Discrete interpolation norms with applications

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**ABSTRACT**
We derive discrete norm representations associated with projections of interpolation spaces onto finite dimensional subspaces. These norms are products of integer and noninteger powers of the Grammian matrices associated with the generating pair of spaces for the interpolation space. We include a brief description of some of the algorithms which allow the efficient computation of matrix powers. We consider in some detail the case of fractional Sobolev spaces both for positive and negative indices together with applications arising in preconditioning techniques. Numerical experiments are included.

**Keywords:** Generalized Laplace operators, Fractional Sobolev spaces, Krylov methods, Generalised Lanczos method, Domain decomposition, Matrix square root.

**AMS(MOS) subject classifications:** 35J15, 35J50, 47G30, 65F10, 65F15, 65F30, 65N30.

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1 Introduction

Interpolation spaces arise naturally in the formulation of many modeling applications ranging from domain decomposition methods (Dryja 1982), (Brezzi and Marini 1994), (Bertoluzza and Kunoth 2000), (Quarteroni and Valli 1999), (Bramble, Pasciak and Vassilevski 2000), (Burstedde. 2007), to image processing (Natterer 1980), (Neubauer 1988), (Hegland 1995), (Tautenhahn 1996), (Egger and Neubauer 2005), (Egger 2006), advection-diffusion problems (Sangalli 2003, Sangalli 2005, Sangalli 2008), and elasticity (Glowinski and Pironneau 1979). It is therefore desirable to characterise the corresponding discrete spaces with the aim to enable new numerical approaches and algorithms. In particular, we are interested in defining discrete norm representations for projections of interpolation spaces onto suitable finite dimensional subspaces. As a major application we derive discrete norms corresponding to conforming finite element discretisations of fractional Sobolev spaces. In particular, we show that they involve fractional powers of products of certain Grammian matrices associated with the bases of the finite element spaces employed. For the case where wavelet spaces are of interest, some recent results can be found in (Burstedde. 2007), (Burstedde 2005), where matrix representations in wavelet bases are given for Sobolev spaces of non-integer orders.

One limitation associated with discrete interpolation norms is that for large scale computations they are seemingly expensive to compute. We show that this need not be so in the case of finite element norms for which inexpensive factorisations can be devised using some standard approximation procedures such as projection onto Krylov subspaces.

The paper is organised as follows. In section 2 we introduce the concept of interpolation between abstract Hilbert spaces as described in (Lions and Magenes 1968). We also consider the projection onto finite dimensional subspaces and derive expressions for the associated norm representations. In section 3 we derive the discrete norms resulting from projection onto finite element spaces of fractional Sobolev spaces. The following section contains a brief review of existing algorithms for computation of matrix non-integer powers which arise in the definition of discrete interpolation norms; we also include in this section the details of a Lanczos procedure employed for the computation of interpolation norms. Section 5 includes applications arising in preconditioning of domain decomposition methods and methods for the biharmonic equation. The paper concludes with some numerical experiments.

2 Interpolation spaces

We review here the presentation from Lions and Magenes (1968). Let $X, Y$ denote two Hilbert spaces with $X \subset Y$, $X$ dense and continuously embedded in $Y$. Let $(\cdot, \cdot)_X, (\cdot, \cdot)_Y$ denote the corresponding inner products, and $\|\cdot\|_X, \|\cdot\|_Y$ the respective norms. By the Riesz representation theory (see for example Riesz and Sz-Nagy, 1956) there exists an operator $\mathcal{J} : X \to Y$ which is positive and self-adjoint with respect to $(\cdot, \cdot)_Y$ such that

\[(u, v)_X = (u, \mathcal{J}v)_Y.\]  \hspace{1cm} (1)

Using the spectral decomposition of $\mathcal{J}$ we define the operator $\mathcal{E} = \mathcal{J}^{1/2} : X \to Y$, which in turn is positive self-adjoint. Moreover (see Lions and Magenes, 1968, Chapter 1, Section 2.2), the space $X$ can be defined to be the domain $D(\mathcal{E})$ of $\mathcal{E}$ and the norm of $X$ is equivalent to the graph norm $\|\cdot\|_\mathcal{E}$

\[\|u\|_X \sim \|u\|_\mathcal{E} := (\|u\|_Y^2 + \|\mathcal{E}u\|_Y^2)^{1/2}.\]
Similarly, the spectral decomposition of \( E \) can be used to define any real power of \( E \). Let \( \theta \in [0, 1] \) and let \( \| \cdot \|_\theta \) denote the scale of graph norms

\[
\| u \|_\theta := \left( \| u \|_Y^2 + \| E^{1-\theta} u \|_Y^2 \right)^{1/2}. \tag{2}
\]

One can then show that the domain of \( E^{1-\theta} \) endowed with the inner-product

\[
(u, v)_\theta = (u, v)_Y + (u, E^{1-\theta} v)_Y
\]

is a Hilbert space (Lions and Magenes, 1968). This is an interpolation space of index \( \theta \) for the pair \([X, Y]\) and is denoted by \([X, Y]_\theta\)

\[
[X, Y]_\theta := D(E^{1-\theta}), \quad 0 \leq \theta \leq 1.
\tag{3}
\]

Note that \([X, Y]_0 = X\) and \([X, Y]_1 = Y\). Moreover, if \(0 < \theta_1 < \theta_2 < 1\) then

\[
X \subset [X, Y]_{\theta_1} \subset [X, Y]_{\theta_2} \subset Y.
\]

Let now \( \mathcal{L}(A; B) \) denote the space of continuous linear operators from \( A \) into \( B \). The following classic interpolation theorem can be found in (Lions and Magenes 1968, Theorem 5.1).

**Theorem 2.1** Let \( X, Y \) be defined as above and let \( X, Y \) satisfy similar properties. Let \( \pi \in \mathcal{L}(X; X) \cap \mathcal{L}(Y; Y) \). Then for all \( \theta \in (0, 1) \),

\[
\pi \in \mathcal{L}([X, Y]_\theta; [X, Y]_\theta).
\]

We turn now to the case where the spaces generating the scale of interpolation spaces are finite-dimensional. In particular, we are interested in the discrete norms associated with these spaces.

### 2.1 Finite dimensional interpolation spaces.

Let \( H \in \mathbb{R}^{n \times n} \) be a Diagonalisable matrix with a real Positive-definite Spectrum (DPS). Let the eigenvalue decomposition of \( H \) be denoted by \( H = V^{-1}D_H V \) where \( D_H \) is a diagonal matrix with entries \( \lambda_i \) which satisfy

\[
0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots \lambda_n.
\]

The following definition (Lancaster and Tismenetsky, 1985, and Higham, 2008) will be useful in the subsequent discussion.

**Definition 1** Let \( \theta \in \mathbb{R} \). The \( \theta \) power of a DPS matrix \( H = V^{-1}D_H V \) is defined via the decomposition

\[
H^\theta := V^{-1}D_H^\theta V.
\]

If \( H \) is symmetric, then \( H = H^{1/2}H^{1/2} \) where \( H^{1/2} = V^{-1}D_H^{1/2} V \) is also a symmetric and positive-definite matrix and \( V^{-1} = V^T \). We also remark that a DPS matrix \( H \) is a self-adjoint operator in the scalar product \((\cdot, \cdot)_V\) and that \( D_H \) is also the spectrum of the positive definite pencil \( \{V^T V, V^T D_H V\} \).

Furthermore, \( H \) induces a vector-norm \( \| \cdot \|_H \) with dual-norm \( \| \cdot \|_H^* \) defined as ((Horn and Johnson. 1985, p. 275))

\[
\|z\|_{H^*} := \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T z}{\|v\|_H} = \|z\|_{H^{-1}}. \tag{4}
\]
Remark 2.1 Let \( z \) be defined to be the vector of coefficients of a function \( z = \sum_{i=1}^{n} z_i \psi_i \) expanded in a given basis \( \{ \psi_i \}_{1 \leq i \leq n} \). Let \( \{ \phi_i \}_{1 \leq i \leq n} \) be another linearly independent set defined via
\[
\phi_i = \sum_{j=1}^{n} K_{ij} \psi_j,
\]
with \( K \in \mathbb{R}^{n \times n} \) nonsingular. Then \( z = \sum_{i=1}^{n} z_i \psi_i = \sum_{i=1}^{n} w_i \phi_i \) so that \( z = Kw \) and
\[
\| z \|_{H'} = \| Kw \|_{H^{-1}} = \| w \|_{K^T H^{-1} K}. \tag{5}
\]
The matrix \( K^T H^{-1} K \) is the representation of \( H' \) with respect to a new basis which is related to the original basis via \( K \).

Given two symmetric and positive-definite matrices \( H_1 \in \mathbb{R}^{n \times n}, H_2 \in \mathbb{R}^{m \times m} \) we define the following matrix norm for matrices \( M \in \mathbb{R}^{m \times n} \) (see (Horn and Johnson. 1985, p. 311))
\[
\| M \|_{H_1, H_2} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\| Mv \|_{H_2}}{\| v \|_{H_1}}. \tag{6}
\]
Finally, we define the \( H \)-condition number of a matrix \( M \) to be the quantity
\[
\kappa_H (M) = \| M \|_{H, H^{-1}} \| M^{-1} \|_{H^{-1}, H}.
\]

Let now \( X_h \subset X, Y_h \subset Y \) denote two finite-dimensional subspaces of \( X, Y \) respectively, with \( n = \dim X_h = \dim Y_h \). They are Hilbert spaces when endowed with the inner-products \( (\cdot, \cdot)_X, (\cdot, \cdot)_Y \). We can similarly define corresponding positive, self-adjoint operators \( J_h, E_h : X_h \to Y_h \)
\[
(u_h, v_h)_X = (u_h, J_h v_h)_Y \quad u_h, v_h \in X_h \tag{7}
\]
where \( J_h \) is positive self-adjoint and \( E_h = J_h^{1/2} \). We define the discrete interpolation spaces
\[
[X_h, Y_h]_\theta := D(E_h^{1-\theta}).
\]
Furthermore, we define the scale of discrete norms
\[
\| u_h \|_{\theta, h} := \left( \| u_h \|_Y^2 + \| E_h^{1-\theta} u_h \|_Y^2 \right)^{1/2}. \tag{8}
\]
We are interested in describing the set of symmetric and positive definite matrices
\[
\{ H_\theta \in \mathbb{R}^{n \times n} : n \in \mathbb{N}, 0 \leq \theta \leq 1 \}
\]
which induce norms equivalent to \( \| \cdot \|_{\theta, h} \) with constants of equivalence independent of \( n \). Let \( H_X, H_Y \) denote the Grammian (or Riesz) matrices corresponding to the inner products \( (\cdot, \cdot)_X, (\cdot, \cdot)_Y \), respectively. More precisely,
\[
(H_X)_{ij} = (\phi_i, \phi_j)_X, \quad (H_Y)_{ij} = (\phi_i, \phi_j)_Y, \quad 1 \leq i, j \leq n,
\]
where \( \{ \phi_i \}_{1 \leq i \leq n} \) denotes a basis of \( X_h \), so that
\[
\| u_h \|_X = \| u \|_{H_X}, \quad \| u_h \|_Y = \| u \|_{H_Y},
\]

where \( \mathbf{u} \) denotes the vector of coefficients of \( u_h \) expanded in the basis \( \{ \phi_i \} \). We first note that the discrete Riesz representation (7) becomes

\[
\mathbf{u}^T H_X \mathbf{v} = \mathbf{u}^T H_Y J \mathbf{v}
\]

so that \( J = H_Y^{-1} H_X \) is a product of two symmetric and positive definite matrices. The matrix \( J \) is self-adjoint and positive definite in the discrete \( H_Y \)-inner-product, as

\[
(u, J v)_{H_Y} = \mathbf{u}^T H_Y (H_Y^{-1} H_X) v = \mathbf{v}^T H_X \mathbf{u} = (\mathbf{v}, T \mathbf{u})_{H_Y}
\]

and

\[
(u, J u)_{H_Y} = \mathbf{u}^T H_X \mathbf{u} > 0 \quad \text{for all } \mathbf{u} \neq \mathbf{0}.
\]

One can also write explicitly the eigenvalue decomposition of \( J \) : since \( H_X, H_Y \) are symmetric and positive-definite, there exists a matrix \( Q \) such that ((Horn and Johnson. 1985, Cor 7.6.2))

\[
H_X = Q^T D Q, \quad H_Y = Q^T Q,
\]

where \( D \) is a diagonal matrix with positive entries, so that

\[
J = H_Y^{-1} H_X = Q^{-1} D Q. \quad (9)
\]

Note that this implies that \( J \) is DPS, so that real powers of \( J \) can be defined via Definition 1.

It is evident that the matrix representation of \( E_h \) in the basis \( \{ \phi_i \} \) is the DPS matrix

\[
E = Q^{-1} D^{1/2} Q;
\]

furthermore, the matrix representation of \( E_h^{1-\theta} \) is similarly

\[
E^{1-\theta} = Q^{-1} D^{(1-\theta)/2} Q.
\]

We now turn to the matrix representation of the norm \( \| \cdot \|_{\theta,h} \) which we denote by \( H_{\theta,h} \). Definition (8) yields

\[
\| \mathbf{u} \|^2_{H_{\theta,h}} = \| \mathbf{u} \|^2_{H_Y} + \| E^{1-\theta} \mathbf{u} \|^2_{H_Y}
\]

so that

\[
H_{\theta,h} = H_Y + (E^{1-\theta})^T H_Y E^{1-\theta} = H_Y + Q^T D^{1-\theta} Q = H_Y + H_Y J^{1-\theta} = Q^T (I + D^{1-\theta}) Q.
\]

Now, by definition, the norm of \( X_h \) is equivalent to the discrete graph norm (8) with \( \theta = 0 \):

\[
Q^T D Q = H_X \sim H_{0,h} = H_Y + H_Y J = Q^T (I + D) Q.
\]

Hence, there exist two positive real constants \( \alpha_1, \alpha_2 \) independent of \( n \) such that

\[
\alpha_1 D_{ii} \leq (1 + D_{ii}) \leq \alpha_2 D_{ii} \quad 1 \leq i \leq n.
\]

It follows that, setting \( \tilde{\alpha}_1 = 1, \tilde{\alpha}_2 = \max \{ \alpha_2, 2 \} \), there holds

\[
\tilde{\alpha}_1 D^{1-\theta}_{ii} \leq (1 + D^{1-\theta}_{ii}) \leq \tilde{\alpha}_2 D^{1-\theta}_{ii} \quad 1 \leq i \leq n
\]

and we deduce that

\[
H_{\theta,h} \sim H_{\theta} := Q^T D^{1-\theta} Q = H_Y J^{1-\theta}. \quad (9)
\]

The matrix \( H_{\theta} \) is the reduced form of \( H_{\theta,h} \) and is equivalent to it.
Remark 2.2 If the matrices $H_X, H_Y$ are simultaneously diagonalisable, one can derive a simpler expression for $H_{\theta,h}$. Assuming there exists an invertible matrix $Z$ such that
\[
Z^{-1}H_XZ = D_X, \quad Z^{-1}H_YZ = D_Y,
\]
we find
\[
J = H_Y^{-1}H_X = ZD_Y^{-1}D_XZ^{-1}, \quad J^{1-\theta} = H_Y^{\theta-1}H_X^{1-\theta}
\]
and thus
\[
H_{\theta,h} = H_Y + H_YJ^{1-\theta} = H_Y + H_YH_Y^{\theta-1}H_X^{1-\theta} = H_Y + H_YH_X^{1-\theta}
\]
and
\[
H_{\theta} = H_YH_X^{1-\theta} = H_Y^{\theta/2}H_X^{1-\theta}H_Y^{\theta/2}.
\]
Let now $i_h : \mathcal{L}(X_h; X) \cap \mathcal{L}(Y_h; Y)$ denote the continuous injection operator between the indicated spaces. By Thm 2.1,
\[
i_h \in \mathcal{L}([X_h, Y_h]_\theta; [X, Y]_\theta).
\]
In particular, for all $u_h \in [X_h, Y_h]_\theta$,
\[
\|i_h u_h\|_\theta = \|u_h\|_\theta \leq C_1\|u_h\|_{\theta,h}.
\]
Assume now that there exists an interpolation operator $I_h$ such that $I_h : \mathcal{L}(X; X_h) \cap \mathcal{L}(Y; Y_h)$ and $I_hu = u_h$ for all $u_h \in X_h$. Again, by Thm 2.1,
\[
I_h \in \mathcal{L}([X, Y]_\theta; [X_h, Y_h]_\theta)
\]
and we have for all $u \in [X, Y]_\theta$,
\[
\|I_h u\|_{\theta,h} \leq C_2\|u\|_\theta.
\]
Since $[X_h, Y_h]_\theta \subset [X, Y]_\theta$ (the set inclusion follows from $X_h \subset X$ and $Y_h \subset Y$), Eqns (10), (11) yield
\[
\frac{1}{C_1}\|u_h\|_\theta \leq \|u_h\|_{\theta,h} \leq C_2\|u_h\|_\theta.
\]
We summarise this result in the following Lemma.

Lemma 2.2 Let $X_h \subset Y_h, X \subset Y$ be Hilbert spaces with inner-products $(\cdot, \cdot)_X, (\cdot, \cdot)_Y$ and let $\| \cdot \|_\theta, \| \cdot \|_{\theta,h}$ be defined by (2), (8), respectively. Let us assume that there exists an interpolation operator $I_h$ such that $I_h : \mathcal{L}(X; X_h) \cap \mathcal{L}(Y; Y_h)$ and $I_hu = u_h$ for all $u_h \in X_h$. Then the norms $\| \cdot \|_\theta, \| \cdot \|_{\theta,h}$ are equivalent on $[X_h, Y_h]_\theta$ for all $\theta \in (0, 1)$.

Corollary 2.3 Let the assumptions of Lemma 2.2 hold. Let $H_{\theta}$ be defined as in (9). Then the norms $\| \cdot \|_\theta, \| \cdot \|_{H_{\theta}}$ are equivalent on $[X_h, Y_h]_\theta$ for all $\theta \in (0, 1)$.

Finally, we end with the following result concerning the conditioning of $H_{\theta}$.

Lemma 2.4 Let $\kappa := \kappa_{H_Y}(H_X)$ denote the $H_Y$-condition number of $H_X$. Then
\[
\kappa_{H_Y}(H_{\theta}) = \kappa^{1-\theta}.
\]
\textbf{Proof:} Using the decompositions $H_X = Q^T D Q$, $H_Y = Q^T Q$, we find
\[
\|H_X\|_{H_Y, H_Y^{-1}} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T H_X v}{v H_Y v} = \max_i D_{hi}, \quad \|H_X^{-1}\|_{H_Y^{-1}, H_Y} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T H_0^{-1} v}{v H_Y^{-1} v} = \left( \min_i D_{hi} \right)^{-1}
\]
so that
\[
\kappa_{H_Y}(H_X) = \kappa_2(D).
\]
Similarly we find
\[
\|H_0\|_{H_Y, H_Y^{-1}} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T H_0 v}{v H_Y v} = \max_i D_{hi}^{1-\theta}, \quad \|H_0^{-1}\|_{H_Y^{-1}, H_Y} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T H_0^{-1} v}{v H_Y^{-1} v} = \left( \min_i D_{hi}^{1-\theta} \right)^{-1}
\]
so that
\[
\kappa_{H_Y}(H_0) = (\kappa_2(D))^{1-\theta},
\]
which is the stated result.

\subsection{2.2 Dual spaces}

The dense inclusion (3) leads to the following inclusion corresponding to the respective dual spaces:
\[
Y' \subset [X, Y]'_0 \subset X'.
\]
Moreover, the following duality result can be found in (Lions and Magenes 1968, Thm 6.2)
\[
[X, Y]'_0 = [Y', X']^{1-\theta}.
\]

Let $X'_h \subset X', Y'_h \subset Y'$ denote two finite-dimensional subspaces of $X', Y'$ respectively. They are Hilbert spaces when endowed with the inner-products $(\cdot, \cdot)_{X'}$, $(\cdot, \cdot)_{Y'}$. We can similarly define corresponding positive, self-adjoint operators $J'_h, E'_h : Y'_h \to X'_h$
\[
(u_h, v_h)_{Y'} = (u_h, J'_h v_h)_{X'}, \quad u_h, v_h \in Y'_h
\]
where $J'_h$ is positive self-adjoint and $E'_h = (J'_h)^{1/2}$. We define the discrete interpolation spaces
\[
[Y'_h, X'_h]_0 := D((E'_h)^{1-\theta}).
\]
Furthermore, setting $\theta' = 1 - \theta$ we define the scale of discrete norms
\[
\|u_h\|_{\theta', h} := \left( \|u_h\|_{X'}^2 + \| (E'_h)^{1-\theta'} u_h \|_{X'}^2 \right)^{1/2}.
\]

Since the matrices corresponding to norms $\| \cdot \|_{Y'}$, $\| \cdot \|_{X'}$ are $H'_X = H_X^{-1}$ and $H'_Y = H_Y^{-1}$, respectively, we find
\[
J' = (H'_Y)^{-1} H'_X = H_Y H_X^{-1} = Q^T D^{-1} Q^{-T}
\]
and hence
\[
(J')^{1-\theta} = Q^T D^{\theta-1} Q^{-T}.
\]
As before, the norm $\| \cdot \|_{\theta', h}$, with matrix representation $H'_{\theta', h}$, can be shown to be equivalent to a reduced norm with matrix representation $H'_{\theta'}$ which in turn can be seen to be simply the inverse of $H_{\theta'}$
\[
H'_{\theta', h} = H'_Y + H'_Y (J')^{1-\theta'} \sim H'_{\theta'} := H'_Y (J')^{1-\theta'} = Q^{-1} D^{\theta-1} Q^{-T} = H_{\theta'}^{-1}.
\]
Hence
\[
H_{\theta'}^{-1} = Q^{-1} D^{\theta-1} Q^{-T} = J^{-1} H_Y^{-1}
\]
can be taken to be the matrix representation of a norm on $[X_h, Y_h]'_0$.  

6
3 Fractional Sobolev spaces

In this section we consider the case where $X,Y$ are Sobolev spaces. In particular, we are interested in deriving the matrix representations of fractional Sobolev norms with a view to designing optimal iterative solution methods for finite element discretisations of PDE. We start by reviewing some standard definitions and results.

Let $\Omega$ denote an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\Gamma$ and let $\alpha$ denote a multi-index of order $m$ where $m$ is a positive integer. Let

$$H^m(\Omega) = \{ u : D^\alpha u \in L^2(\Omega), \ |\alpha| \leq m \}$$

denote the usual Sobolev space of order $m$, with the convention that $H^0(\Omega) = L^2(\Omega)$. Sobolev spaces of real index $0 \leq s \leq m$ are defined as interpolation spaces of index $\theta = 1 - s/m$ for the pair $[H^m(\Omega), H^0(\Omega)]$:

$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_{\theta}.$$

One can use this definition to characterise interpolation spaces for pairs of Sobolev spaces of real-index:

$$[H^{s_1}(\Omega), H^{s_2}(\Omega)]_{\theta} = H^{(1-\theta)s_1 + \theta s_2}(\Omega). \quad (16)$$

Let now $C^\infty_0(\Omega)$ denote the space of infinitely differentiable functions with compact support in $\Omega$ and let $H^s_0(\Omega)$ denote the completion of $C^\infty_0(\Omega)$ in $H^m(\Omega)$, where $s > 0$. Then

$$\begin{cases} 
H^s_0(\Omega) = H^s(\Omega) & s \leq 1/2 \\
H^s_0(\Omega) \subset H^s(\Omega) & s > 1/2 
\end{cases}$$

In fact, given $0 \leq s_2 < s_1$, one has the following characterisations based on interpolation

$$\begin{cases} 
[H^s_0(\Omega), H^{s_2}_0(\Omega)]_{\theta} = H^{(1-\theta)s_1 + \theta s_2}(\Omega) & \text{if } (1-\theta)s_1 + \theta s_2 \neq k + 1/2 \ (k \in \mathbb{N}), \\
[H^s_0(\Omega), H^{s_2}_0(\Omega)]_{\theta} \subset H^{k+1/2}_0(\Omega) & \text{if } (1-\theta)s_1 + \theta s_2 = k + 1/2 \ (k \in \mathbb{N}). 
\end{cases}$$

Finally, we define for $s > 0$

$$H^{-s}(\Omega) = (H^s_0(\Omega))^\prime.$$

Hence if $(1-\theta)s_1 + \theta s_2 \neq k + 1/2 \ (k \in \mathbb{N})$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_{\theta} = H^{-(1-\theta)s_1 - \theta s_2}(\Omega),$$

while if $(1-\theta)s_1 + \theta s_2 = k + 1/2 \ (k \in \mathbb{N})$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_{\theta} = \left(H^{k+1/2}_0(\Omega)\right)^\prime.$$

3.1 Special domains

The open $\Omega$ can be replaced by a regular manifold and the Sobolev spaces are built using a variational formulation based on the Laplace-Beltrami operator. The resulting Hilbert spaces can be used to build by the same techniques presented in Section 2 the corresponding fractional Sobolev spaces (see Lions and Magenes, 1968 [page 42]).

Another choice for $\Omega$ is to be a metric graph (in the literature these are also named quantum graphs). This choice will be used in Section 6.1 (see Remark 6.1). A metric graph is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ made of vertices $v \in \mathcal{V}$ and edges $e \in \mathcal{E}$ where at each edge $e$ is associate its length $\ell(e)$. For sake of simplicity, we assume that $\ell(e) < \infty \ \forall e$ and that number of vertices and edges is finite.
The degree of a vertex is the number of incident edges on it and the boundary of \( \mathcal{G} \) consists of the vertices of degree one. Moreover, an edge can be identified with a finite segment of the real line such that we can introduce a coordinate \( x(e) \) and a preferred direction of the edge. This defines a natural topology on the graph and makes it a 1D simplicial complex. As a consequence the points on the graph are not only vertices but all the intermediate points \( x \) on the edges and, thus, we can define a function \( f(x) \) on \( \mathcal{G} \) as a function defined on all points of the graph and not only on its vertices.

Having the coordinate \( x \), we can define the Lebesgue measure on \( \mathcal{G} \) and, then, the concept of integrability. This allows the introduction of some standard Sobolev function spaces on the graph.

**Definition 2** The space \( L^2(\mathcal{G}) \) consists of all measurable and square integrable functions on each edge \( e \) and such that 
\[
||f||^2_{L^2(\mathcal{G})} = \sum_{e \in E} ||f||^2_{L^2(e)} < \infty.
\]

**Definition 3** The Sobolev space \( H^1(\mathcal{G}) \) consists of all continuous functions on \( \mathcal{G} \) that belong to \( H^1(e) \) for each edge \( e \) and such that 
\[
\sum_{e \in E} ||f||^2_{H^1(e)} < \infty.
\]

We can also define the space \( H^1_0(\mathcal{G}) \) adding the condition that the functions \( f(x) \in H^1(\mathcal{G}) \) take zero values at the boundary vertices.

A friendly but accurate introduction and survey on quantum graphs is given by Kuchment 2006. In (Kuchment 2006) is also observed that there is no natural definition of Sobolev spaces \( H^k(\mathcal{G}) \) of order \( k \) higher than 1, due to the lack of natural conditions at vertices. For the Laplace operators and the case \( k = 2 \) is possible to introduce Neumann-Kirchoff conditions at the internal vertices that allow a consistent definition of the eigenvalue problem (see Friedlander, 2005).

However, taking into account the two previous definitions and the Hilbert structure of \( L^2(\mathcal{G}) \) and \( H^1(\mathcal{G}) \) we can still introduce the corresponding fractional Sobolev spaces by the interpolation method.

### 3.2 Discrete fractional Sobolev norms

The results from the previous section can be used to derive discrete Sobolev norms for some standard pairs. We consider below some examples which arise naturally in the formulation of elliptic PDE. In particular, Sobolev spaces of index \( \text{integer}+1/2 \) will be the focus of our discussion since they arise naturally as codomains of boundary operators.

We start by deriving the matrix representation of a norm defined on a discrete subspace of the interpolation space
\[
H^{k+1/2}_0(\Omega) = [H^k_0(\Omega), H^k_0(\Omega)]_{1/2}.
\]

Let \( X = H^{k+1}_0(\Omega), Y = H^{k}_0(\Omega) \) and let \( X_h \subset X, Y_h \subset Y \). Let \( \{\phi_i\}_{1 \leq i \leq n} \subset X_h \) be a spanning set for \( Y_h \) and let \( L_k \in \mathbb{R}^{n \times n} \) denote the Grammian matrices corresponding to the \( (\cdot, \cdot)_{H^k_0(\Omega)} \)-inner product:
\[
(L_k)_{ij} = (\phi_i, \phi_j)_{H^k_0(\Omega)}.
\]

Using the results of section 2 we find \( J = L_k^{-1}L_{k+1} \) and a norm for the discrete interpolation space \([X_h, Y_h]_\theta \) is given by
\[
H_{k+\theta} = L_kJ^{1-\theta} = L_k(L_k^{-1}L_{k+1})^{1-\theta}.
\]
In particular, if $\theta = 1/2$, we find

$$H_{k+1/2} = L_k(L_k^{-1}L_{k+1})^{1/2}$$

which simplifies in the case when the matrices $L_k$ are simultaneously diagonalisable to

$$H_{k+1/2} = L_k^{1/4}L_{k+1}^{1/4}L_k^{1/4}.$$  

Note that different choices of $L_k$ lead to different discrete norms, though they are all equivalent with constants of equivalence independent of $n$. Let us consider some examples to illustrate the above derivation.

**Example 3.1** Let $X = H^1_0(\Omega), Y = L^2(\Omega)$. We wish to derive the matrix representation of a norm defined on a discrete subspace of the interpolation space

$$H^{1/2}_0(\Omega) = [H^1_0(\Omega), L^2(\Omega)]_{1/2}.$$  

Let $X_h \subset X$ denote the finite element space spanned by a basis $\{\phi_i\}_{1 \leq i \leq n}$ of piecewise polynomials defined on a subdivision of $\Omega$. Using the notation introduced above we let $L_1, L_0 \in \mathbb{R}^{n \times n}$ denote the Grammian matrices with respect to the set $\{\phi_i\}$ corresponding to the following inner products:

$$(L_1)_{ij} = (\nabla \phi_i, \nabla \phi_j)_{L^2(\Omega)} ,\quad (L_0)_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)} .$$

The matrix $L_0$ is known as the mass matrix, while the matrix $L_1$ is a discrete Dirichlet Laplacian. Therefore, $T = L_0^{-1}L_1$ and a norm for the interpolation space $[X_h, Y_h]_\theta$ is given by

$$H_{0+\theta} = L_0 T^{1-\theta} = L_0 (L_0^{-1}L_1)^{1-\theta}.$$  

If $L_0, L_1$ are simultaneously diagonalisable (as is the case for a uniform subdivision of $\Omega$), the expression for $H_\theta$ becomes

$$H_\theta = L_0^\theta L_1^{1-\theta} = L_0^{\theta/2} L_1^{1-\theta/2} L_0^{-\theta/2}.$$  

In particular, if $\theta = 1/2$, we recover the expression derived by Burstedde., 2007:

$$H_{1/2} = L_0^{1/4} L_1^{1/2} L_0^{1/4} .$$

**Example 3.2** Let $X = H^2_0(\Omega), Y = H^1_0(\Omega)$. We can derive in a similar way the matrix representation of a norm for a discrete subspace of

$$H^{3/2}_0(\Omega) = [H^2_0(\Omega), H^1_0(\Omega)]_{1/2}.$$  

Working as before with subspaces $X_h, Y_h$ of $X, Y$ respectively, which are spanned by a set $\{\phi_i\}_{1 \leq i \leq n}$ of suitable piecewise polynomials we define the following Grammian matrices

$$(L_2)_{ij} = (\Delta \phi_i, \Delta \phi_j)_{L^2(\Omega)} ,\quad (L_1)_{ij} = (\nabla \phi_i, \nabla \phi_j)_{L^2(\Omega)} .$$

The matrix $L_1$ is a discrete Dirichlet Laplacian while the matrix $L_2$ is the discretisation of a biharmonic operator with homogeneous boundary conditions. A discrete norm is therefore induced by the matrix

$$H_{1+\theta} = L_1 (L_1^{-1}L_2)^{1-\theta} .$$

If $\theta = 1/2$ we find

$$H_{3/2} = L_1 (L_1^{-1}L_2)^{1/2} .$$

which, in the case of simultaneous diagonalisation simplifies to

$$H_{3/2} = L_1^{1/4} L_2^{1/2} L_1^{1/4} .$$
Remark 3.1 In both examples above, the choice of norms for the spaces \(H_0^k(\Omega)\) is not unique. One could choose to define \(L_1\) in terms of the (equivalent) full norm of \(H_0^k(\Omega)\)

\[
(L_1)_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)} + (\nabla \phi_i, \nabla \phi_j)_{L^2(\Omega)}.
\]

The same can be said about \(L_2\). The resulting discrete norms will therefore be different from, but equivalent to, the norms derived above.

We consider now the case of fractional Sobolev spaces of negative index and the corresponding discrete norms. Consider the dual of the interpolation space \(H_{00}^{k+1/2}(\Omega)\):

\[
\left( H_{00}^{k+1/2}(\Omega) \right)' = [H^{-k+1}(\Omega), H^{-k}(\Omega)]_{1/2} = \left( [H_{00}^{k+1}(\Omega), H_{00}^{k}(\Omega)]_{1/2} \right)'.
\]

Let \(X_h, Y_h\) be defined as above. We are interested in deriving a matrix representation for a norm on \([X_h, Y_h]'\) with respect to the basis \(\{\phi_i\}_{1 \leq i \leq n}\). Let \(Y_h' \subset X_h' = \text{span} \{\psi_i\}_{1 \leq i \leq n}\) where \(\psi_i\) are basis functions dual to \(\phi_i\), i.e.,

\[
(\psi_i, \phi_j)_{H^{-k}(\Omega) \times H_0^k(\Omega)} = \delta_{ij}.
\]

If we let

\[
Y_h' \ni z = \sum_{i=1}^n z_i \psi_i = \sum_{i=1}^n w_i \phi_i, \quad \phi_l = \sum_{i=1}^n K_l \psi_i
\]

we can use the above duality property to deduce that

\[
\delta_{ij} = (\psi_i, \phi_j)_{H^{-k}(\Omega) \times H_0^k(\Omega)} = \sum_{i=1}^n K^{-1}_{il} (\phi_i, \phi_j)_{H_0^k(\Omega)} = \sum_{i=1}^n K^{-1}_{il} (L_k)_{ij}
\]

so that \(z = L_k w\) and (cf. Remark 2.1)

\[
||z||_{H_Y'} = ||w||_{L_k^{-1}}, \quad ||z||_{H_X'} = ||w||_{L_k L_k^{-1} L_k}
\]

and the matrix representation of \(H_Y', H_X'\) are respectively \(L_k\) and \(L_k L_k^{-1} L_k\).

The matrix representation for a norm on \([X_h, Y_h]'\) is therefore (see also section 2)

\[
H_{k+\theta} = L_k^{\theta - 1} = L_k (L_k^{-1} L_{k+1})^{\theta - 1} = H_{k-1+\theta}.
\]

In particular, in the case \(\theta = 1/2\) we find

\[
H_{k+1/2} = L_k (L_k^{-1} L_{k+1})^{-1/2} = H_{k-1/2}.
\]

Example 3.3 Let \(X = H_0^1(\Omega), Y = L^2(\Omega)\) and let \(X_h, Y_h\) be defined as above. Then a discrete norm on \([X_h, Y_h]'_{1/2} \subset H^{-1/2}(\Omega)\) is given by

\[
H_{1/2} = H_{-1/2} = L_0 (L_0^{-1} L_1)^{-1/2},
\]

which in the case of simultaneous diagonalisation of \(L_0, L_1\) can be written as

\[
H_{-1/2} = L_0^{3/4} L_1^{-1/2} L_0^{3/4}.
\]

We end this section with a remark which will be useful in the context of preconditioning applications.
Remark 3.2 Let the basis functions \( \{ \phi_i \}_{1 \leq i \leq n} \) for \( X_h \subset Y_h \) be continuous piecewise polynomials defined on a given subdivision of a polyhedral domain \( \Omega \) into simplices of maximum diameter \( h \). In this case, it is known (see (Brenner and Scott 1994)) that the piecewise polynomial interpolant \( I_h \) for which \( I_h v = v_h \) for \( v \in X \subset H^m(\Omega) \) satisfies
\[
\|v - I_h v\|_{H^m(\Omega)} \leq C\|v\|_{H^m(\Omega)}
\]
so that
\[
\|I_h v\|_{H^m(\Omega)} \leq (1 + C)\|v\|_{H^m(\Omega)}
\]
and therefore \( I_h \in \mathcal{L}(X, X_h) \). Setting \( X_h \subset X \subset H^{k+1}(\Omega) \), \( Y_h \subset Y \subset H^k \) it follows that \( I_h \in \mathcal{L}(X, X_h) \cap \mathcal{L}(Y, Y_h) \) and hence Lemma 2.2 applies for general conforming finite element discretizations under standard regularity conditions on \( \Omega \) and its corresponding subdivision into simplices. In particular, the fractional Sobolev scale of norms \( \| \cdot \|_{k+\theta} \) is equivalent on \([X_h, Y_h]_{\theta}\) to the discrete norms induced by the family of matrices \( H_{k+\theta} \) introduced above for \( \theta \in (0, 1) \).

4 Evaluation of \( H_{\theta} \)

In order to construct and apply in a practical application any of the discrete norms derived in the previous discussion we are required to evaluate (non-integer) powers of a matrix. This task may be achieved in different ways for different applications. In general, if the dimension of the problem is low, one can employ a direct method based on a generalised eigenvalue decomposition. This approach has complexity of order \( O(n^3) \). Another direct approach is available in the case when the matrices involved have a Toeplitz structure; in this case the evaluation can be achieved via an FFT (see Peisker, 1988) and the complexity is \( O(n \log n) \). In both cases the storage requirements are of order \( O(n^2) \). For larger problems, iterative techniques may represent a cheaper alternative. An example in case is Newton’s method which has attractive convergence properties under a suitable implementation. However, the complexity of the method is that of a direct method, even if the original matrix is sparse. One expects that for sparse matrices one can devise efficient techniques for evaluating \( H_{\theta} \) or its action applied to a given vector. This is indeed the case. In (Hale, Higham and Trefethen 2008) the authors propose a method based on representing a function of a matrix as an integral which they go on to evaluate using efficient quadrature rules. The method can be adapted to provide a sparse algorithm for the evaluation of \( H_{\theta} \). Another approach is to construct approximations of Krylov type which are known to take advantage of the sparsity of the matrices involved. Several authors have considered this approach for general matrix functions (Druskin and Knizhnerman 1989), (Cabos 1997), (Druskin and Knizhnerman 1998), (Druskin, Greenbaum and Knizhnerman 1998), (Saad 1992), (Allen, Baglama and Boyd 2000), (Eiermann and Ernst 2006) and some convergence analysis is available for certain algorithms proposed for the computation of the matrix square root function (Druskin and Knizhnerman 1989). We illustrate below the Krylov subspace approximation for the case \( \theta = 1/2 \).

4.1 A generalised Lanczos algorithm

Given a pair of symmetric and positive-definite matrices \( (M, A) \), the generalised Lanczos algorithm constructs a set of \( M \)-orthogonal vectors \( v_i \) such that
\[
AV_k = MV_k T_k + \beta_{k+1} M v_{k+1} e_k^T, \quad V_k^T M V_k = I_k
\]
where the columns \( v_i \) of \( V_k = [v_1, v_2, \ldots, v_k] \) are known as the Lanczos vectors and \( I_k \in \mathbb{R}^{k \times k} \) is the identity matrix with \( k \)th column denoted by \( e_k \), while the matrix \( T_k \in \mathbb{R}^{k \times k} \) is a symmetric
and tridiagonal matrix (Parlett 1998). The standard algorithm corresponds to the case \( M = I \). Note that \( T_k \) can be seen as a projection of \( A \) onto the space spanned by the \( M \)-orthogonal columns of \( V_k \)

\[
V_k^T A V_k = T_k, \quad V_k^T M V_k = I_k.
\]

(18)

In exact arithmetic, when \( k = n \), the algorithm can be seen as providing simultaneous factorisations of the matrix pair \( (M, A) \) as

\[
A = V_n^{-T} T_n V_n^{-1}, \quad M = V_n^{-T} V_n^{-1}.
\]

We recall the algorithm below (Parlett 1998).

**Algorithm 1. Generalised Lanczos Algorithm**

**Input:** \( A, M \in \mathbb{R}^{n \times n} \) (spd), \( v \in \mathbb{R}^n \)

**Output:** \( V_k \in \mathbb{R}^{n \times k}, T_k \in \mathbb{R}^{k \times k} \)

Set \( \beta_1 = 0, v_0 = 0, v_1 = v / \| v \|_M \)

for \( i = 1 : k \)

\[
w_i = M^{-1} A v_i - \beta_i v_{i-1}
\]

\[
\alpha_i = (w_i, v_i)_M
\]

\[
w_i = w_i - \alpha_i v_i
\]

\[
\beta_{i+1} = \| w_i \|_M
\]

if \( \beta_{i+1} = 0 \) stop

\[
v_{i+1} = w_i / \beta_{i+1}
\]

end

\( T_k = \text{tridiag}[\beta, \alpha, \beta] \)

The explicit form of \( T_k \) is given below

\[
T_k = \text{tridiag}[\beta, \alpha, \beta] = \begin{pmatrix}
\alpha_1 & \beta_2 & 0 \\
\beta_2 & \alpha_2 & \ddots \\
& \ddots & \ddots & \beta_k \\
0 & \beta_k & \alpha_k
\end{pmatrix}.
\]

Consider now the generalised Lanczos factorisation for the matrix-pair \( (H_Y, H_X) \):

\[
H_X V = H_Y V T, \quad V^T H_Y V = I.
\]

(19)

where we used the notation \( V = V_n, T = T_n \). We can immediately derive the following result.

**Lemma 4.1** Let (19) hold and let \( H_{\theta, h} = H_Y + H_Y J^{1-\theta} \) and \( H_{\theta} = H_Y J^{1-\theta} \) with \( J = H_Y^{-1} H_X \).

Then

\[
H_{\theta} = \left( V T^{\theta-1} V^T \right)^{-1} = H_Y V T^{1-\theta} V^T H_Y
\]

(20)

and

\[
H_{\theta, h} = \left( V (I + T^{1-\theta})^{-1} V^T \right)^{-1} = H_Y V (I + T^{1-\theta}) V^T H_Y.
\]

(21)

**4.2 Sparse evaluation of \( H_{\theta} z \)**

The complexity of the full \( (k = n) \) generalised Lanczos algorithm is in general \( O(n^3) \). However, in many applications of interest we do not need to compute \( H_{\theta} \), but simply apply it (or its inverse) to a given vector \( z \in \mathbb{R}^n \). In such cases, a truncated version of the algorithm is used in practice
with only \( k \) Lanczos vectors being constructed. As we are interested in approximations of \( H_\theta z \) we note first that if we start the Lanczos process with \( v = z \) then

\[
V_k^T H_Y z = e_1 \| z \|_{H_Y}
\]

where \( e_1 \in \mathbb{R}^k \) is the first column of the identity \( I_k \). This leads us to consider the following approximations of the matrix-vector products:

\[
H_\theta z \approx H_Y V_k T_k^{1-\theta} e_1 \| z \|_{H_Y},
\]

and

\[
H_{\theta,h} z \approx H_Y V_k (I_k + T_k^{1-\theta}) e_1 \| z \|_{H_Y}.
\]

Similarly, if we wish to apply the inverse of \( H_\theta \) to a given vector \( z \) we first note that if we start the iteration with \( v = H_Y^{-1} z \) then

\[
V_k^T z = V_k^T H_Y (H_Y)^{-1} z = e_1 \| H_Y^{-1} z \|_{H_Y} = e_1 \| z \|_{H_Y^{-1}}.
\]

This leads us to consider the following approximations (cf. Lemma 4.1)

\[
H_\theta^{-1} z \approx V_k T_k^{\theta-1} V_k^T z = V_k T_k^{\theta-1} e_1 \| z \|_{H_Y^{-1}},
\]

and

\[
H_{\theta,h}^{-1} z \approx V_k \left( I_k + T_k^{1-\theta} \right)^{-1} V_k^T z = V_k \left( I_k + T_k^{1-\theta} \right)^{-1} e_1 \| z \|_{H_Y^{-1}}.
\]

The complexity of the above operations depends on the complexity corresponding to the application of the inverse of \( H_Y \). If this operation can be achieved in \( O(n) \) operations, then the overall complexity of computing \( H_\theta z, H_\theta^{-1} z \) is of order \( O(kn) \) for \( k \ll n \), with storage requirements of the same order.

5 Applications

5.1 Preconditioners for the Steklov-Poincaré operator

Domain decomposition methods (DD) require the resolution of a problem which involves a pseudo-differential operator defined on the set of boundaries defined by the decomposition of the domain. This operator is generally known as the Steklov-Poincaré operator or the Dirichlet-Neumann map, though this definition was introduced for DD methods applied to second-order problems involving the Laplacian operator. A great number of iterative approaches have been proposed in the literature over the last two decades; classical algorithms include Dirichlet-Neumann, Neumann-Neumann, FETI methods, Scharzw methods together with two-level and overlapping variants. Complete descriptions and analyses can be found in a range of references, see, for example, (Toselli and Widlund 2005), and (Quarteroni and Valli 1999). We present below an alternative which has not been considered to date and which is based on preconditioning in a fractional Sobolev space of index \( 1/2 \).

5.1.1 A model problem

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with boundary \( \partial \Omega \) and consider the model problem

\[
\begin{align*}
-\Delta u &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
\]  

(22)
Let $H^1_0(\Omega)$ denote the space spanned by the restriction of polynomials in two variables of degree $r$ on some subdivision $\Sigma_h$ of $\Omega$ into simplices $t$ of maximum diameter $h$. Let further $V^h_\Omega, V^h_B \subset V^h$ satisfy $V^h_\Omega \oplus V^h_B \equiv V^h$ where $V^h_\Omega = \{ w \in V^h : w|_{\partial \Omega} = 0 \}$. Let $X_h \subset H^1_0(\Gamma)$ denote the space spanned by the restriction
of the basis functions of \( V_I^h \) to the internal boundary \( \Gamma \). The discrete variational formulation of model problem (22) reads

\[
\begin{align*}
& \text{Find } u_h \in V_I^h \text{ such that for all } v_h \in V_I^h, \\
& a(u_h, v_h) = f(v_h),
\end{align*}
\]

where

\[ a(v, w) = (\nabla v, \nabla w), \quad f(v) = (v, f). \]

This formulation give rise to a linear system involving a matrix which is structured in the following way

\[
\begin{pmatrix}
A_{II,1} & 0 & A_{IB,1} \\
0 & A_{II,2} & A_{IB,2} \\
A_{IB,1}^T & A_{IB,2}^T & A_{BB,1} + A_{BB,2}
\end{pmatrix}
\begin{pmatrix}
u_{I,1} \\
u_{I,2} \\
u_B
\end{pmatrix}
= 
\begin{pmatrix}
f_{I,1} \\
f_{I,2} \\
f_B
\end{pmatrix}
\]

where \( A_{II,i}, i = 1, 2 \) are discrete Laplacians corresponding to the interior nodes of the computational domain \( \Omega_i \) and \( A_{BB,i}, i = 1, 2 \) are the corresponding interior boundary contributions from each domain. The above system can be ‘decoupled’ into three problems

(i) \( A_{II,i} u^{(1)}_i = f_{I,i} \),

(ii) \( S u_B = f_B - A_{IB,1}^T u^{(1)}_i - A_{IB,2}^T u^{(2)}_i \),

(iii) \( A_{II,i} u^{(2)}_i = -A_{IB,1}^T u_B - A_{IB,2}^T u_B \),

where \( S \) is the Schur complement corresponding to the boundary nodes

\[ S = S_1 + S_2, \quad S_i = A_{BB,i} - A_{IB,i}^T A_{II,i}^{-1} A_{IB,i}. \]

The resulting solution is \((u_{I,1}, u_{I,2}, u_B)\) where

\[ u_{I,i} = u^{(1)}_i + u^{(2)}_i. \]

It is evident that these algebraic problems are finite element discretisations of the three continuous problems listed above (see Quarteroni and Valli, 1999 for full details). In particular, the Schur complement \( S \) is the finite element discretisation of the variational definition (23) of the Steklov-Poincaré operator \( S \). Since \( s(\cdot, \cdot) \) is \( H^1_0(\Gamma) \)-elliptic, we deduce that for any \( \lambda_h \in X_h \) there holds

\[ c_1 \|\lambda_h\|_{H^1/2(\Gamma)}^2 \leq s(\lambda_h, \lambda_h) \leq c_2 \|\lambda_h\|_{H^1/2(\Gamma)}^2. \]  \hfill (26)

Note that if we denote by \( \lambda \) the vector of coefficients of \( \lambda_h \) when expanded in a finite element basis, then

\[ s(\lambda_h, \lambda_h) = \lambda^T S \lambda. \]

### 5.1.3 \( H^{1/2}_{00} \)-preconditioners

For large problems, constructing and applying the inverse of the Schur complement \( S \) in step (ii) above is computationally prohibitive. Instead, the problem can be solved using a preconditioned iterative technique. For symmetric and positive-definite problems it is known that optimal acceleration of an iterative method is achieved when a spectrally-equivalent preconditioner \( P_S \) is employed. We show below that the matrix representation of a \( H^{1/2}_{00} \)-norm has this property and can therefore be employed as a preconditioner for domain decomposition methods of scalar elliptic problems.

The inequalities (26) describe essentially the spectral equivalence between the discrete operator induced by the bilinear form \( s(\cdot, \cdot) \) acting on a finite-dimensional subspace of \( H^{1/2}_{00}(\Gamma) \times H^{1/2}_{00}(\Gamma) \) and a discrete representation of the \( H^{1/2}_{00} \)-norm. We make this statement precise below.
Proposition 5.1 Let $X_h = \text{span}\{\phi_i, 1 \leq i \leq m\}$ be defined as above and let $(L_k)_{ij} = (√φ_i, √φ_j)_{H^1(\Gamma)}$ for $k = 0, 1$. Let

$$H_{1/2} := L_0(L_0^{-1}L_1)^{1/2}.$$ 

Then for all $λ \in \mathbb{R}^m \setminus \{0\}$

$$κ_1 ≤ \frac{λ^TSλ}{λ^TH_{1/2}λ} ≤ κ_2$$

with $κ_1, κ_2$ independent of $h$.

Proof: Since Lemma (2.2) holds (see Remark 3.2) there exist constants $η_1, η_2$ such that for all $λ_h ∈ X_h$

$$η_1∥λ∥_1/2 ≤ ∥λ_h∥_{θ,h} ≤ η_2∥λ∥_1/2.$$ 

The norm $∥·∥_{θ,h}$ has matrix representation $H_{1/2,h}$ which, by (9), is spectrally equivalent to $H_{1/2}$. Hence, there exist constants $\tilde{η}_1, \tilde{η}_2$ such that

$$\tilde{η}_1∥λ∥_1/2 ≤ ∥λ∥_{H_1/2} ≤ \tilde{η}_2∥λ∥_1/2.$$ 

Using the $H_{1/2}^{1/2}$-ellipticity (26) of $s(·, ·)$ we get

$$\frac{c_1}{\tilde{η}_2}∥λ∥_H^2_{1/2} ≤ λ^TSλ ≤ \frac{c_2}{\tilde{η}_1}∥λ∥_H^2_{1/2}$$

which is the required result. ■

Remark 5.1 The matrix $H_{1/2}$ is the reduced version of the matrix representation for the norm $∥·∥_{1/2,h}$. It is evident that the above result holds with $H_{1/2}$ replaced with $L_0 + L_0(L_0^{-1}L_1)^{1/2}$.

The above result indicates that $S$ and $H_{1/2}$ exhibit the same spectral properties. In particular, the following result holds.

Proposition 5.2 Let $L_0, L_1 ∈ \mathbb{R}^m$ be defined as in Proposition 5.1. Then

$$κ_{L_0}(H_{1/2}) = O(h^{−1}) \quad \text{and} \quad κ_{L_0}(S) = O(h^{−1}).$$

Proof: Since $L_0$ is a mass matrix and $L_1$ is a discrete Laplacian, by the Poincaré inequality there exists a constant $γ_1$ independent of $h$ such that

$$∥λ∥_{L_0} ≤ γ_1^{−1}∥λ∥_{L_1}.$$ 

Furthermore, the following standard discrete inverse inequality is assumed to hold

$$∥λ∥_{L_1}^2 ≤ γ_2 h^{−2}∥λ∥_{L_0}^2,$$

where $γ_2$ is also independent of $h$. Hence,

$$γ_1 ≤ \frac{λ^TL_1λ}{λ^TL_0λ} ≤ γ_2 h^{−2}$$

and therefore $κ_{L_0}(L_1) = O(h^{−2})$. Hence, by Lemma (2.4),

$$κ_{L_0}(H_{1/2}) = (κ_{L_0}(L_1))^{1/2} = O(h^{−1}).$$

The second statement follows from the spectral equivalence of $H_{1/2}$ and $S$ derived in Proposition 5.1.

Since $∥·∥_{L_0}$ is equivalent to the Euclidean ($l_2$-) norm, we conclude that $κ_2(S) = O(h^{−1})$ which is the standard result on the condition number of the discrete Steklov-Poincaré operator (Quarteroni and Valli 1999). This indicates that an iterative technique which ignores the Schur complement problem will be suboptimal. We verify in the numerics section below that this is indeed the case and we demonstrate that $H_{1/2}$ is a suitable preconditioner in this sense.
5.2 Boundary preconditioners for the biharmonic operator

Consider the biharmonic problem in a polygonal convex open domain \( \Omega \subset \mathbb{R}^2 \) with boundary \( \Gamma = \bigcup_{i=1}^{K} \Gamma_i \).

\[
\begin{align*}
\Delta^2 u &= f & \text{in } \Omega, \\
u &= \partial u/\partial n = 0 & \text{on } \Gamma.
\end{align*}
\]  

(27)

A standard approach to solving (27) is to introduce another variable \( v = -\Delta u \) and solve the resulting system:

\[
\begin{align*}
-\Delta u &= f & \text{in } \Omega, \\
v + \Delta u &= 0 & \text{in } \Omega, \\
u &= \partial u/\partial n = 0 & \text{on } \Gamma.
\end{align*}
\]  

(28)

There is a considerable literature on the topic of this model problem, both from the approximation point of view (Monk, 1987, Glowinski and Pironneau, 1979, Osborn, Babuška and Pitkäranta, 1980, Brezzi and Raviart, 1976, Ciarlet and Raviart, 1974, and Falk, 1978) and also an algorithmic one (Glowinski and Pironneau, 1979, Peisker, 1988, Braess and Peisker, 1986, and Silvester and Mihajlović, 2004). A notable approach is provided by Glowinski and Pironneau 1979; this approach was made efficient by Peisker 1988 who suggested for the first time preconditioning with a discrete \( H^{-1/2}(\Gamma) \)-norm. We review briefly this approach below.

5.2.1 The Pironneau-Glowinski method

The following re-formulation of the biharmonic problem was introduced in (Glowinski and Pironneau 1979). Let \( \lambda = v \mid \Gamma \). The solution \((u, v)\) of system (28) can be obtained by solving the following three problems

\[
\begin{align*}
\text{(i)} & \quad \begin{cases}
-\Delta v_0 &= f & \text{in } \Omega, \\
v_0 &= 0 & \text{on } \Gamma, \\
-\partial v_0/\partial n &= \lambda_1 & \text{on } \Gamma,
\end{cases} & \quad \text{(ii)} & \quad S\lambda = \partial u_0/\partial n & \text{on } \Gamma, \\
\text{(iii)} & \quad \begin{cases}
-\Delta v_1 &= 0 & \text{in } \Omega, \\
v_1 &= \lambda & \text{on } \Gamma, \\
-\partial v_1/\partial n &= \lambda_1 & \text{on } \Gamma,
\end{cases}
\end{align*}
\]  

(29)  

(30)  

(31)

the final solution being \((u, v) = (u_0 + u_1, v_0 + v_1)\). The aim of considering this formulation is to split the problem into smaller, easier to solve problems. While (i) and (iii) may indeed be classified as easy from a computational point of view, the crux of the problem becomes equation (ii). As in the case of domain decomposition methods, \( S \) is a boundary operator which is defined on \( H^{-1/2}(\Gamma) \) and which induces a bilinear form \( s(\cdot, \cdot) : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \) via

\[
(S\lambda_1, \lambda_2) = (\Delta \psi_1, \Delta \psi_2) := s(\lambda_1, \lambda_2).
\]

The functions \( \psi_i \) are biharmonic extensions of \( \lambda_i \in H^{-1/2}(\Gamma) \) into \( \Omega \), i.e., they are solutions of the biharmonic problems

\[
\begin{align*}
\Delta^2 \psi_i &= 0 & \text{in } \Omega, \\
\psi_i &= 0 & \text{on } \Gamma, \\
-\partial \psi_i/\partial n &= \lambda_i & \text{on } \Gamma.
\end{align*}
\]  

(32)

It is shown in (Glowinski and Pironneau 1979) that the bilinear form \( s(\cdot, \cdot) \) is symmetric, positive-definite and \( H^{-1/2}(\Gamma) \)-elliptic, i.e., there exist constants \( c_1, c_2 \) such that for all \( \lambda \in H^{-1/2}(\Gamma) \),

\[
c_1 \|\lambda\|_{H^{-1/2}(\Gamma)} \leq s(\lambda, \lambda) \leq c_2 \|\lambda\|_{H^{-1/2}(\Gamma)}^2.
\]

(33)
5.2.2 Discrete formulation

Consider now the following standard mixed finite element method for (28). Let \( V^h, V_I^h, V_B^h \) be defined as above (see (25)). The discrete weak formulation is then

\[
\begin{aligned}
\text{Find } (u_h, v_h) \in V_I^h \times V^h \text{ such that } \forall (w_h, z_h) \in V_I^h \times V^h \\
\begin{cases}
l(v_h, w_h) = (f, w_h) \\
l(u_h, z_h) - m(v_h, z_h) = 0
\end{cases}
\end{aligned}
\tag{34}
\]

where

\[
l(z, w) := (\nabla z, \nabla w), \quad m(z, w) := (z, w).
\]

As described in (Glowinski and Pironneau 1979), (34) is equivalent to the discrete versions of (29–31) given by the following three weak formulations

(i) Find \((u_{0h}, v_{0h}) \in V_I^h \times V_I^h \) such that \( \forall (w_h, z_h) \in V_I^h \times V_I^h \)

\[
\begin{aligned}
\begin{cases}
l(v_{0h}, w_h) = (f, w_h) \\
l(u_{0h}, z_h) - m(v_{0h}, z_h) = 0
\end{cases}
\end{aligned}
\tag{35}
\]

(ii) Find \( \lambda_h \in V_B^h \) such that \( \forall \mu_h \in V_B^h \)

\[
\begin{aligned}
\begin{cases}
s(\lambda_h, \mu_h) = -s(\lambda_{0h}, \mu_h)
\end{cases}
\end{aligned}
\tag{36}
\]

(iii) Find \((u_{1h}, v_{1h}) \in V_I^h \times V^h, \ v_{1h} - \lambda_h \in V_I^h \) such that \( \forall (w_h, z_h) \in V_I^h \times V^h \)

\[
\begin{aligned}
\begin{cases}
l(v_{1h}, w_h) = 0 \\
l(u_{1h}, z_h) - m(v_{1h}, z_h) = 0
\end{cases}
\end{aligned}
\tag{37}
\]

Let now \( \text{span}\{\phi_i, 1 \leq i \leq n\} = V^h \) so that \( w_h \in V^h, z_h \in V_I^h \) can be written

\[
w_h = \sum_{i=1}^{n} w_i \phi_i, \quad z_h = \sum_{i=1}^{n_I} z_i \phi_i
\]

where \( n = |V^h|, n_I = |V_I^h| \). Problem (34) is then equivalent to the following linear system of equations

\[
\begin{pmatrix}
0 & L_{II} & L_{IB} \\
L_{II} & -M_{II} & -M_{IB} \\
L_{IB} & -M_{IB} & -M_{BB}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_I \\
\mathbf{v}_I \\
\mathbf{v}_B
\end{pmatrix}
= \begin{pmatrix}
f \\
0 \\
0
\end{pmatrix}
\]

where

\[
(L_{II})_{ij} = l(\phi_j, \phi_i), \quad (L_{IB})_{ik} = l(\phi_k, \phi_i),
\]

and

\[
(M_{II})_{ij} = m(\phi_j, \phi_i), \quad (M_{IB})_{ik} = m(\phi_k, \phi_i), \quad (M_{BB})_{kl} = m(\phi_l, \phi_k)
\]

for \( 1 \leq i, j, k, l \leq m, \ 1 \leq k, l \leq n - n_I \). We also write (38) in the more compact form

\[
\begin{pmatrix}
L & Z \\
Z^T & -M_{BB}
\end{pmatrix}
\begin{pmatrix}
\mathbf{x} \\
\mathbf{v}_B
\end{pmatrix}
= \begin{pmatrix}
g \\
0
\end{pmatrix}
\]

where

\[
L = \begin{pmatrix} 0 & L_{II} \\ L_{II} & -M_{II} \end{pmatrix}, \ Z = \begin{pmatrix} L_{IB} \\ -M_{IB} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{v}_I \end{pmatrix}.
\]
It can be seen (see for example (Arioli and Loghin 2008)) that the discrete problems (35–37) represent a boundary Schur complement approach to solving (38). As before, the task is therefore the efficient solution of problem (ii), and in particular the derivation of optimal preconditioners for this step. The Schur complement associated with $L$ in the matrix of (39) is

$$S = -M_{BB} - Z^T L^{-1} Z.$$ 

Let $X_h \subset H^1(\Gamma)$ denote the space spanned by the restriction of the basis functions of $V^h_I$ to the boundary $\Gamma$. As in the case of domain decomposition methods, the Schur complement is the matrix representation of the bilinear form $s(\cdot, \cdot)$ with respect to the basis $\{\phi_i\}$. In particular, if $\lambda_h \in X_h$ has a vector of coefficients $\lambda$, then

$$s(\lambda_h, \lambda_h) = \lambda^T S \lambda.$$ 

5.2.3 $H^{-1/2}(\Gamma)$-preconditioners

A discrete $H^{-1/2}$-norm on $X_h$ can be defined as a sum of norms corresponding to each open segment of the polygonal boundary $\Gamma$:

$$\|\lambda_h\|_{H^{-1/2}(\Gamma)} := \left( \sum_{i=1}^K \|\lambda_h\|_{H^{-1/2}(\Gamma_i)}^2 \right)^{1/2}.$$ 

In particular, $H^{-1/2}(\Gamma_i)$ is understood here to be the dual of $H_0^{1/2}(\Gamma)$. For this space, a matrix representation for its norm was derived in section 3 (see (17)). However, Peisker uses a different, algebraic, definition of the norm-matrices corresponding to each boundary segment, based on linear and uniform discretisations of Laplacian and mass matrices. This results in a matrix representation for the discrete norm $\|\cdot\|_{H^{-1/2}(\Gamma)}$ which is a direct sum of norm-matrices corresponding to the interior of each boundary segment. There are two drawbacks to this approach. First, the linear case ($r = 1$) does not yield stable mixed finite element discretisations of the biharmonic problem, (Shaidurov 1995). Second, the resulting preconditioner is not defined for non-uniform meshes. Following the presentation from Section 3 we introduce the following representation of a discrete $H^{-1/2}(\Gamma)$ norm which is based on the above broken norm

$$\|\lambda_h\|_{H^{-1/2}(\Gamma)}^2 = \|\lambda\|_{H^{-1/2}}^2$$

for $\lambda_h \in X_h$ where

$$H_{-1/2} = \bigoplus_{i=1}^K H_{-1/2}^{(i)}, \quad (40)$$

where

$$H_{-1/2}^{(i)} = L_0, i (L_{0,i}^{-1} L_{1,i})^{-1/2}$$

with $(L_{k,r})_{ij} = (\phi_i, \phi_j)_{H^k_0(\Gamma_r)}, k = 0, 1$ discrete operators defined on the interior of each boundary $\Gamma_i$. The following result can be proved immediately.

**Proposition 5.3** Let $X_h = \text{span}\{\phi_i, 1 \leq i \leq m\}$ be defined as above and let $H_{-1/2}^{(i)}$ be defined as in (40). Then for all $\lambda \in \mathbb{R}^m \setminus \{0\}$

$$\kappa_1 \leq \frac{\lambda^T S \lambda}{\lambda^T H_{-1/2} \lambda} \leq \kappa_2$$

with constants $\kappa_1, \kappa_2$ independent of $h$.

**Proof:** Similar to the proof of Proposition (5.1).
6 Numerical experiments

We present in this section numerical experiments corresponding to the two applications of the previous section. In both cases the solutions are obtained using preconditioned iterative methods with a combination of 2D and 1D preconditioners. The latter type are norm-matrices for fractional Sobolev spaces of index 1/2 which are constructed on the boundaries of the computational domain.

6.1 Domain decomposition for elliptic problems

We solve the test problem (22) as well as the following convection-diffusion problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
-\nu \Delta u + \vec{b} \cdot \nabla u = f & \text{in } \Omega, \\
u u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\end{align*}
\]

We note here that Proposition 5.1 only applies in the symmetric case (22). However, one can derive suitable convergence results for the nonsymmetric case also which we did not include here. We refer the reader to (Arioli, Kouronis and Loghin 2008) for further details. The domain is the unit square which was subdivided into squares. We used the finite element method to discretise the problems as well as the norm-matrices required. The choice of finite-dimensional space was \(V_h\) defined in (25) with \(r = 1\) and also \(r = 2\). The choice \(k = 2\) is relevant in the context of preconditioning discrete convection-diffusion operators arising from so-called P2-P1 discretisations of the Stokes (Silvester and Wathen 1994), (Golub and Wathen. 1998), and Oseen problems (Elman and Silvester 1996), (Kay, Loghin and Wathen. 2002) which employ quadratic piecewise polynomial spaces for the approximation of the momentum equations.

Due to non-symmetry, we choose to work with nonsymmetric iterative methods (flexible GM-RES), coupled with nonsymmetric preconditioners of the form

\[P = \begin{pmatrix}
A_{II} & A_{IB} \\
0 & P_S
\end{pmatrix}\]

with \(A_{II} = \nu L_{II} + N_{II}\) where \(L_{II}\) is the direct sum of Laplacians assembled on each subdomain and \(N_{II}\) is the direct sum of the convection operator \(\vec{b} \cdot \nabla\) assembled also on each subdomain. This choice of preconditioner is known to be useful provided we have a good approximation \(P_S\) to the Schur complement. Thus, if \(P_S\) is replaced by \(S\) convergence is achieved in 2 iterations (Murphy, Golub and Wathen 2000). Our choice of preconditioner will never achieve this, since the norm-matrices derived above do not approximate \(S\) itself but are equivalent operators. However, we will see that the resulting performance remains attractive.

The Schur complement preconditioner \(P_S\) is chosen to be each of \(H_{1/2}\) and \(H_{1/2,h}\), the discrete \(H^{1/2}(\Gamma)\)-norms defined in Proposition 5.1 and Remark 5.1, respectively. We also chose to work with a simplified version \(\hat{H}_{1/2}\) of \(H_{1/2}\) obtained by replacing the mass matrix \(L_0\) by a lumped version \(\hat{L}_0\):  

\[\hat{H}_{1/2} := \hat{L}_0(\hat{L}_0^{-1}L_1)^{1/2}.\]

The action of the inverses of \(H_{1/2,h}, H_{1/2}, \hat{H}_{1/2}\) was computed using the iterative method presented in section 4.2 which uses the Lanczos algorithm with \(k = O(m^{1/2})\).

Remark 6.1 In the case where the domain is subdivided into several subdomains the boundary \(\Gamma\) will be the union of internal faces or boundaries (a so-called skeleton or wirebasket)

\[\Gamma = \bigcup_{i=1}^{K} \Gamma_i.\]
One can generalize the definition of a $H^{1/2}(\Gamma)$-norm to a broken $H^{1/2}(\Gamma)$-norm which results in a direct sum of norm-matrices as in 40 for the biharmonic problem. However, we choose to work with a related generalization which involves assembling the Grammians $L_0, L_1$ on the whole wirebasket $\Gamma$. In particular, $L_1$ will incorporate Dirichlet conditions corresponding to the set $\partial \Omega \cap \Gamma$ and will include additional contributions at each internal vertex. The resulting matrix is a norm-matrix on $[X_h, Y_h]_{1/2} \subset H^{1/2}(\Gamma)$ with improved spectral properties (Arioli et al. 2008).

### 6.1.1 The Poisson problem

The number of iterations is displayed in Table 1 for the cases $r = 1, r = 2$ (linear and quadratic finite elements) respectively. The size $m$ of the skeleton is also displayed; it is obvious that a direct calculation of the matrix square-root function is becoming prohibitive for an increasing number of domains and an increasing mesh-size. As expected, the number of iterations is independent of the size of the problem. Moreover, the preconditioning procedure appears to be quasi-scalable with only a slight, possibly logarithmic dependence on the number of subdomains. A notable result is the performance of the preconditioner $\hat{H}_{1/2}$ which is a simplified version of the other two which employs a lumped approximation of the mass matrix.

<table>
<thead>
<tr>
<th>#dom</th>
<th>$n$</th>
<th>$m$</th>
<th>$H_{1/2, h}$</th>
<th>$H_{1/2}$</th>
<th>$\hat{H}_{1/2}$</th>
<th>$\hat{H}_{1/2}$</th>
<th>$\hat{H}_{1/2}$</th>
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<td>10</td>
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<td>11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
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<td>1793</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
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</tr>
<tr>
<td>16</td>
<td>45,953</td>
<td>1149</td>
<td>13</td>
<td>12</td>
<td>12</td>
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<td>13</td>
</tr>
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<td>2301</td>
<td>13</td>
<td>13</td>
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<td>13</td>
<td>13</td>
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</tr>
<tr>
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<td></td>
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<tr>
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<td>14</td>
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<td>15</td>
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<td>7133</td>
<td>16</td>
<td>15</td>
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<td>15</td>
<td>15</td>
</tr>
<tr>
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<td>14,301</td>
<td>17</td>
<td>16</td>
<td>15</td>
<td>17</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 1: $FGMRES$ iterations for model problem (22) for $r = 1, 2$.

### 6.1.2 The convection-diffusion problem

We solved test problem (41) for the choice of ‘rotating wind’

$$\vec{b} = (2(2y - 1)(1 - (2x - 1)^2), -2(2x - 1)(1 - (2y - 1)^2)).$$

We chose to approximate only the action of $\hat{H}_{1/2}^{-1}$ using the generalised Lanczos process with the same choice of $k$. The range of diffusion coefficients was $\nu = 1, 0.1, 0.01$. The results are displayed in Table 2. In all cases the number of iterations is independent of the size of the problem, though it grows with reducing $\nu$. The dependence on the number of subdomain also grows with reducing $\nu$. This reflects the inability of our symmetric preconditioner to remain equivalent in some sense to an increasingly more nonsymmetric Schur complement.
### 6.2 Biharmonic problem

We solved the biharmonic problem using the formulation (28), which was discretised using the weak formulation (34), where $V^h$ is the finite element space defined in (25) with $r = 2$ (quadratic approximation). This is known to be a stable mixed finite element method for the biharmonic problem (Shaidurov 1995). As the Glowinski-Pironneau method is a boundary Schur complement approach, we choose to work again with a block-triangular preconditioner of the form (cf. (39))

$$P = \begin{pmatrix} L & Z \\ & P_S \end{pmatrix}$$

where $P_S = H_{1/2}$ was defined in (40). We ignore the symmetry of our problem and use again flexible GMRES given the changing nature of our preconditioner due to the Lanczos approximation. As in the case of the previous example, we consider an approximation $\hat{H}_{1/2}$ resulting from replacing the mass matrix $L_0$ with a lumped version $\hat{L}_0$. The results are displayed in Table 3. As expected, the number of iterations is independent of $h$; moreover, the preconditioner reduces greatly the iteration count compared to the case where no preconditioner is employed. We notice that in the unpreconditioned case the dependence on $h$ is evident as predicted by Braess and Peisker. (1986), though mild ($O(h^{1/2})$); we also notice that in this case there is a considerable additional computational effort, particularly compared to the minimal effort that the above preconditioners require. The same behaviour can be noticed for un-isotropic meshes. Table 4 displays the iteration count corresponding to a finite element discretisation on an exponentially stretched mesh, with nodes clustered near the boundary and mesh aspect ratio ranging from 1 to 20.

### 7 Summary

We presented a derivation of norm representations of norms associated with interpolation spaces. In particular, we focused on projections onto conforming finite element spaces of fractional
Table 3: FGMRES iterations for model problem (27) for a range of preconditioners $P_S$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$I$</th>
<th>$H_{(-1/2,h)}$</th>
<th>$H_{(-1/2)}$</th>
<th>$\hat{H}_{(-1/2)}$</th>
</tr>
</thead>
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<tr>
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<td>36</td>
<td>9</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 4: FGMRES iterations for model problem (27) for a range of preconditioners $P_S$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$I$</th>
<th>$H_{(-1/2,h)}$</th>
<th>$H_{(-1/2)}$</th>
<th>$\hat{H}_{(-1/2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31,250</td>
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<td>1008</td>
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</tr>
<tr>
<td>518,162</td>
<td>2032</td>
<td>52</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Sobolev norms. A notable result is that interpolation norms can be represented as products of generally real powers of Grammian matrices associated with the pair of spaces generating the scale of interpolation spaces. The issue of algorithmic complexity in the construction of discrete interpolation norms was also considered with the presentation of some sparse matrix algorithms for the approximation of real powers of matrices. Some applications arising from PDE modeling were considered to illustrate the usefulness of interpolation norms in large-scale computing.
References


