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On the Relevance of the BFKL Equation to Deep-Inelastic Scattering at HERA.

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Abstract

We examine the validity of using the BFKL equation for deep-inelastic scattering calculations in the HERA region. In particular, we discuss the theoretical uncertainties which arise from the sensitivity to infra-red physics and from the imposition of energy conservation. We concentrate on restoring energy conservation to the equation via the introduction of an ultra-violet cut-off, the importance of which is demonstrated in terms of the k_T diffusion of emitted gluons. Our results show that, at low to intermediate values of Q^2, the ultra-violet cut-off leads to a suppression of the growth (with decreasing x) of the gluon structure function. This effect is of special relevance to the DESY ep collider, HERA.

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1 Introduction

The perturbative pomeron, generated by the Balitsky, Fadin, Kuraev, Lipatov (BFKL) equation [1] has provoked much interest in the area of 'semi-hard' physics ($\Lambda^2_{QCD} \ll Q^2 \ll E^2_{CM}$). It is expected that, over the next few years, data from the DESY ep collider HERA will comprehensively probe this region. In particular, deep-inelastic scattering (DIS) at small Bjorken-x, heavy-quark production, minijet production and large-t elastic/diffractive scattering are all expected to yield important information [2]. Encouragingly, the first measurements from HERA on the deep-inelastic structure function, $F_2(x,Q^2)$, at small x point to a large perturbative component [3].

In this paper we focus on DIS at small x. Theoretical predictions for the small-x region have been made [4, 5] using the formalism developed by BFKL. The procedure is to sum the leading logarithms in $1/x$ which become large in the semi-hard regime, i.e. the leading log $1/x$ approximation (LL($1/x$)A). The resulting equation describes the behaviour of the unintegrated gluon distribution function, $f(x,k^2)$, in terms of the transverse momentum, k^2, and the longitudinal momentum fraction, x, of the probed gluon. The equation takes the form

$$f(x,k^2) = f^{(0)}(x,k^2) + \int_x^1 \frac{dx'}{x'} \int_0^\infty \frac{dk'^2}{k'^2} K(k'^2,k^2) f(x',k'^2). \tag{1}$$

The gluon distribution function, $xg(x,Q^2)$, is defined in terms of $f(x,k^2)$ through the equation:

$$xg(x,Q^2) = \int_0^{Q^2} \frac{dk'^2}{k'^2} f(x,k'^2) \tag{2}$$

and the kernel, $K(k'^2,k^2)$, is defined by

$$K(k'^2,k^2) f(x,k'^2) = \frac{3\alpha_s}{\pi} k^2 \left(\frac{f(x,k'^2) - f(x,k^2)}{|k'^2 - k^2|} + \frac{f(x,k^2)}{(4k'^4 + k^4)^{1/2}} \right). \tag{3}$$

Notice that, for fixed coupling, the resulting equation is infra-red finite. The introduction of a running coupling is a sub-leading correction to the LL($1/x$)A and we shall have more to say on this issue in section 2.

The term $f^{(0)}(x,k^2)$ in eq.(1) specifies the gluon content of the parent hadron in the absence of any BFKL evolution. If we suppose it to be driven by the non-perturbative pomeron then it is expected to be rather flat for small x [6]. This (unknown) inhomogeneous function is often eliminated by re-writing eq.(1) in the form

$$f(x,k^2) = f(x_0,k^2) + \int_x^{x_0} \frac{dx'}{x'} \int_0^\infty \frac{dk'^2}{k'^2} K(k'^2,k^2) f(x',k'^2), \tag{4}$$

with x_0 chosen to be small enough to ensure that $f^{(0)}(x \leq x_0, k^2) \simeq f^{(0)}(x_0,k^2)$. Equivalently, we can write an evolution equation of the form

$$\frac{\partial f(x,k^2)}{\partial \ln(1/x)} = \int_0^\infty \frac{dk'^2}{k'^2} K(k'^2,k^2) f(x,k'^2)\,. \tag{5}$$

The evolution then begins at $x = x_0$ and determines the structure function at smaller x. Again, the necessary boundary condition is $f(x_0,k^2)$ for $0 \leq k^2 \leq \infty$. Equation (1) can be solved analytically for asymptotically small x [7] resulting in the behaviour

$$f(x,k^2) \sim x^{-\lambda} \left(\frac{k^2}{\ln 1/x}\right)^{1/2} \exp\left(\frac{-\ln(k^2/k_0^2)}{\Delta\lambda \ln 1/x}\right) \tag{6}$$

where λ is the leading eigenvalue of the kernel, K, and is given by

$$\lambda = \frac{3\alpha_s}{\pi} 4 \log 2, \tag{7}$$

and Δ (which determines the width of the k distribution) is

$$\Delta = \frac{14\zeta(3)}{\ln 2} \tag{8}$$

(ζ is the Riemann zeta function). The parameter k_0^2 determines the centre of the k_T distribution and is completely specified by the boundary condition, i.e.

$$\ln k_0^2 = -\frac{d}{d(i\nu)} \left[\ln \int_0^\infty dk^2\, (k^2)^{-3/2-i\nu} f(x_0,k^2)\right]\Bigg|_{\nu=0}. \tag{9}$$

In particular, $k_0^2 = m^2$ for the case $f(x_0,k^2) = \delta(k^2 - m^2)$.

It is unclear whether the behaviour described by eq.(6) will manifest itself in DIS measurements at HERA. In order to understand why, it is necessary to examine the possible failings of the BFKL formalism, when applied to DIS. Therefore, in the next section we examine the uncertainties that arise from sensitivity to unknown infra-red physics and from allowing the coupling to run. In section 3, we discuss a modification to the BFKL equation which arises on imposing energy conservation. Importantly, we find that this modification can lead to significant deviations from the LL($1/x$)A prediction of eq.(6). In section 4, we discuss the significance of our modification in terms of the diffusion in k_T of the emitted gluons and return to the issue of infra-red sensitivity, discussing it in the context of this diffusion. In section 5 we present numerical solutions to our modified BFKL equation and compare them with those of the formal LL($1/x$)A. Finally, in section 6, we present our conclusions.

2 Infra-Red Sensitivity and a Running Coupling

The BFKL equation is derived from standard perturbation theory and so is not capable of describing the soft physics of the infra-red regime. As a result, the contribution to the solution from the region with transverse momenta below values of order 1 GeV cannot be trusted. It is only when this contribution is small that it is possible to obtain a physically meaningful solution. A simple way of determining the importance of the infra-red contribution is to impose a lower cut-off (μ^2) on the integral over the transverse momenta. If the solution is insensitive to μ^2 then the contribution from the infra-red is small. Unfortunately for DIS calculations, the values of the cut-off at which the solution is no longer sensitive tend to be much smaller then 1 GeV. This can be seen from fig.(1), where we have used eq.(5), with a lower cut-off, to evolve down from $x = 0.01$ (using the boundary condition specified by eq.(15)). Only for very small values of μ^2 (i.e. $\mu^2 \lesssim 10^{-3}$ GeV^2) are the results insensitive to its value, implying that a large contribution is coming from the infra-red regime.

In order to regain some predictive power, one can try to modify the infra-red region so as to include what one believes to be the relevant physics. For example the inclusion of a dynamical gluon mass and/or a form factor suppression of the low-k^2 physics might well improve the situation.

As well as the problems of the infra-red physics, one should also worry about the possible importance of sub-leading corrections. To be sure that the LL($1/x$)A provides sensible answers one really needs to check that the next-to-leading logarithmic (NLL) terms are indeed small. In the absence of a NLL calculation, it is natural to try modifying the LL($1/x$)A by introducing some well motivated sub-leading corrections. If such corrections turn out to be significant, one might then worry as to the size of the remaining sub-leading terms and hence the usefulness of the LL($1/x$)A.

One of the most natural modifications that has been tried in the past is the inclusion of a running coupling. The scale of the coupling is usually chosen to ensure that the BFKL equation reduces to the double-leading-log form when strong ordering of the transverse momenta is imposed. This requires the choice $\alpha_s \to \alpha_s(k^2)$. However, it is now necessary to modify further the BFKL equation in order that it remain infra-red finite. The minimal modification would be either to freeze α_s at some scale, κ^2, or to impose an infra-red cut-off, μ^2, on the transverse momentum integral. Such modifications result in a leading eigenvalue which depends quite strongly upon the value of the new parameter (κ^2 or μ^2) [5, 8]. Again the only hope to improve things is to modify the theory in the infra-red region [8, 9].

3 Energy Conservation and an Ultra-Violet Cut-off

The main aim of this paper is to study the effect of imposing an energy conserving ultra-violet cut-off upon the BFKL equation. The need to impose an ultra-violet cut-off is not entirely new: Collins and Landshoff considered the case of fixed upper and lower limits on the BFKL equation [11]. They found that the leading eigenvalue of their modified equation was within 0.1 of that of eq.(7) only when the ratio of ultra-violet to infra-red cut-offs was in excess of 10^4. In this paper, we wish to extend the work of ref. [11] to make definite statements regarding DIS at HERA.

Referring to fig.(2), energy conservation dictates that $\hat{s} \geq 0$, where

$$\begin{aligned} \hat{s} &= (q+k)^2 \\ &\simeq Q^2(x/x_B - 1) - k_T^2 \end{aligned} \qquad (10)$$

and we have used the small-x relation $k_\mu k^\mu \simeq -k_T^2$. The variable x_B is the Bjorken-x of the probed gluon, defined by

$$x_B = \frac{Q^2}{2p.q} \qquad (11)$$

where Q^2 is the modulus of the probe virtuality.

In addition to this constraint, we ought to impose energy conservation to the bottom part of the ladder, i.e. $(p-k)^2 \geq 0$. However, this is a far more difficult constraint to include and we therefore neglect it.

Thus, the modified equation now reads:

$$\begin{aligned} f(x,k^2;x_B,Q^2) &= f(x_0,k^2;x_B,Q^2) \\ &+ \int_x^{x_0} \frac{dx'}{x'} \int_0^{Q^2(x'/x_B-1)} \frac{dk'^2}{k'^2} K(k'^2,k^2) f(x',k'^2;x_B,Q^2). \end{aligned} \qquad (12)$$

As in the case of the LL($1/x$)A we can write this in the form of an evolution equation:

$$\frac{\partial f(x,k^2;x_B,Q^2)}{\partial \ln(1/x)} = \int_0^{Q^2(x/x_B-1)} \frac{dk'^2}{k'^2} K(k'^2,k^2) f(x,k'^2;x_B,Q^2) \qquad (13)$$

with an initial distribution specified at $x = x_0$. This equation differs in an essential way from eq.(5). The variable x determines the position along the ladder (between x_0 and x_B) and $f(x, k^2; x_B, Q^2)$ determines the momentum distribution at some position (defined by x) along the ladder which is being probed by some scale Q^2. Hence to determine the gluon structure function we need to use

$$x_B g(x_B, Q^2) = \int_0^{Q^2} \frac{dk'^2}{k'^2} f(x_B, k'^2; x_B, Q^2). \tag{14}$$

In section 5 we shall solve this equation numerically. First, we illustrate the potential significance of our ultra-violet cut-off by considering the diffusion in the gluon transverse momentum as one moves along a ladder which is determined by the traditional BFKL equation (i.e. with no ultra-violet or infra-red cut-offs and fixed α_s). Considering this diffusion will also lead us to understand the observed sensitivity to the infra-red physics.

4 Gluon Diffusion in the BFKL Equation

The nature of the diffusion of transverse momentum within the gluon ladder is well known to take the form of a random walk [7]. Indeed, the result of eq.(6) illustrates the effect of diffusion: as x falls, so the width of the distribution in k^2 broadens. This is the result of diffusion along a gluon ladder which is unbounded, i.e. the ladder does not terminate with, for example, a quark box. The recent work of Bartels and Lotter has shown how to quantify this diffusion for the case of DIS with an extra identifiable jet [12]. Here we use their approach for the case of simple DIS which requires the k_T distribution to be constrained by the quark box at one end. At the proton end, we use the distribution:

$$f(x_0, k^2; x_B, Q^2) = \frac{k^2}{k^2 + m^2} \tag{15}$$

with $m^2 = 1$ GeV^2. This expression describes a gluon distribution at the bottom of the ladder which has some spread in k_T which is centred on values of order 1 GeV. We expect this to provide a reasonable description of DIS, where there is no hard scale to prevent the input transgressing into the dangerous infra-red regime.

In fig.(3), we show the diffusion in transverse momentum as one moves along the ladder. The resulting cigar-like region defines the RMS spread of the k_T distribution along the ladder, i.e. the width of the cigar at some x determines the width of the gluon k_T distribution at that x. Specifically, we follow ref.[12] in determining the relevant k_T distribution. It is specified by a product of distributions, in our case, obtained by evolving from the distribution of eq.(15) at one end (to give $f(x, k^2)$) and from the quark box at the other end (to give $\tilde{f}(x/x_B, k^2)$), i.e. the product which determines the k_T distribution is

$$\sim \frac{f(x, k^2)}{\sqrt{k^2}} \frac{\tilde{f}(x/x_B, k^2)}{\sqrt{k^2}}. \tag{16}$$

It is clear that, for any scenario where the k_T distribution at the proton end is peaked in the low-k^2 region (as we expect for DIS and reflected in eq.(15)), the subsequent diffusion is such that one is always sensitive to the infra-red physics. This expected sensitivity is illustrated by the shaded region below 1 GeV in fig.(3) and is qualitatively independent of whether the coupling is fixed or allowed to run. Thus we are able to understand the observed sensitivity to the infra-red regulators discussed earlier, and illustrated in fig.(1). One might hope to reduce sensitivity to the infra-red physics by working at high Q^2, i.e. by tilting the cigar. But the need to sum logarithms of Q^2 in this region means that one would now not trust the BFKL approach. There is clearly a need for a formalism which sums both the logarithms in Q^2 and those in $1/x$. Towards this goal, one might consider in more detail the 'angular ordering' approach of ref.[10], which has the present disadvantage that it cannot be formulated as an evolution equation in x.

Also shown in fig.(3) is the curve corresponding to our kinematical ultra-violet cut-off. Importantly we see that it cuts into the cigar as $x \to x_B$. We expect this cut-off to be important if there is a significant area between the cigar and this curve in the region of not-too-high k^2. This observation suggests that such a cut-off will noticeably influence the solution of the BFKL equation at low values of Q^2. The numerical results of the next section confirm this.

5 Results

We now turn to our numerical analysis. We have solved eq.(13) with the form factor of eq.(15) providing the boundary condition at $x = x_0$. By taking the boundary condition to be x_B independent we are implicitly assuming that there is a sufficiently large rapidity gap between the bottom of the ladder (x_0) and the top (x_B). Thus we cannot claim to be making predictions in the region where $x_B \sim x_0$. To understand this, consider performing evolution from some known $f(x_B^{(1)}, k^2; x_B^{(1)}, Q^2)$ and $f(x_B^{(2)}, k^2; x_B^{(2)}, Q^2)$. The two evolution paths will converge as x rises towards x_0 and will, for $x_B^{(i)} \ll x_0$, result in the same $f(x_0, k^2; x_B^{(i)}, k^2)$. Throughout, our results are computed with a fixed coupling, i.e. $\alpha_s = 0.18$.

Fig.(4) shows our results for the gluon structure function at three different values of Q^2. Also shown are the results of LL$(1/x)$A evolution. The effect of imposing energy conservation is clearly significant for Q^2 not too high. In particular, we see that the onset of the traditional BFKL behaviour is delayed to smaller x, resulting in a suppression of the gluon structure function in the $x - Q^2$ range which is to be probed at HERA.

In fig.(5) we show how the evolution with an ultra-violet cut-off approaches that of the LL$(1/x)$A as Q^2 rises. What is shown is the variation of the slope of the gluon structure function at $x = 10^{-4}$ with Q^2. Notice that, even in the strictly leading log

case, the slope does not approach a constant as Q^2 rises, rather it slowly rises with Q^2 over the range considered. This is not unexpected – for example consider the double logarithm case where the slope rises exponentially with the square root of $\log Q^2$. We see that, for x as small as 10^{-4}, the slope calculated with an ultra-violet cut-off is almost identical to that obtained by the BFKL equation. Thus the main effect of our ultra-violet cut-off, for $x \lesssim 10^{-3}$, is to suppress the gluon structure function relative to the BFKL prediction whilst preserving the slope.

6 Conclusions

Small-x DIS at HERA presents a significant theoretical challenge. There are other processes which are conceptually better tests of semi-hard QCD but none which is so easy to measure. As a result, the difficulties which have been emphasised here need to be addressed. In particular, in light of the recent HERA measurements of the structure function, $F_2(x, Q^2)$, we feel that perturbative QCD is in evidence and that it is a worthwhile endeavour to modify (phenomenologically) the infra-red behaviour of the BFKL kernel, whilst at the same time paying close attention to the effects of energy conservation.

It is clear that there remains much theoretical work to be done before we can claim to have serious QCD predictions for DIS in the small-x regime. We have highlighted the importance of including sub-leading corrections through a kinematical cut-off. It is certainly desirable to have a full next-to-leading order BFKL equation, to enable predictions in the semi-hard regime to be made with rather more confidence than hitherto.

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Figure Captions

Figure 1 Sensitivity of BFKL evolution to the value of the infra-red cut-off, μ^2, at $Q^2 = 20$ GeV^2.

Figure 2 A gluon ladder defining the DIS kinematics.

Figure 3 : Figure illustrating the diffusion in transverse momentum as one moves along the gluon ladder (shown above the graph). The region below 1 GeV is shaded to illustrate the potential sensitivity to infra-red physics, and the upper shaded region is that into which the gluons are forbidden from diffusing by energy conservation.

Figure 4 : Comparison of BFKL evolution and that of eqn.(13) for three values of Q^2 and with $\mu^2 = 10^{-4}$ GeV^2.

Figure 5 : Comparison of $\lambda = \partial \ln(x_B g)/\partial \ln(1/x_B)$ as a function of Q^2, as calculated using BFKL evolution and eqn.(13) and at $x_B = 10^{-4}$.

Figure 1

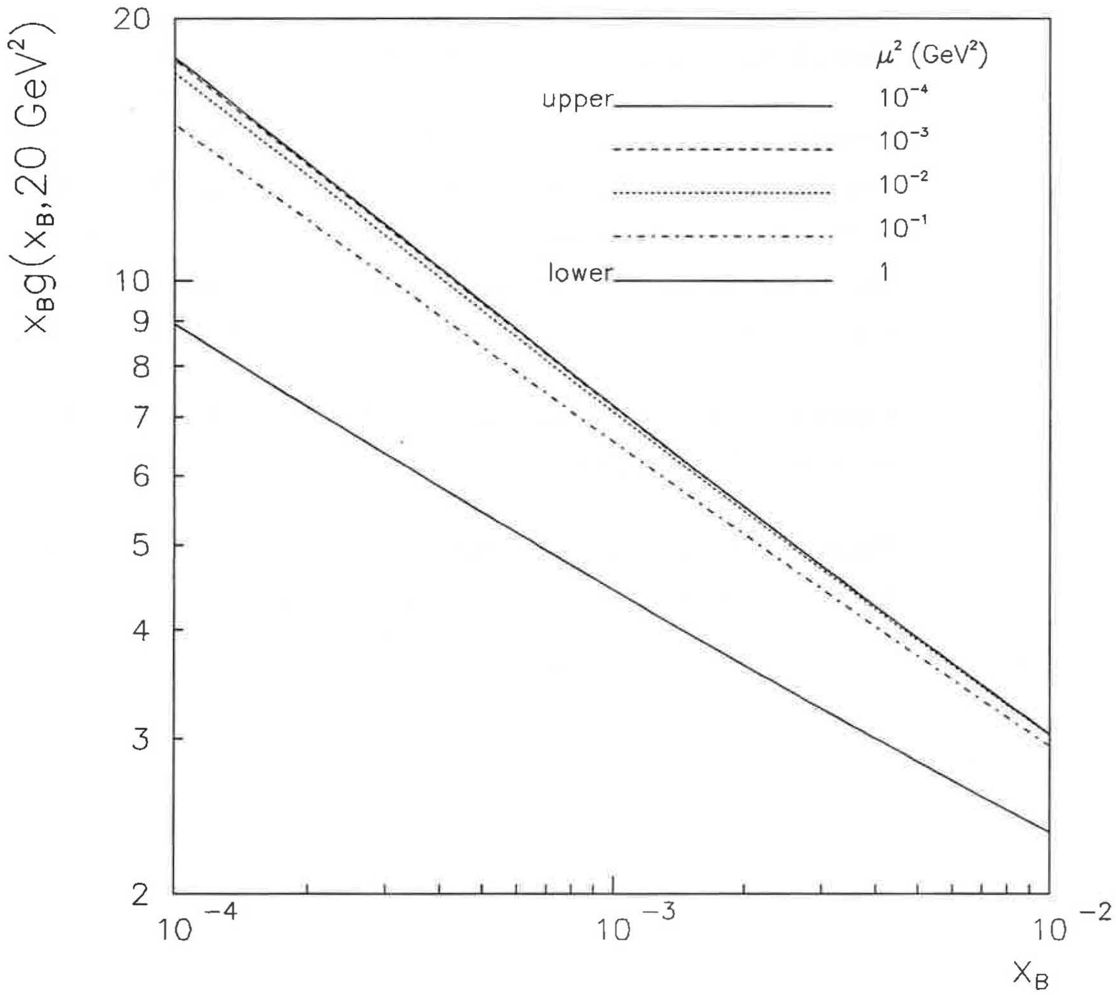


Figure 2

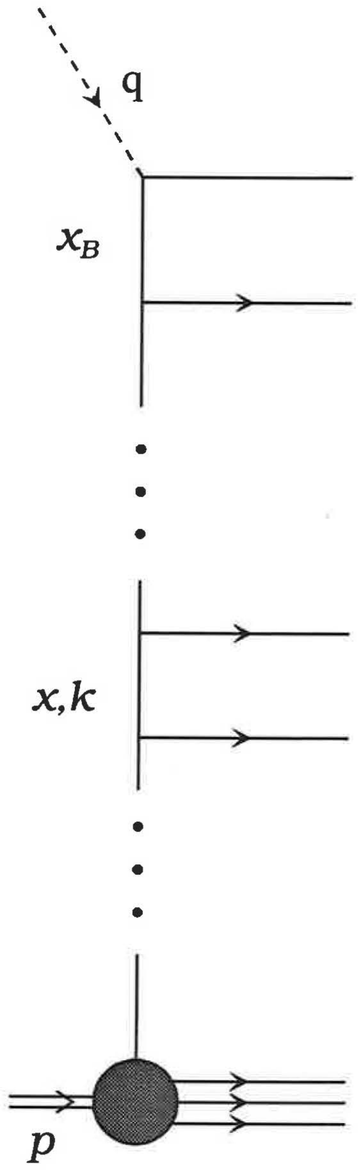


Figure 3

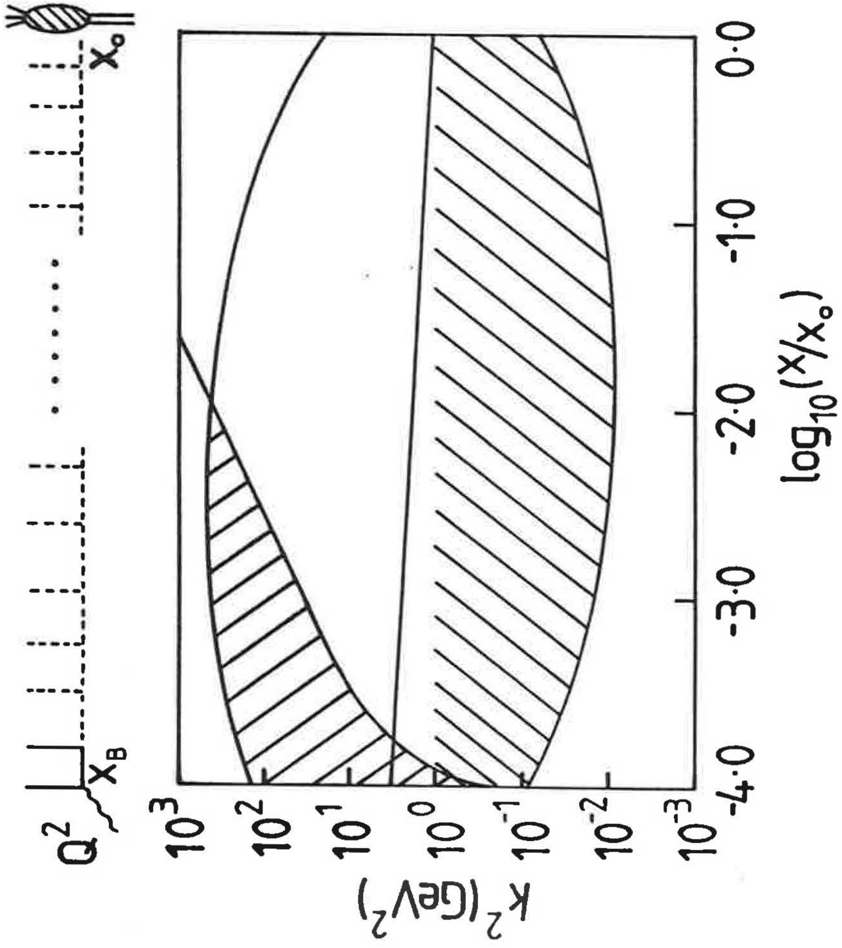


Figure 4

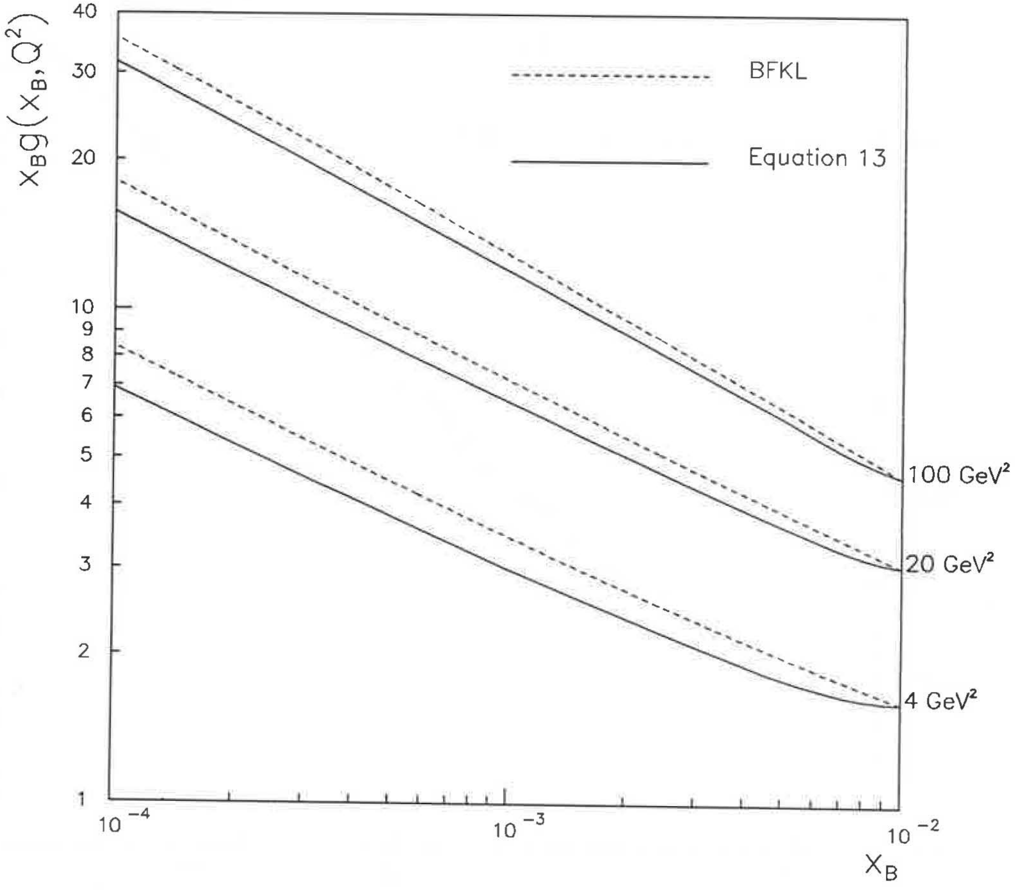
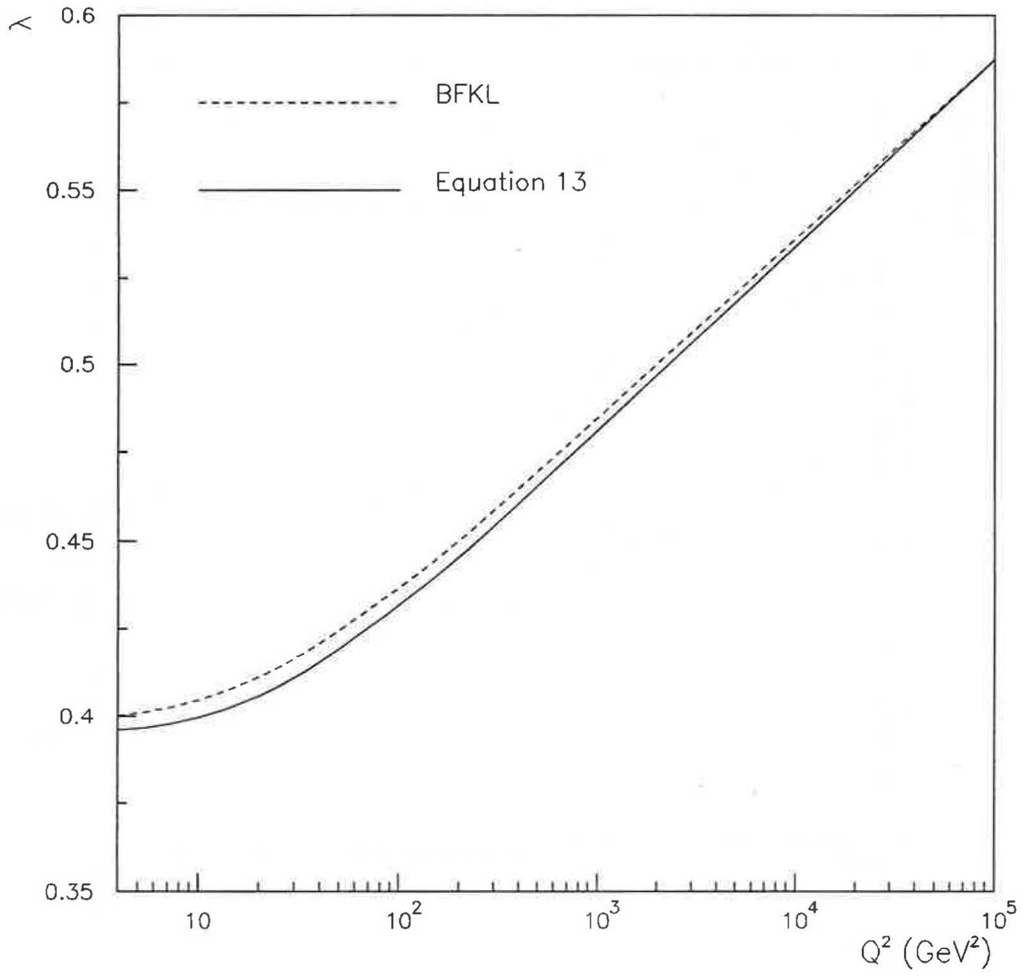


Figure 5



λ
0.6
0.55
0.5
0.45
0.4
0.35
BFKL
Equation 13
10
10^2
10^3
10^4
10^5
Q^2 (GeV^2)