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A New Theory of Critical Flow Velocities in Superfluid ^{4}He

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Abstract

It is shown that the critical flow velocities of superfluid ^{4}He through narrow channels can be understood using only simple quantum mechanics. The predicted variation of critical velocity with channel size is in good agreement with experimental data over range of channel sizes covering three orders of magnitude.

Superfluids are so named because of their ability to flow through narrow tubes or thin films without dissipation of energy. Experimental data on superfluid flow [1] of liquid ^{4}He through narrow channels shows that the liquid is only a superfluid at flow velocities below a certain 'critical velocity' v_C and that v_C increases as the channel gets narrower. Landau[2] showed that the critical velocity is determined by the nature of the collective excitations (ie those involving all atoms) in the superfluid. In particular he showed that the critical velocity is determined by the relation

$$v_C = (\varepsilon / q)_{\min} \tag{1}$$

where $(\varepsilon / q)_{\min}$ is the minimum value of the ratio ε / q, ε is the energy and q the momentum of the excitation. However the critical velocity derived from the excitation spectrum measured by neutron scattering is ~60 m/sec, orders of magnitude higher than experimental values, which are typically less than 1cm/sec. Atkins[3] has attributed the discrepancy to the creation of excitations in the form of vortex motions in which the liquid rotates about a vortex line in a similar way to the rotation of a whirlpool in an ordinary liquid. Feynman[4] derived a similar expression to Atkins by considering the flow of helium from the end of a slit into a vessel of still liquid. However the theory is in an unsatisfactory state. For example it is asssumed that the vortex motion carries no linear momentum and that the momentum q in equation 1 is the "integrated impulse" required to create the angular momentum of rotation in the fluid. The way in which the linear momentum required to reduce the flow rate is transferred to the liquid is not known. The critical velocity is calculated in terms of the maximum diameter of the vortex ring and the core radius, both of which can only be estimated.

Superfluid helium has strikingly similar properties to superconductors and it has long been evident that these two phenomena must have an underlying common origin. It has recently been shown[5] that the original postulate of London[6], that the 'Bose Condensation' of a macroscopic number of particles into a single momentum state is responsible for both superfluidity and superconductivity, is correct and that when there is a Bose condensate present, a macroscopic wavefunction of the form,

$$\Psi(\vec{r}) = \exp[i\vec{p}.\vec{r}] \tag{2}$$

exists, where $\vec{p}$ is the atomic momentum. The motion of all atoms in the fluid is determined by the phase $S(\vec{r}) = \vec{p}.\vec{r}$ of this macroscopic wavefunction and the momentum is $\vec{p} = \hbar\nabla S$, as is postulated in standard text book treatments of superfluid ^{4}He[7]. In this letter it is shown that the observed critical velocities can be explained by the form of the wavefunction in equation 2, using only simple quantum mechanics.

The macroscopic wavefunction Ψ must satisfy the usual Bohr-Sommerfeld quantisation rule which states that the only orbits allowed are those which include an integral number of de Broglie wavelengths (otherwise the wavefunction will destructively interfere with itself and give zero amplitude for the orbit). Using the de Broglie relation $p = h / \lambda$, where λ is the wavelength and h is Planck's constant, this condition is,

$$\oint \bar{p}.d\bar{r} = nh \tag{3}$$

This condition is satisfied for every path if the wavefunction has the form, (in cylindrical coordinates r, ϕ)

$$\Psi(\bar{r}) = g(r)\exp(in\phi) \tag{4}$$

In this case the momentum is ,

$$\bar{p}\Psi = -i\hbar\nabla\Psi = -i\hbar\left(\frac{\partial}{\partial r}\hat{\bar{r}} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\bar{\phi}}\right)\Psi = \hbar\left(-i\frac{\partial\Psi}{\partial r}\hat{\bar{r}} + \frac{n\Psi}{r}\hat{\bar{\phi}}\right) \tag{5}$$

The term parallel to the unit vector $\hat{\bar{r}}$ does not contibute to the integral in equation 3 , whereas the term parallel to $\hat{\bar{\phi}}$ gives nh, in agreement with equation 3. Providing that $g(r)$ is real, there is no particle flux along the direction $\hat{\bar{r}}$.

For simplicity, we consider flow along a tube of rectangular cross-section and assume that the tube wall interacts with the fluid to create a rotational motion of the atoms around a vortex line. The minimum velocity for this to occur will be when n=1 in equation 3 and when the vortex line is created perpendicular to the widest face of the tube and perpendicular to the direction of flow, since the torque exerted by the tube walls will be a maximum for this case. The problem then reduces to a two-dimensional one (see figure 2) with the vortex core centred at some position (x_0, y_0) and with radius a. It is assumed that the wavefunction is given by equation 4 over the whole volume of the tube, with $g(r)$ a constant except at the tube walls and in the vortex core. The exact form of $g(r)$ in the region near the walls and within the core is unimportant in macroscopically large samples as these regions are of microscopic dimensions and can be neglected. It follows from equation 5 that a wavefunction of the form $\Psi = \exp(i\phi)$ implies that the velocity distribution in the fluid is of the form $v = h / (Mr)$, with the velocity always directed perpendicular to the radius vector as shown in figure 1.

The Landau condition requires that the energy and momentum of the excitation are calculated in a frame of reference in which the liquid is stationary. The integrated momentum component parallel to the flow over the tube volume in this frame is,

$$q = \int_\Omega \rho v d\Omega = \frac{\rho h}{M}\int_\Omega \frac{\sin\phi}{r} dr d\phi \tag{6}$$

where the integration volume Ω excludes the core region of radius a. The kinetic energy of the excitation is,

$$\varepsilon = \int_\Omega \frac{\rho v^2}{2} d\Omega = \frac{\rho h^2}{2M^2}\int_\Omega \frac{dr d\phi}{r} \tag{7}$$

We assume that the critical velocity is given by equation 1 and that ε and q can be calculated using the last two equations.

Assuming a tube of length L and width d, (see Figure 2) the vortex line must be situated midway along the tube at $x_0 = L/2$, otherwise the liquid will gain a net momentum perpendicular to the direction of flow. This is not possible since any viscous forces must be parallel to the direction of flow. The ratio ε / q is a minimum when the vortex is as close to the tube walls as possible (ie when $y_0 = a$ or $y_0 = d - a$). This is easily shown numerically, but is also evident since virtually all the liquid will be flowing in the same direction in this case (ie to the left in figure 3) and the momentum q will be a maximum. The minimum distance of approach of the vortex line must such that the flow velocity at the walls is less than the velocity of sound in the liquid, otherwise sound waves will be generated at the tube walls and the friction between the walls and the vortex will prevent the generation of the vortex. This suggests a value for the core radius of $a = \hbar / (Mc)$=0.794Å, where $c \sim 200$ m/sec, is the velocity of sound and this value was used in the calculations. This is relatively close to the value a=1.2Å derived by Rayfield and Reif[8] from measurements on the velocities of ions trapped in vortex rings. The results are insensitive to the exact values of a or x_0.

Assuming that the vortex core is at $(L/2, a)$, the integrals in equations 6 and 7 are

$$q = \frac{2h\rho}{M}\left[I_q(L/2, d-a, a) + I_q(L/2, a, a)\right] \tag{8}$$

where

$$I_q(L, w, a) = \left[\frac{w^2}{L} + \frac{\pi w}{2} - w\tan^{-1}\left(\frac{w}{L}\right) - a\right] \tag{9}$$

and

$$\varepsilon = \frac{h^2}{M^2}\left[I_\varepsilon(L/2, d-a, a) + I_\varepsilon(L/2, a, a)\right] \qquad (10)$$

where

$$I_\varepsilon(L, w, a) = \int_0^{\tan^{-1}(w/L)} \log\left(\frac{L}{a\cos\phi}\right)d\phi + \int_{\tan^{-1}(w/L)}^{\pi/2} \log\left(\frac{w}{a\sin\phi}\right)d\phi \qquad (9)$$

The experimental values and the values calculated at $L=10d$, $L=100d$ *and* $L=1000d$ are given in table 1. It can be seen that ε / q is insensitive to the value of L for $L>10d$, varying by less than than 5% for values of L between $10d$ and $1000d$. Figure 3 shows measured values of v_C and values calculated from equations 1, 8 and 9, assuming $L=100d$. It can be seen that the agreement between theory and experiment is excellent when the tube diameter is greater than ~ 10^{-3} cm. The disagreement between theory and data for d values below 10^{-3} cm is probably due to the fact that the relevant d value required by the theory for rectangular apertures is the largest dimension perpendicular to the flow direction, whereas the experimental d values quoted are the minimum aperture dimensions. Points 3 and 4 were measured by flow through slits and 10 by film flow and in these cases the slit height or the film width rather than thickness determines the critical velocity.

To summarise the results of this work, a new first principles theory of critical velocities of flow in superfluid ^{4}He has been presented which gives excellent agreement with experimental data. The theory has the advantage over previous theories that it contains no unknown parameters. A further advantage is that the necessity to estimate the 'integrated impulse' required by previous theories disappears and the way in which the liquid flow is reduced by transfer of linear momentum between the tube walls and the liquid can be seen. The theory relies essentially only on the quantisation of circulation (equation 3) a result which has been known for the last 40 years. The change intoduced by equation 2 is only conceptual, but suggests that it may be possible to explain other striking phenomena in superfluids and superconductors in a simple way.

Table 1. Measured and Calculated Critical Flow Velocites.

The measured critical velocites (which were compiled by Wilks[1]) are given as a function of channel width d in columns 1 to 3. Columns 3 to 6 contain the values calculated from equations 1,8 and 10 for three ratios of L/d.

Measurement	d (c.m.)	Measurement (cm/sec)	Theory $L=10d$	Theory $L=100d$	Theory $L=1000d$
1. Staas and Taconis [9]	0.026	0.42	0.333	0.323	0.320
2. Kidder and Fairbank[10]	0.11	0.12	8.57×10^{-2}	8.32×10^{-2}	8.24×10^{-2}
3. Winkel et al[11]	4.3×10^{-5}	13	122	119	118
4. " "	3.1×10^{-4}	8	20.3	19.8	19.6
5. Brewer and Edwards[12]	5.2×10^{-3}	1.0	1.50	1.46	1.44
6. " "	0.011	0.5	0.746	0.724	0.717
7. Vinen[13]	0.24	0.051	0.0410	0.0398	0.0394
8. "	0.4	0.033	0.0253	0.0245	0.0243
9. Peshkov and Stryukov[14]	0.38	0.02	0.0265	0.0258	0.0255
10. Wilks	2×10^{-6}	30	1820	1778	1764
11. Atkins	0.021	0.62	0.406	0.395	0.391
12. "	0.069	0.26	0.133	0.129	0.128
13. Heikkila and Hallet[15]	0.1	0.07	0.0937	0.0910	0.0902

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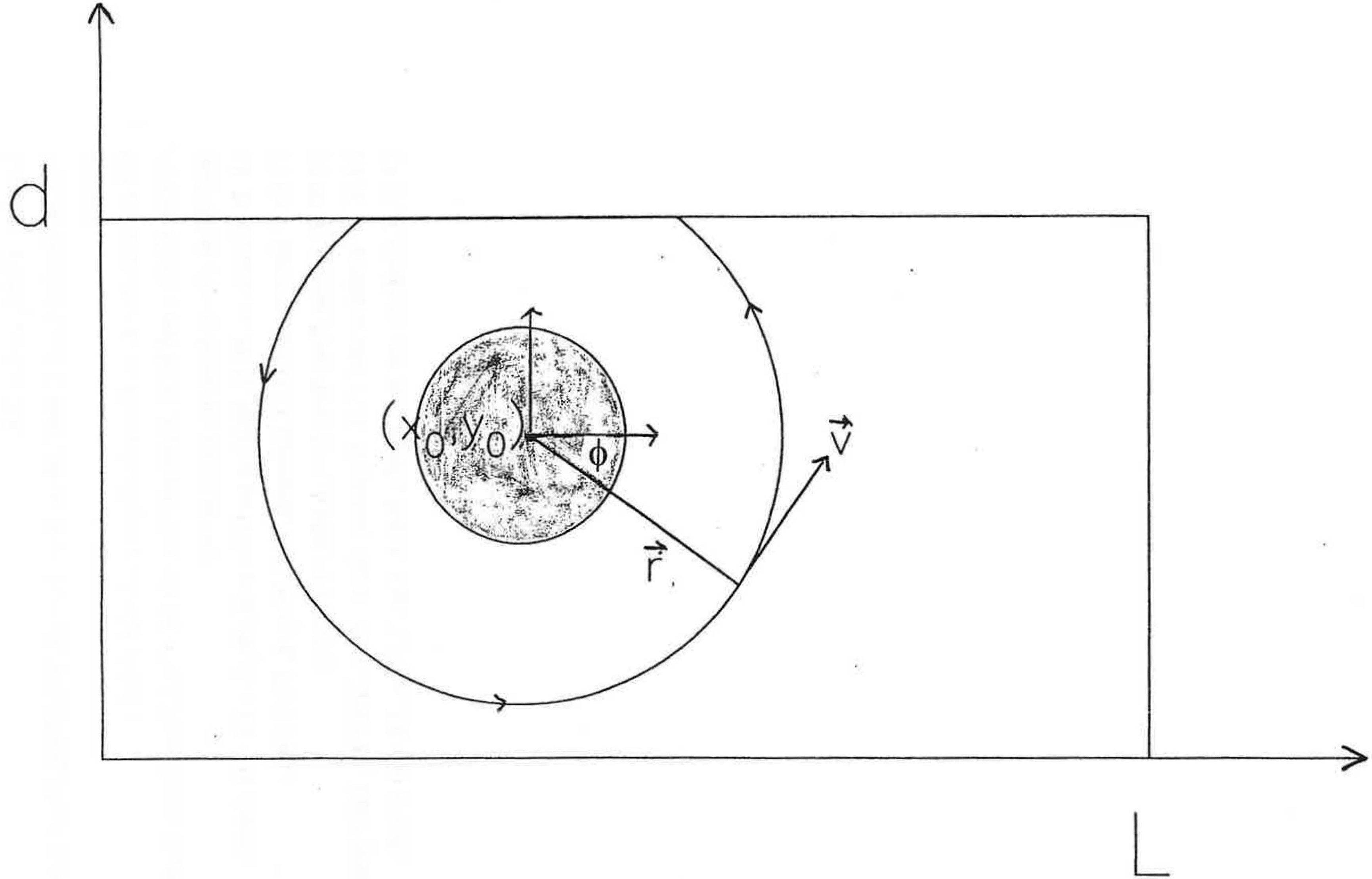

Figure 1. Flow of helium around a vortex line. The line is perpendicular to the plane and centred at the point (x_0, y_0). The flow velocity is perpendicular to the radius vector r and the component of velocity along the tube length is $v \sin \vartheta$. The shaded core region is excluded from the integrations in equations 6 and 7.

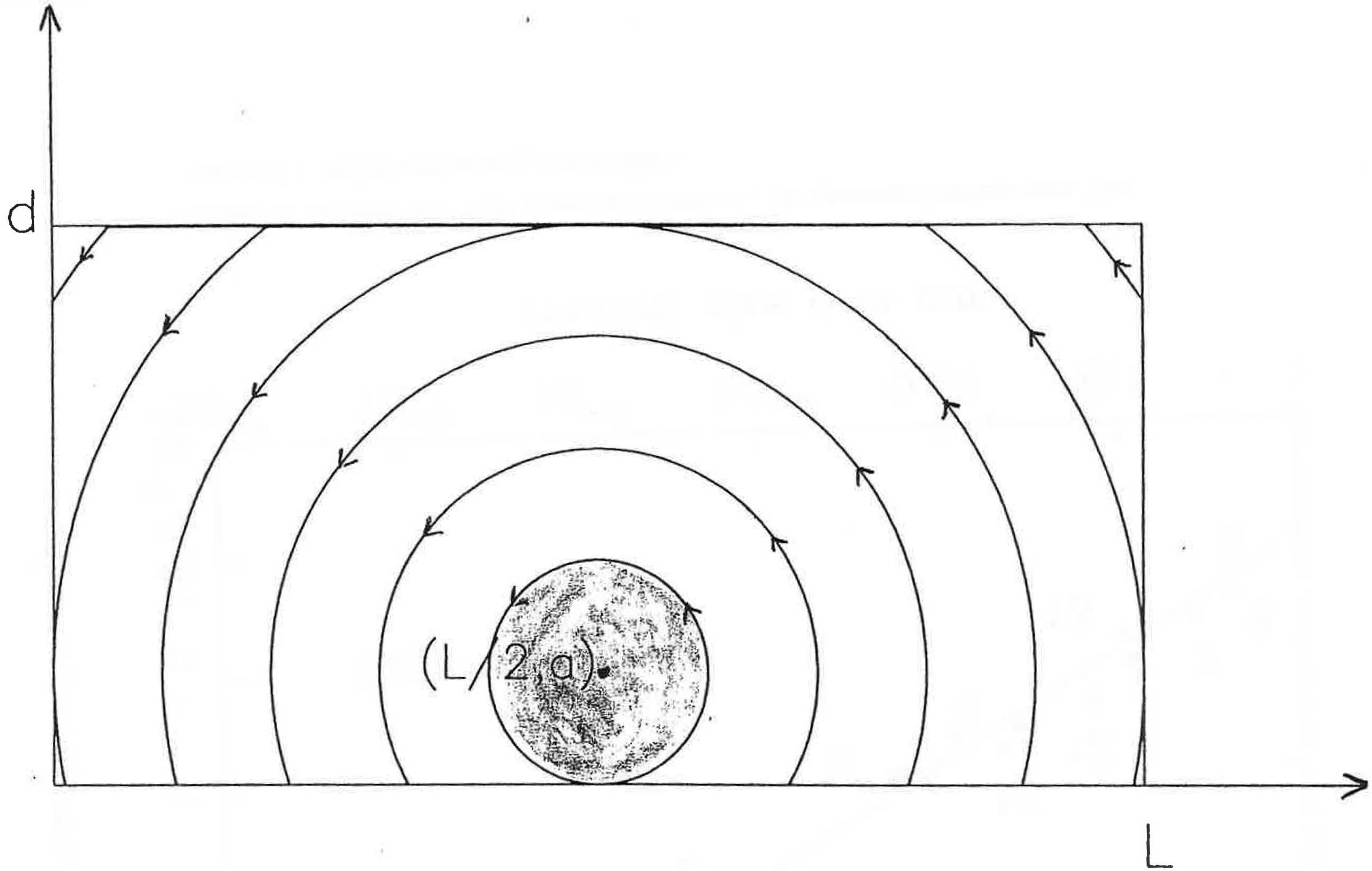

Figure 2. Position of vortex line for minimum value of ε / q. The line is centred at $(L/2, a)$. The direction of the flow of the liquid is also shown.

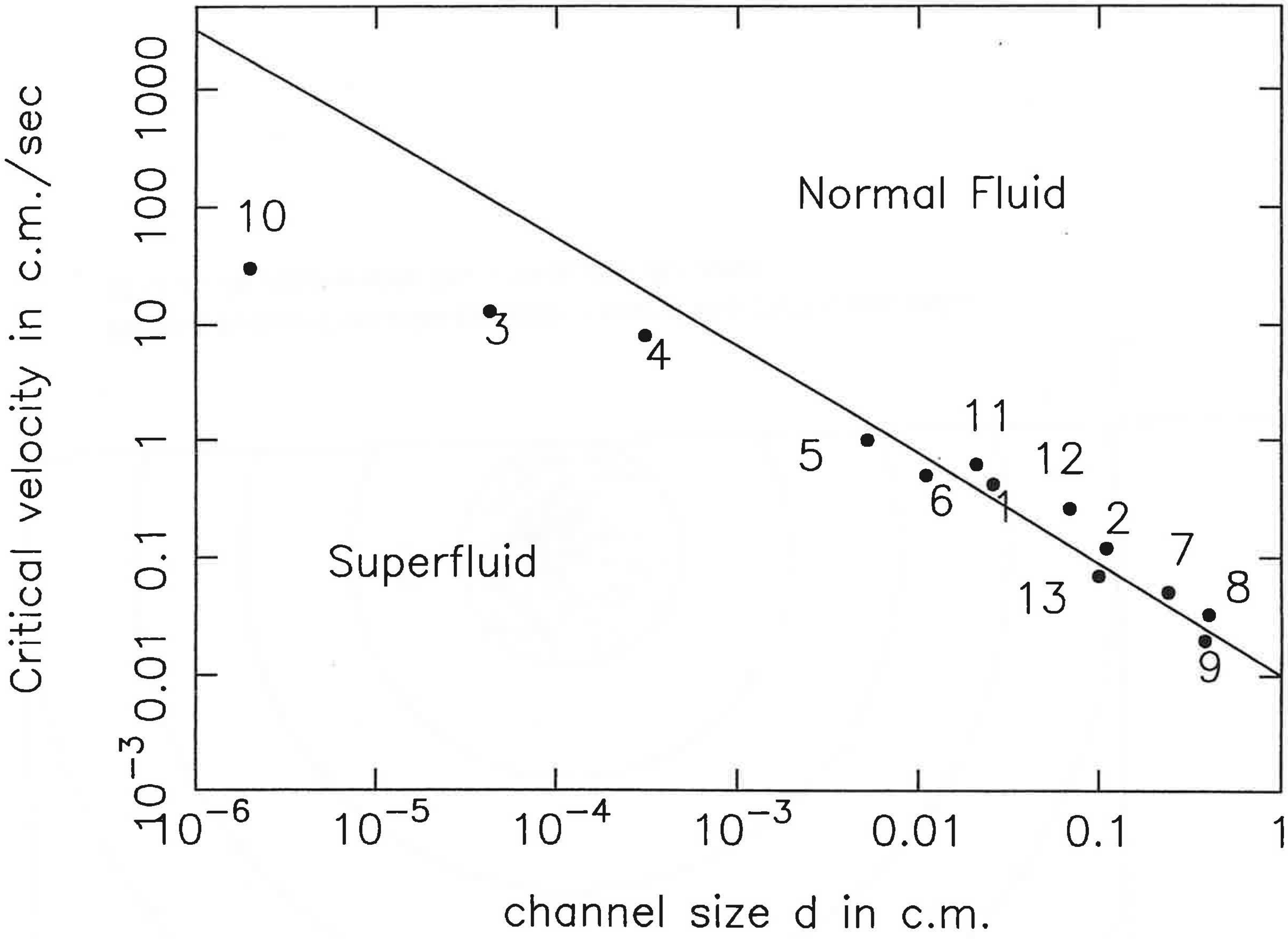

Figure 3. Experimental values of critical velocities v_c for apertures of various sizes. The values of v_c and references are given in table 1.