A novel approach to level-based preconditioners

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Introduction

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Key target is robustness.

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Potential disadvantage of $IC(\tau)$ and $IC(m)$:

structural information may be lost.

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Notation: $L = \{l_{ij}\}$ denotes complete factor of A ($A = LL^T$);
$\hat{L} = \{\hat{l}_{ij}\}$ denotes incomplete factor.

Incomplete factorizations

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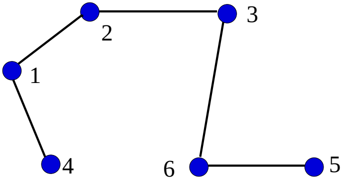

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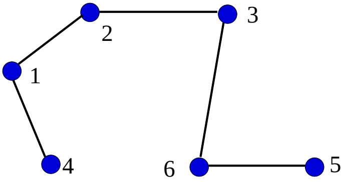

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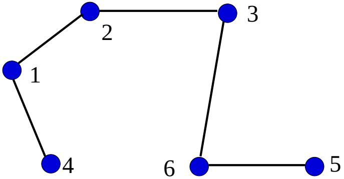

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- l_{ij} $(i > j)$ is nonzero if and only if there is a fill path joining i and j in $\mathcal{G}$ (Rose, Tarjan and Leuker, 1978).

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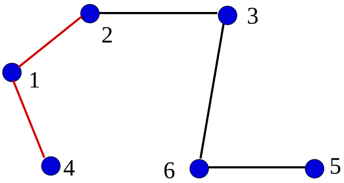

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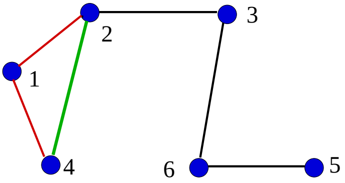

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Level-based incomplete factorizations

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- In practice: entries of $\hat{L}$ corresponding to nonzero entries of A assigned level 0 and $\hat{l}_{ij}$ allowed in $IC(\ell)$ if $level(i,j) \le \ell$, where

$$level(i,j) = \min_{1 \le l \le \min\{i,j\}} \{level(i,l) + level(l,j) + 1\}$$

(sum rule is one of several definitions).

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Major break through

Incomplete fill path theorem (Hysom and Pothen, 2002).

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- Nice feature: the structure of each column can be computed independently (and hence in parallel).
- Once sparsity pattern known, separate factorization phase needed to compute entries of $IC(\ell)$.

Preassigning levels

Our aim:
restrict small entries of A to contributing to fewer levels of fill than larger entries.

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- Sparsify A by dropping any very small entries in group 1.
- Preassign $level(i,j)$ for individual entries according to which group they belong to. Small entries are assigned level $l-1$ and large entries assigned level 0.

Notes on preassigning levels

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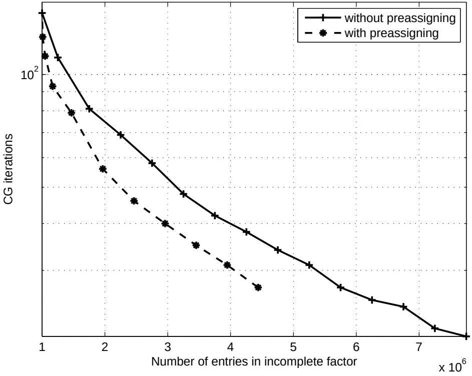
- Straightforward to modify Hysom and Pothen algorithm to accommodate initial levels > 0.
- Extra cost for symbolic factorization but saves on time to compute and apply $\hat{L}$ if $nz(\hat{L})$ is less than for standard approach.

Effect of preassigning levels

Example:

Kohn-Sham equation (`carsten3`) $n = 250500$.



Test set

- Problems taken from University of Florida Sparse Matrix Collection.
- Selected all SPD matrices of order > 1000.
- CG used with $x_0 = 0$, b computed so that $x = 1$, and stopping criteria

$$\|Ax_k - b\| \leq 10^{-6}\|b\|$$

  with limit of 800 iterations.
- Ran with $\ell = 3$ and removed problems that failed to converge both with and without preassigning initial levels.

Set $\mathcal{T}$ comprises 120 problems.

Performance profile

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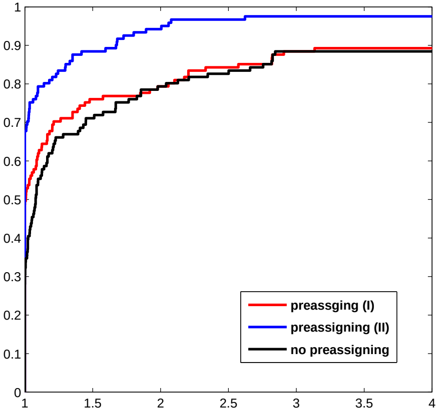
- Suppose P_i run on problem $j \in \mathcal{T}$ has efficiency $e_{ij} \ge 0$.
- Let $\hat{e}_j = \min\{e_{ij}\}$.
- For $\alpha \ge 1$ and each i define

$$k(e_{ij}, \hat{e}_j, \alpha) = \begin{cases} 1 & \text{if } e_{ij} \le \alpha \hat{e}_j \\ 0 & \text{otherwise.} \end{cases}$$

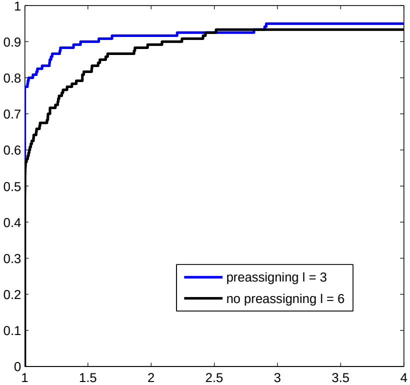
- The performance profile for P_i is then

$$\frac{1}{|\mathcal{T}|} * \sum_{j \in \mathcal{T}} k(e_{ij}, \hat{e}_j, \alpha), \quad \alpha \ge 1.$$

Performance profile $\ell = 3$



Performance profile $\ell = 3$ and 6



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Note: in rest of talk, always use preassigning of levels.

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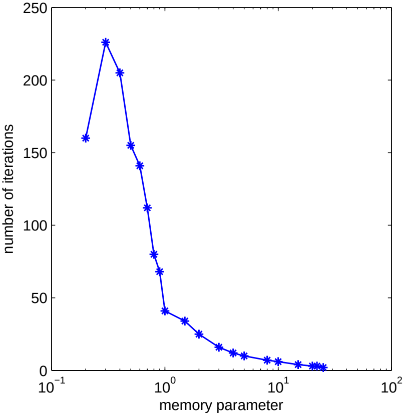
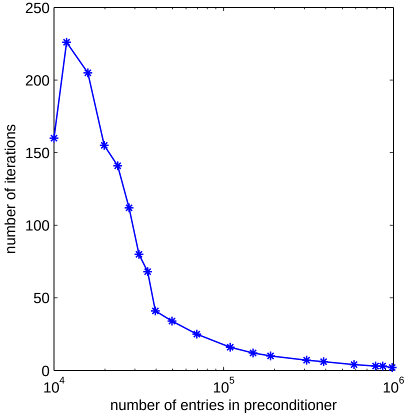
- $m < 1$ keep only largest entries.

Simple Laplace equation





HB/bcsstk17

For small $\ell > 1$, need $m > 2.7$ to get convergence.

$nz = nz(\hat{L})$ in thousands.

m	$\ell = 0$		$\ell = 1$		$\ell = 2$	
	nz	*iter*	*nz*	*iter*	*nz*	*iter*
2.7	592	†	945	†	1271	11
2.75	603	†	966	23	1292	10
3	657	†	1054	17	1412	8
3.5	767	†	1228	12	1596	1
4	876	†	1402	9	1596	1

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- If there is insufficient space to accommodate all such extra entries, sort and retain only the largest.

$IC(\ell, \tau, 1)$ versus $IC(\tau)$

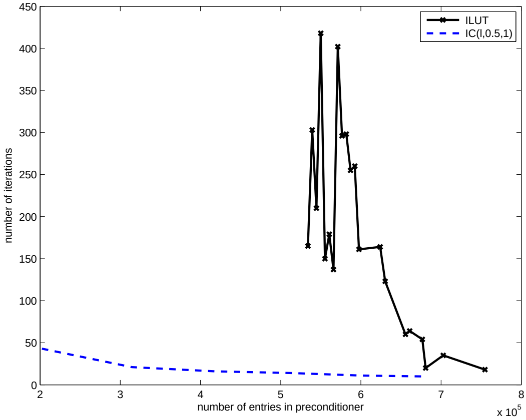
TUBE1, cylindrical shell

$IC(\ell, \tau, 1)$	$\tau = 0.0$		τ =1e-7	
ℓ	*nz*	iter	*nz*	iter
5	2188	283	2105	287
6	2863	223	2711	197
7	3705	159	3431	159
10	7383	230	6346	239
12	10532	158	8527	159
15	13667	83	10404	59

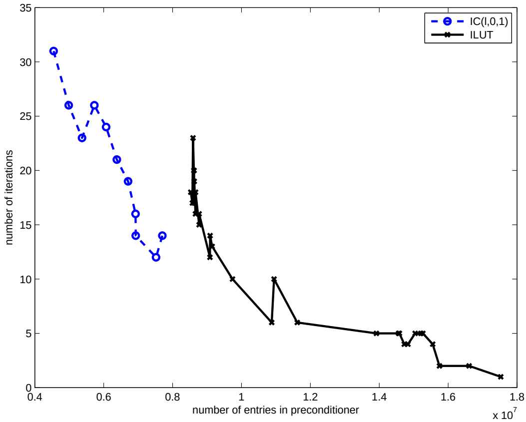
IC(τ)	*nz*	its
5e-5	9649	471
2e-5	9611	87
1e-5	10050	18
5e-6	10741	6
1e-6	12451	2
0	21803	1

- Uni-parameter preconditioner hard to tune.
- Level-based approach converged with much sparser $\hat{L}$.
- Choosing τ is problem and strategy dependent.

Comparison of $ILUT(p, \tau)$ (Saad, 1994) with variable parameters and $IC(\ell, 0.5, 1)$ for `Cylshell/s1rmt3m1`



Comparison of $ILUT(p, \tau)$ with variable parameters and $IC(\ell, 0, 1)$ for `Nasa/nasasrb`



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