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Iterative Methods for Symmetric Quasi-Definite Linear Systems

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Overview of talk

- Symmetric Quasi Positive Definite matrices

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- G-K bidiagonalization
- Generalized LSQR and Craig (Stopping criteria)
- Numerical examples

Linear operators

Let $\mathbf{M} \in \mathrm{R}^{m \times m}$ and $\mathbf{N} \in \mathrm{R}^{n \times n}$ be symmetric positive definite matrices, and let $\mathbf{A} \in \mathrm{R}^{m \times n}$ be a full rank matrix.

$$\mathcal{M} = \{\mathbf{v} \in \mathrm{R}^m; \|\mathbf{u}\|^2_{\mathbf{M}} = \mathbf{v}^T\mathbf{M}\mathbf{v}\},\ \mathcal{N} = \{\mathbf{q} \in \mathrm{R}^n; \|\mathbf{q}\|^2_{\mathbf{N}} = \mathbf{q}^T\mathbf{N}\mathbf{q}\}$$

$$\mathcal{M}' = \{\mathbf{w} \in \mathrm{R}^m; \|\mathbf{w}\|^2_{\mathbf{M}^{-1}} = \mathbf{w}^T\mathbf{M}^{-1}\mathbf{w}\},$$
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$$\langle \mathbf{v}, \mathbf{A}\mathbf{q}\rangle_{\mathcal{M},\mathcal{M}'} = \mathbf{v}^T\mathbf{A}\mathbf{q}, \quad \mathbf{A}\mathbf{q} \in \mathcal{L}(\mathcal{M})\ \forall \mathbf{q} \in \mathcal{N}.$$

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The adjoint operator $\mathbf{A}^\star$ of $\mathbf{A}$ can be defined as

$$\langle \mathbf{A}^\star\mathbf{g}, \mathbf{f}\rangle_{\mathcal{N}',\mathcal{N}} = \mathbf{f}^T\mathbf{A}^T\mathbf{g}, \quad \mathbf{A}^T\mathbf{g} \in \mathcal{L}(\mathcal{N})\ \forall \mathbf{g} \in \mathcal{M}.$$

Generalized SVD

Given $\mathbf{q} \in \mathcal{M}$ and $\mathbf{v} \in \mathcal{N}$, the critical points for the functional

$$\frac{\mathbf{v}^T \mathbf{A} \mathbf{q}}{\|\mathbf{q}\|_{\mathbf{N}} \; \|\mathbf{v}\|_{\mathbf{M}}}$$

are the “*generalized singular values and singular vectors*” of $\mathbf{A}$.

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are the "*generalized singular values and singular vectors*" of **A**.
The saddle-point conditions are

$$\begin{cases} \mathbf{A}\mathbf{q}_i = \sigma_i \mathbf{M}\mathbf{v}_i & \mathbf{v}_i^T\mathbf{M}\mathbf{v}_j = \delta_{ij} \\ \mathbf{A}^T\mathbf{v}_i = \sigma_i \mathbf{N}\mathbf{q}_i & \mathbf{q}_i^T\mathbf{N}\mathbf{q}_j = \delta_{ij} \end{cases}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

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$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

The generalized singular values are the standard singular values of $\tilde{\mathbf{A}} = \mathbf{M}^{-1/2}\mathbf{A}\mathbf{N}^{-1/2}$. The generalized singular vectors $\mathbf{q}_i$ and $\mathbf{v}_i$, $i = 1, \ldots, n$ are the transformation by $\mathbf{M}^{-1/2}$ and $\mathbf{N}^{-1/2}$ respectively of the left and right standard singular vector of $\tilde{\mathbf{A}}$.

Quadratic programming

The general problem

$$\min_{\mathbf{A}^T\mathbf{w}=\mathbf{r}} \frac{1}{2}\mathbf{w}^T\mathbf{W}\mathbf{w} - \mathbf{g}^T\mathbf{w}$$

where the matrix $\mathbf{W}$ is positive semidefinite and $\ker(\mathbf{W}) \cap \ker(\mathbf{A}^T) = 0$ can be reformulated by choosing

$$\left.\begin{array}{l} \mathbf{M} = \mathbf{W} + \nu \mathbf{A}\mathbf{N}^{-1}\mathbf{A}^T \\ \mathbf{u} = \mathbf{w} - \mathbf{M}^{-1}\mathbf{g} \\ \mathbf{b} = \mathbf{r} - \mathbf{A}^T\mathbf{M}^{-1}\mathbf{g}. \end{array}\right\}$$

as a projection problem

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \|\mathbf{u}\|^2_{\mathbf{M}}$$

If $\mathbf{W}$ is non singular then we can choose $\nu = 0$.

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Augmented system

The augmented system that gives the optimality conditions for the projection problem:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.$$

Generalized Golub-Kahan bidiagonalization

In Golub Kahan (1965), Paige Saunders (1982), several algorithms for the bidiagonalization of a $m \times n$ matrix are presented. All of them can be theoretically applied to $\tilde{\mathbf{A}}$ and their generalization to $\mathbf{A}$ is straightforward as shown by Bembow (1999). Here, we want specifically to analyse one of the variants known as the "Craig"-variant (see Paige Saunders (1982), Saunders (1995,1997)).

Generalized Golub-Kahan bidiagonalization

$$\begin{cases} \mathbf{A}\tilde{\mathbf{Q}} = \mathbf{M}\tilde{\mathbf{V}} \begin{bmatrix} \tilde{\mathbf{B}} \\ 0 \end{bmatrix} & \tilde{\mathbf{V}}^T\mathbf{M}\tilde{\mathbf{V}} = \mathbf{I}_m \\ \mathbf{A}^T\tilde{\mathbf{V}} = \mathbf{N}\tilde{\mathbf{Q}} \left[\tilde{\mathbf{B}}^T; 0\right] & \tilde{\mathbf{Q}}^T\mathbf{N}\tilde{\mathbf{Q}} = \mathbf{I}_n \end{cases}$$

where

$$\tilde{\mathbf{B}} = \begin{bmatrix} \tilde{\alpha}_1 & 0 & 0 & \cdots & 0 \\ \tilde{\beta}_2 & \tilde{\alpha}_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \tilde{\beta}_{n-1} & \tilde{\alpha}_{n-1} & 0 \\ 0 & \cdots & 0 & \tilde{\beta}_n & \tilde{\alpha}_n \\ 0 & \cdots & 0 & 0 & \tilde{\beta}_{n+1} \end{bmatrix}.$$

Generalized Golub-Kahan bidiagonalization

$$\left\{ \begin{array}{lcl} \mathbf{AQ} & = & \mathbf{MV} \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \\ \mathbf{A}^T\mathbf{V} & = & \mathbf{NQ} \left[\mathbf{B}^T; 0 \right] \end{array} \right. \qquad \begin{array}{l} \mathbf{V}^T\mathbf{MV} = \mathbf{I}_m \\ \mathbf{Q}^T\mathbf{NQ} = \mathbf{I}_n \end{array}$$

where

$$\mathbf{B} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \beta_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & \alpha_n \end{bmatrix}.$$

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Algorithm

The augmented system that gives the optimality conditions for $\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \|\mathbf{u}\|_{\mathbf{M}}^2$

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

can be transformed by the change of variables

$$\begin{cases} \mathbf{u} = \mathbf{V}\mathbf{z} \\ \mathbf{p} = \mathbf{Q}\mathbf{y} \end{cases}$$

Algorithm

$$\begin{bmatrix} \mathbf{I}_n & 0 & \mathbf{B} \\ 0 & \mathbf{I}_{m-n} & 0 \\ \mathbf{B}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{Q}^T\mathbf{b} \end{bmatrix}.$$

Algorithm

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{Q}^T\mathbf{b} \end{bmatrix}.$$

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Algorithm

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{Q}^T \mathbf{b} \end{bmatrix}.$$

$$\mathbf{Q}^T \mathbf{b} = \mathbf{e}_1 \|\mathbf{b}\|_{\mathbf{N}}$$

the value of $\mathbf{z}_1$ will correspond to the first column of the inverse of $\mathbf{B}$ multiplied by $\|\mathbf{b}\|_{\mathbf{N}}$.

Algorithm

Thus, we can compute the first column of $\mathbf{B}$ and of $\mathbf{V}$:
$\alpha_1 \mathbf{M}\mathbf{v}_1 = \mathbf{A}\mathbf{q}_1$, such as

$$\begin{aligned} &\mathbf{w} = \mathbf{M}^{-1}\mathbf{A}\mathbf{q}_1 \\ &\alpha_1 = \mathbf{w}^T\mathbf{M}\mathbf{w} = \mathbf{w}\mathbf{A}\mathbf{q}_1 \\ &\mathbf{v}_1 = \mathbf{w}/\sqrt{\alpha_1}. \end{aligned}$$

Algorithm

Thus, we can compute the first column of **B** and of **V**:
$\alpha_1 \mathbf{M}\mathbf{v}_1 = \mathbf{A}\mathbf{q}_1$, such as

$$\mathbf{w} = \mathbf{M}^{-1}\mathbf{A}\mathbf{q}_1$$
$$\alpha_1 = \mathbf{w}^T\mathbf{M}\mathbf{w} = \mathbf{w}\mathbf{A}\mathbf{q}_1$$
$$\mathbf{v}_1 = \mathbf{w}/\sqrt{\alpha_1}.$$

Finally, knowing $\mathbf{q}_1$ and $\mathbf{v}_1$ we can start the recursive relations

$$\mathbf{g}_{i+1} = \mathbf{N}^{-1}\left(\mathbf{A}^T\mathbf{v}_i - \alpha_i\mathbf{N}\mathbf{q}_i\right)$$
$$\beta_{i+1} = \mathbf{g}^T\mathbf{N}\mathbf{g}$$
$$\mathbf{q}_{i+1} = \mathbf{g}\ \sqrt{\beta_{i+1}}$$
$$\mathbf{w} = \mathbf{M}^{-1}\left(\mathbf{A}\mathbf{q}_{i+1} - \beta_{i+1}\mathbf{M}\mathbf{v}_i\right)$$
$$\alpha_{i+1} = \mathbf{w}^T\mathbf{M}\mathbf{w}$$
$$\mathbf{v}_{i+1} = \mathbf{w}/\sqrt{\alpha_{i+1}}.$$

$\mathbf{u}$

Thus, the value of $\mathbf{u}$ can be approximated when we have computed the first k columns of $\mathbf{U}$ by

$$\mathbf{u}^{(k)} = \mathbf{V}_k \mathbf{z}_k = \sum_{j=1}^{k} \zeta_j \mathbf{v}_j .$$

u

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$$\mathbf{u}^{(k)} = \mathbf{V}_k \mathbf{z}_k = \sum_{j=1}^{k} \zeta_j \mathbf{v}_j.$$

The entries ζ_j of $\mathbf{z}_k$ can be easily computed recursively starting with

$$\zeta_1 = -\frac{\|\mathbf{b}\|_{\mathbf{N}}}{\alpha_1}$$

as

$$\zeta_{i+1} = -\frac{\beta_i}{\alpha_{i+1}}\zeta_i \qquad i = 1, \ldots, n$$

p

Approximating $\mathbf{p} = \mathbf{Q}\mathbf{y}$ by $\mathbf{p}^{(k)} = \mathbf{Q}_k \mathbf{y}_k = \sum_{j=1}^{k} \psi_j \mathbf{q}_j$, we have that

$$\mathbf{y}_k = -\mathbf{B}_k^{-1} \mathbf{z}_k.$$

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Following an observation made by Paige and Saunders, we can easily transform the previous relation into a recursive one where only one extra vector is required.

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$$\mathbf{y}_k = -\mathbf{B}_k^{-1}\mathbf{z}_k.$$

From $\mathbf{p}^{(k)} = -\mathbf{Q}_k\mathbf{B}_k^{-1}\mathbf{z}_k = -\left(\mathbf{B}_k^{-T}\mathbf{Q}_k^T\right)^T \mathbf{z}_k$ and $\mathbf{D}_k = \mathbf{B}_k^{-T}\mathbf{Q}_k^T$

$$\mathbf{d}_i = \frac{\mathbf{q}_i - \beta_i \mathbf{d}_{i-1}}{\alpha_i} \qquad i = 1, \ldots, n \ \left(\mathbf{d}_0 = 0\right)$$

where $\mathbf{d}_j$ are the columns of $\mathbf{D}$.
Starting with $\mathbf{p}^{(1)} = -\zeta_1\mathbf{d}_1$ and $\mathbf{u}^{(1)} = \zeta_1\mathbf{v}_1$

$$\left.\begin{aligned} \mathbf{u}^{(i+1)} &= \mathbf{u}^{(i)} + \zeta_{i+1}\mathbf{v}_{i+1} \\ \mathbf{p}^{(i+1)} &= \mathbf{p}^{(i)} - \zeta_{i+1}\mathbf{d}_{i+1} \end{aligned}\right\} \qquad i = 1, \ldots, n$$

Stopping criteria

$$\|\mathbf{u} - \mathbf{u}^{(k)}\|_{\mathbf{M}}^2 = \|\mathbf{e}^{(k)}\|_{\mathbf{M}}^2 = \sum_{j=k+1}^{n} \zeta_j^2 = \left\| \mathbf{z} - \begin{bmatrix} \mathbf{z}_k \\ 0 \end{bmatrix} \right\|_2^2.$$

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Stopping criteria

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$$\|\mathbf{A}^T \mathbf{u}^{(k)} - \mathbf{b}\|_{\mathbf{N}^{-1}} = |\beta_{k+1}\, \zeta_k| \le \sigma_1 |\zeta_k| = \|\tilde{\mathbf{A}}\|_2 |\zeta_k|.$$

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Stopping criteria

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$$\|\mathbf{A}^T\mathbf{u}^{(k)} - \mathbf{b}\|_{\mathbf{N}^{-1}} = |\beta_{k+1}\,\zeta_k| \le \sigma_1|\zeta_k| = \|\tilde{\mathbf{A}}\|_2|\zeta_k|.$$

$$\|\mathbf{p}-\mathbf{p}^{(k)}\|_{\mathbf{N}} = \left\| \mathbf{Q}\mathbf{B}^{-1}\left(\mathbf{z} - \begin{bmatrix} \mathbf{z}_k \\ 0 \end{bmatrix}\right) \right\|_{\mathbf{N}} \le \frac{\|\mathbf{e}^{(k)}\|_{\mathbf{M}}}{\sigma_n}.$$

Error bound

Lower bound We can estimate $\|\mathbf{e}^{(k)}\|_{\mathbf{M}}^2$ by the lower bound

$$\xi_{k,d}^2 = \sum_{j=k+1}^{k+d+1} \zeta_j^2 < \|\mathbf{e}^{(k)}\|_{\mathbf{M}}^2.$$

Given a threshold $\tau < 1$ and an integer d, we can stop the iterations when

$$\xi_{k,d}^2 \le \tau \sum_{j=1}^{k+d+1} \zeta_j^2 < \tau \|\mathbf{u}\|_{\mathbf{M}}^2.$$

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Error bound

Lower bound We can estimate $\|\mathbf{e}^{(k)}\|^2_{\mathbf{M}}$ by the lower bound

$$\xi^2_{k,d} = \sum_{j=k+1}^{k+d+1} \zeta_j^2 < \|\mathbf{e}^{(k)}\|^2_{\mathbf{M}}.$$

Given a threshold $\tau < 1$ and an integer d, we can stop the iterations when

$$\xi^2_{k,d} \leq \tau \sum_{j=1}^{k+d+1} \zeta_j^2 < \tau \|\mathbf{u}\|^2_{\mathbf{M}}.$$

Upper bound Despite being very inexpensive, the previous estimator is still a lower bound of the error. We can use an approach inspired by the Gauss-Radau quadrature algorithm and similar to the one described in Golub-Meurant (2010).

Symmetric Quasi-Definite Systems

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad \text{where} \qquad \mathbf{M} = \mathbf{M}^T \succ 0, \ \mathbf{N} = \mathbf{N}^T \succ 0.$$

- Interior-point methods for LP, QP, NLP, SOCP, SDP, . . .
- Regularized/stabilized PDE problems
- Regularized least squares
- How to best take advantage of the structure?

Main Property

Theorem (Vanderbei, 1995)
If $\mathbf{K}$ *is SQD, it is* **strongly factorizable**, *i.e., for any permutation matrix* $\mathbf{P}$, *there exists a unit lower triangular* $\mathbf{L}$ *and a diagonal* $\mathbf{D}$ *such that* $\mathbf{P}^T\mathbf{K}\mathbf{P} = \mathbf{L}\mathbf{D}\mathbf{L}^T$.

- Cholesky-factorizable
- Used to speed up factorization in regularized least-squares (Saunders) and interior-point methods (Friedlander and O.)
- Stability analysis by Gill, Saunders, Shinnerl (1996).

Iterative Methods I

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Fact: . . . none exploits the SQD structure.

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- SYMMLQ
- (F)GMRES??
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Fact: ... none exploits the SQD structure.

If the system were definite, we would like to use CG.

Related Problems: an example

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

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Related Problems: an example

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

are the optimality conditions of

$$\min_{\mathbf{y}\in\mathbb{R}^m} \frac{1}{2} \left\| \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{y} - \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right\|^2_{E_+^{-1}} \equiv \min_{y\in\mathbb{R}^m} \frac{1}{2} \left\| \begin{bmatrix} \mathbf{M}^{-\frac{1}{2}} & 0 \\ 0 & \mathbf{N}^{\frac{1}{2}} \end{bmatrix} \left(\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{y} - \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right) \right\|_2^2$$

Generalized Least Squares

Normal equations: $(\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A} + \mathbf{N})\mathbf{y} = \mathbf{A}^T\mathbf{M}^{-1}\mathbf{b}$.

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At k-th iteration, seek $y \approx \mathbf{y}_k := \tilde{\mathbf{V}}_k\bar{\mathbf{y}}_k$:

$$(\tilde{\mathbf{B}}_k^T\tilde{\mathbf{B}}_k+\mathbf{I})\bar{\mathbf{y}}_k=\tilde{\mathbf{B}}_k^T\beta_1\mathbf{e}_1$$

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i.e.:

$$\min_{\bar{\mathbf{y}}\in\mathrm{R}^k} \ \tfrac{1}{2}\left\| \begin{bmatrix}\tilde{\mathbf{B}}_k \\ \mathbf{I}\end{bmatrix}\bar{\mathbf{y}} - \begin{bmatrix}\beta_1\mathbf{e}_1 \\ 0\end{bmatrix}\right\|_2^2$$

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i.e.:

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or:

$$\begin{bmatrix} \mathbf{I} & \tilde{\mathbf{B}}_k \\ \tilde{\mathbf{B}}_k^T & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_k \\ \bar{\mathbf{y}}_k \end{bmatrix} = \begin{bmatrix} \beta_1\mathbf{e}_1 \\ 0 \end{bmatrix}.$$

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Generalized LSQR

Solve

$$\min_{\bar{\mathbf{y}} \in \mathbb{R}^k} \ \tfrac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{B}}_k \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix} \right\|_2^2$$

by specialized Givens Rotations (Eliminate $\mathbf{I}$ first and $\tilde{\mathbf{R}}_k$ will be upper bidiagonal)

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by specialized Givens Rotations (Eliminate $\mathbf{I}$ first and $\tilde{\mathbf{R}}_k$ will be upper bidiagonal)

$$\min_{\bar{\mathbf{y}}\in \mathrm{R}^k} \ \frac{1}{2}\left\| \begin{bmatrix} \tilde{\mathbf{R}}_k \\ 0 \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \right\|_2^2 .$$

Generalized LSQR

Solve

$$\min_{\bar{\mathbf{y}}\in\mathrm{R}^k} \frac{1}{2}\left\|\begin{bmatrix}\tilde{\mathbf{B}}_k\\ \mathbf{I}\end{bmatrix}\bar{\mathbf{y}}-\begin{bmatrix}\beta_1\mathbf{e}_1\\ 0\end{bmatrix}\right\|_2^2$$

by specialized Givens Rotations (Eliminate $\mathbf{I}$ first and $\tilde{\mathbf{R}}_k$ will be upper bidiagonal)

$$\min_{\bar{\mathbf{y}}\in\mathrm{R}^k} \frac{1}{2}\left\|\begin{bmatrix}\tilde{\mathbf{R}}_k\\ 0\end{bmatrix}\bar{\mathbf{y}}-\begin{bmatrix}\phi_k\\ 0\end{bmatrix}\right\|_2^2.$$

As in Paige-Saunders '82 we can build recursive expressions of $\mathbf{y}_k$

$$\mathbf{y}_{k+1}=\mathbf{y}_k+\mathbf{d}_k\phi_k \quad \left(\mathbf{D}_k=\tilde{\mathbf{V}}_k\tilde{\mathbf{R}}_k^{-1}\right)$$

and we have that

$$||\mathbf{y}||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}=\sum_{j=1}^{m}\phi_j^2 \quad \text{and} \quad ||\mathbf{y}-\mathbf{y}_k||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}=\sum_{j=k+1}^{m}\phi_j^2$$

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- A. and Orban "Iterative methods for symmetric quasi definite systems" in preparation. WORK IN PROGRESS

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