Science & Technology Facilities Council

Technical Report

Discrete interpolation norms with applications

Mario Arioli and Daniel Loghin

July 7, 2008

RAL-TR-2008-012

© **Science and Technology Facilities Council**

Enquires about copyright, reproduction and requests for additional copies of this report should be addressed to:

Library and Information Services
SFTC Rutherford Appleton Laboratory
Harwell Science and Innovation Campus
Didcot
OX11 0QX
UK
Tel: +44 (0)1235 445384
Fax: +44(0)1235 446403
Email: library@rl.ac.uk

The STFC ePublication archive (epubs), recording the scientific output of the Chilbolton, Daresbury, and Rutherford Appleton Laboratories is available online at:
http://epubs.cclrc.ac.uk/

ISSN 1358-6254

Neither the Council nor the Laboratory accept any responsibility for loss or
damage arising from the use of information contained in any of their reports
or in any communication about their tests or investigation

RAL-TR-2008-012

Discrete interpolation norms with applications

M. Arioli[1] and D. Loghin[2]

ABSTRACT
We derive discrete norm representations associated with projections of interpolation spaces onto finite dimensional subspaces. These norms are products of integer and noninteger powers of the Grammian matrices associated with the generating pair of spaces for the interpolation space. We include a brief description of some of the algorithms which allow the efficient computation of matrix powers. We consider in some detail the case of fractional Sobolev spaces both for positive and negative indices together with applications arising in preconditioning techniques. Numerical experiments are included.

Keywords: Generalized Laplace operators, Fractional Sobolev spaces, Krylov methods, Generalised Lanczos method, Domain decomposition, Matrix square root.

AMS(MOS) subject classifications: 35J15, 35J50, 47G30, 65F10, 65F15, 65F30, 65N30.

Current reports available by anonymous ftp to ftp.numerical.rl.ac.uk in directory pub/reports.
[1] m.arioli@rl.ac.uk, Rutherford Appleton Laboratory.
[2] d.loghin@bham.ac.ukm.arioli@rl.ac.uk, School of Mathematics
University of Birmingham, UK.
The work was supported by EPSRC grant GR/S42170/01.

Computational Science and Engineering Department
Atlas Centre
Rutherford Appleton Laboratory
Oxon OX11 0QX

July 7, 2008

Contents

1 Introduction

Interpolation spaces arise naturally in the formulation of many modeling applications ranging from domain decomposition methods (Dryja 1982), (Brezzi and Marini 1994), (Bertoluzza and Kunoth 2000), (Quarteroni and Valli 1999), (Toselli and Widlund 2005) and multilevel methods (Bramble, Pasciak and Vassilevski 2000), (Burstedde. 2007), to image processing (Natterer 1980), (Neubauer 1988), (Hegland 1995), (Tautenhahn 1996), (Egger and Neubauer 2005), (Egger 2006), advection-diffusion problems (Sangalli 2003, Sangalli 2005, Sangalli 2008), and elasticity (Glowinski and Pironneau 1979). It is therefore desirable to characterise the corresponding discrete spaces with the aim to enable new numerical approaches and algorithms. In particular, we are interested in defining discrete norm representations for projections of interpolation spaces onto suitable finite dimensional subspaces. As a major application we derive discrete norms corresponding to conforming finite element discretisations of fractional Sobolev spaces. In particular, we show that they involve fractional powers of products of certain Grammian matrices associated with the bases of the finite element spaces employed. For the case where wavelet spaces are of interest, some recent results can be found in (Burstedde. 2007), (Burstedde 2005), where matrix representations in wavelet bases are given for Sobolev spaces of non-integer orders.

One limitation associated with discrete interpolation norms is that for large scale computations they are seemingly expensive to compute. We show that this need not be so in the case of finite element norms for which inexpensive factorisations can be devised using some standard approximation procedures such as projection onto Krylov subspaces.

The paper is organised as follows. In section 2 we introduce the concept of interpolation between abstract Hilbert spaces as described in (Lions and Magenes 1968). We also consider the projection onto finite dimensional subspaces and derive expressions for the associated norm representations. In section 3 we derive the discrete norms resulting from projection onto finite element spaces of fractional Sobolev spaces. The following section contains a brief review of existing algorithms for computation of matrix non-integer powers which arise in the definition of discrete interpolation norms; we also include in this section the details of a Lanczos procedure employed for the computation of interpolation norms. Section 5 includes applications arising in preconditioning of domain decomposition methods and methods for the biharmonic equation. The paper concludes with some numerical experiments.

2 Interpolation spaces

We review here the presentation from Lions and Magenes (1968). Let X, Y denote two Hilbert spaces with $X \subset Y$, X dense and continuously embedded in Y. Let $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$ denote the corresponding inner products, and $\|\cdot\|_X, \|\cdot\|_Y$ the respective norms. By the Riesz representation theory (see for example Riesz and Sz-Nagy, 1956) there exists an operator $\mathcal{J} : X \to Y$ which is positive and self-adjoint with respect to $(\cdot,\cdot)_Y$ such that

$$(u, v)_X = (u, \mathcal{J} v)_Y \,. \tag{1}$$

Using the spectral decomposition of $\mathcal{J}$ we define the operator $\mathcal{E} = \mathcal{J}^{1/2} : X \to Y$, which in turn is positive self-adjoint. Moreover (see Lions and Magenes, 1968, Chapter 1, Section 2.2), the space X can be defined to be the domain $D(\mathcal{E})$ of $\mathcal{E}$ and the norm of X is equivalent to the graph norm $\|\cdot\|_{\mathcal{E}}$

$$\|u\|_X \sim \|u\|_{\mathcal{E}} := \left(\|u\|_Y^2 + \|\mathcal{E}u\|_Y^2\right)^{1/2} .$$

Similarly, the spectral decomposition of $\mathcal{E}$ can be used to define any real power of $\mathcal{E}$. Let $\theta \in [0,1]$ and let $\|\cdot\|_\theta$ denote the scale of graph norms

$$\|u\|_\theta := \left(\|u\|_Y^2 + \|\mathcal{E}^{1-\theta}u\|_Y^2\right)^{1/2}. \tag{2}$$

One can then show that the domain of $\mathcal{E}^{1-\theta}$ endowed with the inner-product

$$(u,v)_\theta = (u,v)_Y + \left(u, \mathcal{E}^{1-\theta}v\right)_Y$$

is a Hilbert space (Lions and Magenes, 1968). This is an *interpolation space of index θ for the pair* $[X,Y]$ and is denoted by $[X,Y]_\theta$

$$[X,Y]_\theta := D(\mathcal{E}^{1-\theta}), \qquad 0 \le \theta \le 1.$$

Note that $[X,Y]_0 = X$ and $[X,Y]_1 = Y$. Moreover, if $0 < \theta_1 < \theta_2 < 1$ then

$$X \subset [X,Y]_{\theta_1} \subset [X,Y]_{\theta_2} \subset Y. \tag{3}$$

Let now $\mathcal{L}(A;B)$ denote the space of continuous linear operators from A into B. The following classic interpolation theorem can be found in (Lions and Magenes 1968, Theorem 5.1).

Theorem 2.1 *Let X, Y be defined as above and let $\mathcal{X}, \mathcal{Y}$ satisfy similar properties. Let $\pi \in \mathcal{L}(X;\mathcal{X}) \cap \mathcal{L}(Y;\mathcal{Y})$. Then for all $\theta \in (0,1)$,*

$$\pi \in \mathcal{L}([X,Y]_\theta; [\mathcal{X},\mathcal{Y}]_\theta).$$

We turn now to the case where the spaces generating the scale of interpolation spaces are finite-dimensional. In particular, we are interested in the discrete norms associated with these spaces.

2.1 Finite dimensional interpolation spaces.

Let $H \in \mathbb{R}^{n\times n}$ be a Diagonalisable matrix with a real Positive-definite Spectrum (DPS). Let the eigenvalue decomposition of H be denoted by $H = V^{-1}D_HV$ where D_H is a diagonal matrix with entries λ_i which satisfy

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots \lambda_n.$$

The following definition (Lancaster and Tismenetsky, 1985, and Higham, 2008) will be useful in the subsequent discussion.

Definition 1 *Let $\theta \in \mathbb{R}$. The θ power of a DPS matrix $H = V^{-1}D_HV$ is defined via the decomposition*

$$H^\theta := V^{-1}D_H^\theta V.$$

If H is symmetric, then $H = H^{1/2}H^{1/2}$ where $H^{1/2} = V^{-1}D_H^{1/2}V$ is also a symmetric and positive-definite matrix and $V^{-1} = V^T$. We also remark that a DPS matrix H is a self-adjoint operator in the scalar product $(\ ,\ )_{V^TV}$ and that D_H is also the spectrum of the positive definite pencil $\{V^TV, V^TD_HV\}$.

Furthermore, H induces a vector-norm $\|\cdot\|_H$ with dual-norm $\|\cdot\|_{H'}$ defined as ((Horn and Johnson. 1985, p. 275))

$$\|\mathbf{z}\|_{H'} := \max_{\mathbf{v}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\mathbf{v}^T\mathbf{z}}{\|\mathbf{v}\|_H} = \|\mathbf{z}\|_{H^{-1}}. \tag{4}$$

Remark 2.1 *Let* $\mathbf{z}$ *be defined to be the vector of coefficients of a function* $z = \sum_{i=1}^{n} \mathbf{z}_i \psi_i$ *expanded in a given basis* $\{\psi_i\}_{1\le i\le n}$. *Let* $\{\phi_i\}_{1\le i\le n}$ *be another linearly independent set defined via*

$$\phi_i = \sum_{j=1}^{n} K_{ij}\psi_j,$$

with $K \in \mathbb{R}^{n\times n}$ *nonsingular. Then* $z = \sum_{i=1}^{n} \mathbf{z}_i\psi_i = \sum_{i=1}^{n} \mathbf{w}_i\phi_i$ *so that* $\mathbf{z} = K\mathbf{w}$ *and*

$$\|\mathbf{z}\|_{H'} = \|K\mathbf{w}\|_{H^{-1}} = \|\mathbf{w}\|_{K^T H^{-1} K}. \tag{5}$$

The matrix $K^T H^{-1} K$ *is the representation of* H' *with respect to a new basis which is related to the original basis via* K.

Given two symmetric and positive-definite matrices $H_1 \in \mathbb{R}^{n\times n}, H_2 \in \mathbb{R}^{m\times m}$ we define the following matrix norm for matrices $M \in \mathbb{R}^{m\times n}$ (see (Horn and Johnson. 1985, p. 311))

$$\|M\|_{H_1,H_2} = \max_{\mathbf{v}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|M\mathbf{v}\|_{H_2}}{\|\mathbf{v}\|_{H_1}} \tag{6}$$

Finally, we define the H-condition number of a matrix M to be the quantity

$$\kappa_H(M) = \|M\|_{H,H^{-1}}\|M^{-1}\|_{H^{-1},H}.$$

Let now $X_h \subset X, Y_h \subset Y$ denote two finite-dimensional subspaces of X, Y respectively, with $n = \dim X_h = \dim Y_h$. They are Hilbert spaces when endowed with the inner-products $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$. We can similarly define corresponding positive, self-adjoint operators $\mathcal{J}_h, \mathcal{E}_h : X_h \to Y_h$

$$(u_h, v_h)_X = (u_h, \mathcal{T}_h v_h)_Y \qquad u_h, v_h \in X_h \tag{7}$$

where $\mathcal{J}_h$ is positive self-adjoint and $\mathcal{E}_h = \mathcal{J}_h^{1/2}$. We define the discrete interpolation spaces

$$[X_h, Y_h]_\theta := D(\mathcal{E}_h^{1-\theta}).$$

Furthermore, we define the scale of discrete norms

$$\|u_h\|_{\theta,h} := \left(\|u_h\|_Y^2 + \|\mathcal{E}_h^{1-\theta}u_h\|_Y^2\right)^{1/2}. \tag{8}$$

We are interested in describing the set of symmetric and positive definite matrices

$$\left\{H_\theta \in \mathbb{R}^{n\times n} : n \in \mathbb{N}, 0 \le \theta \le 1\right\}$$

which induce norms equivalent to $\|\cdot\|_{\theta,h}$ with constants of equivalence independent of n. Let H_X, H_Y denote the Grammian (or Riesz) matrices corresponding to the inner products $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$, respectively. More precisely,

$$(H_X)_{ij} = (\phi_i, \phi_j)_X, \quad (H_Y)_{ij} = (\phi_i, \phi_j)_Y, \qquad 1 \le i, j \le n,$$

where $\{\phi_i\}_{1\le i\le n}$ denotes a basis of X_h, so that

$$\|u_h\|_X = \|\mathbf{u}\|_{H_X}, \quad \|u_h\|_Y = \|\mathbf{u}\|_{H_Y},$$

where $\mathbf{u}$ denotes the vector of coefficients of u_h expanded in the basis $\{\phi_i\}$. We first note that the discrete Riesz representation (7) becomes

$$\mathbf{u}^T H_X \mathbf{v} = \mathbf{u}^T H_Y J \mathbf{v}$$

so that $J = H_Y^{-1} H_X$ is a product of two symmetric and positive definite matrices. The matrix J is self-adjoint and positive definite in the discrete H_Y-inner-product, as

$$(\mathbf{u}, T\mathbf{v})_{H_Y} = \mathbf{u}^T H_Y (H_Y^{-1} H_X)\mathbf{v} = \mathbf{v}^T H_X \mathbf{u} = \mathbf{v}^T H_Y (H_Y^{-1} H_X)\mathbf{u} = (\mathbf{v}, T\mathbf{u})_{H_Y}$$

and

$$(\mathbf{u}, J\mathbf{u})_{H_Y} = \mathbf{u}^T H_X \mathbf{u} > 0 \quad \text{for all } \mathbf{u} \neq \mathbf{0}.$$

One can also write explicitly the eigenvalue decomposition of J : since H_X, H_Y are symmetric and positive-definite, there exists a matrix Q such that ((Horn and Johnson. 1985, Cor 7.6.2))

$$H_X = Q^T D Q, \quad H_Y = Q^T Q,$$

where D is a diagonal matrix with positive entries, so that

$$J = H_Y^{-1} H_X = Q^{-1} D Q.$$

Note that this implies that J is DPS, so that real powers of J can be defined via Definition 1.

It is evident that the matrix representation of $\mathcal{E}_h$ in the basis $\{\phi_i\}$ is the DPS matrix

$$E = Q^{-1} D^{1/2} Q;$$

furthermore, the matrix representation of $\mathcal{E}_h^{1-\theta}$ is similarly

$$E^{1-\theta} = Q^{-1} D^{(1-\theta)/2} Q.$$

We now turn to the matrix representation of the norm $\|\cdot\|_{\theta,h}$ which we denote by $H_{\theta,h}$. Definition (8) yields

$$\|\mathbf{u}\|_{H_{\theta,h}}^2 = \|\mathbf{u}\|_{H_Y}^2 + \|E^{1-\theta}\mathbf{u}\|_{H_Y}^2$$

so that

$$H_{\theta,h} = H_Y + (E^{1-\theta})^T H_Y E^{1-\theta} = H_Y + Q^T D^{1-\theta} Q = H_Y + H_Y J^{1-\theta} = Q^T (I + D^{1-\theta}) Q.$$

Now, by definition, the norm of X_h is equivalent to the discrete graph norm (8) with $\theta = 0$:

$$Q^T D Q = H_X \sim H_{0,h} = H_Y + H_Y J = Q^T (I + D) Q.$$

Hence, there exist two positive real constants α_1, α_2 independent of n such that

$$\alpha_1 D_{ii} \leq (1 + D_{ii}) \leq \alpha_2 D_{ii} \qquad 1 \leq i \leq n.$$

It follows that, setting $\tilde{\alpha}_1 = 1, \tilde{\alpha}_2 = \max\{\alpha_2, 2\}$, there holds

$$\tilde{\alpha}_1 D_{ii}^{1-\theta} \leq (1 + D_{ii}^{1-\theta}) \leq \tilde{\alpha}_2 D_{ii}^{1-\theta} \qquad 1 \leq i \leq n$$

and we deduce that

$$H_{\theta,h} \sim H_\theta := Q^T D^{1-\theta} Q = H_Y J^{1-\theta}. \tag{9}$$

The matrix H_θ is the reduced form of $H_{\theta,h}$ and is equivalent to it.

Remark 2.2 *If the matrices H_X, H_Y are simultaneously diagonalisable, one can derive a simpler expression for $H_{\theta,h}, H_\theta$. Assuming there exists an invertible matrix Z such that*

$$Z^{-1}H_X Z = D_X, \quad Z^{-1}H_Y Z = D_Y,$$

we find

$$J = H_Y^{-1}H_X = ZD_Y^{-1}D_X Z^{-1}, \quad J^{1-\theta} = H_Y^{\theta-1}H_X^{1-\theta}$$

and thus

$$H_{\theta,h} = H_Y + H_Y J^{1-\theta} = H_Y + H_Y H_Y^{\theta-1}H_X^{1-\theta} = H_Y + H_Y^\theta H_X^{1-\theta}$$

and

$$H_\theta = H_Y^\theta H_X^{1-\theta} = H_Y^{\theta/2}H_X^{1-\theta}H_Y^{\theta/2}.$$

Let now $i_h : \mathcal{L}(X_h; X) \cap \mathcal{L}(Y_h; Y)$ denote the continuous injection operator between the indicated spaces. By Thm 2.1,

$$i_h \in \mathcal{L}([X_h, Y_h]_\theta; [X, Y]_\theta). \tag{10}$$

In particular, for all $u_h \in [X_h, Y_h]_\theta$,

$$\|i_h u_h\|_\theta = \|u_h\|_\theta \leq C_1 \|u_h\|_{\theta,h}.$$

Assume now that there exists an interpolation operator I_h such that $I_h : \mathcal{L}(X; X_h) \cap \mathcal{L}(Y; Y_h)$ and $I_h u = u_h$ for all $u_h \in X_h$. Again, by Thm 2.1,

$$I_h \in \mathcal{L}([X, Y]_\theta; [X_h, Y_h]_\theta) \tag{11}$$

and we have for all $u \in [X, Y]_\theta$,

$$\|I_h u\|_{\theta,h} \leq C_2 \|u\|_\theta.$$

Since $[X_h, Y_h]_\theta \subset [X, Y]_\theta$ (the set inclusion follows from $X_h \subset X$ and $Y_h \subset Y$), Eqns (10), (11) yield

$$\frac{1}{C_1}\|u_h\|_\theta \leq \|u_h\|_{\theta,h} \leq C_2 \|u_h\|_\theta.$$

We summarise this result in the following Lemma.

Lemma 2.2 *Let $X_h \subset Y_h, X \subset Y$ be Hilbert spaces with inner-products $(\cdot,\cdot)_X, (\cdot,\cdot)_Y$ and let $\|\cdot\|_\theta, \|\cdot\|_{\theta,h}$ be defined by* (2), (8)*, respectively. Let us assume that there exists an interpolation operator I_h such that $I_h : \mathcal{L}(X; X_h) \cap \mathcal{L}(Y; Y_h)$ and $I_h u = u_h$ for all $u_h \in X_h$. Then the norms $\|\cdot\|_\theta, \|\cdot\|_{\theta,h}$ are equivalent on $[X_h, Y_h]_\theta$ for all $\theta \in (0, 1)$.*

Corollary 2.3 *Let the assumptions of Lemma* 2.2 *hold. Let H_θ be defined as in* (9)*. Then the norms $\|\cdot\|_\theta, \|\cdot\|_{H_\theta}$ are equivalent on $[X_h, Y_h]_\theta$ for all $\theta \in (0, 1)$.*

Finally, we end with the following result concerning the conditioning of H_θ.

Lemma 2.4 *Let $\kappa := \kappa_{H_Y}(H_X)$ denote the H_Y-condition number of H_X. Then*

$$\kappa_{H_Y}(H_\theta) = \kappa^{1-\theta}.$$

Proof: Using the decompositions $H_X = Q^T D Q, H_Y = Q^T Q$, we find

$$\|H_X\|_{H_Y,H_Y^{-1}} = \max_{\mathbf{v}\in\mathbb{R}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{v}^T H_X \mathbf{v}}{\mathbf{v} H_Y \mathbf{v}} = \max_i D_{ii}, \quad \|H_X^{-1}\|_{H_Y^{-1},H_Y} = \max_{\mathbf{v}\in\mathbb{R}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{v}^T H_0^{-1} \mathbf{v}}{\mathbf{v} H_1^{-1} \mathbf{v}} = \left(\min_i D_{ii}\right)^{-1}$$

so that

$$\kappa_{H_Y}(H_X) = \kappa_2(D).$$

Similarly we find

$$\|H_\theta\|_{H_Y,H_Y^{-1}} = \max_{\mathbf{v}\in\mathbb{R}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{v}^T H_\theta \mathbf{v}}{\mathbf{v} H_Y \mathbf{v}} = \max_i D_{ii}^{1-\theta}, \quad \|H_\theta^{-1}\|_{H_Y^{-1},H_Y} = \max_{\mathbf{v}\in\mathbb{R}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{v}^T H_0^{-1} \mathbf{v}}{\mathbf{v} H_1^{-1} \mathbf{v}} = \left(\min_i D_{ii}^{1-\theta}\right)^{-1}$$

so that

$$\kappa_{H_Y}(H_\theta) = (\kappa_2(D))^{1-\theta},$$

which is the stated result. ■

2.2 Dual spaces

The dense inclusion (3) leads to the following inclusion corresponding to the respective dual spaces:

$$Y' \subset [X, Y]'_\theta \subset X'. \tag{12}$$

Moreover, the following duality result can be found in (Lions and Magenes 1968, Thm 6.2)

$$[X, Y]'_\theta = [Y', X']_{1-\theta}. \tag{13}$$

Let $X'_h \subset X', Y'_h \subset Y'$ denote two finite-dimensional subspaces of X', Y' respectively. They are Hilbert spaces when endowed with the inner-products $(\cdot,\cdot)_{X'}, (\cdot,\cdot)_{Y'}$. We can similarly define corresponding positive, self-adjoint operators $\mathcal{J}'_h, \mathcal{E}'_h : Y'_h \to X'_h$

$$(u_h, v_h)_{Y'} = \left(u_h, \mathcal{J}'_h v_h\right)_{X'} \qquad u_h, v_h \in Y'_h \tag{14}$$

where $\mathcal{J}'_h$ is positive self-adjoint and $\mathcal{E}'_h = (\mathcal{J}'_h)^{1/2}$. We define the discrete interpolation spaces

$$[Y'_h, X'_h]_\theta := D(\left(\mathcal{E}'_h\right)^{1-\theta}).$$

Furthermore, setting $\theta' = 1 - \theta$ we define the scale of discrete norms

$$\|u_h\|_{\theta',h} := \left(\|u_h\|^2_{X'} + \|\left(\mathcal{E}'_h\right)^{1-\theta'} u_h\|^2_{X'}\right)^{1/2}. \tag{15}$$

Since the matrices corresponding to norms $\|\cdot\|_{Y'}, \|\cdot\|_{X'}$ are $H'_X = H_X^{-1}$ and $H'_Y = H_Y^{-1}$, respectively, we find

$$J' = (H'_Y)^{-1} H'_X = H_Y H_X^{-1} = Q^T D^{-1} Q^{-T}$$

and hence

$$(J')^{1-\theta} = Q^T D^{\theta-1} Q^{-T}.$$

As before, the norm $\|\cdot\|_{\theta',h}$, with matrix representation $H'_{\theta',h}$, can be shown to be equivalent to a reduced norm with matrix representation $H'_{\theta'}$ which in turn can be seen to be simply the inverse of H_θ

$$H'_{\theta',h} = H'_Y + H'_Y(J')^{1-\theta'} \sim H'_{\theta'} := H'_Y(J')^{1-\theta'} = Q^{-1} D^{\theta'-1} Q^{-T} = H_{\theta'}^{-1}.$$

Hence

$$H_\theta^{-1} = Q^{-1} D^{\theta-1} Q^{-T} = J^{\theta-1} H_Y^{-1}$$

can be taken to be the matrix representation of a norm on $[X_h, Y_h]'_\theta$.

3 Fractional Sobolev spaces

In this section we consider the case where X, Y are Sobolev spaces. In particular, we are interested in deriving the matrix representations of fractional Sobolev norms with a view to designing optimal iterative solution methods for finite element discretisations of PDE. We start by reviewing some standard definitions and results.

Let Ω denote an open bounded subset of $\mathbb{R}^n$ with smooth boundary Γ and let $\boldsymbol{\alpha}$ denote a multi-index of order m where m is a positive integer. Let

$$H^m(\Omega) = \left\{u : D^{\boldsymbol{\alpha}} u \in L^2(\Omega), \quad |\boldsymbol{\alpha}| \leq m\right\}$$

denote the usual Sobolev space of order m, with the convention that $H^0(\Omega) = L^2(\Omega)$. Sobolev spaces of real index $0 \leq s \leq m$ are defined as interpolation spaces of index $\theta = 1 - s/m$ for the pair $[H^m(\Omega), H^0(\Omega)]$:

$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_\theta.$$

One can use this definition to characterise interpolation spaces for pairs of Sobolev spaces of real-index:

$$[H^{s_1}(\Omega), H^{s_2}(\Omega)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\Omega). \tag{16}$$

Let now $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω and let $H_0^s(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, where $s > 0$. Then

$$\begin{cases} H_0^s(\Omega) = H^s(\Omega) & s \leq 1/2 \\ H_0^s(\Omega) \subset H^s(\Omega) & s > 1/2 \end{cases}$$

In fact, given $0 \leq s_2 < s_1$, one has the following characterisations based on interpolation

$$\begin{cases} [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_0^{(1-\theta)s_1+\theta s_2}(\Omega) & \text{if } (1-\theta)s_1 + \theta s_2 \neq k + 1/2 \quad (k \in \mathbb{N}), \\ [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_{00}^{k+1/2}(\Omega) \subset H_0^{k+1/2} & \text{if } (1-\theta)s_1 + \theta s_2 = k + 1/2 \quad (k \in \mathbb{N}). \end{cases}$$

Finally, we define for $s > 0$

$$H^{-s}(\Omega) = (H_0^s(\Omega))'.$$

Hence if $(1-\theta)s_1 + \theta s_2 \neq k + 1/2 \quad (k \in \mathbb{N})$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = H^{-(1-\theta)s_1-\theta s_2}(\Omega),$$

while if $(1-\theta)s_1 + \theta s_2 = k + 1/2 \quad (k \in \mathbb{N})$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = \left(H_{00}^{k+1/2}(\Omega)\right)'.$$

3.1 Special domains

The open Ω can be replaced by a regular manifold and the Sobolev spaces are built using a variational formulation based on the Laplace-Beltrami operator. The resulting Hilbert spaces can be used to build by the same techniques presented in Section 2 the corresponding fractional Sobolev spaces (see Lions and Magenes, 1968[page 42]).

An other choice for Ω is to be a metric graph (in the literature these are also named quantum graphs). This choice will be used in Section 6.1 (see Remark 6.1). A metric graph is a graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ made of vertices $\mathfrak{v} \in \mathfrak{V}$ and edges $\mathfrak{e} \in \mathfrak{E}$ where at each edge $\mathfrak{e}$ is associate its lenght $\ell(\mathfrak{e})$. For sake of simplicity, we assume that $\ell(\mathfrak{e}) < \infty \ \ \forall \mathfrak{e}$ and that number of vertices and edges is finite.

The degree of a vertex is the number of incident edges on it and the boundary of $\mathfrak{G}$ consists of the vertices of degree one. Moreover, an edge can be identified with a finite segment of the real line such that we can introduce a coordinate $x(\mathfrak{e})$ and a preferred direction of the edge. This define a natural topology on the graph and makes it a 1D simplicial complex. As a consequence the points on the graph are not only is vertices but all the intermediate points x on the edges and, thus, we can define a function $f(x)$ on $\mathfrak{G}$ as a function defined on all points of the graph and not only on its vertices.

Having the coordinate x, we can define the Lebesgue measure on $\mathfrak{G}$ and, then, the concept of integrability. This allows the introduction of some standard Sobolev function spaces on the graph.

Definition 2 *The space $L^2(\mathfrak{G})$ consists of all measurable and square integrable functions on each edge $\mathfrak{e}$ and such that*

$$||f||^2_{L^2(\mathfrak{G})} = \sum_{\mathfrak{e}\in\mathfrak{E}} ||f||^2_{L^2(\mathfrak{e})} < \infty.$$

Definition 3 *The Sobolev space $H^1(\mathfrak{G})$ consists of all continuous functions on $\mathfrak{G}$ that belong to $H^1(\mathfrak{e}$ for each edge $\mathfrak{e}$ and such that*

$$\sum_{\mathfrak{e}\in\mathfrak{E}} ||f||^2_{H^1(\mathfrak{e})} < \infty.$$

We can also define the space $H^1_0(\mathfrak{G})$ adding the condition that the functions $f(x) \in H^1(\mathfrak{G})$ take zero values at the boundary vertices.

A friendly but accurate introduction and survey on quantum graphs is given by Kuchment 2006. In (Kuchment 2006) is also observed that there is no natural definition of Sobolev spaces $H^k(\mathfrak{G})$ of order k higher than 1, due to the lack of natural conditions at vertices. For the Laplace operators and the case $k = 2$ is possible to introduce Neumann-Kirchoff conditions at the internal vertices that allow a consistent definition of the eigenvalue problem (see Friedlander, 2005).

However, taking into account the two previous definitions and the Hilbert structure of $L^2(\mathfrak{G})$ and $H^1(\mathfrak{G})$ we can still introduce the corresponding fractional Sobolev spaces by the interpolation method.

3.2 Discrete fractional Sobolev norms

The results from the previous section can be used to derive discrete Sobolev norms for some standard pairs. We consider below some examples which arise naturally in the formulation of elliptic PDE. In particular, Sobolev spaces of index 'integer+1/2' will be the focus of our discussion since they arise naturally as codomains of boundary operators.
We start by deriving the matrix representation of a norm defined on a discrete subspace of the interpolation space

$$H^{k+1/2}_{00}(\Omega) = [H^{k+1}_0(\Omega), H^k_0(\Omega)]_{1/2}.$$

Let $X = H^{k+1}_0(\Omega), Y = H^k_0(\Omega)$ and let $X_h \subset X, Y_h \subset Y$. Let $\{\phi_i\}_{1\le i\le n} \in X_h$ be a spanning set for Y_h and let $L_k \in \mathbb{R}^{n\times n}$ denote the Grammian matrices corresponding to the $(\cdot,\cdot)_{H^k_0(\Omega)}$-inner product:

$$(L_k)_{ij} = (\phi_i, \phi_j)_{H^k_0(\Omega)}.$$

Using the results of section 2 we find $J = L_k^{-1} L_{k+1}$ and a norm for the discrete interpolation space $[X_h, Y_h]_\theta$ is given by

$$H_{k+\theta} = L_k J^{1-\theta} = L_k (L_k^{-1} L_{k+1})^{1-\theta}.$$

In particular, if $\theta = 1/2$, we find

$$H_{k+1/2} = L_k(L_k^{-1}L_{k+1})^{1/2}$$

which simplifies in the case when the matrices L_k are simultaneously diagonalisable to

$$H_{k+1/2} = L_k^{1/4}L_{k+1}^{1/2}L_k^{1/4}.$$

Note that different choices of L_k lead to different discrete norms, though they are all equivalent with constants of equivalence independent of n. Let us consider some examples to illustrate the above derivation.

Example 3.1 *Let $X = H_0^1(\Omega), Y = L^2(\Omega)$. We wish to derive the matrix representation of a norm defined on a discrete subspace of the interpolation space*

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2}.$$

Let $X_h \subset X$ denote the finite element space spanned by a basis $\{\phi_i\}_{1\leq i\leq n}$ of piecewise polynomials defined on a subdivision of Ω. Using the notation introduced above we let $L_1, L_0 \in \mathbb{R}^{n\times n}$ denote the Grammian matrices with respect to the set $\{\phi_i\}$ corresponding to the following inner products:

$$(L_1)_{ij} = (\nabla\phi_i, \nabla\phi_j)_{L^2(\Omega)}, \qquad (L_0)_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)}.$$

The matrix L_0 is known as the mass matrix, while the matrix L_1 is a discrete Dirichlet Laplacian. Therefore, $T = L_0^{-1}L_1$ and a norm for the interpolation space $[X_h, Y_h]_\theta$ is given by

$$H_{0+\theta} = L_0T^{1-\theta} = L_0(L_0^{-1}L_1)^{1-\theta}.$$

If L_0, L_1 are simultaneously diagonalisable (as is the case for a uniform subdivision of Ω), the expression for H_θ becomes

$$H_\theta = L_0^\theta L_1^{1-\theta} = L_0^{\theta/2}L_1^{1-\theta}L_0^{\theta/2}.$$

In particular, if $\theta = 1/2$, we recover the expression derived by Burstedde., 2007:

$$H_{1/2} = L_0^{1/4}L_1^{1/2}L_0^{1/4}.$$

Example 3.2 *Let $X = H_0^2(\Omega), Y = H_0^1(\Omega)$. We can derive in a similar way the matrix representation of a norm for a discrete subspace of*

$$H_{00}^{3/2}(\Omega) = [H_0^2(\Omega), H_0^1(\Omega)]_{1/2}.$$

Working as before with subspaces X_h, Y_h of X, Y respectively, which are spanned by a set $\{\phi_i\}_{1\leq i\leq n}$ of suitable piecewise polynomials we define the following Grammian matrices

$$(L_2)_{ij} = (\Delta\phi_i, \Delta\phi_j)_{L^2(\Omega)}, \qquad (L_1)_{ij} = (\nabla\phi_i, \nabla\phi_j)_{L^2(\Omega)}.$$

The matrix L_1 is a discrete Dirichlet Laplacian while the matrix L_2 is the discretisation of a biharmonic operator with homogeneous boundary conditions. A discrete norm is therefore induced by the matrix

$$H_{1+\theta} = L_1(L_1^{-1}L_2)^{1-\theta}.$$

If $\theta = 1/2$ we find

$$H_{3/2} = L_1(L_1^{-1}L_2)^{1/2}.$$

which, in the case of simultaneous diagonalisation simplifies to

$$H_{3/2} = L_1^{1/4}L_2^{1/2}L_1^{1/4}.$$

Remark 3.1 *In both examples above, the choice of norms for the spaces $H_0^k(\Omega)$ is not unique. One could choose to define L_1 in terms of the (equivalent) full norm of $H_0^1(\Omega)$*

$$(L_1)_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)} + (\nabla\phi_i, \nabla\phi_j)_{L^2(\Omega)}\,.$$

The same can be said about L_2. The resulting discrete norms will therefore be different from, but equivalent to, the norms derived above.

We consider now the case of fractional Sobolev spaces of negative index and the corresponding discrete norms. Consider the dual of the interpolation space $H_{00}^{k+1/2}(\Omega)$:

$$\left(H_{00}^{k+1/2}(\Omega)\right)' = [H^{-(k+1)}(\Omega), H^{-k}(\Omega)]_{1/2} = \left([H_0^{k+1}(\Omega), H_0^k(\Omega)]_{1/2}\right)'.$$

Let X_h, Y_h be defined as above. We are interested in deriving a matrix representation for a norm on $[X_h, Y_h]'_\theta$ *with respect to the basis* $\{\phi_i\}_{1\le i\le n}$. Let $Y'_h \subset X'_h = \operatorname{span}\{\psi_i\}_{1\le i\le n}$ where ψ_i are basis functions dual to ϕ_i, i.e.,

$$\langle\psi_i, \phi_j\rangle_{H^{-k}(\Omega)\times H_0^k(\Omega)} = \delta_{ij}.$$

If we let

$$Y'_h \ni z = \sum_{i=1}^n \mathbf{z}_i\psi_i = \sum_{i=1}^n \mathbf{w}_i\phi_i, \quad \phi_l = \sum_{i=1}^n K_{li}\psi_i$$

we can use the above duality property to deduce that

$$\delta_{ij} = \langle\psi_i, \phi_j\rangle_{H^{-k}(\Omega)\times H_0^k(\Omega)} = \sum_{i=1}^n K_{il}^{-1}\,(\phi_l, \phi_j)_{H_0^k(\Omega)} = \sum_{i=1}^n K_{il}^{-1}(L_k)_{lj}$$

so that $\mathbf{z} = L_k\mathbf{w}$ and (cf. Remark 2.1)

$$\|\mathbf{z}\|_{H'_Y} = \|\mathbf{w}\|_{L_k^{-1}}, \quad \|\mathbf{z}\|_{H'_X} = \|\mathbf{w}\|_{L_k L_{k+1}^{-1} L_k}$$

and the matrix representation of H'_Y, H'_X are respectively L_k and $L_k L_{k+1}^{-1} L_k$.
The matrix representation for a norm on $[X_h, Y_h]'_\theta$ is therefore (see also section 2)

$$H'_{k+\theta} = L_k J^{\theta-1} = L_k(L_k^{-1}L_{k+1})^{\theta-1} = H_{k-1+\theta}.$$

In particular, in the case $\theta = 1/2$ we find

$$H'_{k+1/2} = L_k(L_k^{-1}L_{k+1})^{-1/2} = H_{k-1/2}.$$

Example 3.3 *Let $X = H_0^1(\Omega), Y = L^2(\Omega)$ and let X_h, Y_h be defined as above. Then a discrete norm on $[X_h, Y_h]'_{1/2} \subset H^{-1/2}(\Omega)$ is given by*

$$H'_{1/2} = H_{-1/2} = L_0(L_0^{-1}L_1)^{-1/2}, \tag{17}$$

which in the case of simultaneous diagonalisation of L_0, L_1 can be written as

$$H_{-1/2} = L_0^{3/4}L_1^{-1/2}L_0^{3/4}.$$

We end this section with a remark which will be useful in the context of preconditioning applications.

Remark 3.2 *Let the basis functions* $\{\phi_i\}_{1\le i\le n}$ *for* $X_h \subset Y_h$ *be continuous piecewise polynomials defined on a given subdivision of a polyhedral domain* Ω *into simplices of maximum diameter* h*. In this case, it is known (see (Brenner and Scott 1994)) that the piecewise polynomial interpolant* I_h *for which* $I_h v = v_h$ *for* $v \in X \subset H^m(\Omega)$ *satisfies*

$$\|v - I_h v\|_{H^m(\Omega)} \le C\|v\|_{H^m(\Omega)}$$

so that

$$\|I_h v\|_{H^m(\Omega)} \le (1 + C)\|v\|_{H^m(\Omega)}$$

and therefore $I_h \in \mathcal{L}(X, X_h)$*. Setting* $X_h \subset X \subset H^{k+1}(\Omega)$*,* $Y_h \subset Y \subset H^k$ *it follows that* $I_h \in \mathcal{L}(X, X_h) \cap \mathcal{L}(Y, Y_h)$ *and hence Lemma 2.2 applies for general conforming finite element discretizations under standard regularity conditions on* Ω *and its corresponding subdivision into simplices. In particular, the fractional Sobolev scale of norms* $\|\cdot\|_{k+\theta}$ *is equivalent on* $[X_h, Y_h]_\theta$ *to the discrete norms induced by the family of matrices* $H_{k+\theta}$ *introduced above for* $\theta \in (0, 1)$*.*

4 Evaluation of H_θ

In order to construct and apply in a practical application any of the discrete norms derived in the previous discussion we are required to evaluate (non-integer) powers of a matrix. This task may be achieved in different ways for different applications. In general, if the dimension of the problem is low, one can employ a direct method based on a generalised eigenvalue decomposition. This approach has complexity of order $O(n^3)$. Another direct approach is available in the case when the matrices involved have a Toeplitz structure; in this case the evaluation can be achieved via an FFT (see Peisker, 1988) and the complexity is $O(n \log n)$. In both cases the storage requirements are of order $O(n^2)$. For larger problems, iterative techniques may represent a cheaper alternative. An example in case is Newton's method which has attractive convergence properties under a suitable implementation. However, the complexity of the method is that of a direct method, even if the original matrix is sparse. One expects that for sparse matrices one can devise efficient techniques for evaluating H_θ or its action applied to a given vector. This is indeed the case. In (Hale, Higham and Trefethen 2008) the authors propose a method based on representing a function of a matrix as an integral which they go on to evaluate using efficient quadrature rules. The method can be adapted to provide a sparse algorithm for the evaluation of H_θ. Another approach is to construct approximations of Krylov type which are known to take advantage of the sparsity of the matrices involved. Several authors have considered this approach for general matrix functions (Druskin and Knizhnerman 1989), (Cabos 1997), (Druskin and Knizhnerman 1998), (Druskin, Greenbaum and Knizhnerman 1998), (Saad 1992), (Allen, Baglama and Boyd 2000), (Eiermann and Ernst 2006) and some convergence analysis is available for certain algorithms proposed for the computation of the matrix square root function (Druskin and Knizhnerman 1989). We illustrate below the Krylov subspace approximation for the case $\theta = 1/2$.

4.1 A generalised Lanczos algorithm

Given a pair of symmetric and positive-definite matrices (M, A), the generalised Lanczos algorithm constructs a set of M-orthogonal vectors $\mathbf{v}_i$ such that

$$AV_k = MV_kT_k + \beta_{k+1}M\mathbf{v}_{k+1}\mathbf{e}_k^T, \quad V_k^TMV_k = I_k$$

where the columns $\mathbf{v}_i$ of $V_k = [\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_k]$ are known as the Lanczos vectors and $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix with kth column denoted by $\mathbf{e}_k$, while the matrix $T_k \in \mathbb{R}^{k\times k}$ is a symmetric and tridiagonal matrix (Parlett 1998). The standard algorithm corresponds to the case $M = I$. Note that T_k can be seen as a projection of A onto the space spanned by the M-orthogonal columns of V_k

$$V_k^T A V_k = T_k, \quad V_k^T M V_k = I_k. \tag{18}$$

In exact arithmetic, when $k = n$, the algorithm can be seen as providing simultaneous factorisations of the matrix pair (M, A) as

$$A = V_n^{-T} T_n V_n^{-1}, \quad M = V_n^{-T} V_n^{-1}.$$

We recall the algorithm below (Parlett 1998).

Algorithm 1. Generalised Lanczos Algorithm

Input: $A, M \in \mathbb{R}^{n\times n}(\text{spd}), \mathbf{v} \in \mathbb{R}^n$
Output: $V_k \in \mathbb{R}^{n\times k}, T_k \in \mathbb{R}^{k\times k}$

```
Set β_1 = 0, v_0 = 0, v_1 = v/||v||_M
for i = 1 : k
  w_i = M^{-1} A v_i − β_i v_{i−1}
  α_i = (w_i, v_i)_M
  w_i = w_i − α_i v_i
  β_{i+1} = ||w_i||_M
  if β_{i+1} = 0 stop
  v_{i+1} = w_i/β_{i+1}
end
T_k = tridiag[β, α, β]
```

The explicit form of T_k is given below

$$T_k = \text{tridiag}[\boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\beta}] = \begin{pmatrix} \alpha_1 & \beta_2 & & 0 \\ \beta_2 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_k \\ 0 & & \beta_k & \alpha_k \end{pmatrix}.$$

Consider now the generalised Lanczos factorisation for the matrix-pair (H_Y, H_X):

$$H_X V = H_Y V T, \qquad V^T H_Y V = I. \tag{19}$$

where we used the notation $V = V_n, T = T_n$. We can immediately derive the following result.

Lemma 4.1 *Let* (19) *hold and let* $H_{\theta,h} = H_Y + H_Y J^{1-\theta}$ *and* $H_\theta = H_Y J^{1-\theta}$ *with* $J = H_Y^{-1} H_X$. *Then*

$$H_\theta = \left(V T^{\theta-1} V^T\right)^{-1} = H_Y V T^{1-\theta} V^T H_Y \tag{20}$$

and

$$H_{\theta,h} = \left(V(I + T^{1-\theta})^{-1} V^T\right)^{-1} = H_Y V(I + T^{1-\theta}) V^T H_Y. \tag{21}$$

4.2 Sparse evaluation of $H_\theta \mathbf{z}$

The complexity of the full ($k = n$) generalised Lanczos algorithm is in general $O(n^3)$. However, in many applications of interest we do not need to compute H_θ, but simply apply it (or its inverse) to a given vector $\mathbf{z} \in \mathbb{R}^n$. In such cases, a truncated version of the algorithm is used in practice with only k Lanczos vectors being constructed. As we are interested in approximations of $H_\theta \mathbf{z}$ we note first that if we start the Lanczos process with $\mathbf{v} = \mathbf{z}$ then

$$V_k^T H_Y \mathbf{z} = \mathbf{e}_1 \|\mathbf{z}\|_{H_Y}$$

where $\mathbf{e}_1 \in \mathbb{R}^k$ is the first column of the identity I_k. This leads us to consider the following approximations of the matrix-vector products:

$$H_\theta \mathbf{z} \approx H_Y V_k T_k^{1-\theta} \mathbf{e}_1 \|\mathbf{z}\|_{H_Y}$$

and

$$H_{\theta,h} \mathbf{z} \approx H_Y V_k (I_k + T_k^{1-\theta}) \mathbf{e}_1 \|\mathbf{z}\|_{H_Y}.$$

Similarly, if we wish to apply the inverse of H_θ to a given vector $\mathbf{z}$ we first note that if we start the iteration with $\mathbf{v} = H_Y^{-1}\mathbf{z}$ then

$$V_k^T \mathbf{z} = V_k^T H_Y (H_Y)^{-1} \mathbf{z} = \mathbf{e}_1 \|H_Y^{-1}\mathbf{z}\|_{H_Y} = \mathbf{e}_1 \|\mathbf{z}\|_{H_Y^{-1}}.$$

This leads us to consider the following approximations (cf. Lemma 4.1)

$$H_\theta^{-1} \mathbf{z} \approx V_k T_k^{\theta-1} V_k^T \mathbf{z} = V_k T_k^{\theta-1} \mathbf{e}_1 \|\mathbf{z}\|_{H_Y^{-1}}$$

and

$$H_{\theta,h}^{-1} \mathbf{z} \approx V_k \left(I_k + T_k^{1-\theta}\right)^{-1} V_k^T \mathbf{z} = V_k \left(I_k + T_k^{1-\theta}\right)^{-1} \mathbf{e}_1 \|\mathbf{z}\|_{H_Y^{-1}}.$$

The complexity of the above operations depends on the complexity corresponding to the application of the inverse of H_Y. If this operation can be achieved in $O(n)$ operations, then the overall complexity of computing $H_\theta \mathbf{z}, H_\theta^{-1}\mathbf{z}$ is of order $O(kn)$ for $k \ll n$, with storage requirements of the same order.

5 Applications

5.1 Preconditioners for the Steklov-Poincaré operator

Domain decomposition methods (DD) require the resolution of a problem which involves a pseudo-differential operator defined on the set of boundaries defined by the decomposition of the domain. This operator is generally known as the Steklov-Poincaré operator or the Dirichlet-Neumann map, though this definition was introduced for DD methods applied to second-order problems involving the Laplacian operator. A great number of iterative approaches have been proposed in the literature over the last two decades; classical algorithms include Dirichlet-Neumann, Neumann-Neumann, FETI methods, Scharwz methods together with two-level and overlapping variants. Complete descriptions and analyses can be found in a range of references, see, for example, (Toselli and Widlund 2005), and (Quarteroni and Valli 1999). We present below an alternative which has not been considered to date and which is based on preconditioning in a fractional Sobolev space of index 1/2.

5.1.1 A model problem

Let Ω be an open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and consider the model problem

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{22}$$

Given a partition of Ω into two subdomains $\Omega \equiv \Omega_1 \cup \Omega_2$ with common boundary Γ this problem can be equivalently written as

$$\begin{cases} -\Delta u_1 &= f \quad \text{in } \Omega_1, \\ \quad u_1 &= 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_2 &= f \quad \text{in } \Omega_2, \\ \quad u_2 &= 0 \quad \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases}$$

with the 'interface conditions'

$$\begin{cases} u_1 &= u_2 \\ \frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} \end{cases} \quad \text{on } \Gamma$$

Let now $\lambda_1, \lambda_2 \in H^{1/2}_{00}(\Gamma)$ and, correspondingly, let ψ_1, ψ_2 denote the harmonic extensions of λ_1, λ_2 respectively into Ω_1, Ω_2, i.e., for $i = 1, 2$, ψ_i satisfy

$$\begin{cases} -\Delta \psi_i &= 0 \quad \text{in } \Omega_i, \\ \quad \psi_i &= \lambda_i \quad \text{on } \Gamma, \\ \quad \psi_i &= 0 \quad \text{on } \partial\Omega_i \setminus \Gamma. \end{cases}$$

Let $H^{1/2}_{00}(\Gamma) = [H^1_0(\Gamma), L_2(\Gamma)]_{1/2}$. We define the Steklov-Poincaré operator $\mathcal{S} : H^{1/2}_{00}(\Gamma) \to H^{-1/2}(\Gamma)$ via

$$(\mathcal{S}\lambda_1, \lambda_2)_{H^{1/2}(\Gamma)} = (\nabla\psi_1, \nabla\psi_2)_{L^2(\Omega)} =: s(\lambda_1, \lambda_2). \tag{23}$$

Note that we also defined a bilinear form $s(\cdot,\cdot) : H^{1/2}_{00}(\Gamma) \times H^{1/2}_{00}(\Gamma) \to 0$ which can be seen to be symmetric and positive-definite. One can show further that this bilinear form is also $H^{1/2}_{00}(\Gamma)$-elliptic, i.e., there exist constants c_1, c_2 such that for all $\lambda \in H^{1/2}_{00}(\Gamma)$,

$$c_1 \|\lambda\|^2_{H^{1/2}(\Gamma)} \le s(\lambda, \lambda) \le c_2 \|\lambda\|^2_{H^{1/2}(\Gamma)}. \tag{24}$$

With this definition of $\mathcal{S}$ our model problem can be recast as an ordered sequence of three decoupled problems involving Poisson problems on each subdomain together with a problem set on the interface Γ.

$$\text{(i)} \quad \begin{cases} -\Delta u_i^{\{1\}} &= f \ \text{ in } \Omega_i, \\ \quad u_i^{\{1\}} &= 0 \ \text{ on } \partial\Omega_i, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \mathcal{S}\lambda = -\dfrac{\partial u_1^{\{1\}}}{\partial n_1} - \dfrac{\partial u_2^{\{1\}}}{\partial n_2} \ \text{ on } \Gamma, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} -\Delta u_i^{\{2\}} &= 0 \ \text{ in } \Omega_i, \\ \quad u_i^{\{2\}} &= \lambda \ \text{ on } \partial\Omega_i. \end{cases}$$

The resulting solution is

$$u|_{\Omega_i} = u_i^{\{1\}} + u_i^{\{2\}}.$$

5.1.2 Discrete formulation

Let $P_r(\mathfrak{t})$ denote the space of polynomials in two variables of degree r defined on a set $\mathfrak{t} \subset \mathbb{R}^2$. Let

$$V^h = V^{h,r} := \left\{ w \in C^0(\Omega) : w|_{\mathfrak{t}} \in P_k \ \ \forall \mathfrak{t} \in \mathfrak{T}_h \right\} \subset H^1(\Omega), \tag{25}$$

be a finite-dimensional space of piecewise polynomial functions defined on some subdivision $\mathfrak{T}_h$ of Ω into simplices $\mathfrak{t}$ of maximum diameter h. Let further $V_I^h, V_B^h \subset V^h$ satisfy $V_I^h \oplus V_B^H \equiv V^h$ where $V_I^h = \left\{ w \in V^h : w|_{\partial\Omega} = 0 \right\}$. Let $X_h \subset H^1_0(\Gamma)$ denote the space spanned by the restriction of the basis functions of V_I^h to the internal boundary Γ. The discrete variational formulation of model problem (22) reads

$$\begin{cases} \text{Find } u_h \in V_I^h \text{ such that for all } v_h \in V_I^h \\ \qquad a(u_h, v_h) = f(v_h) \end{cases},$$

where

$$a(v, w) = (\nabla v, \nabla w)\,, \quad f(v) = (v, f)\,.$$

This formulation give rise to a linear system involving a matrix which is structured in the following way

$$\begin{pmatrix} A_{II,1} & 0 & A_{IB,1} \\ 0 & A_{II,2} & A_{IB,2} \\ A_{IB,1}^T & A_{IB,2}^T & A_{BB,1} + A_{BB,2} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{I,1} \\ \mathbf{u}_{I,2} \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{I,1} \\ \mathbf{f}_{I,2} \\ \mathbf{f}_B \end{pmatrix}$$

where $A_{II,i}, i = 1, 2$ are discrete Laplacians corresponding to the interior nodes of the computational domain Ω_i and $A_{BB,i}, i = 1, 2$ are the corresponding interior boundary contributions from each domain. The above system can be 'decoupled' into three problems

$$\begin{array}{ll} \text{(i)} & A_{II,i}\mathbf{u}_i^{\{1\}} = \mathbf{f}_{I,i}, \\ \text{(ii)} & S\mathbf{u}_B = \mathbf{f}_B - A_{IB,1}^T\mathbf{u}_1^{\{1\}} - A_{IB,2}^T\mathbf{u}_2^{\{2\}}, \\ \text{(iii)} & A_{II,i}\mathbf{u}_i^{\{2\}} = -A_{IB,1}^T\mathbf{u}_B - A_{IB,2}^T\mathbf{u}_B, \end{array}$$

where S is the Schur complement corresponding to the boundary nodes

$$S = S_1 + S_2, \quad S_i = A_{BB,i} - A_{IB,i}^T A_{II,i}^{-1} A_{IB,i}.$$

The resulting solution is $(\mathbf{u}_{I,1},\ \mathbf{u}_{I,2},\ \mathbf{u}_B)$ where

$$\mathbf{u}_{I,i} = \mathbf{u}_i^{\{1\}} + \mathbf{u}_i^{\{2\}}.$$

It is evident that these algebraic problems are finite element discretisations of the three continuous problems listed above (see Quarteroni and Valli, 1999 for full details). In particular, the Schur complement S is the finite element discretisation of the variational definition (23) of the Steklov-Poincaré operator $\mathcal{S}$. Since $s(\cdot,\cdot)$ is $H_{00}^{1/2}(\Gamma)$-elliptic, we deduce that for any $\lambda_h \in X_h$ there holds

$$c_1\|\lambda_h\|_{H^{1/2}(\Gamma)}^2 \leq s(\lambda_h, \lambda_h) \leq c_2\|\lambda_h\|_{H^{1/2}(\Gamma)}^2. \tag{26}$$

Note that if we denote by $\boldsymbol{\lambda}$ the vector of coefficients of λ_h when expanded in a finite element basis, then

$$s(\lambda_h, \lambda_h) = \boldsymbol{\lambda}^T S \boldsymbol{\lambda}.$$

5.1.3 $H_{00}^{1/2}$-preconditioners

For large problems, constructing and applying the inverse of the Schur complement S in step (ii) above is computationally prohibitive. Instead, the problem can be solved using a preconditioned iterative technique. For symmetric and positive-definite problems it is known that optimal acceleration of an iterative method is achieved when a spectrally-equivalent preconditioner P_S is employed. We show below that the matrix representation of a $H_{00}^{1/2}-$norm has this property and can therefore be employed as a preconditioner for domain decomposition methods of scalar elliptic problems.

The inequalities (26) describe essentially the spectral equivalence between the discrete operator induced by the bilinear form $s(\cdot,\cdot)$ acting on a finite-dimensional subspace of $H_{00}^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma)$ and a discrete representation of the $H_{00}^{1/2}$-norm. We make this statement precise below.

Proposition 5.1 *Let $X_h = \text{span}\,\{\phi_i, 1 \le i \le m\}$ be defined as above and let $(L_k)_{ij} = (\phi_i, \phi_j)_{H_0^k(\Gamma)}$ for $k = 0, 1$. Let*

$$H_{1/2} := L_0(L_0^{-1}L_1)^{1/2}.$$

Then for all $\boldsymbol{\lambda} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$

$$\kappa_1 \le \frac{\boldsymbol{\lambda}^T S \boldsymbol{\lambda}}{\boldsymbol{\lambda}^T H_{1/2} \boldsymbol{\lambda}} \le \kappa_2$$

with κ_1, κ_2 independent of h.

Proof: Since Lemma (2.2) holds (see Remark 3.2) there exist constants η_1, η_2 such that for all $\lambda_h \in X_h$

$$\eta_1 \|\lambda_h\|_{1/2} \le \|\lambda_h\|_{\theta,h} \le \eta_2 \|\lambda_h\|_{1/2}.$$

The norm $\|\cdot\|_{\theta,h}$ has matrix representation $H_{1/2,h}$ which, by (9), is spectrally equivalent to $H_{1/2}$. Hence, there exist constants $\tilde{\eta}_1, \tilde{\eta}_2$ such that

$$\tilde{\eta}_1 \|\lambda_h\|_{1/2} \le \|\boldsymbol{\lambda}\|_{H_{1/2}} \le \tilde{\eta}_2 \|\lambda_h\|_{1/2}.$$

Using the $H_{00}^{1/2}$-ellipticity (26) of $s(\cdot,\cdot)$ we get

$$\frac{c_1}{\tilde{\eta}_2^2} \|\boldsymbol{\lambda}\|_{H_{1/2}}^2 \le \boldsymbol{\lambda}^T S \boldsymbol{\lambda} \le \frac{c_2}{\tilde{\eta}_1^2} \|\boldsymbol{\lambda}\|_{H_{1/2}}^2$$

which is the required result. ∎

Remark 5.1 *The matrix $H_{1/2}$ is the reduced version of the matrix representation for the norm $\|\cdot\|_{1/2,h}$. It is evident that the above result holds with $H_{1/2}$ replaced with $L_0 + L_0(L_0^{-1}L_1)^{1/2}$.*

The above result indicates that S and $H_{1/2}$ exhibit the same spectral properties. In particular, the following result holds.

Proposition 5.2 *Let $L_0, L_1 \in \mathbb{R}^m$ be defined as in Proposition* (5.1)*. Then*

$$\kappa_{L_0}(H_{1/2}) = O(h^{-1}) \quad \text{and } \kappa_{L_0}(S) = O(h^{-1}).$$

Proof: Since L_0 is a mass matrix and L_1 is a discrete Laplacian, by the Poincaré inequality there exists a constant γ_1 independent of h such that

$$\|\boldsymbol{\lambda}\|_{L_0} \le \gamma_1^{-1} \|\boldsymbol{\lambda}\|_{L_1}.$$

Furthermore, the following standard discrete inverse inequality is assumed to hold

$$\|\boldsymbol{\lambda}\|_{L_1}^2 \le \gamma_2 h^{-2} \|\boldsymbol{\lambda}\|_{L_0}^2,$$

where γ_2 is also independent of h. Hence,

$$\gamma_1 \le \frac{\boldsymbol{\lambda}^T L_1 \boldsymbol{\lambda}}{\boldsymbol{\lambda}^T L_0 \boldsymbol{\lambda}} \le \gamma_2 h^{-2}$$

and therefore $\kappa_{L_0}(L_1) = O(h^{-2})$. Hence, by Lemma (2.4),

$$\kappa_{L_0}(H_{1/2}) = (\kappa_{L_0}(L_1))^{1/2} = O(h^{-1}).$$

The second statement follows from the spectral equivalence of $H_{1/2}$ and S derived in Proposition 5.1. ∎

Since $\|\cdot\|_{L_0}$ is equivalent to the Euclidean (l_2-) norm, we conclude that $\kappa_2(S) = O(h^{-1})$ which is the standard result on the condition number of the discrete Steklov-Poincaré operator (Quarteroni and Valli 1999). This indicates that an iterative technique which ignores the Schur complement problem will be suboptimal. We verify in the numerics section below that this is indeed the case and we demonstrate that $H_{1/2}$ is a suitable preconditioner in this sense.

5.2 Boundary preconditioners for the biharmonic operator

Consider the biharmonic problem in a polygonal convex open domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \cup_{i=1}^{K} \Gamma_i$.

$$\begin{cases} \Delta^2 u &= f & \text{in } \Omega, \\ u = \partial u/\partial n &= 0 & \text{on } \Gamma. \end{cases} \tag{27}$$

A standard approach to solving (27) is to introduce another variable $v = -\Delta u$ and solve the resulting system:

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ v + \Delta u &= 0 & \text{in } \Omega, \\ u = \partial u/\partial n &= 0 & \text{on } \Gamma. \end{cases} \tag{28}$$

There is a considerable literature on the topic of this model problem, both from the approximation point of view (Monk, 1987, Glowinski and Pironneau, 1979, Osborn, Babuška and Pitkäranta, 1980, Brezzi and Raviart, 1976, Ciarlet and Raviart., 1974, and Falk, 1978) and also an algorithmic one (Glowinski and Pironneau, 1979, Peisker, 1988, Braess and Peisker., 1986, and Silvester and Mihajlović., 2004). A notable approach is provided by Glowinski and Pironneau 1979; this approach was made efficient by Peisker 1988 who suggested for the first time preconditioning with a discrete $H^{-1/2}(\Gamma)$-norm. We review briefly this approach below.

5.2.1 The Pironneau-Glowinski method

The following re-formulation of the biharmonic problem was introduced in (Glowinski and Pironneau 1979). Let $\lambda = v\mid_\Gamma$. The solution (u, v) of system (28) can be obtained by solving the following three problems

$$\text{(i)} \quad \begin{cases} -\Delta v_0 &= f & \text{in } \Omega, \\ v_0 &= 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_0 &= v_0 & \text{in } \Omega, \\ u_0 &= 0 & \text{on } \Gamma, \end{cases} \tag{29}$$

$$\text{(ii)} \qquad \mathcal{S}\lambda = \partial u_0/\partial \nu \qquad \text{on } \Gamma, \tag{30}$$

$$\text{(iii)} \quad \begin{cases} -\Delta v_1 &= 0 & \text{in } \Omega, \\ v_1 &= \lambda & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_1 &= v_1 & \text{in } \Omega, \\ u_1 &= 0 & \text{on } \Gamma, \end{cases} \tag{31}$$

the final solution being $(u, v) = (u_0 + u_1, v_0 + v_1)$. The aim of considering this formulation is to split the problem into smaller, easier to solve problems. While (i) and (iii) may indeed be classified as easy from a computational point of view, the crux of the problem becomes equation (ii). As in the case of domain decomposition methods, $\mathcal{S}$ is a boundary operator which is defined on $H^{-1/2}(\Gamma)$ and which induces a bilinear form $s(\cdot, \cdot) : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ via

$$(S\lambda_1, \lambda_2) = (\Delta\psi_1, \Delta\psi_2) := s(\lambda_1, \lambda_2).$$

The functions ψ_i are biharmonic extensions of $\lambda_i \in H^{-1/2}(\Gamma)$ into Ω, i.e., they are solutions of the biharmonic problems

$$\begin{cases} \Delta^2 \psi_i &= 0 & \text{in } \Omega, \\ \psi_i &= 0 & \text{on } \Gamma, \\ -\partial\psi_i/\partial n &= \lambda_i & \text{on } \Gamma. \end{cases} \tag{32}$$

It is shown in (Glowinski and Pironneau 1979) that the bilinear form $s(\cdot, \cdot)$ is symmetric, positive-definite and $H^{-1/2}(\Gamma)-$elliptic, i.e., there exist constants c_1, c_2 such that for all $\lambda \in H^{-1/2}(\Gamma)$,

$$c_1 \|\lambda\|^2_{H^{-1/2}(\Gamma)} \leq s(\lambda, \lambda) \leq c_2 \|\lambda\|^2_{H^{-1/2}(\Gamma)}. \tag{33}$$

5.2.2 Discrete formulation

Consider now the following standard mixed finite element method for (28). Let V^h, V_I^h, V_B^h be defined as above (see (25)). The discrete weak formulation is then

$$\text{Find } (u_h, v_h) \in V_I^h \times V^h \text{ such that } \forall (w_h, z_h) \in V_I^h \times V^h$$

$$\left\{ \begin{array}{rl} l(v_h, w_h) & = (f, w_h) \\ l(u_h, z_h) - m(v_h, z_h) & = 0 \end{array} \right. \tag{34}$$

where

$$l(z, w) := (\nabla z, \nabla w), \quad m(z, w) := (z, w).$$

As described in (Glowinski and Pironneau 1979), (34) is equivalent to the discrete versions of (29–31) given by the following three weak formulations

$$\text{(i) Find } (u_{0h}, v_{0h}) \in V_I^h \times V_I^h \text{ such that } \forall (w_h, z_h) \in V_I^h \times V_I^h$$

$$\left\{ \begin{array}{rl} l(v_{0h}, w_h) & = (f, w_h) \\ l(u_{0h}, z_h) - m(v_{0h}, z_h) & = 0 \end{array} \right. \tag{35}$$

$$\text{(ii) Find } \lambda_h \in V_B^h \text{ such that } \forall \mu_h \in V_B^h$$

$$\left\{ \; s(\lambda_h, \mu_h) \quad = -s(\lambda_{0h}, \mu_h) \right. \tag{36}$$

$$\text{(iii) Find } (u_{1h}, v_{1h}) \in V_I^h \times V^h,\ v_{1h} - \lambda_h \in V_I^h, \text{ such that } \forall (w_h, z_h) \in V_I^h \times V^h$$

$$\left\{ \begin{array}{rl} l(v_{1h}, w_h) & = 0 \\ l(u_{1h}, z_h) - m(v_{1h}, z_h) & = 0 \end{array} \right. \tag{37}$$

Let now $\mathrm{span}\{\phi_i, 1 \leq i \leq n\} = V^h$ so that $w_h \in V^h, z_h \in V_I^h$ can be written

$$w_h = \sum_{i=1}^{n} \mathbf{w}_i \phi_i, \quad z_h = \sum_{i=1}^{n_I} \mathbf{z}_i \phi_i$$

where $n = |V^h|, n_I = |V_I^h|$. Problem (34) is then equivalent to the following linear system of equations

$$\begin{pmatrix} 0 & L_{II} & L_{IB} \\ L_{II} & -M_{II} & -M_{IB} \\ L_{IB}^T & -M_{IB}^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

where

$$(L_{II})_{ij} = l(\phi_j, \phi_i), \quad (L_{IB})_{ik} = l(\phi_k, \phi_i),$$

and

$$(M_{II})_{ij} = m(\phi_j, \phi_i), \quad (M_{IB})_{ik} = m(\phi_k, \phi_i), \quad (M_{BB})_{kl} = m(\phi_l, \phi_k)$$

for $1 \leq i, j, \leq m,\ 1 \leq k, l \leq n - n_I$. We also write (38) in the more compact form

$$\begin{pmatrix} L & Z \\ Z^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ 0 \end{pmatrix} \tag{39}$$

where

$$L = \begin{pmatrix} 0 & L_{II} \\ L_{II} & -M_{II} \end{pmatrix}, \quad Z = \begin{pmatrix} L_{IB} \\ -M_{IB} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \end{pmatrix}.$$

It can be seen (see for example (Arioli and Loghin 2008)) that the discrete problems (35–37) represent a boundary Schur complement approach to solving (38). As before, the task is therefore the efficient solution of problem (ii), and in particular the derivation of optimal preconditioners for this step. The Schur complement associated with L in the matrix of (39) is

$$S = -M_{BB} - Z^T L^{-1} Z.$$

Let $X_h \subset H^1(\Gamma)$ denote the space spanned by the restriction of the basis functions of V_I^h to the boundary Γ. As in the case of domain decomposition methods, the Schur complement is the matrix representation of the bilinear form $s(\cdot,\cdot)$ with respect to the basis $\{\phi_i\}$. In particular, if $\lambda_h \in X_h$ has a vector of coefficients $\boldsymbol{\lambda}$, then

$$s(\lambda_h, \lambda_h) = \boldsymbol{\lambda}^T S \boldsymbol{\lambda}.$$

5.2.3 $H^{-1/2}(\Gamma)$-preconditioners

A discrete $H^{-1/2}$-norm on X_h can be defined as a sum of norms corresponding to each *open* segment of the polygonal boundary Γ:

$$\|\lambda_h\|_{H^{-1/2}(\Gamma)} := \left(\sum_{i=1}^{K} \|\lambda_h\|^2_{H^{-1/2}(\Gamma_i)} \right)^{1/2}.$$

In particular, $H^{-1/2}(\Gamma_i)$ is understood here to be the dual of $H_{00}^{1/2}(\Gamma_i)$. For this space, a matrix representation for its norm was derived in section 3 (see (17)). However, Peisker uses a different, algebraic, definition of the norm-matrices corresponding to each boundary segment, based on *linear and uniform* discretisations of Laplacian and mass matrices. This results in a matrix representation for the discrete norm $\|\cdot\|_{H^{-1/2}(\Gamma)}$ which is a direct sum of norm-matrices corresponding to the interior of each boundary segment. There are two drawbacks to this approach. First, the linear case ($r = 1$) does not yield stable mixed finite element discretisations of the biharmonic problem, (Shaidurov 1995). Second, the resulting preconditioner is not defined for non-uniform meshes. Following the presentation from Section 3 we introduce the following representation of a discrete $H^{-1/2}(\Gamma)$ norm which is based on the above broken norm

$$\|\lambda_h\|^2_{H^{-1/2}(\Gamma)} = \|\boldsymbol{\lambda}\|^2_{H_{\{-1/2\}}}$$

for $\lambda_h \in X_h$ where

$$H_{\{-1/2\}} = \bigoplus_{i=1}^{K} H^{\{i\}}_{-1/2}, \tag{40}$$

where

$$H^{\{i\}}_{-1/2} = L_{0,i}(L_{0,i}^{-1} L_{1,i})^{-1/2}$$

with $(L_{k,r})_{ij} = (\phi_i, \phi_j)_{H_0^k(\Gamma_r)}$, $k = 0, 1$ discrete operators defined on the interior of each boundary Γ_i. The following result can be proved immediately.

Proposition 5.3 *Let $X_h = span\{\phi_i, 1 \le i \le m\}$ be defined as above and let $H_{\{-1/2\}}$ be defined as in (40). Then for all $\boldsymbol{\lambda} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$*

$$\kappa_1 \le \frac{\boldsymbol{\lambda}^T S \boldsymbol{\lambda}}{\boldsymbol{\lambda}^T H_{\{-1/2\}} \boldsymbol{\lambda}} \le \kappa_2$$

with constants κ_1, κ_2 independent of h.

Proof: Similar to the proof of Proposition (5.1). ■

6 Numerical experiments

We present in this section numerical experiments corresponding to the two applications of the previous section. In both cases the solutions are obtained using preconditioned iterative methods with a combination of 2D and 1D preconditioners. The latter type are norm-matrices for fractional Sobolev spaces of index 1/2 which are constructed on the boundaries of the computational domain.

6.1 Domain decomposition for elliptic problems

We solve the test problem (22) as well as the following convection-diffusion problem

$$\begin{cases} -\nu\Delta u + \vec{b}\cdot\nabla u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{41}$$

We note here that Proposition 5.1 only applies in the symmetric case (22). However, one can derive suitable convergence results for the nonsymmetric case also which we did not include here. We refer the reader to (Arioli, Kouronis and Loghin 2008) for further details. The domain is the unit square which was subdivided into squares. We used the finite element method to discretise the problems as well as the norm-matrices required. The choice of finite-dimensional space was V^h defined in (25) with $r = 1$ and also $r = 2$. The choice $k = 2$ is relevant in the context of preconditioning discrete convection-diffusion operators arising from so-called P2-P1 discretisations of the Stokes (Silvester and Wathen 1994), (Golub and Wathen. 1998), and Oseen problems (Elman and Silvester 1996), (Kay, Loghin and Wathen. 2002) which employ quadratic piecewise polynomial spaces for the approximation of the momentum equations.
Due to non-symmetry, we choose to work with nonsymmetric iterative methods (flexible GMRES), coupled with nonsymmetric preconditioners of the form

$$P = \begin{pmatrix} A_{II} & A_{IB} \\ 0 & P_S \end{pmatrix}$$

with $A_{II} = \nu L_{II} + N_{II}$ where L_{II} is the direct sum of Laplacians assembled on each subdomain and N_{II} is the direct sum of the convection operator $\vec{b}\cdot\nabla$ assembled also on each subdomain. This choice of preconditioner is known to be useful provided we have a good approximation P_S to the Schur complement. Thus, if P_S is replaced by S convergence is achieved in 2 iterations (Murphy, Golub and Wathen 2000). Our choice of preconditioner will never achieve this, since the norm-matrices derived above do not approximate S itself but are equivalent operators. However, we will see that the resulting performance remains attractive.
The Schur complement preconditioner P_S is chosen to be each of $H_{1/2}$ and $H_{1/2,h}$, the discrete $H^{1/2}(\Gamma)$-norms defined in Proposition 5.1 and Remark 5.1, respectively. We also chose to work with a simplified version $\widehat{H}_{1/2}$ of $H_{1/2}$ obtained by replacing the mass matrix L_0 by a lumped version $\widehat{L}_0$:

$$\widehat{H}_{1/2} := \widehat{L}_0(\widehat{L}_0^{-1}L_1)^{1/2}.$$

The action of the inverses of $H_{1/2,h}, H_{1/2}, \widehat{H}_{1/2}$ was computed using the iterative method presented in section 4.2 which uses the Lanczos algorithm with $k = O(m^{1/2})$.

Remark 6.1 *In the case where the domain is subdivided into several subdomains the boundary* Γ *will be the union of internal faces or boundaries (a so-called skeleton or wirebasket)*

$$\Gamma = \bigcup_{i=1}^{K} \Gamma_i.$$

One can generalize the definition of a $H^{1/2}(\Gamma)$-norm to a broken $H^{1/2}(\Gamma)$-norm which results in a direct sum of norm-matrices as in 40 for the biharmonic problem. However, we choose to work with a related generalization which involves assembling the Grammians L_0, L_1 on the whole wirebasket Γ. In particular, L_1 will incorporate Dirichlet conditions corresponding to the set $\partial\Omega \cap \Gamma$ and will include additional contributions at each internal vertex. The resulting matrix is a norm-matrix on $[X_h, Y_h]_{1/2} \subset H^{1/2}(\Gamma)$ with improved spectral properties (Arioli et al. 2008).

6.1.1 The Poisson problem

The number of iterations is displayed in Table 1 for the cases $r = 1, r = 2$ (linear and quadratic finite elements) respectively. The size m of the skeleton is also displayed; it is obvious that a direct calculation of the matrix square-root function is becoming prohibitive for an increasing number of domains and an increasing mesh-size. As expected, the number of iterations is independent of the size of the problem. Moreover, the preconditioning procedure appears to be quasi-scalable with only a slight, possibly logarithmic dependence on the number of subdomains. A notable result is the performance of the preconditioner $\widehat{H}_{1/2}$ which is a simplified version of the other two which employs a lumped approximation of the mass matrix.

			$r=1$			$r=2$		
#dom	n	m	$H_{1/2,h}$	$H_{1/2}$	$\widehat{H}_{1/2}$	$H_{1/2,h}$	$H_{1/2}$	$\widehat{H}_{1/2}$
4	45,377	449	10	9	9	11	11	11
	180,865	897	10	10	10	11	11	11
	722,177	1793	11	11	11	11	11	11
16	45,953	1149	13	12	12	13	13	13
	183,041	2301	13	13	13	13	13	13
	730,625	4605	13	13	13	13	13	13
64	66,049	3549	16	14	14	16	15	15
	263,169	7133	16	15	15	16	15	15
	1,050,625	14,301	17	16	15	17	15	15

Table 1: *FGMRES iterations for model problem* (22) *for* $r = 1, 2$.

6.1.2 The convection-diffusion problem

We solved test problem (41) for the choice of 'rotating wind'

$$\vec{b} = (2(2y-1)(1-(2x-1)^2), -2(2x-1)(1-(2y-1)^2)).$$

We chose to approximate only the action of $\widehat{H}_{1/2}^{-1}$ using the generalised Lanczos process with the same choice of k. The range of diffusion coefficients was $\nu = 1, 0.1, 0.01$. The results are displayed in Table 2. In all cases the number of iterations is independent of the size of the problem, though it grows with reducing ν. The dependence on the number of subdomain also grows with reducing ν. This reflects the inability of our symmetric preconditioner to remain equivalent in some sense to an increasingly more nonsymmetric Schur complement.

			$r = 1$			$r = 2$		
#dom	n	m	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$
	45,377	449	10	12	21	12	13	20
4	180,865	897	11	11	20	12	13	19
	722,177	1793	11	11	19	12	12	18
	45,953	1149	12	17	37	13	17	35
16	183,041	2301	13	17	35	13	16	32
	730,625	4605	12	15	32	12	15	30
	66,049	3549	16	22	55	17	21	51
64	263,169	7133	17	22	52	16	20	46
	1,050,625	14,301	15	19	47	16	19	43

Table 2: *FGMRES iterations for model problem* (41) *for* $r = 1, r = 2$, $\nu = 1, 0.1, 0.01$ *and* $P_S = \widehat{H}_{1/2}$.

6.2 Biharmonic problem

We solved the biharmonic problem using the formulation (28), which was discretised using the weak formulation (34), where V^h is the finite element space defined in (25) with $r = 2$ (quadratic approximation). This is known to be a stable mixed finite element method for the biharmonic problem (Shaidurov 1995). As the Glowinski-Pironneau method is a boundary Schur complement approach, we choose to work again with a block-triangular preconditioner of the form (cf. (39))

$$P = \begin{pmatrix} L & Z \\ & P_S \end{pmatrix}$$

where $P_S = H_{\{-1/2\}}$ was defined in (40). We ignore the symmetry of our problem and use again flexible GMRES given the changing nature of our preconditioner due to the Lanczos approximation. As in the case of the previous example, we consider an approximation $\tilde{H}_{\{-1/2\}}$ resulting from replacing the mass matrix L_0 with a lumped version $\widehat{L}_0$. The results are displayed in Table 3. As expected, the number of iterations is independent of n; moreover, the preconditioner reduces greatly the iteration count compared to the case where no preconditioner is employed. We notice that in the unpreconditioned case the dependence on h is evident as predicted by Braess and Peisker. (1986), though mild ($O(h^{1/2})$); we also notice that in this case there is a considerable additional computational effort, particularly compared to the minimal effort that the above preconditioners require. The same behaviour can be noticed for un-isotropic meshes. Table 4 displays the iteration count corresponding to a finite element discretisation on an exponentially stretched mesh, with nodes clustered near the boundary and mesh aspect ratio ranging from 1 to 20.

7 Summary

We presented a derivation of norm representations of norms associated with interpolation spaces. In particular, we focused on projections onto conforming finite element spaces of fractional

n	m	I	$H_{\{-1/2,h\}}$	$H_{\{-1/2\}}$	$\widehat{H}_{\{-1/2\}}$
84,610	640	26	10	12	12
337,154	1,280	30	9	11	11
1,346,050	2,560	36	9	11	11

Table 3: *FGMRES iterations for model problem* (27) *for a range of preconditioners* P_S.

n	m	I	$H_{\{-1/2,h\}}$	$H_{\{-1/2\}}$	$\widehat{H}_{\{-1/2\}}$
31,250	496	37	8	9	9
128,018	1008	43	8	9	9
518,162	2032	52	7	8	8

Table 4: *FGMRES iterations for model problem* (27) *for a range of preconditioners* P_S.

Sobolev norms. A notable result is that interpolation norms can be represented as products of generally real powers of Grammian matrices associated with the pair of spaces generating the scale of interpolation spaces. The issue of algorithmic complexity in the construction of discrete interpolation norms was also considered with the presentation of some sparse matrix algorithms for the approximation of real powers of matrices. Some applications arising from PDE modeling were considered to illustrate the usefulness of interpolation norms in large-scale computing.

References

Allen, E., Baglama, J. and Boyd, S. (2000), 'Numerical approximation of the product of the square root of a matrix with a vector', *Linear Algebra and its Applications* **310**, 167–181.

Arioli, M. and Loghin, D. (2008), Boundary preconditioners for fourth-order elliptic problems, Technical report, RAL. In preparation.

Arioli, M., Kouronis, D. and Loghin, D. (2008), Matrix square root algorithms for domain decomposition methods, Technical report, RAL. In preparation.

Bertoluzza, S. and Kunoth, A. (2000), 'Wavelet stabilization and preconditioning for domain decomposition', *IMA J. Numer. Anal.* **20**, 533–559.

Braess, D. and Peisker., P. (1986), 'On the numerical solution of the biharmonic equation and the role of squaring matrices for preconditioning', *IMA J. Numer. Anal.* **6**, 393–404.

Bramble, J. H., Pasciak, J. E. and Vassilevski, P. S. (2000), 'Computational scales of Sobolev norms with applications to preconditioning', *Math. Comp.* **69**, 463–480.

Brenner, S. C. and Scott, L. R. (1994), *The Mathematical Theory of Finite Element Methods*, Springer.

Brezzi, F. and Marini, L. (1994), 'A three-field domain decomposition method', *Contemp. Math.*

Brezzi, F. and Raviart, P. A. (1976), *Topics in Numerical Analysis III*, Academic Press, chapter Mixed finite element methods for 4-th order elliptic problems, pp. 33–36.

Burstedde, C. (2005), Fast Optimised Wavelet Methods for Control Problems Constrained by Elliptic PDEs, PhD thesis, Rheinischen Friedrich-Wilhelms-Universität Bonn.

Burstedde., C. (2007), 'On the numerical evaluation of fractional Sobolev norms', *Comm. Pure and Appl. Analysis* **6**(3), 587–605.

Cabos, C. (1997), 'Evaluation of matrix functions with the block Lanczos algorithm', *Comput. Math. Appl.* **33**(1-2), 45–57. Approximation theory and applications.

Ciarlet, P. G. and Raviart., P. A. (1974), *Mathematical Aspects of Finite Elements in Partial Differential Equations*, Academic Press, New York, chapter "A mixed finite element method for the biharmonic equation", pp. 125–145.

Druskin, V. and Knizhnerman, L. (1989), 'Two polynomial methods of calculating functions of symmetric matrices', *U.S.S.R. Comput. Maths. Math. Phys.* **29**(6), 112–121.

Druskin, V. and Knizhnerman, L. (1998), 'Extended Krylov subspaces: approximations of the matrix square root and related functions', *SIAM J. Matrix Anal. Appl.* **19**(3), 755–771.

Druskin, V., Greenbaum, A. and Knizhnerman, L. (1998), 'Using nonorthogonal Lanczos vectors in the computation of matrix functions', *SIAM J. Sci. Comput.* **19**(1), 38–54 (electronic). Special issue on iterative methods (Copper Mountain, CO, 1996).

Dryja, M. (1982), 'A capacitance matrix method for Dirichlet problem on polygon region', *Numer. Math.* **39**, 51–64.

Egger, H. (2006), 'Semi-iterative regularization in Hilbert scales', *SIAM J. Numer. Anal.* **44**(1), 66–81.

Egger, H. and Neubauer, A. (2005), 'Preconditioning landweber iteration in Hilbert scales', *Numer. Anal.* **101**, 643–662.

Eiermann, M. and Ernst, O. G. (2006), 'A restarted Krylov subspace method for the evaluation of matrix functions', *SIAM J. Numer. Anal.* **44**(6), 2481–2504 (electronic).

Elman, H. C. and Silvester, D. J. (1996), 'Fast nonsymmetric iterations and preconditioning for Navier-Stokes equations', *SIAM J. Sci. Comp.* **17**, 33–46.

Falk, R. S. (1978), 'Approximation of the biharmonic equation by a mixed finite element method', *SIAM J. Numer. Anal.* **15**(3), 556–567.

Friedlander, L. (2005), 'Extremal properties of eigenvalues for a metric graph', *Ann. Inst. Fourier* **55**(1), 199–211.

Glowinski, R. and Pironneau, O. (1979), 'Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem', *SIAM Rev.* **21**(2), 167–212.

Golub, G. H. and Wathen., A. J. (1998), 'An iteration for indefinite systems and its application to the Navier–Stokes equations', *SIAM J. Sci. Comput.* **19**(2), 530–539.

Hale, N., Higham, N. J. and Trefethen, L. N. (2008), 'Computing A^α , $\log A$ and related matrix functions by contour integrals', *SIAM J. Numer. Anal.* To appear.

Hegland, M. (1995), 'Variable Hilbert scales and their interpolation inequalities with applications to Tikhonov regularization', *Appl. Anal.* **59**(1-4), 207–223.

Higham, N. J. (2008), *Functions of Matrices: Theory and Computation*, SIAM, Philadelphia, USA.

Horn, R. A. and Johnson., C. R. (1985), *Matrix Analysis*, Cambridge University Press.

Kay, D., Loghin, D. and Wathen., A. J. (2002), 'A preconditioner for the Steady-State Navier-Stokes equations', *SIAM J. Sci. Comput.* **24**(1), 237–256.

Kuchment, P. (2006), Quantum graphs: an introduction and a brief survay, Technical report, arXiv.0802.3442v1 [math-ph].

Lancaster, P. and Tismenetsky, M. (1985), *The Theory of Matrices, 2nd ed.*, Academic Press, New York.

Lions, J. L. and Magenes, E. (1968), *Problèmes aux limites non homogènes et applications. I*, Dunod, Paris.

Monk, P. (1987), 'A mixed finite element method for the biharmonic equation', *SIAM J. Numer. Anal.*

Murphy, M. F., Golub, G. H. and Wathen, A. J. (2000), 'A note on preconditioning for indefinite linear systems', *SIAM J. Sci. Comp.* **21**, 1969–1972.

Natterer, F. (1980), 'A Sobolev space analysis of picture reconstruction', *SIAM J. Appl. Math.* **39**(3), 402–411.

Neubauer, A. (1988), 'An a posteriori parameter choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates', *SIAM J. Numer. Anal.*

Osborn, J., Babuška, I. and Pitkäranta, J. (1980), 'Analysis of mixed methods using mesh dependent norms', *Math. Comp.* **35**, 1039–1062.

Parlett, B. N. (1998), *The Symmetric Eigenvalue Problem*, Classics in Applied Mathematics 20, SIAM, Philadelphia, USA.

Peisker, P. (1988), 'On the numerical solution of the first biharmonic equation', *Mathematical Modelling and Numerical Analysis* **22**(4), 655–676.

Quarteroni, A. and Valli, A. (1999), *Domain Decomposition Methods for Partial Differential Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, UK.

Riesz, F. and Sz-Nagy, B. (1956), *Functional Analysis*, Blackie and Son Limited.

Saad, Y. (1992), 'Analysis of some Krylov suspace approximations to the matrix exponential operator', *SIAM J. Numer. Anal.* **29**(1), 209–228.

Sangalli, G. (2003), 'Quasi-optimality of the supg method for the one-dimensional advection-diffusion problem', *SIAM J. Numer. Anal.* **41**(4), 1528–1542.

Sangalli, G. (2005), 'A uniform analysis of non-symmetric and coercive linear operators', *SIAM J. Math. Anal.* **36**(6), 2033–2048.

Sangalli, G. (2008), 'Robust a-posteriori estimator for advection-diffusion-reaction problems', *Math. Comput.* **77**, 41–70.

Shaidurov, V. V. (1995), *Multigrid Methods for Finite Elements*, Kluwer, Dordrecht.

Silvester, D. J. and Mihajlović., M. D. (2004), 'A black-box multigrid preconditioner for the biharmonic equation', *BIT* **44**(1), 151–163.

Silvester, D. J. and Wathen, A. J. (1994), 'Fast iterative solution of stabilised Stokes systems. II. Using general block preconditioners', *SIAM J. Num. Anal.* **31**(5), 1352–1367.

Tautenhahn, U. (1996), 'Error estimates for regularization methods in Hilbert scales', *SIAM J. Numer. Anal.* **33**(6), 2120–2130.

Toselli, A. and Widlund, O. (2005), *Domain Decomposition Methods – Algorithms and Theory*, Springer, Berlin-Heildeberg-New York.