^P &> \|s_k^P\|_\infty \geq \frac{1}{2} \Delta_{\max}^L(x_k, 1) \min\left(\frac{1}{LM}, \frac{1}{n(1+\Delta_u)\|B_k\|_2}\right) \\ &\geq \min\left(\frac{\delta}{2LM}, \frac{\delta}{2n(1+\Delta_u)M_k}\right) \\ &\geq \min\left(\frac{c_3}{M_0}, \frac{c_3}{M_k}\right) = \frac{c_3}{M_k} \geq \frac{c_3}{M_{k+1}} \end{aligned}$$

so that the result is true for $k+1$. Finally, we suppose that iterate k is unsuccessful. Inequality (2.18), Corollary 2.2, inequality (4.47), and the assumptions of this lemma, imply

$$\Delta M_k^H(s_k) \geq \eta \Delta M_k^H(s_k^{\text{CP}}) \geq \frac{\eta \Delta_{\max}^L(x_k, 1)}{4} \min(S_k) \geq \kappa_2 \min\left(\Delta_k^P, \frac{1}{M_k}\right). \quad (4.51)$$

We now observe that since $\|s_k^P\|_\infty < \Delta/(1+\tau_f) \leq \Delta \leq \Delta_*$, it follows from the first sentence of the proof that $\Delta_k^P = \|s_k^P\|_\infty$ and therefore $\Delta_k^S = \tau_f \|s_k^P\|_\infty$. We may then conclude that

$$\|s_k\|_\infty \leq \|s_k^P\|_\infty + \|s_k^S\|_\infty \leq \|s_k^P\|_\infty + \tau_f \|s_k^P\|_\infty = (1+\tau_f) \|s_k^P\|_\infty \leq \Delta. \quad (4.52)$$

We then have from the definitions of ϕ and ΔM_k^L , equation (4.2), equation (4.52), and equation (4.48), that

$$\begin{aligned}
\phi(x_k + s_k) - \phi_k &= f(x_k + s_k) + \sigma \| [c(x_k + s_k)]^- \|_1 - (f_k + \sigma \| [c_k]^- \|_1) \\
&= f(x_k + s_k) + \sigma \| [c(x_k + s_k)]^- \|_1 - (f_k + \sigma \| [c_k]^- \|_1) \\
&\quad + f_k + g_k^T s_k + \sigma \| [c_k + J_k s_k]^- \|_1 - (f_k + g_k^T s_k + \sigma \| [c_k + J_k s_k]^- \|_1) \\
&= f(x_k + s_k) + \sigma \| [c(x_k + s_k)]^- \|_1 - (f_k + g_k^T s_k + \sigma \| [c_k + J_k s_k]^- \|_1) - \Delta M_k^L(s_k) \\
&\leq L \left\| \begin{pmatrix} f(x_k + s_k) - f_k - g_k^T s_k \\ c(x_k + s_k) - c_k - J_k s_k \end{pmatrix} \right\|_\infty - \Delta M_k^L(s_k) \\
&\leq \frac{(1 - \eta_s) \kappa_2}{2} \|s_k\|_\infty - \Delta M_k^L(s_k). \tag{4.53}
\end{aligned}$$

Since iteration k is assumed to be unsuccessful, we have

$$\phi_k - \phi(x_k + s_k) < \eta_s \left(f_k + \sigma \| [c_k]^- \|_1 - (f_k + g_k^T s_k + \frac{1}{2} s_k^T H_k s_k + \sigma \| [c_k + J_k s_k]^- \|_1) \right). \tag{4.54}$$

Rearrangement, equation (4.53), the definition of ΔM_k^L , and simplification yields

$$\begin{aligned}
\frac{\eta_s}{2} s_k^T H_k s_k &< \eta_s \left(f_k + \sigma \| [c_k]^- \|_1 - (f_k + g_k^T s_k + \sigma \| [c_k + J_k s_k]^- \|_1) \right) + \phi(x_k + s_k) - \phi_k \\
&\leq \eta_s \Delta M_k^L(s_k) + \frac{(1 - \eta_s) \kappa_2}{2} \|s_k\|_\infty - \Delta M_k^L(s_k) \\
&= (1 - \eta_s) \left[\frac{\kappa_2}{2} \|s_k\|_\infty - \Delta M_k^L(s_k) \right].
\end{aligned}$$

“Adding” this inequality to $(1 - \eta_s)$ times equation (4.51) and then using the definitions of ΔM_k^L and ΔM_k^H , we have

$$\frac{\eta_s}{2} s_k^T H_k s_k + \kappa_2 (1 - \eta_s) \min \left(\Delta_k^P, \frac{1}{M_k} \right) \leq (1 - \eta_s) \left[\frac{\kappa_2}{2} \|s_k\|_\infty - \Delta M_k^L(s_k) + \Delta M_k^H(s_k) \right] \tag{4.55}$$

$$= (1 - \eta_s) \left[\frac{\kappa_2}{2} \|s_k\|_\infty - \frac{1}{2} s_k^T H_k s_k \right] \tag{4.56}$$

Rearrangement and simplification gives

$$s_k^T H_k s_k \leq \kappa_2 (1 - \eta_s) \left[\|s_k\|_\infty - \min \left(2\Delta_k^P, \frac{2}{M_k} \right) \right]. \tag{4.57}$$

Next, using the Cauchy-Schwarz inequality, simple algebra, and equation (4.45), we have

$$\|H_k\|_2 \|s_k\|_2^2 \geq \kappa_2 (1 - \eta_s) \left[\min \left(2\Delta_k^P, \frac{2}{M_k} \right) - \|s_k\|_\infty \right] \tag{4.58}$$

$$= \kappa_2 (1 - \eta_s) \left[\min \left(2\Delta_k^P - \|s_k\|_\infty, \frac{2}{M_k} - \|s_k\|_\infty \right) \right] \tag{4.59}$$

$$\geq \kappa_2 (1 - \eta_s) \min \left(\frac{1 - \tau_f}{1 + \tau_f} \|s_k\|_\infty, \frac{2}{M_k} - \|s_k\|_\infty \right). \tag{4.60}$$

If $\frac{1-\tau_f}{1+\tau_f}\|s_k\|_\infty \geq \frac{2}{M_k} - \|s_k\|_\infty$, it follows that $\|s_k\|_\infty \geq \frac{1+\tau_f}{M_k}$; otherwise, we have

$$n\|H_k\|_2\|s_k\|_\infty^2 \geq \|H_k\|_2\|s_k\|_2^2 \quad (4.61)$$

$$\geq \frac{\kappa_2(1-\eta_s)(1-\tau_f)}{1+\tau_f}\|s_k\|_\infty, \quad (4.62)$$

which implies

$$\|s_k\|_\infty \geq \frac{\kappa_2(1-\eta_s)(1-\tau_f)}{n(1+\tau_f)\|H_k\|_2} \geq \frac{\kappa_2(1-\eta_s)(1-\tau_f)}{n(1+\tau_f)M_k}. \quad (4.63)$$

Putting these two observations together, we have

$$\|s_k\|_\infty \geq \frac{1}{M_k} \min\left(1+\tau_f, \frac{\kappa_2(1-\eta_s)(1-\tau_f)}{n(1+\tau_f)}\right).$$

This inequality, equation (4.45), definition of c_3 , and monotonicity of the sequence $\{M_k\}$ imply

$$\Delta_{k+1}^p = \eta_c \Delta_k^p \geq \frac{\eta_c}{(1+\tau_f)}\|s_k\|_\infty \quad (4.64)$$

$$\geq \frac{\eta_c}{M_k(1+\tau_f)} \min\left(1+\tau_f, \frac{\kappa_2(1-\eta_s)(1-\tau_f)}{n(1+\tau_f)}\right) \quad (4.65)$$

$$\geq \frac{1}{M_k} \min\left(\eta_c, \frac{\eta_c \kappa_2(1-\eta_s)(1-\tau_f)}{n(1+\tau_f)^2}\right) \geq \frac{c_3}{M_k} \geq \frac{c_3}{M_{k+1}}, \quad (4.66)$$

so that the result is true for $k+1$. We are done since this covers all cases. \square

The next Lemma is due to Powell [21, p. 301].

Lemma 4.3. *Let $\{\Delta_k\}$ and $\{M_k\}$ be two sequences such that $\Delta_k \geq c/M_k$ for all k , where c is a positive constant. Let \mathcal{S} be a subset of the integers such that*

1. $\Delta_{k+1} \leq \eta_e \Delta_k$, for all $k \in \mathcal{S}$;
2. $\Delta_{k+1} \leq \eta_c \Delta_k$, for all $k \notin \mathcal{S}$; and
3. $M_{k+1} \geq M_k$ for all k and that $\sum_{k \in \mathcal{S}} \min(\Delta_k, 1/M_k) < \infty$,

where $\eta_e > 1$, $\eta_c < 1$ are positive constants. Then

$$\sum_{k=1}^{\infty} 1/M_k < \infty.$$

\square

We now establish convergence of Algorithm 3.1 under the second assumption in equation (4.44); the statement and proof are similar to those given by Yuan [26, Theorem 3.5].

Theorem 4.2. *Let f and c be twice continuously differentiable functions, and let $\{x_k\}$, $\{H_k\}$, $\{B_k\}$, $\{\Delta_k^p\}$, and $\{\Delta_k^s\}$, be sequences generated by Algorithm 3.1. Assume that the following conditions hold:*

1. *the sequence $\{x_k\} \in \mathcal{B} \subset \mathbb{R}^n$ for all $k \geq 0$ where \mathcal{B} is a closed and bounded set;*
2. *there exists a positive constant b_B such that $\|B_k\|_2 \leq b_B$; and*
3. *there exist positive constants c_1 and c_2 such that*

$$\|H_k\|_2 \leq c_1 + c_2 k \quad \text{for all } k \geq 0.$$

Then, either x_K is a first-order critical point for problem $(\ell_1-\sigma)$ for some $K \geq 0$, or there exists a subsequence of $\{x_k\}$ that converges to a first-order solution of problem $(\ell_1-\sigma)$.

Proof. If x_K is a first-order critical point for problem $(\ell_1-\sigma)$ for some $K \geq 0$, then we are done. So suppose that this is not the case. For a proof by contradiction, we suppose that there exists a positive number δ such that $\Delta_{\max}^L(x_k, 1) \geq \delta > 0$ holds. It is easily verified that equation (4.47) holds and a similar argument to that used in the proof of Theorem 4.1 shows that

$$\phi(x_0) - \phi(x_{j+1}) \geq \sum_{k \in \mathcal{S}, k \leq j} \eta \eta_s \Delta M_k^H(s_k^{\text{CP}}), \quad (4.67)$$

where \mathcal{S} is the set of those values of k that are successful/very successful iterates.

Since f is continuous on the bounded set \mathcal{B} , it follows that $\{\phi(x_k)\}_{k \geq 0}$ is bounded from below. The previous inequality then implies

$$\sum_{k \in \mathcal{S}} \Delta M_k^H(s_k^{\text{CP}}) < \infty, \quad (4.68)$$

and from Corollary 2.2 and inequality $\Delta_{\max}^L(x_k, 1) \geq \delta > 0$ we conclude that

$$\sum_{k \in \mathcal{S}} \min(S_k) < \infty. \quad (4.69)$$

It follows from equations (4.69) and (4.47) that $\sum_{k \in \mathcal{S}} \min(\Delta_k^p, 1/M_k) < \infty$, and then Lemmas 4.2 and 4.3 imply

$$\sum_{k=1}^{\infty} 1/M_k < \infty.$$

However, the definition of M_k and assumption 3 imply

$$\sum_{k=1}^{\infty} \frac{1}{M_k} = \sum_{k=1}^{\infty} \frac{1}{b_B + \max_{i < k} \{\|H_i\|_2\}} \geq \sum_{k=1}^{\infty} \frac{1}{b_B + c_1 + c_2 k} = \infty, \quad (4.70)$$

which is a contradiction. Thus, there exists a subsequence K_1 such that $\lim_{k \in K_1} \Delta_{\max}^L(x_k, 1) = 0$. \square

An immediate corollary is that Algorithm 3.1 is globally convergent under the first assumption in equation (4.44).

Corollary 4.2. *Let f and c be twice continuously differentiable functions, and let $\{x_k\}$, $\{H_k\}$, $\{B_k\}$, $\{\Delta_k^p\}$, and $\{\Delta_k^s\}$, be sequences generated by Algorithm 3.1. Assume that the following conditions hold:*

1. *the sequence $\{x_k\} \in \mathcal{B} \subset \mathbb{R}^n$ for all $k \geq 0$ where \mathcal{B} is a closed and bounded set;*
2. *there exists a positive constant b_B such that $\|B_k\|_2 \leq b_B$; and*
3. *there exist positive constants c_1 and c_2 such that*

$$\|H_k\|_2 \leq c_1 + c_2 \sum_{i=1}^k \|s_i\|_\infty \quad \text{for all } k \geq 0.$$

Then, either x_K is a first-order critical point for problem $(\ell_1-\sigma)$ for some $K \geq 0$, or there exists a subsequence of $\{x_k\}$ that converges to a first-order solution of problem $(\ell_1-\sigma)$.

Proof. Since $\Delta_k^p \leq \Delta_u$ and $\Delta_k^s = \tau_f \Delta_k^p$, it follows that $\|s_k\|_\infty \leq \Delta_u + \tau_f \Delta_u = (1 + \tau_f) \Delta_u$. Therefore,

$$\|H_k\|_2 \leq c_1 + c_2 \sum_{i=1}^k \|s_i\|_\infty \leq c_1 + (c_2 \Delta_u (1 + \tau_f)) k \quad \text{for all } k \geq 0. \quad (4.71)$$

The result now follows from Theorem 4.2. □

5. Conclusions and future work

Research on second-derivative SQP methods is very active. The optimization community continues to tangle with the difficulties associated with nonconvex subproblems in an attempt to further our understanding of these methods. This paper has provided additional understanding by showing that the relatively simple idea of imposing descent (via an artificial constraint) guarantees that certain pitfalls typically associated with second-derivative SQP algorithms may be avoided.

We presented an SQP method that is based on the work by Fletcher [12]. In Section 2, we described how to compute trial steps using a predictor step, a Cauchy step and an (optional) SQP step. Since there is considerable flexibility in defining the SQP step from various subproblems, we explored three options: Section 2.3.1 discussed an explicitly-constrained SIQP subproblem that was enhanced by an artificial descent-constraint; Section 2.3.2 considered a subproblem based on the so-called SEQP approach; and Section 2.3.3 discussed a class of implicitly-constrained SIQP correction subproblems that were designed to avoid the Maratos effect. We feel that the flexibility in our algorithm provides a natural framework for avoiding the Maratos effect that is less ad-hoc than traditional means. The key to an effective and efficient implementation of our method is the careful utilization of the advantages that each subproblem enjoys.

In Section 4, we proved that our method is globally convergent without having to compute the global minimizer of a nonconvex quadratic program; this is arguably the greatest contribution of this paper. We also proved global convergence under weaker assumptions on the Hessian matrices; these results are analogous to those provided by Yuan [26] for Fletcher’s $S\ell_1QP$ method.

In a companion paper we plan to 1) discuss strategies for updating the penalty parameter; 2) investigate local convergence issues; 3) discuss mechanisms for defining convex approximations to the Hessian of the Lagrangian in the large-scale case; and 4) provide numerical experiments with our evolving GALAHAD package **S2QP**. We note that Byrd, Nocedal, and Waltz [6] and Byrd et al. [4] have published clever techniques for updating the penalty parameter and this will influence our developments.

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