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Dual Yang-Mills Theory — the Quantum Theory of Nonabelian Monopoles

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Abstract

Starting from the definition of monopoles as topological obstructions, one can formulate a universal principle for deriving monopole interactions in any gauge theory whether in classical or quantum mechanics. In this paper, using loop space techniques developed earlier for the classical monopole, equations of motion are derived also for quantum monopoles in nonabelian gauge fields. An additional gauge symmetry appears as a degeneracy in solving the Euler-Lagrange problem, so that starting say from an $SU(N)$ symmetry, a 'parity doubling' of the symmetry to $SU(N) \times SU(N)$ is obtained. A beginning is made in exploring the resultant dynamics.

1 Introduction

Through the work of Dirac[1], Lubkin[2], Wu-Yang[3], Coleman[4] and others, it is now well-known that monopoles occur "naturally" as topological obstructions in many gauge theories with compact gauge groups. However, it is still perhaps not widely recognised that the dynamics of monopoles is then uniquely determined as a consequence of the topology. Intuitively, that this is the case can be seen as follows. The assertion that the particle carries a monopole charge is synonymous with the statement that the gauge field has a certain topological configuration in the spatial region surrounding that particle. If the particle moves, therefore, the field around it will have to rearrange itself so as to maintain a configuration consistent with the particle at the new position still carrying the same monopole charge since this charge, being topological, is by continuity conserved. Hence, it follows that the field must be coupled to the monopole position in some specific manner, or that there is an intrinsic interaction between them.

The problem may be formulated more precisely as follows. Suppose we take the (free) action of a field-particle system as:

$$\mathcal{A}^0 = \mathcal{A}_F^0 + \mathcal{A}_M^0, \tag{1}$$

where $\mathcal{A}_F^0$ depends only on the field variables and $\mathcal{A}_M^0$ only on the variables describing the particle, and stipulate further that the particle carries a monopole charge, which according to the argument above means that there is a constraint relating the field and particle variables. If we now extremise (1) under this constraint defining the monopole charge, the equations of motion will no longer be free equations of the field and particle separately, but coupled equations with interactions between the particle and the field.

Now, this is not the conventional manner in which interactions between fields and particles are formulated. Usually, one introduces into the action in addition to (1) an interaction term depending on both the field and particle variables, thus:

$$\mathcal{A} = \mathcal{A}_F^0 + \mathcal{A}_M^0 + \mathcal{A}_I, \tag{2}$$

so that in extremising the full action $\mathcal{A}$ with respect to the dynamical variables, one obtains explicitly through $\mathcal{A}_I$ coupled equations containing interactions. This conventional formulation involves, first, grafting a new concept of interaction on to the original concept of a free action, and, second, introducing a degree of arbitrariness into the problem through the choice of the interaction term $\mathcal{A}_I$.

Of the two formulations, that of monopole interactions outlined above would seem to be much the more elegant, following, as it did, directly and uniquely from the definition of the monopole charge as a topological obstruction and, in the memorable words of Dirac, "one would be surprised if Nature had made no use of it."[1] In order to answer the question whether Nature has actually availed herself of this possibility, we have first of course to ascertain what sorts of interactions are implied by the above procedure for monopoles, so that we may check the result against experiment. This involves solving a variational problem which is not entirely straightforward since the gauge potential A_μ usually used to describe the field has to be patched in the presence of monopoles, with the patching conditions depending on the monopoles' positions, so that a direct attack using A_μ as variables becomes rather intractable. For this reason, some new tactics will first have to be devised to bypass this difficulty.

The problem for the classical monopole was first solved for the abelian theory in 1976[5], by an ingenious indirect method, and again directly in 1986[6]. The interactions of a monopole were found here to be exactly the dual to those of a classical source of the field, namely those described by the Maxwell and Lorentz equations. Hence, the abelian theory being dual symmetric in the sense that both the Maxwell field and its dual:

$$ {}^*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{3} $$

are gauge fields each derivable, at least locally, from a potential:

$$ F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \tag{4} $$

$$ {}^*F_{\mu\nu} = \partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu, \tag{5} $$

and that a monopole of $F_{\mu\nu}$ (${}^*F_{\mu\nu}$) can equally be considered as a source in the dual field ${}^*F_{\mu\nu}(F_{\mu\nu})$, the above result can be regarded as just an alternative, yet apparently inequivalent, derivation of the standard electromagnetic interactions conventionally formulated in terms of an interaction term in (2) of the form:

$$ \mathcal{A}_I = e\int A_\mu dY^\mu, \tag{6} $$

where Y is the world-path of the particle. In this case, therefore, one can claim, if one is so inclined, that monopole interactions as formulated above do occur in Nature.

The result for the corresponding problem for the classical monopole in a nonabelian Yang-Mills field, solved in 1986 using loop space techniques, is more intriguing. The theory has then ostensibly no dual symmetry in that given a Yang-Mills field $F_{\mu\nu}$ derivable from a gauge potential A_μ:

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu], \tag{7}$$

the dual field defined as in (3) is *not* in general a gauge field derivable from a potential; namely, no $\bar{A}_\mu$ need exist for which:

$$^*F_{\mu\nu} = \partial_\nu \bar{A}_\mu - \partial_\mu \bar{A}_\nu + ig[\bar{A}_\mu, \bar{A}_\nu]. \tag{8}$$

Furthermore, in contrast to the abelian case, a monopole in a nonabelian theory, defined as a topological obstruction in the same way, bears no apparent resemblance to a source in the gauge field. Indeed, even their charges are different, with the charges of a monopole being labelled by homotopy classes of the gauge group and those of a source by representations of the same. Nevertheless, the dynamics for the monopole as deduced from the constrained action principle outlined above is found to be still *almost dual* to that of a classical Yang-Mills source as contained in the so-called Wong equations, which are obtained as the classical limit of the standard Yang-Mills equations. However, this result, though intriguing, is not of much use in answering the practical question of whether monopole interactions actually occur in Nature, since Yang-Mills theory is almost never applied to classical particles in practice. We are therefore pushed further in this paper to investigate the same problem for quantum particles.

To warm up, we shall solve the quantum problem again first for the abelian theory. As in the classical case, we shall find that the equations governing the dynamics of a monopole, as deduced from (1) under the constraint defining the monopole charge, are exactly the dual of the Maxwell equation:

$$\partial_\nu F^{\mu\nu} = -4\pi e \bar{\psi}\gamma^\mu\psi, \tag{9}$$

and the Dirac equation:

$$(i\partial_\mu\gamma^\mu - m)\psi = eA_\mu\gamma^\mu\psi, \tag{10}$$

governing the dynamics of a source. In other words, the monopole equations turn out to be just those obtained by replacing in (9) and (10) the source charge e by the monopole charge $\tilde{e}$, $F_{\mu\nu}$ by its dual $^*F_{\mu\nu}$, A_μ by $\tilde{A}_\mu$ as defined in (5), and the source wave function ψ by the wave function $\tilde{\psi}$ describing the monopole. Bearing in mind then that the system is *dual symmetric* in the sense detailed above, one concludes that, in the quantum theory also, the constrained action principle for abelian monopoles leads to interactions which occur in Nature, since it may be regarded as just an alternative to the conventional approach of defining electromagnetic interactions through an interaction term:

$$\mathcal{A}_I = e \int d^4x \bar{\psi} A_\mu \gamma^\mu \psi, \tag{11}$$

in the action (2). The new approach, however, has the novel feature of giving the result uniquely as a consequence of the definition of the monopole charge without relying on an equivalent of the so-called *minimal coupling* hypothesis required in the conventional approach for choosing the interaction term (11).

There is another notable feature in the result. In extremising the free action (1) under the constraint defining the monopole charge, the system acquires, as a degeneracy in the solution of the Euler-Lagrange problem, and in addition to the original gauge symmetry under the transformations:

$$\psi \longrightarrow (1 + ie\Lambda)\psi, \quad A_\mu \longrightarrow A_\mu - \partial_\mu \Lambda, \tag{12}$$

a further local gauge symmetry represented by the transformations:

$$\tilde{\psi} \longrightarrow (1 + i\tilde{e}\tilde{\Lambda})\tilde{\psi}, \quad \tilde{A}_\mu \longrightarrow \tilde{A}_\mu - \partial_\mu \tilde{\Lambda}. \tag{13}$$

The full symmetry is thus extended to a local $U(1) \times \tilde{U}(1)$ where $\tilde{U}(1)$ represents a second $U(1)$ symmetry carrying a parity opposite to that of the first. This phenomenon which we shall have occasion to discuss further will be referred to in the future as a *chiral doubling* of the symmetry.

The abelian theory being dual symmetric, the above solution of the Euler-Lagrange problem was not technically difficult, nor was the result obtained entirely unexpected. For the nonabelian theory, however, the lack of a dual symmetry made both the answer to the analogous questions much more intriguing and the steps leading to it technically much more complicated. As in the solution to the classical problem mentioned above, because of the patching complication in the gauge potential $A_\mu(x)$ as well as in the field tensor $F_{\mu\nu}(x)$, one is led to reformulate the problem in loop space, which being ∞-dimensional and therefore highly redundant in its description, requires some nontrivial development in technique[7]. The results we have obtained in this paper on the nonabelian theory are therefore not so easily summarised. We can state nevertheless that the equations of motion governing the dynamics of quantum nonabelian monopoles are again uniquely derived from the same principles as before, though now in loop space. Under some assumptions, these loop space equations reduce to an almost local form in which the monopole field $\tilde{\psi}(x)$ is coupled to to a vector field $\tilde{A}_\mu(x)$ in exactly the same way that an ordinary Yang-Mills field $\psi(x)$ is coupled to the ordinary Yang-Mills potential $A_\mu(x)$. However, this $\tilde{A}_\mu(x)$ is not the $\tilde{A}_\mu(x)$ in (8) since it has a relationship to the dual field ${}^*F_{\mu\nu}$ which is different to the standard Yang-Mills relation (7) between the potential and the field. Furthermore, it has been shown that as in the abelian theory, a *chiral doubling* emerges of the local gauge symmetry arising from a degeneracy in the solution of the Euler-Lagrange problem. Thus, starting with a local $SU(N)$ gauge symmetry, we shall arrive at a symmetry of $SU(N) \times \widetilde{SU(N)}$ where $\widetilde{SU(N)}$ represents a second $SU(N)$ symmetry carrying a parity opposite to that of the first. Statements of the results in detail, however, will have to be given in later sections after a fuller development of the technology.

In this paper, we shall concentrate on the development of the theory and the derivation of the equations of motion, the physical consequences of which, however, have not yet been fully investigated. For this reason, only a very preliminary discussion will be given in the last section on the question whether nonabelian monopoles actually occur or are likely to occur in Nature.

2 The Quantum Theory of Abelian Monopoles

To set the stage for an attack later on the target problem of monopoles in a general Yang-Mills theory, let us first investigate the specially simple abelian case.

We recall that for the classical monopole, we chose:

$$\mathcal{A}_F^0 = -\frac{1}{16\pi} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x), \tag{14}$$

with $F_{\mu\nu}$ satisfying (4), and

$$\mathcal{A}_M^0 = -m \int d\tau, \tag{15}$$

with

$$d\tau = \sqrt{dY^2}, \tag{16}$$

for the action in (1), which was to be extremised under the constraint:

$$\partial_\nu {}^*F^{\mu\nu}(x) = -4\pi \tilde{e} \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta^4(x - Y(\tau)), \tag{17}$$

the last being a differential version of the topological condition asserting that there is a monopole charge $\tilde{e}$ all along the particle world-path $Y(\tau)$. Indeed, interpreting the abelian field as the usual Maxwell field, and hence $\tilde{e}$ as a magnetic charge, the r.h.s. of (17) is then just the magnetic current.

Suppose now that our monopole is to be described instead by a wave function $\tilde{\psi}(x)$, then we would choose the free particle action as usual to be:

$$\mathcal{A}_M^0 = -\int d^4x \bar{\tilde{\psi}}(x)(i\partial_\mu \gamma^\mu - m)\tilde{\psi}(x). \tag{18}$$

Instead of the constraint (17), we propose the following:

$$\partial_\nu {}^*F^{\mu\nu}(x) = -4\pi \tilde{e} \bar{\tilde{\psi}}(x)\gamma^\mu \tilde{\psi}(x), \tag{19}$$

in which we have just substituted for the classical current in (17) the quantum current.

This last equation requires some care in its interpretation in that it is still supposed to be a topological condition defining the monopole charge, albeit in a differential form. Hence, if we were to integrate (19) over a finite volume, we would still expect to obtain, for the monopole charge contained in that volume, always some discrete quantised value, say $4\pi\tilde{e}n$ for n an integer, which it will not be in general if $\tilde{\psi}(x)$ is a continuous wave function. For this reason, we believe that a fully consistent interpretation will be obtained only when one regards $\tilde{\psi}(x)$ as a quantised field, so that the integral of the quantity on the right of (19) will give $4\pi\tilde{e}$ times the number of particles inside that volume, which will then be discrete for any volume. In this paper, however, we shall work only with the first quantised theory.

To find the equations of motion, we have now to extremise (1) under the constraint (19) with respect to the field variables $A_\mu(x)$ and the particle variables $\tilde{\psi}(x)$. As explained in the introduction, direct variations with respect to $A_\mu(x)$ are difficult because of patching. A useful tactic that we have learned in the classical case is to adopt instead of $A_\mu(x)$ as field variables the field tensor $F_{\mu\nu}(x)$, which, being gauge invariant, is also patch independent. The beauty in doing so is that, by virtue of the Poincaré lemma, the constraint (17) that we wish to impose ensures that, except at the monopole position, $F_{\mu\nu}$ is a gauge field derivable from a gauge potential as per (4), thus removing exactly the intrinsic redundancy in $F_{\mu\nu}(x)$ which would otherwise be inadmissible as alternative variables. We propose now to adopt the same tactic also in the quantum case.

At first sight, the efficacy of such a tactic may appear dubious. We know already from the Bohm-Aharonov experiment that $F_{\mu\nu}$ is inadequate to describe the electromagnetic interactions of a charged particle in quantum mechanics which require the explicit introduction of a gauge potential. It seems thus foolhardy to give up $A_\mu(x)$ in favour of $F_{\mu\nu}(x)$ as variables in trying to derive the laws of motion of a magnetic monopole which we expect to behave just as the dual of the electric charge. On closer examination, however, this is soon seen not to be the case. The wave function $\tilde{\psi}(x)$ describing the monopole, which though carrying a magnetic charge is electrically neutral, is actually invariant under the usual gauge transformations rotating the phase for which A_μ acts as the connection or parallel transport. What one needs, therefore, for describing the Bohm-Aharonov type of phase effects for $\tilde{\psi}(x)$ will not be A_μ but presumably some other potential $\tilde{A}_\mu$ bearing the same relation (5) to the dual field ${}^*F_{\mu\nu}$ as A_μ bears to the original Maxwell field $F_{\mu\nu}$. We are therefore not losing out in giving up $A_\mu(x)$ in favour of $F_{\mu\nu}(x)$ as variables - the only question is how, in solving the variational problem for the equations of motion, a dual potential will be recouped to take its place. As we shall see, this will occur in a natural manner in consequence of a degeneracy in the solution.

In terms of $F_{\mu\nu}(x)$ and $\tilde{\psi}(x)$ as variables, the variational problem then becomes very easy. Introduce for each component of the constraint (19) a Lagrange multiplier $\lambda_\mu(x)$, one forms the auxiliary action:

$$\mathcal{A}' = \mathcal{A}^0 + \int d^4x \lambda_\mu(x)\{\partial_\nu {}^*F^{\mu\nu}(x) + 4\pi\tilde{e}\bar{\tilde{\psi}}(x)\gamma^\mu\tilde{\psi}(x)\}, \tag{20}$$

which is to be extremised under all variations of $F_{\mu\nu}(x)$ and $\tilde{\psi}(x)$. Extremising $\mathcal{A}'$ w.r.t. $F_{\mu\nu}(x)$, we obtain:

$$F^{\mu\nu}(x) = -2\pi\epsilon^{\mu\nu\rho\sigma}[\partial_\sigma\lambda_\rho(x) - \partial_\rho\lambda_\sigma(x)], \tag{21}$$

and w.r.t. $\bar{\tilde{\psi}}(x)$:

$$(i\partial_\mu\gamma^\mu - m)\tilde{\psi}(x) = 4\pi\tilde{e}\lambda_\mu(x)\gamma^\mu\tilde{\psi}(x). \tag{22}$$

The first equation (21) says that:

$${}^*F_{\mu\nu}(x) = 4\pi[\partial_\nu\lambda_\mu(x) - \partial_\mu\lambda_\nu(x)], \tag{23}$$

or that ${}^*F_{\mu\nu}$ is a gauge field derivable as per (5) from a potential:

$$\tilde{A}_\mu(x) = 4\pi\lambda_\mu(x), \tag{24}$$

while the second (22) also reduces in terms of $\tilde{A}$ into a familiar form:

$$(i\partial_\mu \gamma^\mu - m)\tilde{\psi}(x) = \tilde{e}\tilde{A}_\mu \gamma^\mu \tilde{\psi}(x). \tag{25}$$

Indeed, as anticipated in the introduction, these equations are just exactly the dual of the standard Maxwell and Dirac equations for an electric charge moving in an electromagnetic field.

In spite of its simplicity, the above derivation has two very interesting features both of which have, as we shall see, parallels in the nonabelian case. First, although we have started by eliminating the gauge potential A_μ as variables, a new quantity $\tilde{A}_\mu$ has emerged in the form of the Lagrange multiplier λ_μ, which couples obligingly to the field and the particle wave function in exactly the manner that a gauge potential should. Second, although we have formulated the variational problem entirely in terms of $F_{\mu\nu}$ and $\tilde{\psi}$ both of which are invariant under the original gauge transformations (12), a new local gauge symmetry (13) has arisen by virtue of the degeneracy in the solution of the Euler-Lagrange problem. This new symmetry can be seen directly in the equations of motion (23) and (25), or else in the auxiliary action (20) as follows. Under the transformation (13), $\mathcal{A}'$ in (20) acquires three new terms, the first from the transformation of the wave function $\tilde{\psi}(x)$ in the particle action (18), which cancels with another coming from the constraint term due to the change in the Lagrange multiplier $\lambda_\mu(x)$ multiplied by the current. Finally, there is the term:

$$-\frac{1}{4\pi}\int d^4x \partial_\mu \tilde{\Lambda}(x) \partial_\nu {}^*F^{\mu\nu}(x), \tag{26}$$

coming again from the variation in $\lambda_\mu(x)$ in the constraint term. Integrating (26) by parts w.r.t. ∂_ν, one easily sees from the antisymmetry of ${}^*F_{\mu\nu}(x)$ in its indices that this term vanishes, implying thus the invariance of $\mathcal{A}'$ in total. As we shall see, an invariance of the constrained action will also be obtained in a quite analogous fashion in the nonabelian theory.

3 Formulation of the Nonabelian Theory

The first step is to write down explicitly the free action (1). For the field, the standard Yang-Mills action is:

$$\mathcal{A}_F^0 = -\frac{1}{16\pi}\int d^4x \, \mathrm{Tr}(F_{\mu\nu}(x)F^{\mu\nu}(x)) \tag{27}$$

which is considered as a functional of the gauge potential $A_\mu(x)$ through (7). However, in the presence of monopoles, $A_\mu(x)$ will have to be patched, and to avoid the difficult problem of variations with respect to such patched $A_\mu(x)$, we seek again to replace them as field variables by patch-independent quantities. The field tensor adopted for the abelian theory will no longer be suitable in the nonabelian case since $F_{\mu\nu}$ is here only gauge covariant, not gauge invariant, and hence still patch-dependent. For this reason, techniques were developed in which the problem was recast in a loop space formulation entirely in terms of patch-independent loop variables[7]. Since these techniques have already been applied with success to solve the problem for the classical nonabelian monopole[6], we are tempted to adopt them here also for the quantum case.

The tactic employs as field variables the quantities:

$$F_\mu(C|s) = \frac{i}{g}\Phi^{-1}(C)\frac{\delta}{\delta\xi^\mu(s)}\Phi(C), \tag{28}$$

where $\Phi(C)$ is the Dirac nonintegrable phase factor or Wilson loop. For each parametrised loop passing through a reference point $P_0 = \{\xi_0^\mu\}$:

$$C : \{\xi^\mu(s) : \quad s = 0 \rightarrow 2\pi, \quad \xi^\mu(0) = \xi^\mu(2\pi) = \xi_0^\mu\}, \tag{29}$$

we define a phase factor $\Phi(C)$ as:

$$\Phi(C) = P_s \exp ig \int_0^{2\pi} ds A_\mu(\xi(s))\frac{d\xi^\mu(s)}{ds}, \tag{30}$$

where P_s denotes ordering with respect to s, say from right to left, and $F_\mu(C|s)$, defined for each C and s, is its logarithmic loop derivative. Thus geometrically, $F_\mu(C|s)$ may be interpreted as a sort of loop space connection giving the change in the phase of $\Phi(C)$ in moving from point to neighbouring points in loop space. In terms of ordinary space-time variables, $F_\mu(C|s)$ takes the form:

$$F_\mu(C|s) = \Phi_C^{-1}(s,0)F_{\mu\nu}(\xi(s))\Phi_C(s,0)\frac{d\xi^\nu(s)}{ds}, \tag{31}$$

where by definition:

$$\Phi_C(s_2, s_1) = P_s \exp ig \int_{s_1}^{s_2} ds A_\mu(\xi(s))\frac{d\xi^\mu(s)}{ds}. \tag{32}$$

Using (31), the Yang-Mills action can then be written as:

$$\mathcal{A}_F^0 = -(4\pi\bar{N})^{-1}\int \delta C \int_0^{2\pi} ds \operatorname{Tr}(F_\mu(C|s)F^\mu(C|s))(d\xi(s)/ds)^{-2}, \tag{33}$$

where

$$\int \delta C ... = \int \prod_{s'} d^4\xi(s') ..., \tag{34}$$

and $\bar{N}$ is a normalisation constant:

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s' \neq s} d^4\xi(s'). \tag{35}$$

This is the form of $\mathcal{A}_F^0$ we need in terms of $F_\mu(C|s)$ as variables.

Next we turn to the free action of the particle carrying the monopole charge. Our experience with the abelian case suggests that we again represent the particle by a wave function $\tilde{\psi}(x)$ and specify the action again by (18). The actual form of the wave function, however, requires a closer scrutiny. In analogy to the abelian theory, $\tilde{\psi}(x)$, though carrying a monopole charge, is — say in chromodynamics — "colour neutral", and should not therefore transform under the usual "colour" gauge transformations. However, we have seen in the section above that as the theory develops, the wave function of the abelian monopole picks up a transformation with respect to a new gauge symmetry, behaving then as a charged representation of that symmetry. So, in a similar manner, we expect that in the nonabelian theory, the wave function of a monopole may also pick up a transformation with respect to a new gauge symmetry and behave as a representation of that symmetry, neither of which properties however are we yet ready to specify until we have first dealt with the constraint defining the charge of the monopole.

We turn now to that crucial constraint under which the free action (1) is to be extremised to give interactions between the field and the monopole. We have learned from earlier work[6],[7] that this is most naturally expressed in loop space, which is fine, since we already had the intention above of working in loop space in any case. We have learned further that the constraint can be stated either in a global version in terms of the loop space holonomy, or else in a differential form in terms of the loop space curvature, defined as:[1]

$$G_{\mu\nu}(C|s) = \frac{\delta}{\delta\xi^\nu(s)} F_\mu(C|s) - \frac{\delta}{\delta\xi^\mu(s)} F_\nu(C|s) + ig[F_\mu(C|s), F_\nu(C|s)]. \tag{36}$$

[1]In ref. [6] and [7], a $G_{\mu\nu}(C; s, s')$ was defined depending on two points s, s' on the loop C. This is actually unnecessary since for $s \neq s'$, the quantity represents merely the holonomy Θ_Σ over a loop Σ in parametrised loop space which corresponds only to a reparametrisation of the zero surface at the loop C. Because Θ_Σ is by definition independent of the parametrisation of Σ, $G_{\mu\nu}(C; s, s')$ is then automatically zero when $s \neq s'$.

The two approaches are equivalent and can be used alternatively depending on convenience. In the present case, our aim is to generalise the analogous constraint (19) for the abelian theory, which was in the differential form in ordinary space-time. This suggests that we employ here also a differential version of the constraint, but now in loop space.

To see what specific form this constraint should take, let us recall the analogous constraint for the classical case[6]:

$$G_{\mu\nu}(C|s) = -4\pi\tilde{g}\int d\tau \kappa(C|s)\epsilon_{\mu\nu\rho\sigma}\frac{dY^{\rho}(\tau)}{d\tau}\frac{d\xi^{\sigma}(s)}{ds}\delta^4(\xi(s)-Y(\tau)) \tag{37}$$

where $\kappa(C|s)$ takes values in the gauge Lie algebra. In terms of ordinary space-time variables,

$$G_{\mu\nu}(C|s) = \Phi_C^{-1}(s,0)\epsilon_{\mu\nu\rho\sigma}D_{\alpha}{}^*F^{\rho\alpha}(\xi(s))\Phi_C(s,0)\frac{d\xi^{\sigma}(s)}{ds}, \tag{38}$$

where D_μ is the ordinary covariant derivative:

$$D_\mu = \partial_\mu - ig[A_\mu(x), \ ], \tag{39}$$

and $\Phi_C(s_1,s_2)$ as defined in (32) is an element of the gauge group, being the parallel "phase" transport from s_1 to s_2. One sees then that for the abelian theory, where $\kappa(C|s) = 1$, the constraint (37) reduces simply to (17). For the nonabelian theory, (38) does not, strictly speaking, apply on the monopole world-path, since A_μ which occurs in the covariant derivative D_μ does not exist there. Nevertheless, accepting (38) at face value, we may rewrite (37) similarly as:

$$D_\nu{}^*F^{\mu\nu}(x) = -4\pi\tilde{g}\int d\tau K(\tau)\frac{dY^{\mu}(\tau)}{d\tau}\delta^4(x-Y(\tau)), \tag{40}$$

with:

$$K(\tau) = \Phi_C(s,0)\kappa(C|s)\Phi_C^{-1}(s,0). \tag{41}$$

In analogy to (17), one may then interpret the r.h.s. of (40) as the nonabelian monopole current, and $\tilde{g}K(\tau)$ as some sort of effective monopole charge. Notice that in contrast to the original topological monopole charge defined as a homotopy class of the gauge group, this new effective charge $\tilde{g}K$ is an element of the gauge Lie algebra. Both charges are conserved, but whereas the topological charge is conserved simply by virtue of continuity, the effective charge $\tilde{g}K$ is conserved due to the specific form of the dynamics as defined by the constrained action principle, and will for this reason be referred to as the *Noether charge* in what follows.

Suppose now we proceed to the quantum case. We would be tempted then, as in the abelian theory, to replace the classical current on the right of (40) by a quantum current, say:

$$D_\nu {}^*F^{\mu\nu}(x) \stackrel{?}{=} -4\pi\tilde{g}(\bar{\tilde{\psi}}(x)\gamma^\mu T_i \tilde{\psi}(x))\tau^i, \tag{42}$$

where τ_i represent the generators of the gauge Lie algebra. The quantum "monopole current" on the right of (42) would transform like an element of the algebra as it should if the monopole wave function $\tilde{\psi}(x)$ transforms like a representation of the algebra and T_i are matrices representing the generators in that representation. This suggestion is however not quite correct since, the monopole being *colour* neutral, its wave function $\tilde{\psi}$ should not transform at all under the usual Yang-Mills gauge transformations U, and if it does transform as a representation of the gauge algebra, it should do so only under a new type of, say, $\tilde{U}$-transformations. Hence, as it stands, the l.h.s of (42) is covariant but the r.h.s. invariant under U-transformations, while under $\tilde{U}$-transformations, the l.h.s. is invariant but the r.h.s. covariant.

We can correct the above discrepancy in (42) as follows. Introduce at each space-time point x two local frames for the gauge Lie algebra, one transforming under U- and the other under $\tilde{U}$-transformations. Let $\omega(x)$ be the matrix which transforms from the U-frame to the $\tilde{U}$-frame at x. Then simultaneously under a U-transformation:

$$\psi(x) \longrightarrow (1 + ig\Lambda(x))\psi(x), \tag{43}$$

and a $\tilde{U}$-transformation:

$$\tilde{\psi}(x) \longrightarrow (1 + i\tilde{g}\tilde{\Lambda}(x))\tilde{\psi}(x), \tag{44}$$

the matrix $\omega(\mathrm{x})$ will transform as:

$$\omega(x) \longrightarrow (1 + i\tilde{g}\tilde{\Lambda}(x))\omega(x)(1 - ig\Lambda(x)). \tag{45}$$

Instead of (42), we then write:

$$D_\nu {}^*F^{\mu\nu}(x) = -4\pi\tilde{g}[\bar{\tilde{\psi}}(x)\omega(x)\gamma^\mu T_i \omega^{-1}(x)\tilde{\psi}(x)]\tau^i, \tag{46}$$

which will now have correct transformation properties under both U- and $\tilde{U}$-transformations, being covariant under the first and invariant under the second type of transformations.

Returning now to loop space, we then write in analogy to (37) the defining constraint for the monopole charge as:

$$G_{\mu\nu}(C|s) = -4\pi\tilde{g}\int d^4x\epsilon_{\mu\nu\rho\sigma}\{\bar{\tilde{\psi}}(x)\Omega_C(s,0)\gamma^\rho T_i\Omega_C^{-1}(s,0)\tilde{\psi}(x)\} \times\tau^i\frac{d\xi^\sigma(s)}{ds}\delta^4(x-\xi(s)), \tag{47}$$

where

$$\Omega_C(s,0) = \omega(\xi(s))\;\Phi_C(s,0) \tag{48}$$

is the operator for parallel phase transport from the U-frame at the reference point $\xi_0 = \xi(0) = \xi(2\pi)$ to the $\tilde{U}$-frame at $\xi(s)$ along the loop $C = \{\xi(s)\}$. The constraint (47) is then what we wish to impose on variations in the dynamical variables when extremising the (free) action to deduce the (interacting) equations of motion of the field-monopole system.

Both the action and the defining constraint for the monopole have now been expressed in term of the patch-independent loop variables $F_\mu(C|s)$ suitable for our purpose. These variables, however, have one great disadvantage in that, being labelled by the parametrised loops C as well as points on them as denoted by the parameter s, they are much more numerous than the conventional variables $A_\mu(x)$ which are labelled only by points in space-time. Since $A_\mu(x)$ are known already to be sufficient for a full description of the Yang-Mills field, this means that the loop variables must be highly redundant, and before they can be used exclusively to reformulate the theory, this redundancy has to be removed by imposing the appropriate infinite number of constraints on them. The beauty in the present problem, as also in the parallel problem in the classical case[6], is that the constraints required for removing the redundancy are exactly the conditions (47) that we wish to impose in any case for dynamical reasons. That this is so has been demonstrated in ref.[7]. The situation is then closely analogous to the abelian case in the preceeding section where the constraint (17) on $F_{\mu\nu}$ guarantees via the Poincaré lemma that the gauge potential $A_\mu(x)$ exists everywhere except at the monopole position, which allowed us then to employ the redundant set $F_{\mu\nu}(x)$ as field variables instead of $A_\mu(x)$. Similarly, for the nonabelian theory, it can be shown that there is an extended Poincaré lemma which asserts that the conditions (37) or (47) on $F_\mu(C|s)$ guarantee that a potential $A_\mu(x)$ bearing the appropriate relations (28) and (30) to $F_\mu(C|s)$ exists everywhere except at the monopole position, thus again removing the redundancy from the loop variables.

With the question of redundancy then resolved, we can now formulate our problem as follows. The equations of motion are to be derived by extremising the action:

$$\begin{aligned} \mathcal{A}^0 &= -(4\pi\bar{N})^{-1}\int \delta C \int_0^{2\pi} ds\, \mathrm{Tr}(F_\mu(C|s)F^\mu(C|s))(d\xi(s)/ds)^{-2} \\ &\quad - \int d^4x \bar{\tilde{\psi}}(x)(i\partial_\mu\gamma^\mu - m)\tilde{\psi}(x) \end{aligned} \tag{49}$$

with respect to the variables $F_\mu(C|s)$ and $\tilde{\psi}(x)$, under the constraint (47).

4 Chiral Doubling of Gauge Symmetry

In the process of specifying the constraint defining the monopole charge in the preceding section, we were led to the conclusion that the wave function $\tilde{\psi}(x)$ describing the monopole should belong to some representation of the gauge algebra, but transforming under transformations other than, and in some sense dual to, the usual Yang-Mills gauge transformations. This is intriguing, and we think nontrivial, since a monopole charge was originally defined only as a homotopy class of the gauge group, having thus no apparent similarity to a source charge of the theory which is labelled by a group representation. The assertion that now a monopole also belongs to a group representation means the theory has acquired a more dual symmetric appearance than it at first possessed. Thus, for example, we could put the monopole wave function $\tilde{\psi}(x)$ in chromodynamics into a fundamental triplet representation, in which case it would look almost dual to a quark which is also in a triplet. A hint that such a state of affairs may obtain was already apparent in the classical nonabelian theory where the monopole was found to obey equations almost dual to those for a source, and where, in addition to the original topological charge, the monopole was seen to acquire through the invariance of the dynamics a conserved Noether charge $\tilde{g}K(\tau)$ taking values in the gauge algebra. It now begins to look that also in the quantum theory, the dynamics may also end up with a near dual symmetry.

However, what we have found so far would not mean too much if the $\tilde{U}$-transformations under which the monopole wave function $\tilde{\psi}(x)$ transforms as a representation do not constitute a symmetry of the dynamical system, and it is now our purpose to investigate whether such a symmetry indeed exists. We are encouraged to believe that this may be the case by our experience in the quantum abelian theory of section 2 where the $\tilde{U}$-symmetry appears as a degeneracy in the solution to the Euler-Lagrange problem while extremising the action. However, it should be stressed that whereas the abelian theory is known to be dual symmetric which made a $\tilde{U}$-symmetry an inevitable consequence of the ordinary U-symmetry, the nonabelian theory has no dual symmetry so that the discovery of a $\tilde{U}$-symmetry for nonabelian theory would be far from trivial and ought by no means to be taken for granted.

Let us first remind ourselves how various quantities in the theory transform under U- and $\tilde{U}$-transformations. Consider first U- transformations, i.e. ordinary Yang-Mills gauge transformations. The loop variables $F_\mu(C|s)$ we have adopted to describe the field are by construction invariant under U-transformations up to a constant (i.e. x- independent) *phase* rotation at the reference point ξ_0. This last rotation is trivial to handle, and in what follows it will be ignored — in other words, we shall henceforth restrict ourselves for convenience to only those gauge transformations which leaves the *phase* at the reference point ξ_0 invariant. It follows then that $G_{\mu\nu}(C|s)$ occurring in (47) which is defined in terms of $F_\mu(C|s)$ is also U-invariant. Next, the monopole wave function $\tilde{\psi}(x)$ is also invariant under U-transformations, the monopole being *colour* neutral. Finally, the factors $\Omega_C(s,0)$ which occur in the constraint (47) are also invariant under U for although $\Phi_C(s,0)$ in (48) transforms under U by a *phase* rotation at $\xi(s)$, thus:

$$\Phi_C(s,0) \xrightarrow{U} \{1 + ig\Lambda(\xi(s))\}\Phi_C(s,0) \tag{50}$$

this is cancelled in $\Omega_C(s,0)$ by an inverse transformation in $\omega(x)$, as per (45). Hence, the whole variational problem as formulated in the preceding section is invariant under U-transformations, from which we conclude that the dynamics of monopoles must, not surprisingly, also have the usual Yang-Mills gauge symmetry.

Next, under $\tilde{U}$-transformations, we want $\tilde{\psi}(x)$ to transform as (44) and $\omega(x)$ as (45). The transformation laws of the other quantities under $\tilde{U}$ have yet to be specified, and the idea is, if possible, to choose them such as to exhibit an overall symmetry. In principle, we could make the field variable $F_\mu(C|s)$ transform under $\tilde{U}$, but we would rather not do so. From our experience in the abelian theory, we anticipate the final symmetry to be a chiral doubling of the original U-symmetry, namely $U \times \tilde{U}$, so that the wave function $\psi(x)$ of a source which carries no monopole charge should not transform under a $\tilde{U}$-transformation just as the monopole wave function is invariant under U. If so, for consistency, the

Yang-Mills gauge potential $A_\mu(x)$, and hence also $F_\mu(C|s)$ defined in terms of it through (28) and (30) should also be $\tilde{U}$-invariant. From this, it follows that $\Phi_C(s,0)$ is $\tilde{U}$-invariant too, and that $\Omega_C(s,0)$ defined in (48) should transform as:

$$\Omega_C(s,0) \xrightarrow{\tilde{U}} \{1 + i\tilde{g}\tilde{\Lambda}(\xi(s))\}\Omega_C(s,0). \tag{51}$$

This leaves the r.h.s. of (47) $\tilde{U}$-invariant as it should, since we have already specified that $F_\mu(C|s)$, and hence also $G_{\mu\nu}(C|s)$ to be invariant under $\tilde{U}$.

Applied to the action $\mathcal{A}^0$ in (49), we obtain that under the $\tilde{U}$-transformation (44):

$$\mathcal{A}^0 \xrightarrow{\tilde{U}} \mathcal{A}^0 + \tilde{g}\int d^4x \bar{\tilde{\psi}}(x)\partial_\mu\tilde{\Lambda}(x)\gamma^\mu\tilde{\psi}(x), \tag{52}$$

the action acquires an additional term. Now, in the conventional formulation of the theory for a Yang-Mills source, such an increment of the matter action under a U-transformation also occurs with $\tilde{\psi}(x)$ here replaced by the source wave function $\psi(x)$. There, this extra term is cancelled by another coming from the interaction term in the action analogous to (11) due to the transformation of the gauge potential $A_\mu(x)$. Here, we have no interaction term in the action, whose function has now been usurped by the constraint defining the monopole charge. Hence, the effect of the increment in the action (52) will have to be balanced by one from the constraint itself. This was the case in abelian theory (section 2), where it was the Lagrange multiplier $\lambda_\mu(x)$, playing the role of a dual potential $\tilde{A}_\mu(x)$, which transforms under $\tilde{U}$ so as to maintain overall invariance. Therefore, we expect that also in the present case, we shall have to look for a transformation in the Lagrange multiplier for the constraint to balance the variation (52) in the action.

Introduce therefore the Lagrange multipliers $L^{\mu\nu}(C;s)$, one for each of the constraints in (47) and form the auxiliary action:

$$\mathcal{A}' = \mathcal{A}^0 + \int \delta C ds \, \mathrm{Tr}(L^{\mu\nu}(C;s)\{G_{\mu\nu}(C|s) - J_{\mu\nu}(C|s)\}), \tag{53}$$

where $J_{\mu\nu}(C|s)$ is just the r.h.s. of (47):

$$J_{\mu\nu}(C|s) = -4\pi\tilde{g}\int d^4x \epsilon_{\mu\nu\rho\sigma}\{\bar{\tilde{\psi}}(x)\Omega_C(s,0)\gamma^\rho T_i \Omega_C^{-1}(s,0)\tilde{\psi}(x)\}\tau^i \frac{d\xi^\sigma(s)}{ds}\delta^4(x-\xi(s)). \tag{54}$$

The equations of motion are now to be obtained by extremising $\mathcal{A}'$ w.r.t. unconstrained variations of $F_\mu(C|s)$ and $\tilde{\psi}(x)$. We know already that under U-transformations, both the action (49) and the constraint (47) are invariant so that $L^{\mu\nu}(C;s)$ can be left invariant too. Our aim then is to seek further an appropriate $\tilde{U}$-transformation of $L^{\mu\nu}(C;s)$ so as to leave $\mathcal{A}'$ also invariant under $\tilde{U}$-transformations.

We propose that under the transformation $\tilde{U}$ in (44) $L^{\mu\nu}(C;s)$ should transform as:

$$L^{\mu\nu}(C;s) \xrightarrow{\tilde{U}} L^{\mu\nu}(C;s) + \tilde{\Delta}L^{\mu\nu}(C;s), \tag{55}$$

where:

$$\tilde{\Delta}L^{\mu\nu}(C;s) = \frac{1}{R}\epsilon^{\mu\nu\rho\sigma}\Omega_C^{-1}(s_-,0)\frac{\delta\tilde{\Lambda}(\xi(s))}{\delta\xi^\rho(s)}\Omega_C(s_-,0)\frac{d\xi_\sigma(s)}{ds}, \tag{56}$$

and R is a normalisation constant to be specified later. In the formula (56), the factor $\Omega_C(s_-,0)$, as indicated by the symbol s_-, is to be taken as the value approached from below in s. This means (for $\epsilon > 0$):

$$\Omega_C(s_-,0) = \lim_{\epsilon\to 0}\Omega_C(s-\epsilon,0). \tag{57}$$

The reason for this will be apparent later.

Let the change in $\mathcal{A}'$ under $\tilde{U}$ of (44) be represented by:

$$\tilde{\Delta}\mathcal{A}' = \tilde{\Delta}\mathcal{A}'_1 + \tilde{\Delta}\mathcal{A}'_2 + \tilde{\Delta}\mathcal{A}'_3, \tag{58}$$

where:

$$\tilde{\Delta}\mathcal{A}'_1 = \tilde{g}\int d^4x\bar{\tilde{\psi}}(x)\partial_\mu\tilde{\Lambda}(x)\gamma^\mu\tilde{\psi}(x), \tag{59}$$

$$\tilde{\Delta}\mathcal{A}'_2 = -\int \delta C ds\, \mathrm{Tr}(\tilde{\Delta}L^{\mu\nu}(C;s)J_{\mu\nu}(C|s)), \tag{60}$$

$$\tilde{\Delta}\mathcal{A}'_3 = \int \delta C ds\, \mathrm{Tr}(\tilde{\Delta}L^{\mu\nu}(C;s)G_{\mu\nu}(C|s)). \tag{61}$$

Then substituting into $\tilde{\Delta}\mathcal{A}'_2$ the change $\tilde{\Delta}L_{\mu\nu}(C;s)$ given in (56), we have:

$$\tilde{\Delta}\mathcal{A}'_2 = \frac{4\pi\tilde{g}}{R}\int \delta C ds \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta}\bar{\tilde{\psi}}(\xi(s))\gamma^\rho\frac{\delta\tilde{\Lambda}}{\delta\xi^\alpha(s)}\tilde{\psi}(\xi(s))\frac{d\xi_\beta(s)}{ds}\frac{d\xi^\sigma(s)}{ds}, \tag{62}$$

where we notice that, apart from the factors $d\xi^\beta(s)/ds$ and $d\xi^\sigma(s)/ds$ which depend on the tangent to the loop C at $x = \xi(s)$, the rest of the integrand is a function only of the space-time point $x = \xi(s)$, and can therefore be taken out of the integral over C. The integral can thus be easily performed by averaging over all directions of the tangent to the loop C at that point:

$$\int \delta C ds\frac{d\xi^\beta(s)}{ds}\frac{d\xi^\sigma(s)}{ds} = \tfrac{1}{4}g^{\beta\sigma}\int \delta C ds(d\xi(s)/ds)^2, \tag{63}$$

giving for (62):

$$\tilde{\Delta}\mathcal{A}'_2 = -\frac{3\pi\tilde{g}}{2R}\int d^4x(\bar{\tilde{\psi}}(x)\partial_\mu\tilde{\Lambda}(x)\gamma^\mu\tilde{\psi}(x))\int \delta C ds(d\xi(s)/ds)^2\delta^4(x-\xi(s)). \quad (64)$$

Choosing the normalisation constant R in (56) to be:

$$R = \frac{3\pi}{2}\int \delta C ds(d\xi(s)/ds)^2\delta^4(x-\xi(s)) \quad (65)$$

ensures that $\tilde{\Delta}\mathcal{A}'_2$ will exactly cancel in (58) the increment $\tilde{\Delta}\mathcal{A}'_1$ from the free action $\mathcal{A}^0$.

For the whole $\mathcal{A}'$ in (53) to be invariant under $\tilde{U}$-transformations, we now want $\tilde{\Delta}\mathcal{A}'_3$ to be zero. That this is indeed the case can be seen as follows. We first show that the loop space curvature $G_{\mu\nu}(C|s)$ can be written as:

$$G_{\mu\nu}(C|s) = \lim_{\epsilon'\to 0}\frac{\delta}{\delta\xi_\lambda(s)}\{\Theta^{-1/2}(\pi,\pi-\epsilon')\{F_\mu(C|s)g_{\lambda\nu}-F_\nu(C|s)g_{\lambda\mu}\}\Theta^{1/2}(\pi,\pi-\epsilon')\}, \quad (66)$$

where:

$$\Theta(t_2,t_1) = P_t \exp ig\int_{t_1}^{t_2} dt\int_0^{2\pi} ds F_\mu(C_t|s)\frac{\partial\xi_t^\mu(s)}{\partial t}, \quad (67)$$

P_t means ordering w.r.t. t, increasing in our convention from right to left, and for C as given in (29), one defines:

$$C_t = \{\xi_t^\mu(s) = \frac{t}{\pi}(\xi^\mu(s)-\xi_0^\mu)+\xi_0^\mu, \;\; s = 0\to 2\pi, \; t = 0\to\pi\}. \quad (68)$$

One sees that by construction, C_t varies, for $t = 0 \to \pi$, from the zero loop at the reference point ξ_0 for $t = 0$ to $C_t = C$ for $t = \pi$. Hence, if we choose to call the path traced out by C_t in loop space Σ, (which means of course a surface swept out by C_t in ordinary space-time), the quantity $\Theta(t_2,t_1)$ in (67) is just the parallel transport from t_1 to t_2 along the path Σ in loop space. Now, it is not hard to see that:

$$\left\{\frac{\delta}{\delta\xi^\mu(s)}\Theta(\pi,t')\right\}\Theta^{-1}(\pi,t') = igF_\mu(C|s) \quad (69)$$

for any t', for, since $C_\pi = C$, the derivative is with respect to the upper end-point of the integral in t. Alternatively, one can obtain (69) by writing:

$$\Theta(\pi,t') = \Theta_\Sigma(\pi,0)\;\Theta_\Sigma^{-1}(t',0) \quad (70)$$

in the notation of ref.[7], then using the relation there:

$$\Theta_\Sigma(t,0) = \Phi^{-1}(C_t) \quad (71)$$

to express $\Theta(\pi, t')$ as:

$$\Theta(\pi, t') = \Phi^{-1}(C)\, \Phi(C_{t'}). \tag{72}$$

Hence we have:

$$\left\{\frac{\delta}{\delta\xi^\mu(s)}\Theta(\pi, t')\right\} \Theta^{-1}(\pi, t') = -\{\Phi^{-1}(C)\frac{\delta}{\delta\xi^\mu(s)}\Phi(C)\}, \tag{73}$$

which, from the definition (28) of $F_\mu(C|s)$, is (69) as desired. Similarly, we have:

$$\Theta(\pi, t')\frac{\delta}{\delta\xi^\mu(s)}\Theta^{-1}(\pi, t') = -igF_\mu(C|s). \tag{74}$$

Applying (69) and (74) to the r.h.s of (66), one then easily obtains the result $G_{\mu\nu}(C; s)$ as claimed.

Substituting (56) and (66) into (61), we obtain:

$$\tilde{\Delta}\mathcal{A}'_3 = \frac{1}{R}\int \delta C ds \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\left\{\Omega_C^{-1}(s_-, 0)\frac{\delta\tilde{\Lambda}(\xi(s))}{\delta\xi^\rho(s)}\Omega_C(s_-, 0)\frac{d\xi_\sigma(s)}{ds}\times\right.$$

$$\left.\lim_{\epsilon'\to 0}\frac{\delta}{\delta\xi_\lambda(s)}\{\Theta^{-1/2}(\pi, \pi-\epsilon')\{F_\mu(C|s)g_{\lambda\nu} - F_\nu(C|s)g_{\lambda\mu}\}\Theta^{1/2}(\pi, \pi-\epsilon')\}\right\}, \tag{75}$$

with $\Omega(s_-, 0)$ defined as in (57). Assuming next that both the limits $\epsilon \to 0$ and $\epsilon' \to 0$ can be taken outside the integral, we obtain, on integrating by parts w.r.t. $\xi(s)$:

$$\tilde{\Delta}\mathcal{A}'_3 = \lim_{\epsilon,\epsilon'\to 0}\frac{1}{R}\int \delta C ds\, \mathrm{Tr}\left\{\Omega_C^{-1}(s-\epsilon, 0)\epsilon^{\mu\nu\rho\sigma}\frac{\delta^2\tilde{\Lambda}(\xi(s))}{\delta\xi_\lambda(s)\delta\xi^\rho(s)}\Omega_C(s-\epsilon, 0)\frac{d\xi_\sigma(s)}{ds}\times\right.$$

$$\left.\Theta^{-1/2}(\pi, \pi-\epsilon')\{F_\mu(C|s)g_{\lambda\nu} - F_\nu(C|s)g_{\lambda\mu}\}\Theta^{1/2}(\pi, \pi-\epsilon')\right\}, \tag{76}$$

which by contracting indices with $g_{\lambda\nu}$ and $g_{\lambda\mu}$ in the second factor gives two terms with factors of the form:

$$\epsilon^{\mu\nu\rho\sigma}\frac{\delta^2\tilde{\Lambda}(\xi(s))}{\delta\xi^\nu(s)\delta\xi^\rho(s)},\quad \epsilon^{\mu\nu\rho\sigma}\frac{\delta^2\tilde{\Lambda}(\xi(s))}{\delta\xi^\mu(s)\delta\xi^\rho(s)}, \tag{77}$$

both of which vanish by symmetry. We conclude therefore that $\tilde{\Delta}\mathcal{A}'_3$ is indeed zero, or that our monopole-field system as embodied in the action $\mathcal{A}'$ of (53) admits $\tilde{U}$-transformations as a gauge symmetry.

We note that the arguments presented in the preceding paragraphs, though considerably more elaborate, are basically very similar to those for the abelian theory given at the end of section 2. We are thus led to the very intriguing and rather surprising result that although there is no dual symmetry in the Yang-Mills theory and monopoles start out there as very different objects from sources, the theory nevertheless experiences a chiral doubling of the gauge symmetry in close analogy to the dual symmetric abelian case.

5 Equations of Motion

According to the formulation of section 3, the equations of motion of the monopole-field system are to be obtained by extremising the action (49) w.r.t. the field variables $F_\mu(C|s)$ and the particle wave function $\tilde{\psi}(x)$ under the constraint (47). Equivalently, they are obtained by extremising the auxiliary action (53) w.r.t. unconstrained variations of the same variables.

In $\mathcal{A}'$ of (53), only the following two terms depend on the wave function $\tilde{\psi}(x)$:

$$\mathcal{A}'_M = -\int d^4x \bar{\tilde{\psi}}(x)(i\partial_\mu\gamma^\mu - m)\tilde{\psi}(x) - \int \delta C ds \,\mathrm{Tr}(L^{\mu\nu}(C;s)J_{\mu\nu}(C|s)), \quad (78)$$

with $J_{\mu\nu}(C|s)$ as given in (54). Varying then $\mathcal{A}'_M$ w.r.t. $\bar{\tilde{\psi}}(x)$ and putting the coefficient equal to zero, one obtains as one of the equations of motion:

$$(i\partial_\mu\gamma^\mu - m)\tilde{\psi}(x) = \tilde{g}\tilde{A}_\mu(x)\gamma^\mu\tilde{\psi}(x), \quad (79)$$

with:

$$\tilde{A}_\mu(x) = 4\pi \int \delta C ds \,\epsilon_{\mu\nu\rho\sigma}\Omega_C(s,0)L^{\rho\sigma}(C;s)\Omega_C^{-1}(s,0)\frac{d\xi^\nu(s)}{ds}\delta^4(x-\xi(s)). \quad (80)$$

One sees that equation (79) is exactly the dual of the ordinary Yang-Mills equation for a colour source moving in a gauge field described by a potential $A_\mu(x)$. A new space-time local quantity $\tilde{A}_\mu(x)$ which we may call the *dual potential* has emerged to take the place of the ordinary potential $A_\mu(x)$, and this $\tilde{A}_\mu(x)$ is indeed given in terms of the Lagrange multiplier $L_{\mu\nu}(C;s)$ as we anticipated. Moreover, under the $\tilde{U}$-transformation (44), it is easily shown using (56) and (51) that $\tilde{A}_\mu(x)$ transforms as:

$$\tilde{A}_\mu(x) \longrightarrow \tilde{A}_\mu(x) - \partial_\mu\tilde{\Lambda}(x) + i\tilde{g}[\tilde{\Lambda}(x), \tilde{A}_\mu(x)], \quad (81)$$

exactly as a gauge potential should, and leaves therefore the equation (79) invariant. The equation is invariant also of course under U-transformations.

Next, the terms of $\mathcal{A}'$ in (53) depending on the field variables $F_\mu(C|s)$ are:

$$\begin{aligned}\mathcal{A}'_F &= -(4\pi\bar{N})^{-1}\int \delta C ds \,\mathrm{Tr}(F_\mu(C|s)F^\mu(C|s))(d\xi(s)/ds)^{-2} \\ &\quad + \int \delta C ds \,\mathrm{Tr}(L^{\mu\nu}(C;s)G_{\mu\nu}(C|s)). \end{aligned} \quad (82)$$

Substituting (66) into the second term of (82), and integrating by parts w.r.t. $\delta\xi^\lambda(s)$ before taking the limit $\epsilon' \to 0$, we obtain:

$$\begin{aligned}\mathcal{A}'_F &= -(4\pi\bar{N})^{-1}\int \delta C ds \,\mathrm{Tr}(F_\mu(C;s)F^\mu(C;s))(d\xi(s)/ds)^{-2} \\ &\quad - \int \delta C ds \,\mathrm{Tr}\left[\left(\frac{\delta}{\delta\xi_\lambda(s)}L^{\mu\nu}(C;s)\right)(F_\mu(C|s)g_{\lambda\nu} - F_\nu(C|s)g_{\lambda\mu})\right]. \end{aligned} \quad (83)$$

Varying the above expression w.r.t. $F_\mu(C|s)$ and putting the variation equal to zero, we obtain another of the required equations of motion as:

$$F_\mu(C|s) = -(4\pi\tilde{N})\left(\frac{d\xi(s)}{ds}\right)^2 \frac{\delta}{\delta\xi_\nu(s)} L_{\mu\nu}(C;s) \tag{84}$$

As for the other equation (79) the equation (84) is also invariant under both U- and $\tilde{U}$-transformations, as can be seen in the second case by using (56) with due care in taking the limit $\epsilon \to 0$ only after performing the differentiation w.r.t. $\xi(s)$ as was specified above. Notice that we have actually two equivalent forms (36) and (66) for the loop space curvature $G_{\mu\nu}(C|s)$, either of which can in principle be used in the derivation above. However, had we used (36) instead we would have obtained a form of the equation of motion which is not explicitly covariant, and more work would have been necessary to arrive at a covariant form.

Together then with the constraint equation (47), the equations (79) and (84) represent the complete set of equations governing the motion of our monopole-field system, and all the equations have been shown to be invariant or covariant under both U- and $\tilde{U}$-transformations. We have therefore already achieved the prime objective we have set ourselves at the beginning.

These equations, however, are unfortunately a little unwieldy in that they are expressed in terms of unfamiliar field variables in loop space, the physical significance of which is a little hard to appreciate. We seek therefore to recast the equations, if possible, into a more transparent form in terms of more familiar space-time local variables, as was done with the corresponding classical equations in ref[6] leading to interesting conclusions. We have already done so for two of the equations, namely (79), where the field appears only in terms of the space-time local "dual potential" $\tilde{A}_\mu(x)$, and (47), which was shown to be equivalent to the space-time local equation (46). We shall try to do the same now also for the remaining equation (84).

We note first that by the antisymmetry of $L_{\mu\nu}(C;s)$ in its indices μ and ν, the equation (84) implies the Polyakov equation[8]:

$$\frac{\delta}{\delta\xi^\mu(s)} F_\mu(C|s) = 0, \tag{85}$$

which is just Yang-Mills equation written in loop space notation. Indeed, if we substitute for $F_\mu(C|s)$ the formula (31) and using the fact deducible from (32) that:

$$\left\{\frac{\delta}{\delta\xi^\mu(s)} \Phi_C(s,0)\right\} \Phi_C^{-1}(s,0) = igA_\mu(\xi(s)), \tag{86}$$

we have instead of (85)

$$D^\mu F_{\mu\nu}(\xi(s))\frac{d\xi^\nu(s)}{ds} = 0, \tag{87}$$

which is to hold for all C and s, and hence for all $d\xi^\nu(s)/ds$. This will be true if and only if the Yang-Mills equation:

$$D^\nu F_{\mu\nu}(x) = 0 \tag{88}$$

is satisfied for all x.

More generally, to write the original equation (84) in terms of ordinary space-time local variables, we write it first as:

$$F_\mu(C|s) = -(4\pi\bar{N})\left(\frac{d\xi(s)}{ds}\right)^2 \frac{\delta}{\delta\xi_\nu(s)}\left(\Omega_C^{-1}(s,0)\Omega_C(s,0)L_{\mu\nu}(C;s)\Omega_C^{-1}(s,0)\Omega_C(s,0)\right). \tag{89}$$

We can then rewrite (89) as:

$$F^{\mu\nu}(\xi(s))\frac{d\xi_\nu(s)}{ds} = (2\pi\bar{N})\left(\frac{d\xi(s)}{ds}\right)^2 \epsilon^{\mu\nu\rho\sigma}\Phi_C(s,0)\Omega_C^{-1}(s,0)\times$$

$$\left\{\frac{\delta}{\delta\xi^\nu(s)}\left\{\Omega_C(s,0)\,{}^*L_{\rho\sigma}(C;s)\,\Omega_C^{-1}(s,0)\right\} - [\eta_\nu(\xi(s)),\Omega_C(s,0)\,{}^*L_{\rho\sigma}(C;s)\,\Omega_C^{-1}(s;0)]\right\}$$

$$\times\,\Omega_C(s,0)\Phi_C^{-1}(s,0), \tag{90}$$

with:

$${}^*L_{\mu\nu}(C;s) = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}L^{\rho\sigma}(C;s), \tag{91}$$

and:

$$\eta_\nu(\xi(s)) = \left\{\frac{\delta}{\delta\xi^\nu(s)}\Omega_C(s,0)\right\}\Omega_C^{-1}(s,0), \tag{92}$$

which by the definition (48) of $\Omega_C(s,0)$ and the ordering convention in the definition (32) of $\Phi_C(s,0)$ is a local quantity depending only on the point $\xi(s)$, explicitly expressible in fact as:

$$\eta_\nu(x) = (\partial_\nu\omega(x))\omega^{-1}(x) + ig\omega(x)A_\nu(x)\omega^{-1}(x). \tag{93}$$

Hence, multiplying both sides of (90) by $(d\xi^\sigma(s)/ds)\delta^4(x-\xi(s))$ and integrating w.r.t. C and s, we obtain, using (80) and (48):

$$F^{\mu\nu}(x) = \epsilon^{\mu\nu\rho\sigma}\omega^{-1}(x)\left\{\partial_\sigma\tilde{A}_\rho(x) - [\eta_\sigma(x),\tilde{A}_\rho(x)]\right\}\omega(x), \tag{94}$$

Equivalently, we can rewrite (94) as:

$$ {}^{*}F_{\mu\nu}(x) = \omega^{-1}(x)\left\{\tilde{D}_\nu \tilde{A}_\mu - \tilde{D}_\mu \tilde{A}_\nu\right\}\omega(x), \tag{95} $$

with:

$$ \tilde{D}_\mu = \partial_\mu - [\eta_\mu(x), \quad], \tag{96} $$

being the covariant derivative w.r.t. the connection $\eta_\mu(x)$. Both equations (94) and (95) involve only quantities local in ordinary space-time, and either can be taken as the third equation of motion in addition to (79) and (46). However, because of the complications of patching, their meaning is subject to the same reservations as were mentioned in the derivation of (46) in section 3.

The equation (94) or (95) is covariant w.r.t. U-transformations since on the r.h.s. only $\omega(x)$ and $\omega^{-1}(x)$ transform under U in accordance to (45). Under $\tilde{U}$, however, the l.h.s. is invariant but the r.h.s. does not seem to be an invariant quantity. This would be surprising if it were true since the equations were derived from (84) which was shown already to be invariant. The reason for this apparent discrepancy is that the $\tilde{U}$-transformation on $L^{\mu\nu}(C;s)$ has been defined in loop space with a certain limiting convention which is not easily represented in ordinary space-time local notation. When handled with care, however, the correct transformation properties will obtain, as can be seen as follows. Starting from the formula (80) for $\tilde{A}_\mu(x)$, we deduce using (51) and (55) that the change in $\tilde{A}_\mu(x)$ under the $\tilde{U}$-transformation (44) is given by:

$$ \tilde{\Delta}\tilde{A}_\mu(x) = i\tilde{g}[\tilde{\Lambda}(x), \tilde{A}_\mu(x)] + \lim_{\epsilon\to 0}\int \delta C ds \epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\Omega_C(s,0)\Omega_C^{-1}(s-\epsilon,0) $$
$$ \frac{\delta\tilde{\Lambda}(\xi(s))}{\delta\xi^\alpha(s)}\Omega_C(s-\epsilon,0)\Omega_C^{-1}(s,0)\frac{d\xi^\nu(s)}{ds}\frac{d\xi_\beta(s)}{ds}\delta^4(x-\xi(s)). \tag{97} $$

On letting $\epsilon \to 0$, this yields the standard transformation law for $\tilde{A}_\mu(x)$ as exhibited in (81). Consider next the derivative $\partial_\mu \tilde{A}_\nu(x)$. Usually, one would expect that under a $\tilde{U}$-transformation, the change in $\partial_\mu \tilde{A}_\nu(x)$ would just be $\partial_\mu \tilde{\Delta}\tilde{A}_\nu(x)$. However, this is not what obtains if one applies the rule of taking first the derivative before taking the limit $\epsilon \to 0$ as stipulated above. It can easily be seen from (97) that one obtains instead:

$$ \tilde{\Delta}\{\partial_\mu \tilde{A}_\nu(x)\} = \partial_\mu \tilde{\Delta}\tilde{A}_\nu(x) + [\eta_\mu(x), \partial_\nu \tilde{\Lambda}(x)], \tag{98} $$

with an extra term. With this extra term in the transformation of the derivative of $\tilde{A}_\mu(x)$, it can then easily be checked that the equations (94) and (95) are indeed invariant under $\tilde{U}$-transformations. The occurrence of an extra term in the transformation of the derivative of $\tilde{A}_\mu(x)$ presumably means that the equation (95) and its $\tilde{U}$-transformation which were originally defined in loop space are not really local in space-time in the usual sense. Their true significance in space-time language and the mathematical structure underlying them have yet to be clearly understood.

Nevertheless, being much simpler than their loop-space counterparts, the local-seeming equations (79) and (95), together with the original constraint equation (46) are likely to be of use for a first exploration of monopole-field dynamics. For example, by comparing (79) with the corresponding equation for a Yang-Mills source, which is of the same form, one deduces that the monopole wave function $\tilde{\psi}$ is coupled to the 'dual potential' $\tilde{A}_\mu$ in exactly the same way that a source wave function ψ is coupled to the ordinary potential A_μ, though with a coupling $\tilde{g}$ instead of a coupling g. The equation (46), on the other hand, though seemingly also the dual of the usual Yang-Mills equation:

$$D_\nu F^{\mu\nu}(x) = -4\pi g[\bar{\psi}(x)\gamma^\mu T_i \psi(x)]\tau^i, \tag{99}$$

is in fact quite different in structure in that the covariant derivative D_ν in (46) is still defined as (39) in terms of the ordinary potential A_μ and coupling g just as in (99), not the 'dual potential' $\tilde{A}_\mu$ and coupling $\tilde{g}$ as one would expect for exact duality. Further, the equation (95), which may be regarded as the dual equivalent of (7) in giving the 'dual field' ${}^*F_{\mu\nu}$ in terms of the 'dual potential' $\tilde{A}_\mu$, involves in addition the potential A_μ and the coupling g, spoiling thus again exact dual symmetry. At a preliminary level, therefore, it would appear that a primary difference between the dual and standard Yang-Mills theories is in the gauge-boson self-coupling. Apart from duality, the gauge-boson's coupling to matter is similar in the two cases, but its self-coupling is not. In the standard theory there is just one coupling constant coupling the gauge-boson both to matter and to itself, whereas in the dual theory, the gauge-boson couples to itself with one coupling constant g, but couples to matter with another $\tilde{g}$ which is related to g via the Dirac quantisation condition[1], which for the $SO(3)$ theory reads as follows:

$$g\tilde{g} = n/4. \tag{100}$$

This difference in coupling is probably the first signature to look for when asking the practical question whether monopole interactions as deduced above do or do not occur in nature.

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