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Pole Counting and Resonance Classification

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Abstract

$S-$wave resonances occurring close to an inelastic threshold can be classified according to the number of nearby poles they possess. One then has a useful possibility of distinguishing dynamical alternatives by objective appeal to data. Making this quantitative entails developing suitable effective range expansions for various realizations of potential scattering. A key application is deciding the make-up of $f_0(976)(S^*)$.

1 Introduction

A constantly recurring problem in hadron spectroscopy is to decide the composition of the various states that are observed. The majority are commonly attributed to simple configurations of constituent quarks – (qqq) for baryons and $(q\bar{q})$ for mesons. Within QCD, other states involving constituent gluons or additional $(q\bar{q})$ components are also expected. In addition, there could be compounds of any of the foregoing states – molecular configurations akin to the deuteron - if suitably attractive forces operate [1].

Distinguishing these alternatives is never straightforward and usually impossible without recourse to detailed model assumptions. Spectroscopic quantum numbers of the observed states are commonly compatible with all or several of the above mechanisms; moreover, as concerns observable behaviour in their decay channels, resonances 'forget' their dynamic origins. In the case of S-wave resonances (and bound states) occurring close to a threshold, an exception to this 'amnesia' principle can occur [2]. The associated resonance pole structure can be such as to make a molecular composition extremely unlikely – essentially to preclude it. Since the pole structure is directly inferrable by extrapolation from the relevant scattering data we have an objective test which, in suitable circumstances, can be decisive.

The aim of this paper is to amplify and justify the above claim by studying resonance formation within various schemes for describing scattering. In each case we derive an 'effective range expansion' [3] and this provides the basic tool for our classification.

The philosophy, and aspects of the methodology, of the present work have many precedents in particle and nuclear physics [4,5]. What is aimed for here is a systematic and quantitative formulation that can be applied in a variety of cases.

The following sections are mainly devoted to deriving effective range expansions in various contexts. The final section provides a summary and mentions a key application [2,6] - testing the popular $K\bar{K}$ molecule assignment [7] for the $f_0(976)(S^*)$ resonance.

2 Multi-channel effective range expansions and resonance classification

Our object is to classify alternative resonance mechanisms that can operate close to an inelastic threshold, when the opening channel is a two-body S-wave. The aim is to distinguish effects that arise from 'potential' scattering in the explicit, 'open' channels ('molecular resonances') from those that feed through from inaccessible 'hidden' channels. For the $I = 0$ scalar meson system near 1 GeV, the 'open' channels are $\pi\pi$ and $K\bar{K}$; the 'hidden' channels could be inaccessible from having a higher threshold (e.g. $N\bar{N}$), or in principle, typically from confinement (e.g. $q\bar{q}$). In all cases the issue is to which group of channels does the resonance in question predominantly couple – to the open, explicit channels whose scattering properties we study, or to the hidden channels. In relation to the open channels, this latter type of resonance couples just like an extra elementary particle; in the scattering formalism for the open channels it features as a CDD pole [8]. Within the quark model this latter is the norm and the molecular situation the exception.

It is not in general possible to distinguish these alternatives except using models, however for resonances occurring very close to an S-wave threshold there are circumstances where the resonance characteristics are definitely not of the molecular type [2,6]. The hall- mark of these definitely non-potential resonances is the occurrence of associated resonance poles on two of the nearby unphysical sheets of the energy plane (Sheets II and III)[1]. Provided these poles both lie very close to threshold (on a scale defined by the range of the exchange forces), the resonance is not of the potential type. This is not asserted as a theorem - suitably baroque potentials could doubtless furnish counter examples - but as a regularity or 'rule' that should obtain in real physical situations.

To establish this rule, one merely needs to formulate and justify a suitable 'effective range expansion' [3]. The starting point is to write the scattering amplitude, T, for CM momentum k in the opening channel as

$$T = [M - ik]^{-1}. \tag{1}$$

Resonance poles (which may be specified by the associated complex k values) correspond to zeros of $M - ik$. Under a potential scattering regime M can be

[1] For a pair of two-body channels with CM momenta k_1 and k_2, the sheets can be characterized by the signs of $(Im\,k_1, Im\,k_2)$: $(++)$ for Sheet I, $(-+)$ for Sheet II, $(--)$ for Sheet III and $(+-)$ for Sheet IV in the conventional numbering.

expanded in powers of k^2:

$$M = M^{(0)} + \frac{1}{2}M^{(2)}k^2 + 0(k^4). \tag{2}$$

Provided the modulus of $M^{(2)}$ (usually denoted r_{eff}) is bounded above by the range, R, of the potential (and provided the higher terms in (2) behave correspondingly), $M - ik$ cannot have two distinct zeros with $|k|$ appreciably less than R^{-1}. A resonance having such a pair of associated poles is therefore not of molecular type.

Such pole pairs arise when a CDD pole [8] couples weakly to the channels of interest. We shall discuss this possibility below in the context of partial wave dispersion relations and, as we shall see, $M(k^2)$ is then of the form

$$M = (k^2 - k_0^2)/g^2 + \text{ rescattering corrections.}$$

For g very small, this results in a pair of resonance poles at approximately opposite positions ($k \to -k$) in the k plane.

Our task is to show how effective range expansions like (2) arise in various 'potential scattering' situations. Depending on context, 'potential scattering' will denote any iterative exchange mechanism whether described by a non-relativistic Schrödinger equation or by some suitable relativistic formalism (Bethe-Salpeter equation [9], dispersion relations [10] or whatever). The key requirement is that the dynamical features - resonances and bound states - 'arise' from the potential and disappear if it is switched off.

In the following sections we examine in turn non-relativistic potential scattering and a relativistic description using partial wave dispersion relations. In each case we first look at one channel, then at the realistic multi-channel situation.

3 Non-relativistic potential scattering

3.1 The single channel case

As a warm up, recall the familiar derivation for the single-channel case [3]. Scattering parameters are extracted from the corresponding radial wave function, f, defined as a that solution of the (reduced) radial Schrödinger equation

$$\frac{d^2f}{dr^2} + (U(r) + k^2)f = 0 \tag{3}$$

that vanishes at $r = 0$. Where necessary one can write the above general solution as $f\{U(r); k^2\}$ to distinguish it from the corresponding 'outer' solution of the free wave equation, $g \equiv f\{0; k^2\}$, and the counterparts of f and g for $k^2 = 0$, $\bar{f} \equiv f\{u(r); 0\}$ and $\bar{g} \equiv f\{0; 0\}$. The four functions $f, g, \bar{f}$ and $\bar{g}$ are needed for deriving the effective range formula. Normalizations are fixed by equating f to g outside the range, R, of the potential and by requiring $g(0) = 1$ allowing g to be written as

$$g(r) = (M/k) \sin kr + \cos kr. \tag{4}$$

Matching coefficients to the usual expression in terms of the phase shift, δ, yields the standard formula for the one-channel M-matrix and relates this to $g'(0)$:

$$M = k \cot \delta = g'(0). \tag{5}$$

As already noted M determines the scattering amplitude, T, via the equation

$$T = [M - ik]^{-1}.$$

Elementary manipulation of the respective Schrödinger equations for $f, \bar{f}, g$ and $\bar{g}$ yields

$$\frac{d}{dr}[f\bar{f}' - \bar{f}f'] = k^2 f\bar{f}; \quad \frac{d}{dr}[g\bar{g}' - \bar{g}g'] = k^2 g\bar{g} \tag{6}$$

whence by integration

$$\left[(g\bar{g}' - \bar{g}g') - (f\bar{f}' - \bar{f}f')\right]_0^\infty = k^2 \int_0^\infty [g\bar{g} - f\bar{f}]dr. \tag{7}$$

Since the respective f's and g's are equal for $r > R$, the bi-linear expression on the left

$$W(r) \equiv (g\bar{g}' - \bar{g}g') - (f\bar{f}' - \bar{f}f') \tag{8}$$

only receives non-vanishing contributions from the lower limit and the right hand integral only has support from 0 to R. Inserting values for $g(0)$ etc. from (4,5), one thus obtains the **exact relation**

$$-W(0) = M(k^2) - M(0) = k^2 \int_0^R [g\bar{g} - f\bar{f}]dr \tag{9}$$

and the **effective range expansion**

$$\begin{aligned} M(k^2) &= M(0) + k^2 \int_0^R [\bar{g}^2 - \bar{f}^2]dr + 0(k^4) \\ &\equiv M(0) + \frac{1}{2}k^2 r_{eff} + 0(k^4) \end{aligned} \tag{10}$$

with

$$r_{eff} = 2 \int_0^R [\bar{g}^2 - \bar{f}^2]dr. \tag{11}$$

The integrand in (11) takes the values 1 and 0 at its respective end-points; for a weakly bound state and for a scattering potential that is not unduly complicated, it should approximate an inverted S-shaped form form like $\cos^2(\pi r/2R)$ (as for a square-well potential in the limit of very weak binding). Its average is therefore $\approx \frac{1}{2}$ yielding

$$r_{eff} \simeq R. \tag{12}$$

For more general types of potential, (11) **defines** r_{eff}; the above argument shows this to be an appropriate definition.

3.2 The two-channel case

Having demonstrated the machinery, we now move closer to our real problem by allowing a pair of coupled channels, for which however we still adopt a non-relativistic description. Unlike formal nuclear reaction theory we need to specify the inter-channel conversion process in sufficient detail to encompass the concept of range. In place of (3) we thus assume the two-channel reaction to be governed by a two-component radial Schrödinger equation

$$\begin{aligned} \frac{d^2 f_1}{dr^2} + (U_{11} + k_1^2) f_1 + U_{12} f_2 &= 0 \\ \frac{d^2 f_2}{dr^2} + U_{21} f_1 + (U_{22} + k_2^2) f_2 &= 0 \end{aligned} \tag{13}$$

with $f_1(f_2)$ the wave function for channel 1(2) here written as a function of the same inter-particle distance, r.[2] f_1 and f_2 are required to vanish at $r = 0$ and the relative normalizations are chosen such that $U_{21} = U_{12}$ yields $M_{21} = M_{12}$ as dictated by time reversal. The CM momenta of the two channels are related by conservation of energy

$$k_1^2/\mu_1 - k_2^2/\mu_2 = \Delta M \tag{14}$$

with ΔM (by convention taken to be positive) the total rest mass difference and μ_1, μ_2 the respective reduced masses.

Scattering is now described by a (2×2) scattering matrix, T_{ij}, whose unitarity properties allow an $\underset{\sim}{M}$ matrix parameterization

$$\underset{\sim}{T} = [\underset{\sim}{M} - i\,\underset{\sim}{k}]^{-1} \tag{15}$$

[2] We could consider a non-local form for U_{12} and U_{21} with distinct relative coordinates r_1 and r_2 for each channel. The interaction (or conversion) terms would then be of the form $\int dr_2 U_{12}(r_1; r_2) f_2(r_2)$ and likewise for the term in U_{21}. The symmetry requirement from time reversal would then become $U_{21}(r_2; r_1) = U_{12}(r_1; r_2)$.

with M_{ij} real and symmetric and $\underset{\sim}{k}$ the diagonal matrix of CM channel momenta, k_i.

Proceeding as for the one-channel case we focus on a particular T-matrix element, here T_{22}, which we write as

$$T_{22} = [X_{22} - ik_2]^{-1}. \tag{16}$$

From (15) one easily derives

$$X_{22} = M_{22} - M_{12}^2/(M_{11} - ik_1). \tag{17}$$

Our objective is now to derive an effective range expansion for X_{22} as a function of k_2^2. For this we have to delve a little into the solution structure of the coupled Schrödinger equations (13).

As before we manipulate full solutions and the associated 'outer' solutions at non- zero k_2^2 and their counterparts at $k_2^2 = 0$ ($k_1^2 = \bar{k}_1^2 = \mu_1 \Delta M$ (14)). Each is a 2-component object and is conveniently written in vector form:

$$\begin{aligned}
\underset{\sim}{f} &\equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \underset{\sim}{f}\{\underset{\sim}{U}; \underset{\sim}{k}^2\} \\
\underset{\sim}{g} &\equiv \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \equiv \underset{\sim}{f}\{\underset{\sim}{0}; \underset{\sim}{k}^2\} \quad (\underset{\sim}{g} = \underset{\sim}{f} \text{ for } r \geq R) \\
\underset{\sim}{\bar{f}} &\equiv \underset{\sim}{f}(k_2^2 = 0); \quad \underset{\sim}{\bar{g}} \equiv g(k_2^2 = 0).
\end{aligned} \tag{18}$$

There are two linearly independent $\underset{\sim}{g}$'s for which we select

$$\overset{(1)}{\underset{\sim}{g}} \equiv \begin{bmatrix} g_1^{(1)} \\ g_2^{(1)} \end{bmatrix} = \begin{bmatrix} (M_{11}/k_1)\sin k_1 r + \cos k_1 r \\ (M_{12}/k_2)\sin k_2 r \end{bmatrix} \tag{19}$$

and $\overset{(2)}{\underset{\sim}{g}}$ obtained from $\overset{(1)}{\underset{\sim}{g}}$ by interchanging channels 1 and 2. We also use the linear combination of these with no ingoing wave in channel 1:

$$\overset{(3)}{\underset{\sim}{g}} \equiv \overset{(2)}{\underset{\sim}{g}} - (M_{21}/(M_{11} - ik_1)) \overset{(1)}{\underset{\sim}{g}}. \tag{20}$$

Values and derivatives of $g_j^{(k)}(r)$ at $r = 0$ are collected in Table 1.

We derive analogues of (6) and (7) from various combinations of the above solutions:

$$-\frac{d}{dr}[W^{(i,j)}(r)] = (k_1^2 - \bar{k}_1^2)F_1^{(i,j)} + k_2^2 F_2^{(i,j)} \tag{21}$$

with

$$W^{(i,j)}(r) \equiv \frac{1}{2}\left\{\sum_{k=1}^{2}[g_k^{(i)}\bar{g}_k'^{(j)} - \bar{g}_k^{(j)}g_k'^{(i)}] + i \leftrightarrow j \right.$$
$$\left. - \text{corresponding expression in the } f's\right\}. \tag{22}$$

The $F_k^{(i,j)}$ denote the same combination without derivatives (cf (8)).

Integrating with respect to r and expanding in powers of k_2^2 (using $k_1^2 - \bar{k}_1^2 = (\mu_1/\mu_2)k_2^2 + 0(k_2^4)$ (cf (14)) yields

$$W^{(i,j)}(0) = (1/2)R^{(i,j)}k_2^2 + 0(k_2^4), \tag{23}$$

with the coefficients, $R^{(i,j)}$, given by

$$R^{(i,j)} = 2(\mu_1/\mu_2)\int_0^R dr\bar{F}_1^{(i,j)} + 2\int_0^R dr\bar{F}_2^{(i,j)}, \tag{24}$$

where

$$\bar{F}_k^{(i,j)} = \bar{g}_k^{(i)}\bar{g}_k^{(j)} - \bar{f}_k^{(i)}\bar{f}_k^{(j)}. \tag{25}$$

The $W^{(i,j)}$'s that we shall use are those with (i,j) equal to (1,1), (2,2) and (2,3). For all of these, the integrals (24) specifying the corresponding $R^{(i,j)}$'s either have $F_1^{(i,j)}(0) = F_1^{(i,j)}(R) = 0$ with $F_2^{(i,j)}$ going from 1 to 0 or vice versa with 1 and 2 interchanged. All the $R^{(i,j)}$'s are therefore $\simeq R$ by the previous argument (preceding (12)).

Inserting values from Table 1 into (22) and (23) we obtain

$$\begin{aligned}
W^{(1,1)}(0) &= M_{11} - \bar{M}_{11} = \frac{1}{2}R^{(1,1)}k_2^2 + 0(k_2^4), \\
W^{(2,2)}(0) &= M_{22} - \bar{M}_{22} = \frac{1}{2}R^{(2,2)}k_2^2 + 0(k_2^4), \\
W^{(2,3)}(0) &= \frac{1}{2}[X_{22} - \bar{X}_{22} + M_{22} - \bar{M}_{22} + \Delta^{(2,3)}] \\
&= \frac{1}{2}R^{(2,3)}k_2^2 + 0(k_2^4).
\end{aligned} \tag{26}$$

One can show that the residue, $\Delta^{(2,3)}$, in the last of the above formulae is negligible for the presently relevant situation where $Im\bar{X}_{22}$ is very small[3].

[3]To see this, one can manoeuvre $\Delta^{(2,3)}$ into the form

$$\Delta^{(2,3)} = -[Im\bar{X}_{22}/\bar{k}_1][(\bar{M}_{11} + i\bar{k}_1)/(\bar{M}_{11} - i\bar{k}_1)][M_{11} - \bar{M}_{11} - i(k_1 - \bar{k}_1)]$$

and then use the effective range expansion for M_{11}.

Dropping $\Delta^{(2,3)}$ we immediately derive the effective range formula for X_{22} in the form

$$X_{22} - \bar{X}_{22} = (1/2) r_{eff}^{(X)} k_2^2 + 0(k_2^4) \tag{27}$$

with

$$r_{eff}^{(X)} = 2R^{(2,3)} - R^{(2,2)} \simeq R. \tag{28}$$

Despite all the complications from having an extra channel the same effective range prescription applies as for a single channel.

Thus far the discussion has been confined to non-relativistic Schrödinger scattering; the next step is to remove this restriction.

4 Potential scattering and CDD poles in the language of dispersion relations

We now examine our problem in the context of partial wave dispersion relations [10]. These provide a formulation of scattering dynamics that captures much more of the structure of field theory with relativistic kinematics and inter-particle forces arising from 't' and 'u' channel exchanges. The central idea is that scattering amplitudes should be determined by their discontinuities or cuts. The incursion of extra resonances and bound states not arising from the explicit inter-particle forces is signalled by the occurrence of CDD poles [8]. Although all this has long been text-book material, we need to outline some key aspects to motivate and establish our rule. As before, we start with the simpler one-channel case.

4.1 Single channel

We begin by considering the standard 'potential scattering' situation with prescribed exchange forces and no CDD poles[4]. The effect of these is discussed in the following section (4.2). The partial wave amplitude, $T(\nu), (\nu \equiv k^2/m^2)$, extended to complex ν forms a 'real' analytic function in the cut ν-plane[5]. Its discontinuities comprise a left-hand cut due to the exchange forces

$$T(\nu + i\epsilon) - T(\nu - i\varepsilon) = 2i Im T_L(\nu) \quad (-\infty < \nu < \nu_L), \tag{29}$$

[4]The bootstrap programme of the early 1960's [10] attributed all hadrons (including hadron resonances) to this type of mechanism with the same hadrons collectively also providing the exchange forces.

[5]We explicitly consider the case of scattering between particles of equal mass (denoted by m). For the general case of unequal masses a number of details require modification but the end result is the same.

and a right-hand cut from unitarity

$$T^{-1}(\nu + i\epsilon) - T^{-1}(\nu - i\varepsilon) = -2i\rho(\nu) \quad (0 < \nu < \infty). \tag{30}$$

$\rho(\nu)$ denotes the density of final states factor, $2k/E$, which in terms of the above ν is given by

$$\rho(\nu) = [\nu/(\nu+1)]^{1/2}. \tag{31}$$

Together with certain requirements at infinity, these properties specify $T(\nu)$ and this is the basis of dispersion dynamics.

The standard way to solve (29) and (30) is by the 'N/D' method [11] whereby $T(\nu)$ is expressed as a ratio $T(\nu) = N(\nu)/D(\nu)$ with the numerator, N, bearing the left- hand cuts and the denominator, D, the right. The ensuing equations can be cast into the form

$$T^{-1}(\nu) = N^{-1}(\nu) - \frac{(\nu - \nu_0)}{\pi N(\nu)} \int_0^\infty \frac{d\nu' N(\nu')\rho(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}. \tag{32}$$

The function $N(\nu)$ is given by

$$N(\nu) = B(\nu) + \frac{1}{\pi} \int_0^\infty \frac{d\nu' N(\nu')\rho(\nu')}{(\nu' - \nu)} [B(\nu') - (\frac{(\nu - \nu_0)}{\nu' - \nu_0})B(\nu)], \tag{33}$$

where $B(\nu)$ denotes the 'Born' amplitude corresponding to the specified exchange processes,

$$B(\nu) = \frac{1}{\pi} \int_{-\infty}^{\nu_L} d\nu' \frac{Im\, T_L(\nu')}{\nu' - \nu}. \tag{34}$$

ν_0 is an arbitrary subtraction point at which $T = N$ is enforced.

Equations (32) and (33) express the concept of 'potential scattering' in an S- matrix framework. One pictures the scattering process as originating in the Born amplitude, B; iteration through equations (33) and (32) transform B into T, complete with resonances or bound states if B is sufficiently attractive.

We again need to develop an effective range expansion. Writing

$$T^{-1}(\nu) \equiv \tilde{M} - i\theta(\nu)\rho(\nu) \equiv M - i\rho(\nu), \tag{35}$$

we can evolve a formula for M using (32)[6]:

$$\begin{aligned} M(\nu) &= N^{-1}(\nu) + \tilde{R}(\nu) - \tilde{R}(-\nu_N) - \frac{(\nu + \nu_N)}{\pi} \int_0^\infty \frac{(d\nu'[N(\nu')/N(\nu) - 1]\rho(\nu')}{(\nu' + \nu_N)(\nu' - \nu)} \\ &+ i\theta(-\nu)\rho(\nu) \\ &\equiv N^{-1}(\nu) + R(\nu) - R(-\nu_N) + J(\nu;\nu_N) + i\rho(-\nu_N)\theta(\nu_N). \end{aligned} \tag{36}$$

[6]Note the change of dimensions. M now denotes the quantity $\rho \cot\delta$ with $\rho = 2k/E$ and r_{eff} is $\frac{2}{m}\frac{d}{d\nu}M(\nu)\,|_{\nu=0}$.

Several points in (36) require explanation: firstly, the appearance of $\tilde{R}(\nu)$ and its variant $R(\nu)$ which are given by

$$\tilde{R}(\nu) \equiv -\frac{\nu}{\pi} P \int_0^\infty \frac{d\nu' \rho(\nu')}{(\upsilon - \nu)\nu'} \equiv R(\nu) - i\theta(-\nu)\rho(\nu). \tag{37}$$

$\tilde{R}$ and R are the 'dispersive' counterparts of $-i\rho(\nu)$ that commonly appear in inverse scattering formulae; the shift in (35) from $i\theta(\nu)\rho(\nu)$ to plain $i\rho$ without the θ functions is to secure cusp-free forms for M and R, eg

$$R(\nu) = \frac{2}{\pi}\nu + 0(\nu^2). \tag{38}$$

A further point to note regarding (36) is that we have set the subtraction point, ν_0, to the value $-\nu_N$ appearing in the low-energy expansion of N^{-1},

$$N^{-1}(\nu) = (\nu + \nu_N)/\lambda_N + 0(\nu^2). \tag{39}$$

We can relate ν_N (via the analogous ν_B for B^{-1}) to the range of the exchange forces[7]; for exchange of a particle of mass μ

$$\nu_N > \nu_B = \mu^2/2m^2. \tag{40}$$

The integral in (36) (denoted J in the second line) appears in the subtracted form shown because we have extracted the main contribution into the explicit term $R(\nu) - \tilde{R}(-\nu_N)$.

All that remains is to make a linear expansion for $M(\nu)$ and derive r_{eff}. Since we are pre-supposing $M(0)$ to be very small, we can greatly simplify the algebra by forming the combination

$$\frac{2}{m}[\frac{d}{d\nu}M(\nu)\,|_{\nu=0} - (1/\nu_N)M(o)] \simeq r_{eff} \quad (\text{for } M(o) \ll 1). \tag{41}$$

The utility of this construct is that most of the individual contributions to it from (36) vanish in linear approximation leaving just the final contribution from $+i\rho(-\nu_N)$. For ($\mu \ll m$) we infer

$$r_{eff} \simeq 2\sqrt{2}/\mu \tag{42}$$

and verify that r_{eff} is once again bounded by the range of the forces.

[7] $N(\nu)$ has an integral representation like (34) but with an extra factor of $D(\nu')$ in the integrand. For simple, attractive, exchange forces we expect $|D(\nu')|$ to decrease towards $\nu' = \nu_L$ leading to $\nu_N > \nu_B$. For exchange of a particle of mass μ, $B(\nu)$ takes the form $B(\nu)$= (const./ν) $\ln[1 + 4m^2\nu/\mu^2] \simeq$ const.$'/(\nu + \mu^2/2m^2)$ for small ν.

4.2 Modification when a CDD pole couples

The above discussion is for pure potential scattering. CDD poles [8] generalize this by adding extra parameters – two for each CDD pole – corresponding to the mass and coupling constant of each extra particle that couples to the scattering system. Such a possibility is clearly needed, for example to allow for states that merely decay into the external scattering channels but whose composition is quite distinct (eg K^0_S in $\pi^+\pi^-$ scattering).

There are various equivalent ways of expressing the modifications to the N/D equations that a CDD pole induces [12]. Most convenient for the present purpose is to require $ReT^{-1}(\nu)$ to pass through zero with a given slope at a prescribed point $\nu = \nu_P$. Choosing this as the subtraction point and requiring $N(\nu_P) = 1$ leads to a modified version of (33) for $N(\nu)$,

$$N(\nu) = 1 - \frac{(\nu - \nu_P)}{g^2} B(\nu) + \frac{(\nu - \nu_P)}{\pi} \int_0^\infty \frac{d\nu' \rho(\nu') N(\nu') [B(\nu') - B(\nu)]}{(\nu' - \nu_P)(\nu' - \nu)}, \quad (43)$$

and of (36) for $M(\nu)$,

$$\begin{aligned} M(\nu) &= \frac{\nu_P - \nu}{g^2} N^{-1}(\nu) + R(\nu) - \tilde{R}(\nu_P) - \frac{(\nu - \nu_P)}{\pi} \int_0^\infty \frac{d\nu' [N(\nu')/N(\nu) - 1] \rho(\nu')}{(\nu' - \nu)(\nu' - \nu_P)} \\ &= \frac{\nu_P - \nu}{g^2} + \text{ rescattering corrections.} \end{aligned} \quad (44)$$

For a weakly coupled resonance with g^2 small M can have an arbitrary large slope as a function of ν and a bound state near threshold will correspond to two poles with $\sqrt{\nu} \simeq \pm\sqrt{\nu_P}$.

4.3 Two-channel case

As a final step we extend the dispersion discussion to a multi-channel situation, explicitly to the case of two channels. We now have a matrix, $\underset{\sim}{T}$, of scattering amplitudes for which we adopt a matrix $\underset{\sim}{N}\ \underset{\sim}{D}^{-1}$ representation,

$$\underset{\sim}{T} = \underset{\sim}{N}\ \underset{\sim}{D}^{-1}. \quad (45)$$

We again impose the known left and right discontinuities from exchange forces and unitarity. Proceeding as before we derive, in the absence of CDD poles, an equation like (33) relating $\underset{\sim}{N}$ to the corresponding Born matrix, $\underset{\sim}{B}$, and an expression for the D_{ij} elements

$$D_{ij}(s) = \delta_{ij} - \frac{(s - s_0)}{\pi} \int_{s_i}^\infty ds' \frac{\rho_i(s') N_{ij}(s')}{(s' - s_0)(s' - s)}. \quad (46)$$

(We have switched to $s = E^2$ as dispersion variable; s_i is the corresponding i channel threshold and $\rho_i = [1 - s_i/s]^{1/2}$, the associated phase space density.)

Specializing to the two-channel case, we generate a formula for the amplitude T_{22} (in fact for its inverse) as follows:

$$T_{22}^{-1} = \frac{D_{11}D_{22} - D_{12}D_{21}}{D_{11}N_{22} - D_{12}N_{21}} \equiv \frac{\tilde{D}_{22}(s)}{\tilde{N}_{22}(s)} \tag{47}$$

where

$$\tilde{N}_{22}(s) = N_{22}(s) - D_{12}(s)N_{21}(s)/D_{11}(s), \tag{48}$$

and the associated $\tilde{D}_{22}$ is given by

$$\tilde{D}_{22}(s) = 1 - \frac{(s - s_0)}{\pi}\int_{s_2}^{\infty} \frac{ds'\rho_2(s')}{(s' - s_0)(s' - s)}[N_{22}(s') - D_{12}(s)N_{21}(s')/D_{11}(s)] \tag{49}$$

Proceeding as for the one channel case (cf (36)) this can be cast into the form

$$\begin{aligned}\tilde{D}_{22}(s) = 1 - \frac{(s - s_0)}{\pi}\int_{s_2}^{\infty} \frac{ds'\rho_2(s')}{(s' - s_0)(s' - s)}\{&\tilde{N}_{22}(s') - \tilde{N}_{22}(s) \\ &+(\frac{D_{12}(s')}{D_{11}(s')} - \frac{D_{12}(s)}{D_{11}(s)})N_{21}(s')\} \\ &+\tilde{N}_{22}(s)[R_2(s) - i\rho_2(s) - \tilde{R}_2(s_0)]\end{aligned} \tag{50}$$

where R_2 and $\tilde{R}_2$ are the counterparts of the previous R and $\tilde{R}$ (37) with ρ_2 for ρ. We now imitate the one-channel discussion (following (38)) writing

$$\tilde{N}_{22}^{-1}(s) = (s + s_N)/\lambda_N + 0((s - s_2)^2), \tag{51}$$

and taking the subtraction constant s_0 as $-s_N$. Finally, using the analogous construction to (41) we conclude that (42) again applies[8].

In contrast, for the CDD pole case, we will have

$$D_{ij} = (s_P - s)\delta_{ij} - \frac{(s - s_P)}{\pi}\int_{s_i}^{\infty} \frac{ds'\rho_i(s')N_{ij}(s')}{(s' - s_P)(s' - s)} \tag{52}$$

$$N_{ij} = g_i g_j + \frac{(s - s_P)}{\pi}\sum_k \int_{-\infty}^{s_L} \frac{ds' Im\, T_{ik}^L(s')D_{kj}(s')}{(s' - s_P)(s' - s)} \tag{53}$$

leading to the possibility of arbitrarily rapid variation of $\underset{\sim}{M}$ from a weakly coupled CDD pole.

[8]To reach this result, we have to assume somewhat more than for the one-channel case.

5 Summary and conclusions

The various sections of this paper have examined effective range expansions

$$M = M^{(0)} + \frac{1}{2} r_{eff} k^2 + 0(k^4) \tag{54}$$

for different formulations of single and multi-channel 'potential scattering'. In each case, the above quadratic coefficient, r_{eff}, is related to the range of the potential, $|r_{eff}| \approx R$. Guided by these examples, we conjecture that this can be extended into a general rule for potential scattering in all physically realistic situations. This has the effect that a 'potential' or 'molecular' resonance very close to a threshold corresponds to a single resonance pole. When a resonance corresponds to a pair of poles which both lie close to threshold, the resonance is non-molecular and must be attributed to dynamics in the hidden channels.

An application of current interest (from which the present investigation stemmed) is to the question of whether $f_0(976)$ (alias S^*) is a $K\bar{K}$ molecule as suggested by Weinstein and Isgur [7], or is built from confined particles. Answering this along the present lines, means going to the relevant data to find what and how many resonance poles are indicated. Such an investigation has been undertaken by Michael Pennington and the present author [2,6]. The outcome is that two pole descriptions (nearby resonance poles on sheets II and III) are distinctly preferred to those with just a single pole. Using the above rule[9], one infers that $f_0(976)$ is most likely not a molecule. This finding hinges crucially on the input data (which is not without its inconsistencies) being correct in essentials. Time and future experiments will tell.

Two other S-wave resonances which invite a similar treatment in terms of their pole structure are the baryon, $\Lambda(1405)$ and the $I = 1$ meson $a_0(980)$ (alias δ). Investigation of $\Lambda(1405)$ (by methods similar in spirit to those reported here) has a long history [4,13]. A comparable study of $a_0(980)$ (closely parallel to $f_0(S^*)$ in possessing alternative $K\bar{K}$ molecule and $(q\bar{q})$ assignments) has thus far been hampered by the quality of the available data.

[9]Typical pole positions in the preferred two-pole fits have $|k| \leq 0.13$ GeV. The range, R, used in Weinstein and Isgur's $K\bar{K}$ potential calculations ([10] - last reference) is $0.8fm$ yielding $(\hbar c)R^{-1}$ =0.25 GeV.

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k	$g_1^{(k)}(0)$	$g_1^{(k)'}(0)$	$g_2^{(k)}(0)$	$g_2^{(k)'}(0)$
1	1	M_{11}	0	M_{12}
2	0	M_{21}	1	M_{22}
3	$-M_{12}/(M_{11}-ik_1)$	$-ik_1M_{12}/(M_{11}-ik_1)$	1	$M_{22}-M_{12}^2/(M_{11}-ik_1)$ $\equiv X_2$

Table 1: Characteristic properties at $r = 0$ of solutions $g^{(k)}(k = 1, 3)$ (cf. eqs (19) and (20)).