Challenges from PDE-constrained optimization and some methods to circumvent these issues

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In collaboration with

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PDE-constrained optimization

- Given f and boundary condition g, calculate u, where

$$\mathcal{L}u = f, \quad \alpha_1 u + \alpha_2 \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega$$

on some domain Ω

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  on some domain Ω

- Suppose given g and an approximation $\hat{u}$ to u on some domain $\hat{\Omega} \subset \Omega$. Want to calculate f such that $u \approx \hat{u}$: distributed control

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- Suppose given g and an approximation $\hat{u}$ to u on some domain $\hat{\Omega} \subset \Omega$. Want to calculate f such that $u \approx \hat{u}$: distributed control
- Suppose given f and an approximation $\hat{u}$ to u on some domain $\hat{\Omega} \subset \Omega$. Want to calculate g such that $u \approx \hat{u}$: boundary control

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Distributed control

$$\min_{\mathtt{u},\mathtt{f}} \frac{1}{2} \|\omega(x)\,(\mathsf{u} - \hat{\mathsf{u}})\|_2^2 + \beta\,\|\mathsf{f}\|_2^2$$

subject to

$$\begin{aligned} \mathcal{L}\mathsf{u} &= \mathsf{f} \text{ in } \Omega \\ \mathsf{u} &= \mathsf{g} \text{ on } \delta\Omega \end{aligned}$$

Here

$$\omega(x) = \begin{cases} 1 & x \in \hat{\Omega} \\ 0 & \text{otherwise} \end{cases}$$

Distributed control

Discretize:

$$\mathtt{u_h} = \sum u_j \phi_j, \quad \mathtt{f_h} = \sum f_j \phi_j$$

$$\min_{\mathtt{u_h},\mathtt{f_h}} \frac{1}{2} \left\| \omega(x) \left(\mathtt{u_h} - \widehat{u} \right) \right\|_2^2 + \beta \left\| \mathtt{f_h} \right\|_2^2$$

subject to

$$\begin{aligned} -\nabla^2 \mathtt{u_h} &= \mathtt{f_h} \text{ in } \Omega \\ \mathtt{u_h} &= \mathsf{g} \text{ on } \delta\Omega \end{aligned}$$

Distributed control

$$
\begin{aligned}
\|\omega(x)\,(\mathtt{u_h} - \widehat{u})\|_2^2 &= \int_\Omega \omega(x)\,(\mathtt{u_h} - \widehat{u})^2 \\
&= \sum_i \sum_j u_i u_j \int_\Omega \omega_i \omega_j \phi_i \phi_j - 2 \sum_j u_j \int_\Omega \omega_j \phi_j \widehat{u} + \int_{\hat{\Omega}} \widehat{u}^2 \\
&= u^T \bar{M} u - u^T b + c \\
\|\mathtt{f_h}\|_2^2 &= f^T M f \\
K u &= M f
\end{aligned}
$$

where M is the mass matrix, K is the stiffness matrix, $\bar{M} = WMW$ and $W = \mathsf{diag}(\omega_i)$

Distributed control

$$\min_{u,f} \frac{1}{2} u^T \bar{M} u - u^T b + c + \beta f^T M f$$

subject to

$$Ku - Mf \quad = \quad d$$

$$\begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix} \begin{bmatrix} f \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ d \end{bmatrix}$$

Spectral properties of linear system

$$H = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

If A is symmetric and positive definite, then $\lambda(A) \in I^- \cup I^+$, where

$$\begin{aligned} I^- &= \left[\frac{1}{2}\left(\lambda_{\min}(A) - \sqrt{\lambda^2_{\min}(A) + 4\,\|B\|^2}\right), \frac{1}{2}\left(\|A\| - \sqrt{\|A\|^2 + 4\sigma^2_{\min}(B)}\right)\right], \\ I^+ &= \left[\lambda_{\min}(A), \frac{1}{2}\left(\|A\| + \sqrt{\|A\|^2 + 4\,\|B\|^2}\right)\right], \end{aligned}$$

[Rusten and Winther 1992]

Spectral properties of linear system

$$H = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

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$l(A,B)$ defined in Dollar 2009 (revised)

Previously thought that $l(A,B) = Z^T AZ$ (Simoncini talk 2008)

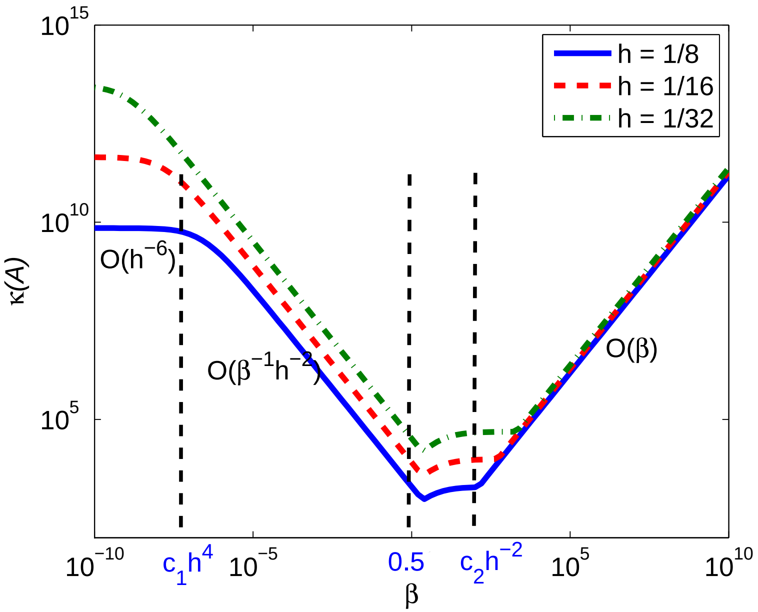
Spectral properties of linear system

$$H_\beta = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix}$$

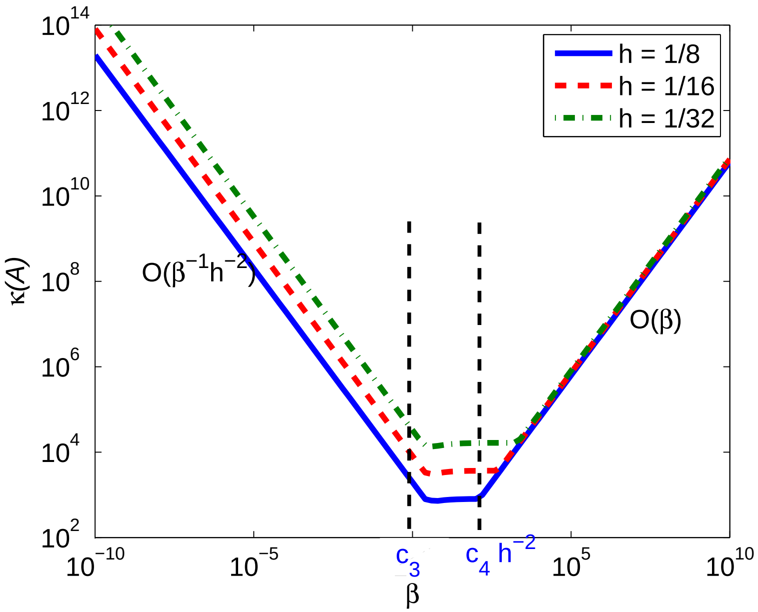
$$= H_0 + \begin{bmatrix} 2\beta M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Spectral properties of linear system

$\hat{\Omega} = \Omega$



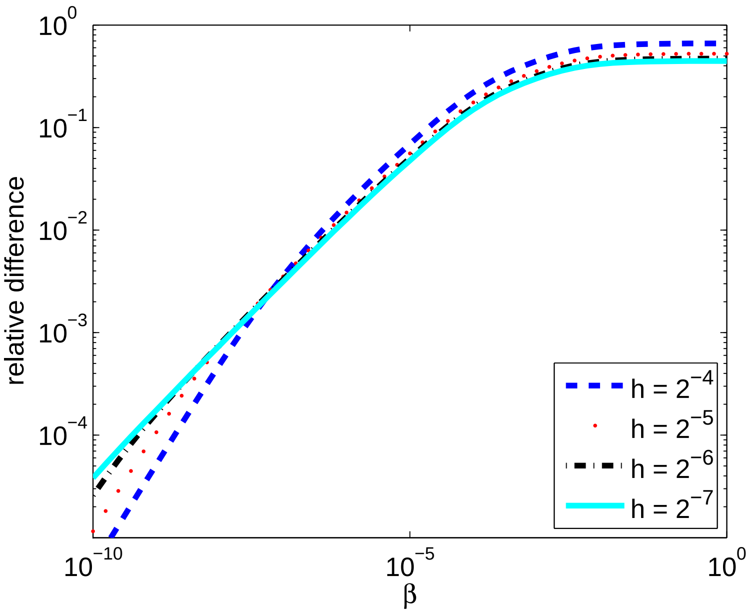
$\hat{\Omega} \neq \Omega$



$\hat{\Omega}$	$\hat{\Omega}_1$	$\hat{\Omega}_2$	$\hat{u}(x,y)\|_{\hat{\Omega}_1}$	$\hat{u}(x,y)\|_{\hat{\Omega}_2}$
$\hat{\Omega}_1 \cup \hat{\Omega}_2$	$\left[0, \frac{1}{2}\right]^2$	$\Omega/\hat{\Omega}_1$	$(2x-1)^2\,(2y-1)^2$	0
$\hat{\Omega}_1 \cup \hat{\Omega}_2$	$\left\{(x,y) : (x-\frac{5}{8})^2 + (y-\frac{3}{4})^2 \leq \frac{1}{25}\right\}$	$\partial\Omega$	2	0

Spectral properties of linear system

$\hat{\Omega} = \Omega$



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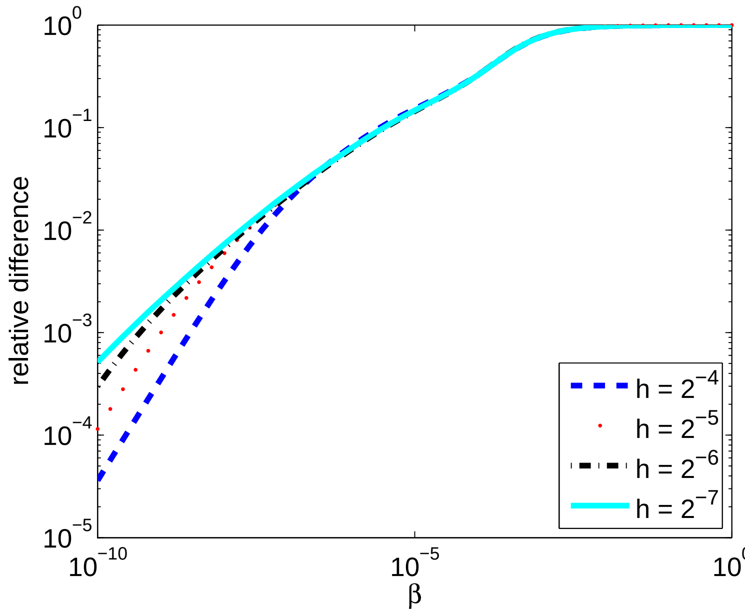

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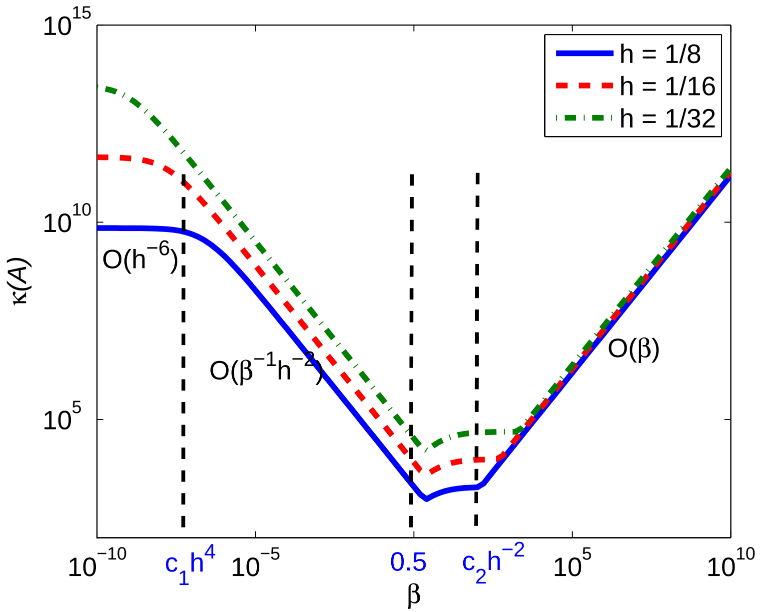

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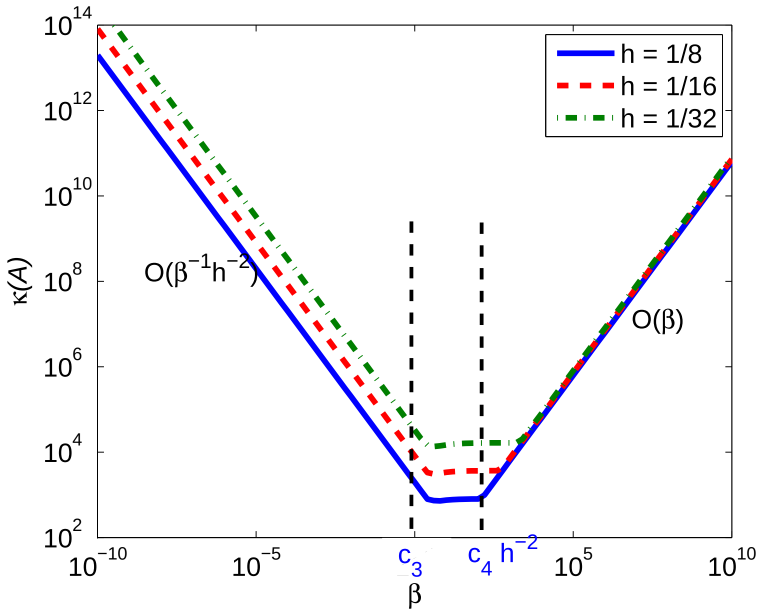

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Distributed control

$$\min_{u,f} \frac{1}{2}u^T M u - u^T b + c + \beta f^T M f$$

subject to

$$Ku - Mf = d$$

$$\begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix} \begin{bmatrix} f \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ d \end{bmatrix}$$

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Projected Preconditioned CG Method

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Write

$$x = Yx_y + Zx_z,$$

where columns Z span nullspace of B and $[Y, Z]$ spans $\mathbb{R}^n$

$$\begin{aligned} BYx_y &= d, \\ Z^T AZx_z &= Z^T \left(b - AYx_y\right), \\ Y^T Bw &= Y^T(b - Ax). \end{aligned}$$

If $Z^T AZ$ is SPD, then use PCG with preconditioner $Z^T GZ$.

Projected Preconditioned CG Method

Remove references to Z by making substitutions (Gould, Hribar, Nocedal, 2001):

Choose initial point x satisfying $Bx = d$

Compute $r = Ax - b$

Solve

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g \\ v \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

Set $p = -g$

repeat

Set $\alpha = r^T g / p^T Ap$

Set $x = x + \alpha p$ and $r^+ = r + \alpha Ap$

Solve

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g^+ \\ v^+ \end{bmatrix} = \begin{bmatrix} r^+ \\ 0 \end{bmatrix}$$

Set $\beta = (r^+)^T g^+ / r^T g$

Set $p = -g^+ + \beta p$, $r = r^+$ and $g = g^+$

until converged

Projected Preconditioned CG Method

Remove references to Z by making substitutions:

Choose initial point x satisfying $Bx = d$

Compute $r = Ax - b$

Solve

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Solve

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Projected Preconditioned CG Method

Solve

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Projected Preconditioned CG Method

Solve

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Solve

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g \\ v \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

Set $p = -g$ and $y = -v$
repeat
Set $\alpha = r^T g / p^T A p$
Set $x = x + \alpha p$ and $r^+ = r + \alpha A p$
Solve

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g^+ \\ v^+ \end{bmatrix} = \begin{bmatrix} r^+ \\ 0 \end{bmatrix}$$

Set $\beta = (r^+)^T g^+ / r^T g$
Set $p = -g^+ + \beta p$, $r = r^+ - B^T v^+$, $y = y - v^+$ and $g = g^+$
until converged

Projected Preconditioned CG Method

(Dollar 2005) Can be generalised to

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Constraint preconditioners

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \qquad \mathcal{P} = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix}$$

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Theorem (Keller, Gould, Wathen, 2000): If $A, G \in \mathbb{R}^{n\times n}$ are symmetric and $B \in \mathbb{R}^{m\times n}$ has full row rank, then $\mathcal{P}^{-1}\mathcal{A}$ has

- $2m$ eigenvalues at 1
- remaining $n-m$ are defined by

$$Z^T A Z x = \lambda Z^T G Z x,$$

where the columns of $Z \in \mathbb{R}^{n\times(n-m)}$ span nullspace of B. If G is nonsingular, then these eigenvalues interlace the eigenvalues of $G^{-1}A$. The Krylov subspace wrt $\mathcal{P}^{-1}\mathcal{A}$ has dimension at most $n-m+2$

Preconditioner

$$\mathcal{A} = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix} \quad Z = \begin{bmatrix} M^{-1}K \\ I \end{bmatrix}$$

$$Z^T A Z = 2\beta K^T M^{-1} K + \bar{M}$$

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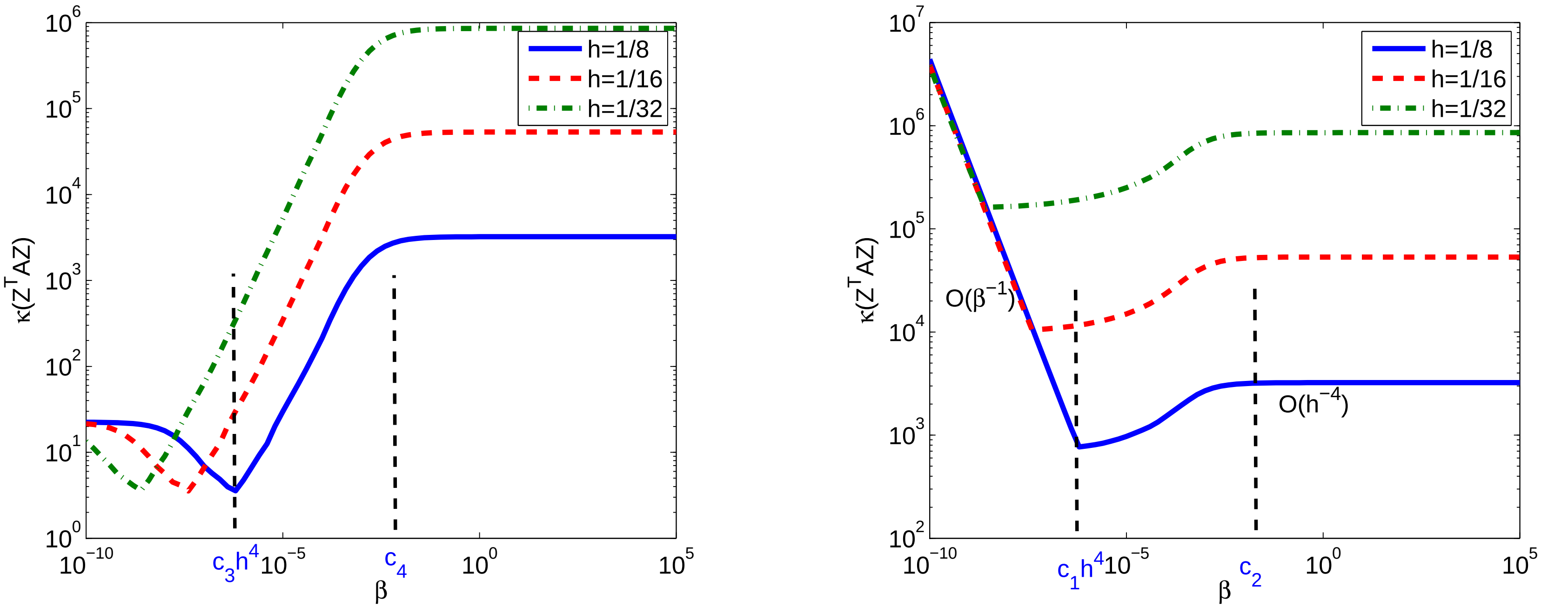

Preconditioner

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$$Z^T A Z = 2\beta K^T M^{-1} K + \bar{M}$$

$$\mathcal{P} = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & 0 & K^T \\ -M & K & 0 \end{bmatrix} ?$$

$$Z^T G Z = 2\beta K^T M^{-1} K$$

$\bar{M} = M$	$\bar{M} \neq M$
$1 + \frac{ch^4}{2\beta} \leq \lambda \leq 1 + \frac{C}{2\beta}$	$1 + \frac{\bar{c}h^4}{2\beta} \leq \lambda \leq 1 + \frac{\bar{C}}{2\beta}$
$c \leq \bar{c} \leq \bar{C} \leq C$	$\lambda = 1$

Biros and Ghattas (2000)

Preconditioner

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$$Z^T A Z = 2\beta K^T M^{-1} K + \bar{M}$$

$$\mathcal{P} = \begin{bmatrix} 0 & 0 & -M \\ 0 & 2\beta K^T M^{-1} K & K^T \\ -M & K & 0 \end{bmatrix} ?$$

$$Z^T G Z = 2\beta K^T M^{-1} K$$

$\bar{M} = M$	$\bar{M} \neq M$
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Numerical Example

Using bilinear **Q1** elements and setting $\beta = 5 \times 10^{-5}$:

$$\mathcal{A} = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0 & 0 & -M \\ 0 & 2\beta K^T M^{-1} K & K^T \\ -M & K & 0 \end{bmatrix}$$

- Solves with M : Direct method (`HSL_MA57`) or 20 Chebyshev semi-iterations
- Solves with K : Direct method (`HSL_MA57`) or two(three) V-cycles of AMG (`HSL_MI20`)
- PPCG: relative tolerance 10^{-9} for $r^T Z(Z^T G Z)^{-1} Z^T r$, `HSL_MI27` (soon to be released)
- Fortran 95, NAG f95 compiler
- Hardware: Dell Precision T340, single Core2 Quad Q9550 processor (2.83GHz, 1333MHz FSB, 12MB L2 Cache), 4GB RAM

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2D

N	n	Direct	PPCG(direct)	PPCG(approx)
8	147	0.002	0.001 (8)	0.003 (9)
16	675	0.01	0.006 (8)	0.011 (9)
32	2883	0.04	0.025 (8)	0.044 (9)
64	11907	0.19	0.12 (8)	0.17 (8)
128	48487	1.59	0.55 (7)	0.72 (8)
256	195075	8.82	3.27 (6)	3.18 (8)
512	783363	53.5	21.5 (6)	14.2 (8)

3D

N	n	Direct	PPCG(direct)	PPCG(approx)
4	81	0.001	0.002 (7)	0.002 (7)
8	1029	0.04	0.02 (8)	0.05 (8)
16	10125	1.25	0.33 (8)	0.64 (8)
32	89373	38.0	6.61 (7)	7.32 (7)
64	750141	1000+	217 (5)	59.0 (6)

Numerical Example

Using bilinear **Q1** elements and setting $\beta = 5 \times 10^{-5}$:

$$\mathcal{A} = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & \bar{M} & K^T \\ -M & K & 0 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0 & 0 & -M \\ 0 & 2\beta K^T M^{-1} K & K^T \\ -M & K & 0 \end{bmatrix}$$

- Solves with M : Direct method (`HSL_MA57`) or 20 Chebyshev semi-iterations
- Solves with K : Direct method (`HSL_MA57`) or two(three) V-cycles of AMG (`HSL_MI20`)
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N	n	Direct	PPCG(direct)	PPCG(approx)
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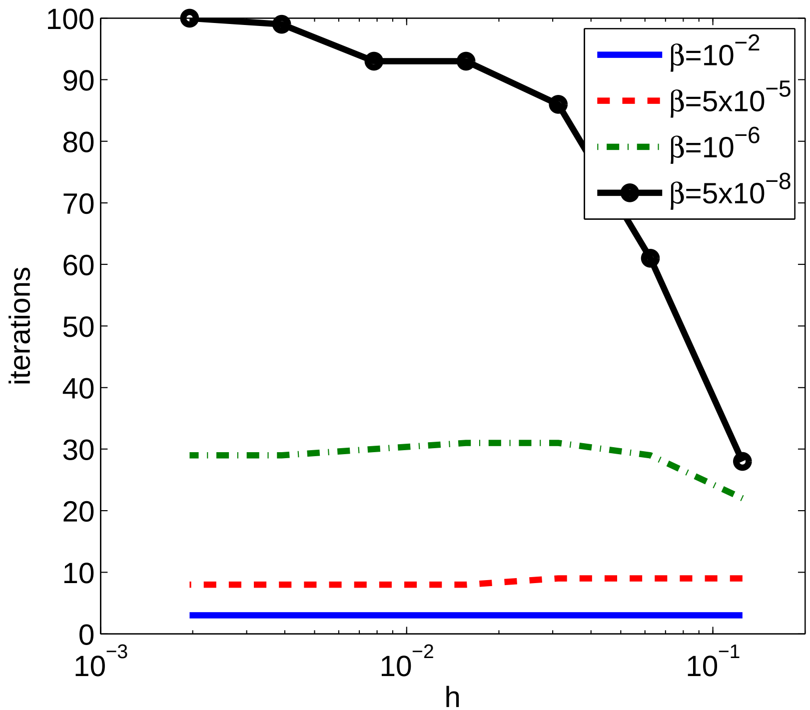
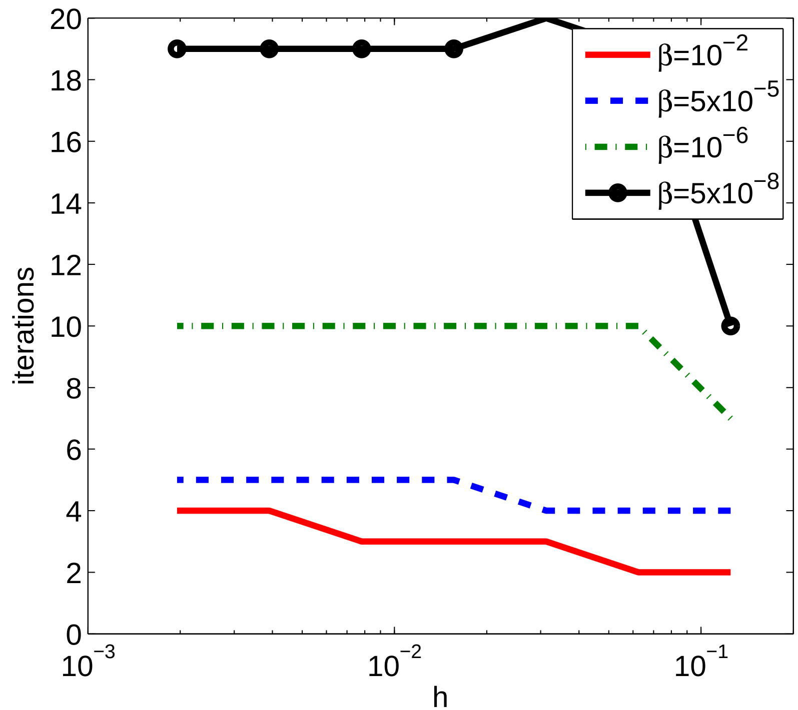
64	11907	0.35	0.10 (4)	0.13 (5)
128	48487	2.78	0.50 (5)	0.53 (5)
256	195075	16.8	3.11 (5)	2.36 (5)
512	783363	147	20.5 (5)	10.3 (5)

Behaviour of preconditioner with β

$\bar{M} = M$	$\bar{M} \neq M$
$1 + \frac{ch^4}{2\beta} \leq \lambda \leq 1 + \frac{C}{2\beta}$ $c \leq \bar{c} \leq \bar{C} \leq C$	$1 + \frac{\bar{c}h^4}{2\beta} \leq \lambda \leq 1 + \frac{\bar{C}}{2\beta}$ $\lambda = 1$





Neumann boundary control

$$\min_{\mathtt{u},\mathtt{g}} \frac{1}{2} \left\| \omega(x) \left(\mathsf{u} - \hat{\mathsf{u}} \right) \right\|_2^2 + \beta \left\| \mathsf{g} \right\|_2^2$$

subject to

$$\begin{aligned} \mathcal{L}\mathsf{u} &= \mathsf{f} \text{ in } \Omega \\ \frac{\partial \mathsf{u}}{\partial n} &= \mathsf{g} \text{ on } \delta\Omega \end{aligned}$$

Here

$$\omega(x) = \begin{cases} 1 & x \in \hat{\Omega} \\ 0 & \text{otherwise} \end{cases}$$

Neumann boundary control

Discretize:

$$\mathtt{u_h} = \sum u_j \phi_j + \sum \hat{u}_j \hat{\phi}_j, \quad \mathtt{g_h} = \sum g_j \hat{\phi}_j, \quad \mathtt{f_h} = \sum f_j \phi_j + \sum \hat{f}_j \hat{\phi}_j$$

$$\min_{\mathtt{u_h},\mathtt{g_h}} \frac{1}{2} \left\| \omega(x) \left(\mathtt{u_h} - \widehat{u} \right) \right\|_2^2 + \beta \left\| \mathtt{g_h} \right\|_2^2$$

subject to

$$\begin{aligned} \mathcal{L}\mathtt{u_h} &= \mathtt{f_h} \text{ in } \Omega \\ \frac{\partial \mathtt{u_h}}{\partial \mathtt{n}} &= \mathtt{g_h} \text{ on } \partial\Omega \end{aligned}$$

Neumann boundary control

$$\|\omega(x)\,(\mathtt{u_h} - \widehat{u})\|_2^2 = \int_\Omega \omega(x)\,(\mathtt{u_h} - \widehat{u})^2$$

$$= \begin{bmatrix} u^T & \hat{u}^T \end{bmatrix} \begin{bmatrix} \bar{M}_{II} & \bar{M}_{IB} \\ \bar{M}_{BI} & \bar{M}_{BB} \end{bmatrix} \begin{bmatrix} u \\ \hat{u} \end{bmatrix} - u^T b - \hat{u}^T \hat{b} + c$$

$$\|\mathtt{g_h}\|_2^2 = g^T M_g g$$

$$\begin{bmatrix} d \\ \hat{d} \end{bmatrix} = \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} u \\ \hat{u} \end{bmatrix} - \begin{bmatrix} 0 \\ M_g \end{bmatrix} g$$

Neumann boundary control

$$\min_{u,\hat{u},f} \begin{bmatrix} u^T & \hat{u}^T \end{bmatrix} \begin{bmatrix} \bar{M}_{II} & \bar{M}_{IB} \\ \bar{M}_{BI} & \bar{M}_{BB} \end{bmatrix} \begin{bmatrix} u \\ \hat{u} \end{bmatrix} - u^T b - \hat{u}^T \hat{b} + c + \beta g^T M_g g$$

subject to

$$\begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} u \\ \hat{u} \end{bmatrix} - \begin{bmatrix} 0 \\ M_g \end{bmatrix} g = \begin{bmatrix} d \\ \hat{d} \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ \hline 0 & K_{II} & K_{IB} & 0 & 0 \\ -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \begin{bmatrix} g \\ u \\ \hat{u} \\ \lambda \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ \hat{b} \\ d \\ \hat{d} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & B_1^T \\ 0 & G_1 & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ \hline 0 & K_{II} & K_{IB} & 0 & 0 \\ -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ 0 & K_{II} & K_{IB} & 0 & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ \hline 0 & K_{II} & K_{IB} & 0 & 0 \\ -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ 0 & K_{II} & K_{IB} & 0 & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right]$$

$$Z^T A Z = \begin{bmatrix} \hat{A} & \hat{B}^T \\ \hat{B} & 0 \end{bmatrix}, \quad \hat{A} = 2\beta \begin{bmatrix} K_{IB} \\ K_{BB} \end{bmatrix} M_g^{-1} \begin{bmatrix} K_{BI} & K_{BB} \end{bmatrix} + \bar{M}, \quad \hat{B} = \begin{bmatrix} K_{II} & K_{IB} \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ \hline 0 & K_{II} & K_{IB} & 0 & 0 \\ -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ 0 & K_{II} & K_{IB} & 0 & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right]$$

$$Z^T A Z = \begin{bmatrix} \hat{A} & \hat{B}^T \\ \hat{B} & 0 \end{bmatrix}, \quad \hat{A} = 2\beta \begin{bmatrix} K_{IB} \\ K_{BB} \end{bmatrix} M_g^{-1} \begin{bmatrix} K_{BI} & K_{BB} \end{bmatrix} + \bar{M}, \quad \hat{B} = \begin{bmatrix} K_{II} & K_{IB} \end{bmatrix}$$

$Z^T A Z$ indefinite $\Rightarrow$ avoid PPCG
Solve $Z^T A Z x_z = Z^T (b - A Y x_y)$ with BICGSTAB, MINRES, SQMR or...

$$\left[\begin{array}{ccc|cc} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ \hline 0 & K_{II} & K_{IB} & 0 & 0 \\ -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cccc|c} 2\beta M_g & 0 & 0 & 0 & -M_g \\ 0 & \bar{M}_{II} & \bar{M}_{IB} & K_{II} & K_{IB} \\ 0 & \bar{M}_{BI} & \bar{M}_{BB} & K_{BI} & K_{BB} \\ 0 & K_{II} & K_{IB} & 0 & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right]$$

$$Z^T AZ = \begin{bmatrix} \hat{A} & \hat{B}^T \\ \hat{B} & 0 \end{bmatrix}, \quad \hat{A} = 2\beta \begin{bmatrix} K_{IB} \\ K_{BB} \end{bmatrix} M_g^{-1} \begin{bmatrix} K_{BI} & K_{BB} \end{bmatrix} + \bar{M}, \quad \hat{B} = \begin{bmatrix} K_{II} & K_{IB} \end{bmatrix}$$

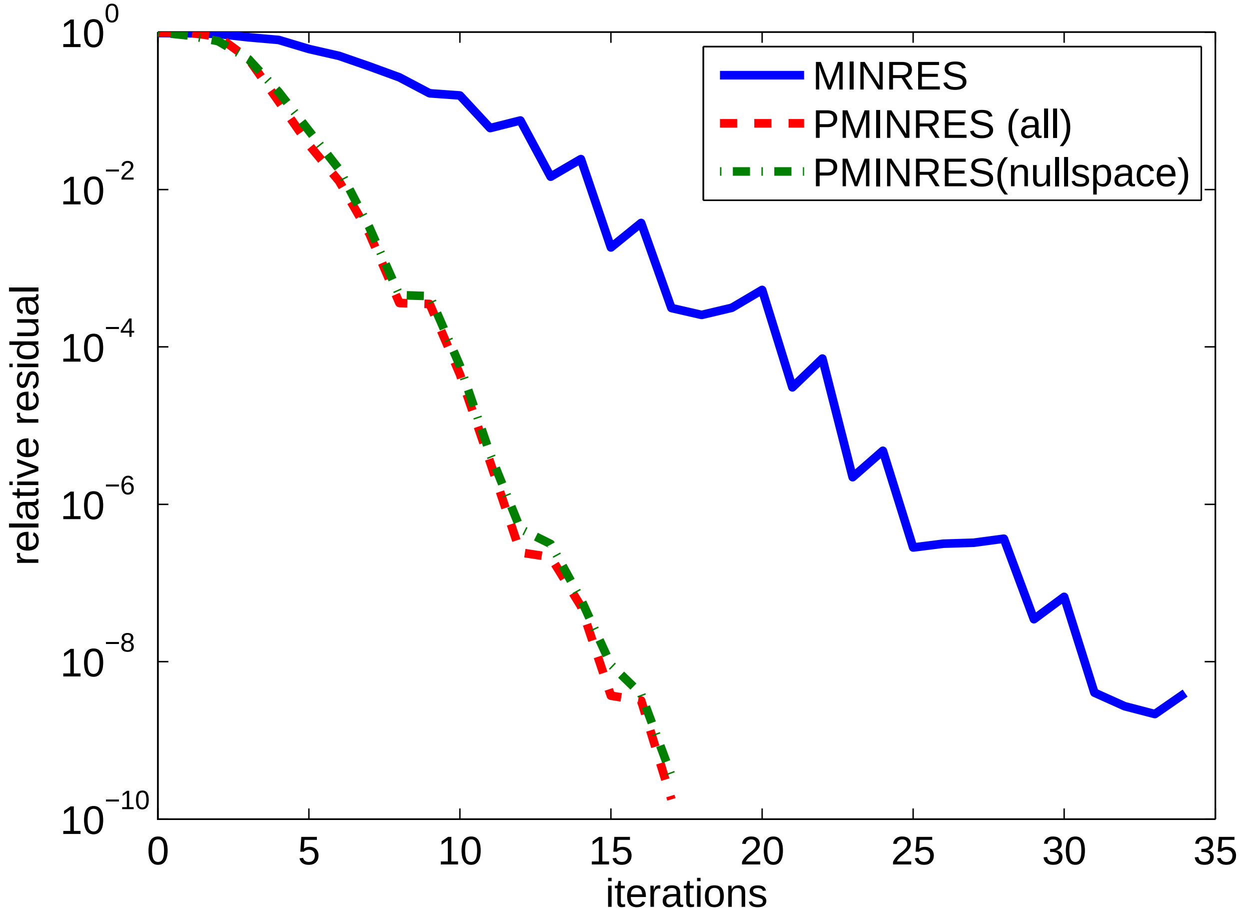
$Z^T AZ$ indefinite $\Rightarrow$ avoid PPCG
Solve $Z^T AZx_z = Z^T (b - AYx_y)$ with BICGSTAB, MINRES, SQMR or...
Solves with

$$\left[\begin{array}{cccc|c} & & & & -M_g \\ & G & & & K_{IB} \\ & & & & K_{BB} \\ & & & & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -M_f^T \\ 0 & I & 0 & 0 & K_{IB} \\ 0 & 0 & I & 0 & K_{BB} \\ 0 & 0 & 0 & I & 0 \\ \hline -M_g & K_{BI} & K_{BB} & 0 & 0 \end{array}\right]$$

PMINRES and Distributed Control

$$\mathcal{A} = \begin{bmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{bmatrix}$$

MINRES	PMINRES
$P = \begin{bmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & KM^{-1}K \end{bmatrix}$	$P = \begin{bmatrix} 0 & 0 & -M \\ 0 & 2\beta K^T M^{-1} K & K^T \\ -M & K & 0 \end{bmatrix}$
$\lambda = 1$ $\frac{1}{2}\left(1+\sqrt{5+\frac{2\alpha_1 h^4}{\beta}}\right) \leq \lambda \leq \frac{1}{2}\left(1+\sqrt{5+\frac{2\alpha_2}{\beta}}\right)$ $\frac{1}{2}\left(1-\sqrt{5+\frac{2\alpha_2}{\beta}}\right) \leq \lambda \leq \frac{1}{2}\left(1-\sqrt{5+\frac{2\alpha_1 h^4}{\beta}}\right)$	$\lambda = 1$ $1+\frac{ch^4}{2\beta} \leq \lambda \leq 1+\frac{C}{2\beta}$
2 solves with M 2 solves with K 5 matrix-vector multiplications with M 2 matrix-vector multiplications with K	2 solves with M 2 solves with K 3 matrix-vector multiplications with M 3 matrix-vector multiplications with K


MINRES
PMINRES (all)
PMINRES(nullspace)
relative residual
iterations
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0
5
10
15
20
25
30
35

Conclusions and Future Work

- PDE-constrained problems difficult to solve
- Avoid any solves with discretized PDE
- Use block structure
- Constraint preconditioners lead to projected iterative methods
- Mesh size independent convergence
- Regularization parameter independent convergence?
- Nonlinear PDEs
- Time-dependent PDEs

- `HSL_MI20` is part of `HSL2007`, which is now free for academics
- `HSL_MI27` will be part of `HSL2007`
- 'Optimal solvers for PDE-constrained optimization' Rees, Dollar, Wathen, SISC 2010
- 'Properties of linear systems in PDE-constrained optimization. Part I: Distributed control' Dollar RAL TR-2009-017
- 'Properties of linear systems in PDE-constrained optimization. Part II: Boundary control' Thorne (in preparation)
- 'PDE-constrained optimization and constraint preconditioners' Thorne (in preparation)