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Can you use MINRES with an indefinite preconditioner?

Tyrone Rees

STFC Rutherford Appleton Laboratory

Nick Gould (RAL), Dominique Orban (École Polytechnique de Montréal)

The problem...

Solve

$$\mathcal{A}x = b$$

where A is symmetric, but indefinite.

MINRES (Paige & Saunders, 1975)

MINimal RESidual

Finds

$$x_k \in x_0 + \mathrm{span}\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$$

which minimizes

$$\|r_k\|_2 = \|b - \mathcal{A}x_k\|_2$$

MINRES (Paige & Saunders, 1975)

$r_k := b - \mathcal{A}x_k$

MINimal RESidual

Finds

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which minimizes

$$\|r_k\|_2 = \|b - \mathcal{A}x_k\|_2$$

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Lanczos othogonalization

Want an orthonormal basis for the set $\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$

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$$v_1 = r_0/\|r_0\|$$

Given an orthonormal set $\{v_1, \ldots, v_m\}$

- set $w = \mathcal{A}v_m$
- $h_{i,m} = {v_i}^T w$
- $w \leftarrow w - \sum h_{i,m} v_i$
- $h_{m+1,m} = \|w\|_2, \;\; v_{m+1} = w/h_{m+1,m}$

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$$\mathcal{A}\underbrace{\begin{bmatrix} \vdots & & \vdots \\ v_1 & \ldots & v_m \\ \vdots & & \vdots \end{bmatrix}}_{V_m} = \underbrace{\begin{bmatrix} \vdots & & \vdots \\ v_1 & \ldots & v_{m+1} \\ \vdots & \vdots & \end{bmatrix}}_{V_{m+1}} \underbrace{\begin{bmatrix} h_{1,1} & & \ldots & h_{1,m} \\ h_{2,1} & h_{2,2} & & h_{2,m} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & h_{m,m-1} & h_{m,m} \\ 0 & \ldots & 0 & h_{m+1,m} \end{bmatrix}}_{\widehat{H}_m}$$

Lanczos othogonalization

Want an orthonormal basis for the set $\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$

$$V_m^T \mathcal{A} V_m = \underbrace{\begin{bmatrix} h_{1,1} & h_{1,2} & & \cdots & h_{1,m} \\ h_{2,1} & h_{2,2} & & & h_{2,m} \\ & & & & \vdots \\ \vdots & & \ddots & \ddots & h_{m-1,m} \\ 0 & & & h_{m,m-1} & h_{m,m} \end{bmatrix}}_{H_m}$$

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Lanczos othogonalization

Want an orthonormal basis for the set $\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$

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Lanczos othogonalization

Want an orthonormal basis for the set $\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$

$$V_m^T \mathcal{A} V_m = \underbrace{\begin{bmatrix} \alpha_1 & \beta_1 & 0 & \ldots & 0 \\ \beta_1 & \alpha_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & & & \beta_{m-1} \\ 0 & \ldots & 0 & \beta_{m-1} & \alpha_m \end{bmatrix}}_{H_m}$$

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From Lanczos to MINRES

$$\|b - \mathcal{A}x_k\|_2 \quad = \quad \|b - \mathcal{A}(x_0 + V_k z_k)\|_2$$

From Lanzcos to MINRES

$$\begin{aligned} \|b - \mathcal{A}x_k\|_2 &= \|b - \mathcal{A}(x_0 + V_k z_k)\|_2 \\ &= \|(b - \mathcal{A}x_0) - \mathcal{A}V_k z_k\|_2 \end{aligned}$$

From Lanzcos to MINRES

$$
\begin{aligned}
\|b - \mathcal{A}x_k\|_2 &= \|b - \mathcal{A}(x_0 + V_k z_k)\|_2 \\
&= \|(b - \mathcal{A}x_0) - \mathcal{A}V_k z_k\|_2 \\
&= \|\|r_0\| v_1 - \mathcal{A}V_k z_k\|_2
\end{aligned}
$$

$v_1 = r_0 / \|r_0\|$

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From Lanzcos to MINRES

$$\begin{aligned}\|b - \mathcal{A}x_k\|_2 &= \|b - \mathcal{A}(x_0 + V_k z_k)\|_2 \\ &= \|(b - \mathcal{A}x_0) - \mathcal{A}V_k z_k\|_2 \\ &= \|\|r_0\|v_1 - \mathcal{A}V_k z_k\|_2 \\ &= \|\|r_0\|v_1 - V_{k+1}\widehat{H}_k z_k\|_2\end{aligned}$$

$$\mathcal{A}V_k = V_{k+1}\widehat{H}_k$$

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From Lanzcos to MINRES

$$\begin{aligned}
\|b - \mathcal{A}x_k\|_2 &= \|b - \mathcal{A}(x_0 + V_k z_k)\|_2 \\
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&= \|\|r_0\|v_1 - \mathcal{A}V_k z_k\|_2 \\
&= \|\|r_0\|v_1 - V_{k+1}\widehat{H}_k z_k\|_2 \\
&= \|V_{k+1}(\|r_0\|e_1 - \widehat{H}_k z_k)\|_2
\end{aligned}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

From Lanzcos to MINRES

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&= \|\|r_0\|e_1 - \widehat{H}_k z_k\|_2
\end{aligned}
$$

Least squares problem: solved by Givens rotations

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What about preconditioning?

Convergence of MINRES depends on the clustering of the eigenvalues.
Suppose $P = MM^T$ is such that the spectrum of

$$M^{-1}\mathcal{A}M^{-T}$$

is 'nice'.
Apply MINRES to

$$M^{-1}\mathcal{A}M^{-T}y = M^{-1}b,$$

where $y = M^T x$.
Expect better convergence.

Algorithm

$$\mathbf{v}_1 = \mathbf{b} - \mathcal{A}\mathbf{x}_0$$
$$P\mathbf{z}_1 = \mathbf{v}_1$$
$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$$
$$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$$

for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j / \beta_j$$
$$\alpha_j = \mathbf{z}_j^T \mathcal{A}\mathbf{z}_j$$
$$\mathbf{v}_{j+1} = \mathcal{A}\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1}$$
$$P\mathbf{z}_{j+1} = \mathbf{v}_{j+1}$$
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$$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$$
$$\gamma_1 = \sqrt{\gamma_0^2 + {\beta_{j+1}}^2}$$
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$$c_{j+1} = \gamma_0/\gamma_1;\ s_{j+1} = \beta_{j+1}/\gamma_1$$
$$\mathbf{w}_{j+1} = (\mathbf{z}_j - \gamma_3\mathbf{w}_{j-1} - \gamma_2\mathbf{w}_j)/\gamma_1$$
$$\mathbf{x}_j = \mathbf{x}_{j-1} + c_{j+1}\eta\mathbf{w}_{j+1}$$
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Algorithm

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Preconditioner. Note M factors not needed.

Algorithm

$\mathbf{v}_1 = \mathbf{b} - \mathcal{A}\mathbf{x}_0$

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Preconditioner. Note M factors not needed.

diagonal entries of H_k

off-diagonal entries of H_k

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Algorithm

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diagonal entries of H_k
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Givens' rotations

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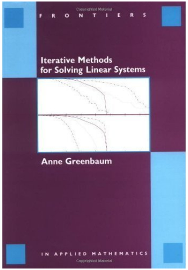
Greenbaum



Chapter 8

Overview and Preconditioned Algorithms

⋮

The same modifications can be made to any of the MINRES implementations, provided that the preconditioner M is positive definite. To obtain a preconditioned version of Algorithm 4, first consider the Lanczos algorithm applied directly to the matrix $L^{-1}AL^{-H}$ with initial vector $\hat{q}_1$. Successive vectors satisfy

$$\hat{v}_j = L^{-1}AL^{-H}\hat{q}_j - \alpha_j\hat{q}_j - \beta_{j-1}\hat{q}_{j-1},$$
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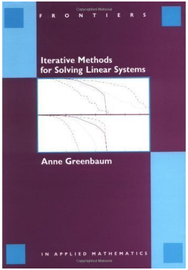
Greenbaum



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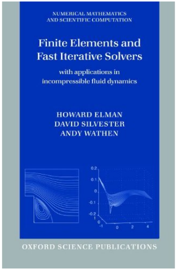
Elman, Silvester, Wathen



6.1 The preconditioned MINRES method

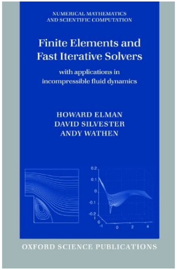
In the generic context of solving a symmetric and indefinite matrix system

$$K\mathbf{x} = \mathbf{b}, \tag{6.3}$$

⋮

It is desirable to ensure that any preconditioner does not destroy the symmetry of the discrete Stokes problem; otherwise iterative methods for nonsymmetric systems would have to be employed as for discrete convection–diffusion problems. To preserve symmetry in the preconditioned system, a symmetric and positive-definite preconditioner $M = HH^T$ is required.

Elman, Silvester, Wathen



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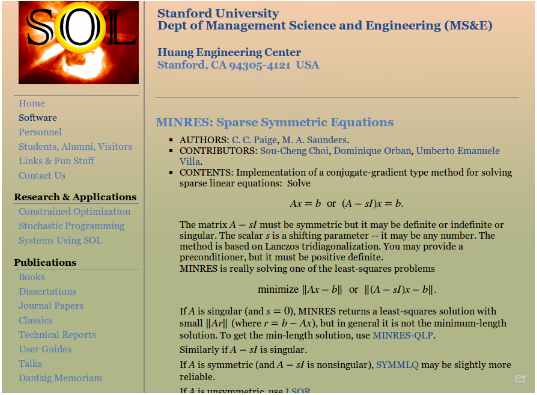
Saunders' implementation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Stanford University
Dept of Management Science and Engineering (MS&E)

Huang Engineering Center
Stanford, CA 94305-4121 USA

MINRES: Sparse Symmetric Equations

- AUTHORS: C. C. Paige, M. A. Saunders.
- CONTRIBUTORS: Sou-Cheng Choi, Dominique Orban, Umberto Emanuele Villa.
- CONTENTS: Implementation of a conjugate-gradient type method for solving sparse linear equations: Solve

$$Ax = b \quad \text{or} \quad (A - sI)x = b.$$

The matrix $A - sI$ must be symmetric but it may be definite or indefinite or singular. The scalar s is a shifting parameter -- it may be any number. The method is based on Lanczos tridiagonalization. You may provide a preconditioner, but it must be positive definite.
MINRES is really solving one of the least-squares problems

$$\text{minimize } \|Ax - b\| \quad \text{or} \quad \|(A - sI)x - b\|.$$

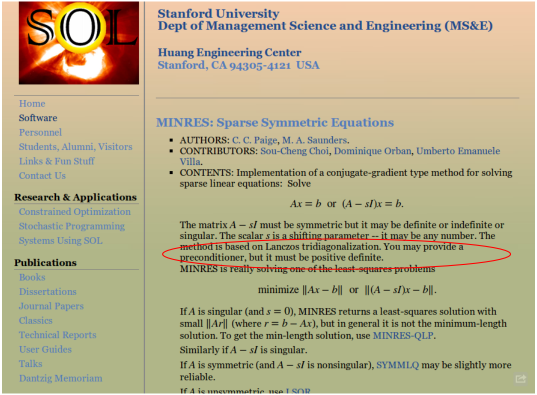
If A is singular (and $s = 0$), MINRES returns a least-squares solution with small $\|Ar\|$ (where $r = b - Ax$), but in general it is not the minimum-length solution. To get the min-length solution, use MINRES-QLP.

Similarly if $A - sI$ is singular.

If A is symmetric (and $A - sI$ is nonsingular), SYMMLQ may be slightly more reliable.

If A is unsymmetric, use LSQR

Saunders' implementation



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Can you use MINRES with an indefinite preconditioner?

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Can you use MINRES with an indefinite preconditioner?

No

Tyrone Rees

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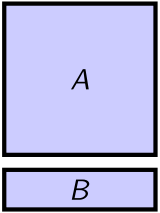
Nick Gould (RAL), Dominique Orban (École Polytechnique de Montréal)

A class of indefinite problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$



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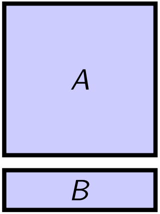

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x + x_0 \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{aligned} \hat{x} &= x + x_0 \\ Bx_0 &= b \end{aligned}$$

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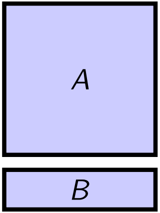
A class of indefinite problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$c = a - Ax_0$$



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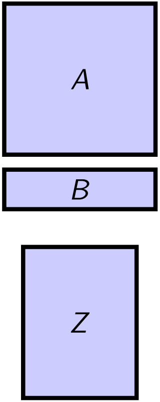
Let Z span the nullspace of B (i.e. $BZ = 0$)

$$BZ = 0 \Rightarrow x = Z\bar{x}$$



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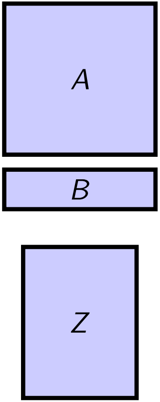
$$AZ\bar{x} + \quad B^T y = \quad c$$



A class of indefinite problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

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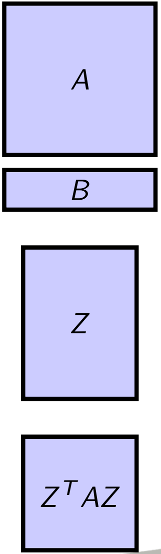

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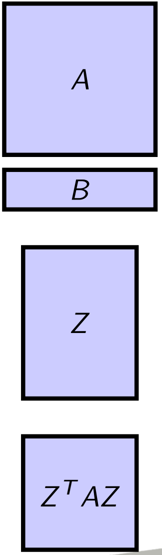

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$$Z^T AZ\bar{x} = Z^T c$$

If A symmetric, $Z^T AZ$ symmetric, but (possibly) indefinite - apply MINRES!

Use as a preconditioner $Z^T GZ$, where G is positive definite on the null space of B.

MINRES applied to the reduced system

$$\bar{\mathbf{v}}_1 = Z^T\mathbf{c} - Z^TAZ\bar{\mathbf{x}}_0$$
$$\bar{\mathbf{z}}_1 = (Z^TGZ)^{-1}\bar{\mathbf{v}}_1$$
$$\beta_1 = \sqrt{\bar{\mathbf{z}}_1^T\bar{\mathbf{v}}_1}$$
$$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$$

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$$\mathbf{z} = Z(Z^T G Z)^{-1} Z^T \mathbf{v}$$
$$\Rightarrow \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}$$

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- $\mathbf{w}_k = Z\bar{\mathbf{w}}_k$

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$$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$$
$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{0} \end{bmatrix}$$
$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$$
$$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$$

for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j / \beta_j$$
$$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$$
$$\mathbf{v}_{j+1} = A\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1}$$
$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{0} \end{bmatrix}$$
$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$$
$$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$$
$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$
$$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$$
$$\gamma_3 = s_{j-1} \beta_j$$
$$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$$
$$\mathbf{w}_{j+1} = (\mathbf{z}_j - \gamma_3 \mathbf{w}_{j-1} - \gamma_2 \mathbf{w}_j) / \gamma_1$$
$$\mathbf{x}_j = \mathbf{x}_{j-1} + c_{j+1} \eta \mathbf{w}_{j+1}$$
$$\eta = -s_{j+1} \eta$$

Projected MINRES

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

MINRES with (indefinite) constraint preconditioner

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0,\ \mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j,\ \mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + 0}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:

If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j$, $\mathbf{g}_j = \mathbf{g}_j/\beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0/\gamma_1$; $s_{j+1} = \beta_{j+1}/\gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

$$\mathbf{z}_j^T A\mathbf{z}_j + \mathbf{z}_j^T B^T \mathbf{g}_j + \mathbf{y}_j^T B\mathbf{z}_j$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

Science & Technology Facilities Council

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j$, $\mathbf{g}_j = \mathbf{g}_j/\beta_j$

$\alpha_j = \mathbf{z}_j^T(A\mathbf{z}_j + B^T\mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j\alpha_j + c_{j-1}c_j\beta_j$

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$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1}\left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1}\eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1}\eta$

$$\mathbf{z}_j^T A\mathbf{z}_j + \mathbf{z}_j^T B^T \mathbf{g}_j + \mathbf{y}_j^T B\mathbf{z}_j$$
$$= \mathbf{z}_j^T A\mathbf{z}_j$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
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Science & Technology Facilities Council

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for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j / \beta_j,\ \mathbf{g}_j = \mathbf{g}_j / \beta_j$$

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$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

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$$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$$

$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$

$$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$$

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$$\eta = -s_{j+1} \eta$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

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$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

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$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

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$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

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for $j = 1, 2 \ldots$ **until** convergence

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$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

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$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

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Assume, $(k \leq j)$:

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- $B\mathbf{z}_k = 0$

  $\mathbf{u}_{j+1} = 0$

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$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + 0}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

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$\eta = -s_{j+1} \eta$

Assume, $(k \le j)$:

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$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

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$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j$, $\mathbf{g}_j = \mathbf{g}_j/\beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

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$\eta = -s_{j+1}\eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

  $\mathbf{u}_{j+1} = 0$

  $B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

problem?

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

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$\eta = -s_{j+1} \eta$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

  $\mathbf{u}_{j+1} = 0$

  $B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

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for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

problem?

unchanged

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

$$\begin{aligned} \beta_{j+1} &= \sqrt{\mathbf{z}_{j+1}^T \left(A\mathbf{z}_j + B^T \mathbf{g}_j - \frac{\alpha_j}{\beta_j} \mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}} \mathbf{v}_{j-1} \right)} \\ &= \sqrt{\mathbf{z}_{j+1}^T \left(A\mathbf{z}_j - \frac{\alpha_j}{\beta_j} \mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}} \mathbf{v}_{j-1} \right)} \\ &= \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}^{\text{PPMINRES}}} \end{aligned}$$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

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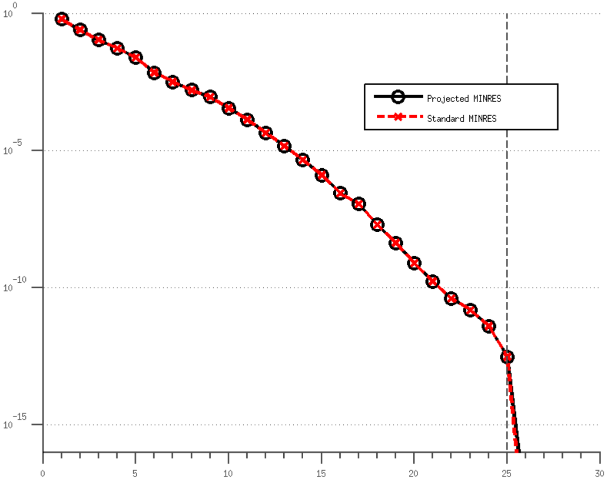
Numerical Comparison



Can you use MINRES with an indefinite preconditioner?

~~No~~ Sometimes[†]

Tyrone Rees

STFC Rutherford Appleton Laboratory

Nick Gould (RAL), Dominique Orban (École Polytechnique de Montréal)

[†] Gould, N.I.M., Orban, D. and Rees, T., *Projected Krylov Methods for Saddle Point Systems* (in preparation)