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Abstract

The thermoviscoelastic behaviour of a composite material is represented as a mixed linear-nonlinear convolution form of the Onsager equations. A vector multidinensional Volterra functional series expansion is used to describe the thermoviscoelastic constitutive relationships. Linear and mixed linear-nonlinear moment hierarchy time series analyses are performed to investigate the nature of the thermoviscoelastic phenomena. A tractable set of simultaneous equations with well behaved coefficients can be generated by taking time series moments of a suitably truncated Volterra series expansion. This moment hierarchy is used to determine the dynamic and steady state thermoviscoelastic properties of the composite. Statistical analyses on out of sample predictions made using the estimated response functions demonstrate that the characterisation provides a good description of the thermoviscoelastic process. The transport coefficients are estimated and used to determine the flux-force surfaces which characteries the behaviour of the composite. These surfaces illustrate that Curie's principle for combined thermodynamic phenomena does not hold for the thermoviscoelastic process. The thermoviscoelastic behaviour of composite materials under fatigue loadings is weakly nonlinear. The estimated response function values can be used to develop either an empirical field theory of the phenomena or alternatively be used in the design process.

Introduction

It has long been know that a relationship exists between the mechanical deformation and the thermal energy in a material [1]. The thermoviscoelastic phenomena was first considered theoretically by Lord Kelvin [2]. This relationship was first used by Belgen [3] to determine the stress field due to the thermoelastic effect in metals. It was assumed that, for an oscillatory fatigue loading the adiabatic approximation is valid and that the thermoviscoelastic behaviour could be detected using infrared thermography. Recent experimental and theoretical work [4] has demonstrated that the adiabatic assumption is not valid for fibre reinforced composites. That work, however, does not consider the finite memory of viscous effects in the material and is only valid for small temperature changes and ignores the effects of fluid interactions, such as convection, at the surface of the body. This work develops a more general formalism for the thermoviscoelastic process and applies novel time series methods to extract the physical properties of the process from the experimental data. The formalism can be appropriately applied to a wide range of material types and is valid for a wide range of applied mechanical and thermal forces.

The main difficulty in characterising thermoviscoelastic behaviour, from an experimental point of view, is the accurate simultaneous measurement of all the variables needed to describe the process. A guide to the choice of observables in an experiment, is that the governing equation used should be in a closed form, for example, the conservation laws in a control volume should contain terms for all of the physical processes which significantly contribute to the process. The characterisation obtained from the data can then be related to the theory underpinning the process. Generally speaking, the ability of current data analysis methods to accurately and consistently quantify all of these interactions under general dynamic boundary conditions is severely limited. Thus, there is a need to develop and refine data analysis techniques that can separate and quantify the thermoviscoelastic processes and their interactions and relate them to an appropriate theoretical description of the process.

The notion of inverse equations and their approximate solution by discretisation is commonly employed for linear ordinary differential equations. A coupled differential representation being developed, which is suitable for the analysis of the thermoviscoelastic process, will be discussed elsewhere. In this work a vector multidimensional convolution form of the Volterra series expansion, suitably truncated, is operated on to obtain a tractable moment hierarchy with well behaved coefficients, from which the dynamical response of the thermoviscoelastic process can be determined.

The solution of the moment hierarchy yields, directly, the Volterra kernel values. These kernel functions are usually known as the linear and non-linear response functions and are fundamental properties of the physical system being studied. The convolution form of representation is an extension of the Taylor's expansion to processes which possess a finite memory. The vector multidimensional convolution form of the Volterra series is used in the present paper to analyse experimental data for the mixed linear and non-linear properties of complex thermoviscoelastic behaviour. The response function values can be extracted at different times of the sample's life cycle and they represent elements of the life history of the thermoviscoelastic material.

The multidimensional convolution representation developed in the present work relates the components of the observed deformation gradient, $\{\varepsilon_{ij}(t)\}$, to the applied mechanical and local thermal forces and is readily extendible to more complex situations. The formalism assumes that a causal relation exists between the deformation gradients, the forces acting on the body and the fluxes flowing through the body; whether they be mechanical, electrical, thermal or chemical in origin. This indicates that each experimental case should be examined, and a characterisation chosen that represents the causal nature of the interactions between the physical processes.

That is, in the present case the deformation induced by the mechanical forcing and the thermal gradients are characterised by the estimated response function values. The formalism is presented in general terms without specific properties being attributed to the functionals and their coefficients, the response function values. The formalism is then used to analyse specific data and for that case meaning is attributed to the functionals and their coefficients. The formalism developed simultaneously characterises the dynamical properties of the thermal and mechanical process and their mutual interactions. The linear and non linear response functions of the formalism are estimated directly from the experimental data [5-7]. The formalism is then applied to experimental data to analyse the thermoviscoelastic process in resin matrix composites under stochastic loading conditions.

Linear thermoviscoelastic materials

Before details of the multidimensional convolution formalism are given it is of value to outline the underlying methodology with a simple example. Ideal materials deform instantaneously in response to an applied load and have the ability to store energy without dissipation, so that all of its stored energy can be recovered. Real materials have the capacity to both store and dissipate energy, and the response to an applied force will be a fast deformation followed by a slow flow process. In a linear viscoelastic material the strain is directly proportional to the strain field and for a given constant applied stress the strain increases with time. This process is known as creep and when the applied force is reduced, or stopped, there is a period of creep recovery when the material experiences strain decay. This is known as relaxation. The phenomena of relaxation and creep are basic characteristics of viscoelastic materials. Any theory that successfully describes the behaviour of thermoviscoelastic materials should be able to characterise the constitutive relationship between the observed deformation and the forces acting and the fluxes flowing. In addition, the theory should be able to characterise the storage and dissipative processes that simultaneously act in the material.

As an example of the basis of the methodology underlying the treatment of complex materials, consider a one dimensional linear thermoviscoelastic material that is submitted to a history of mechanical forces in the absence of other forces and thermodynamic fluxes. Then the most general linear relationship which characterises the local thermodynamic forces, $\{\underline{\sigma}(t)\}$ and $\{\nabla T(t)\}$, and local thermodynamic fluxes, $\{\underline{\varepsilon}(t)\}$ and $\{\underline{q}(t)\}$ is a vector convolution equation.

For a discrete one dimensional process which possesses a local fading memory of duration μ, then the coupled convolution equations can be expressed as

$$\begin{aligned} \varepsilon(t) &= \sum_{\tau_1=0}^{\mu} J_{\sigma\varepsilon}(\tau_1)\sigma(t-\tau_1) + \sum_{\tau_1=0}^{\mu} J_{\nabla T\,\varepsilon}(\tau_1)\nabla T(t-\tau_1) \\ q(t) &= \sum_{\tau_1=0}^{\mu} J_{\sigma\,q}(\tau_1)\sigma(t-\tau_1) + \sum_{\tau_1=0}^{\mu} J_{\nabla T\,q}(\tau_1)\nabla T(t-\tau_1) \end{aligned} \qquad (7)$$

where τ_1 denotes delay with respect to the time t.

Under steady state conditions these coupled convolution equations reduce to a set of Onsager equations [8], with

$$\begin{vmatrix}\varepsilon \\ q\end{vmatrix} = \begin{vmatrix} J^*_{\sigma\varepsilon} & J^*_{\nabla T\varepsilon} \\ J^*_{\sigma q} & J^*_{\nabla T q}\end{vmatrix}\begin{vmatrix}\sigma \\ \nabla T\end{vmatrix} \qquad (8)$$

where $J^*_{\sigma\varepsilon} = \sum_{\tau_1=0}^{\mu} J_{\sigma\varepsilon}(\tau_1)$, $J^*_{\nabla T\varepsilon} = \sum_{\tau_1=0}^{\mu} J_{\nabla T\varepsilon}(\tau_1)$, $J^*_{\sigma q} = \sum_{\tau_1=0}^{\mu} J_{\sigma q}(\tau_1)$ and $J^*_{\nabla T q} = \sum_{\tau_1=0}^{\mu} J_{\nabla T q}(\tau_1)$.

It is straightforward to form and solve a moment hierarchy from the experimental data in order to determine the response function values, $J_{\sigma\varepsilon}(\tau_1)$, $J_{\nabla T\varepsilon}(\tau_1)$, $J_{\sigma q}(\tau_1)$ and $J_{\nabla T q}(\tau_1)$ which characterise the dynamic and steady state thermoelastic process and also the memory properties of the thermoviscoelastic process.

Thus the heat flux at the surface of the solid under steady state conditions will be given by

$$q = J^*_{\sigma q}\varepsilon + J^*_{\nabla T q}\nabla T$$

In the adiabatic limit, the heat flux vanishes, leaving

$$\nabla T = -\frac{J^*_{\sigma q}}{J^*_{\nabla T q}}\sigma$$

The relationship between a change in stress and temperature in the adiabatic limit is

$$-\left\{\frac{J^*_{\sigma q}}{J^*_{\nabla T q}}\right\}(\sigma_1 - \sigma_2) = (\nabla T_1 - \nabla T_2)$$

and the ratio of transport coefficients is a thermoelastic constant, which can be compared with Lord Kelvin's thermoelastic constant [2,9]

Volterra functional series representation of thermoviscoelasticity

There are many physical processes where the form of the differential equations that govern the observed behaviour are not known. In such cases other representations must be used to describe the physical process. For example, in the fields of thermodynamics, fluid dynamics and elasticity use a truncated Taylor's series expansion representations have been used. When the Taylor's series expansion description is used, the physical laws that describe aspects of the observed behaviour can be based on the values of the coefficients of the ascending order terms in the expansion.

Constitutive equations are expressions which characterise the observed behaviour between forces and fluxes and conservation expressions relate a conserved variable to the constituent variables. For example, an observed thermodynamic flux can be characterised in terms of the observed thermodynamic forces. The empirical coefficients of the Taylor's series expansion describe the steady state transport properties of the process. Such empirical coefficients represent the, so called, steady state gains of the dependent variable to the independent variables and cannot be derived from any fundamental theory, but are estimated directly from the experimental data.

If the thermodynamic flux at a given point in space and instant of time depends on a set of local thermodynamic forces, $\{F_i(t)\}$, then the thermodynamic flux can be written as a multidimensional function of the forces, with

$$f_k(t) = f_k(F_1,\ldots,F_I,t) \qquad (12)$$

This multidimensional function can be written as an ascending multivariate Taylor's series expansion with

$$f_k(t) = \sum_{n=1}^{N}\frac{1}{n!}\sum_{i_1=1}^{I}\cdots\sum_{i_n=i_{n-1}}^{I} L_{f_k i_1 \ldots i_n}\prod_{j=1}^{n} F_{i_j}(t) \qquad (13)$$

where N is the order of the process and where the lowest order transport coefficients are given by

$$L_{f_k i_1} = \left(\frac{\partial f_k}{\partial F_{i1}}\right) \quad \text{and} \quad L_{f_k i_1 i_2} = \left(\frac{\partial f_k^2}{\partial F_{i_1}\partial F_{i_2}}\right)$$

In the linear approximation this expansion reduces to the well known Onsager relations [8].

Equally, the phenomena can be described as an inverse problem in terms of an ascending order of functionals which characterises the physical processes as a mapping between functional spaces. If there is a unique solution to Taylor's series expansion, then, formally at least, it will be in the form of the kernel functions of the inverse mapping.

The inverse mapping associates each value of the dependent physical variable to the finite history of a set of independent physical variables. The emphasis of the inverse problem approach is to identify the form of relationship between the observables and hence establish the laws that govern the process. If the process has a finite memory, of duration μ, then the non-linear nonequlibrium behaviour of a macroscopic thermodynamic process can be described as a functional expansion of physically observable causal time series quantities.

For example, the constitutive equations which describe thermodynamic processes, each thermodynamic flux, $f_k(t)$, can be described as a multidimensional convolution expansion in terms of the local thermodynamic forces acting, and defined as

$$f_k(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \int_{t-\mu}^{t} d\tau_1 \cdots \int_{t-\mu}^{t} d\tau_n \, J_{f_k F_{i_1} \cdots F_{i_n}}(\tau_1, \ldots, \tau_n) \prod_{j=1}^{n} F_{i_j}(t-\tau_j) \qquad (14)$$

where N is the order of the system, where t denote time and where the τ_j's denotes time delay with respect to the time t.

A discrete approximation to the multidimensional convolution expansion can be defined as

$$f_k(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{i_1=1}^{I} \cdots \sum_{i_n=i_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{f_k F_{i_1} \cdots F_{i_n}}(\tau_1, \ldots, \tau_n) \prod_{j=1}^{n} F_{i_j}(t-\tau_j) \qquad (15)$$

where N is the order of truncation of the system, where t denote time, where I is the number of forces and where the τ_j's denote time delay with respect to the time t. The kernel function values, $J_{f_k F_{i_1} \cdots F_{i_n}}(\tau_1, \ldots, \tau_n)$, characterise the behaviour of $f_k(t)$, in terms of the forces, $\{F_i(t)\}$.

On discretisation, the truncated Volterra series remains ill posed in the sense that there are too many unknown coefficients to solve for. Thus, the approximate method of discretisation used for the linear case cannot by themselves be used to solve the Volterra series. A tractable set of simultaneous equations with well behaved coefficients can be generated by taking time series moments of a suitably truncated Volterra series expansion.

Integrating each kernel function yields the linear and non-linear gain between the dependent and independent variables [5] , with

$$L_{f_k F_{i_1} \ldots F_{i_n}} = \sum_{\sigma_1=0}^{\mu} \ldots \sum_{\sigma_n=0}^{\mu} J_{f_k F_{i_1} \ldots F_{i_n}} (\tau_1, \ldots, \tau_n) \tag{16}$$

That is, the integral of the kernel function values yields the steady state gain between the observables and is equivalent to the ascending order transport coefficients of the phenomena being characterised.

The conditioning can be improved by statistical averaging and the use of operators allows a set of tractable equations with average variable values to be generated. Equation (15) is operated on with a series of averaging operators, one for each permutation of delayed applied forces $\langle F_i(t-\tau_j)* \rangle$, to give a moment hierarchy of the form [10]

$$\left\langle \prod_{j=1}^{m} F_{r_j}(t-\zeta_j) f_k(t) \right\rangle = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{K} \ldots \sum_{r_n=r_{n-1}}^{K} \sum_{\tau_1=0}^{\mu} \ldots \sum_{\tau_n=0}^{\mu} J_{f_k F_{r_1} \ldots F_{r_n}} (\tau_1, \ldots, \tau_n) \left\langle \prod_{j=1}^{m} F_{r_j}(t-\zeta_j) \prod_{i=1}^{n} F_{r_i}(t-\tau_i) \right\rangle \tag{17}$$

where <*> denotes the averaging operation. The moment hierarchy can be rewritten in the obvious matrix form $\underline{C} = \underline{\underline{M}}\underline{h}$ where $\underline{\underline{M}}$ is a square matrix whose elements are the auto-moments of the applied forces $\{F_i(t-\tau_j)\}$, where $\underline{C}$ is a column vector whose elements are the cross moments between the thermodynamic flux, $\{f_k(t)\}$ and the applied forces $\{F_i(t)\}$ and where $\underline{h}$ is a column vector whose elements are the kernel function values of the mapping between $\{F_i(t)\}$ and $\{f_k(t)\}$.

If the matrix $\underline{\underline{M}}$ is non-singular then $\underline{h} = (\underline{\underline{M}})^{-1}\underline{C}$ has a unique solution. If however $\underline{\underline{M}}$ is singular, then $\underline{\underline{M}}$ is rank deficient and some of its rows will be linearly dependent on the others. If the same relationship holds between the corresponding elements of the column vector $\underline{C}$, the solution will not be unique, indeed an infinity of solutions will exist. If this is not the case then the matrix expression is not consistent and there will not be any solution. Thus, in general, there may be a unique solution, an infinite number of solutions or no solution. However, given the construction of the moment values used in the moment hierarchy, the rows of $\underline{\underline{M}}$ will be linearly independent of each other, thus the matrix will usually be non-singular and have a unique solution. This will be true for many mixed stochastic and deterministic processes.

The thermoviscoelastic effects can arise when simultaneous mechanical and thermal forces act on a solid body. The physical observables of thermoviscoelasticity can be represented as a vector valued functional series expansion which relate the components of the stress field, $\{\varepsilon_i(t)\}$, and heat flux, $\{q_i(t)\}$, to the mechanical, $\{\sigma_j(t)\}$, and thermal, $\{\nabla T_j(t)\}$, forces acting. The discrete form of the coupled functional expansions is defined as

$$q_i(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{I} \cdots \sum_{r_n=r_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{q_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n) \prod_{k=1}^{n} \theta_{r_k}(t-\tau_k)$$

$$\varepsilon_i(t) = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{I} \cdots \sum_{r_n=r_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{\varepsilon_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n) \prod_{k=1}^{n} \theta_{r_k}(t-\tau_k) \quad (18)$$

where $\{\tau_i\}$ denote time delay, where N is the order of the expansion and I is the number of independent observables (two in the present case) and where $\prod_{k=1}^{n} \theta_{r_k}(t-\tau_k)$ denotes the delayed product of the independent variables $\{\sigma_j(t-\tau_k)\}$ and $\{\nabla T_j(t-\tau_k)\}$. The estimated response function values $J_{q_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)$ and $J_{\varepsilon_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)$ not only characterise the observed behaviour, but they also yield the dynamical and steady state properties of the composite under study and represent the solutions to the equations which describe the process [6] and provide physical insight into the nature of the phenomena.

The moment hierarchy in the thermoviscoelastic case is

$$\langle \prod_{p=1}^{m} \theta_{s_p}(t-\varsigma_p) q_i(t) \rangle = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{I} \cdots \sum_{r_n=r_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{q_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n) \langle \prod_{p=1}^{m} \theta_{s_p}(t-\varsigma_p) \prod_{k=1}^{n} \theta_{r_k}(t-\tau_k) \rangle$$

$$\langle \prod_{p=1}^{m} \theta_{s_p}(t-\varsigma_p) \varepsilon_i(t) \rangle = \sum_{n=1}^{N} \frac{1}{n!} \sum_{r_1=1}^{I} \cdots \sum_{r_n=r_{n-1}}^{I} \sum_{\tau_1=0}^{\mu} \cdots \sum_{\tau_n=0}^{\mu} J_{\varepsilon_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n) \langle \prod_{p=1}^{m} \theta_{s_p}(t-\varsigma_p) \prod_{k=1}^{n} \theta_{r_k}(t-\tau_k) \rangle \quad (19)$$

Here the averaged values $\langle \prod_{p=1}^{m} \theta_{s_p}(t - \varsigma_p) q_i(t) \rangle$ and $\langle \prod_{p=1}^{m} \theta_{s_p}(t - \varsigma_p) \varepsilon_i(t) \rangle$ denote the ascending order cross moments and $\langle \prod_{p=1}^{m} \theta_{s_p}(t - \varsigma_p) \prod_{k=1}^{n} \theta_{r_k}(t - \tau_k) \rangle$ denote the ascending order auto moments between the thermodynamic fluxes and forces.

Facilities and analysis of the thermoviscoelasticity experiments

Test coupons were cut from an existing wind turbine blade manufactured by L/M Glassfiber. Their lay up was approximately uniaxial, composed of a resin matrix encapsulating a woven 0°-90° fabric with a 90/10 ratio of weave. In addition a curved plate section, cut from an existing turbine blade was tested. The test coupons and curved plate were studied under uniaxial fatigue loadings. Dynamic fatigue loading is normally used to investigate the nature of the failure mechanisms of the material under study. The most commonly used excitation function is the sinusoid which is used in the present work. Such accelerated lifetime methods provide fatigue life endurance and fatigue limit, but have the disadvantage that the results can be unrealistic, due, for example, to the heating effects in the composite. This particular shortcoming is used to advantage in the present work to study the simultaneous action of the mechanical and thermal forces and fluxes.

The applied force was measured with a balanced load cell, the stress measurements were made with a strain gauge, the heat flux reaching the surface was measured with heat flux mats and the thermal gradient within the solid was measured with thermocouples. During the series of experiments it was observed that the heat flux and temperature gradient were sensitive to the convective and radiative forces acting at the surface of the solid. In the present work these effects are assumed to be small. However, an experimental study is just about to commence that will quantify the effect on the transport coefficients determined from the data. Time series measurements were collected with a personal computer through a data collection card. For each separate experiment some 4000 time series points for each sensor were collected. Samples of the data are shown in figure 1. Of each 4000 data points measured some 500 in sample points were used to estimate the response factor values of the process and a further 3000 out of sample points were compared with the predicted values of the stress field. The stress being predicted with the estimated response function values and the measured applied force and temperature gradient values. This enabled the accuracy and consistency of the estimated response function values to be determined. These response function values were then used to predict the behaviour of the sample under a different loading regime, thus testing the nature of the solutions determined by the moment hierarchy method.

The time series values of the stress field were considered as 1) a linear function of the applied load and the local temperature gradient and 2) a mixed linear and non-linear function of the applied load and temperature within the solid. In each case the properties of the process were characterised with 500 data points. The response functions estimated with these 500 points were then used to predict the future, out of sample, behaviour of the stress field for 3000 points.

These predicted values of stress, $\{\varepsilon_p(t)\}$, were then statistically compared with the observed values, $\{\varepsilon(t)\}$. It should be stressed that during the prediction phase no use was made of the observed stress values. This provides a quantitative measure of the quality of the response function characterisation of the thermoviscoelastic process. The accuracy of the predicting ability was determined by the Student's t-test values for the differences between the actual, $\{\varepsilon(t)\}$, and predicted $\{\varepsilon_p(t)\}$, time series sequences. An example of a sample prediction with the measured values is shown in figure 2 together with the differences with no obvious sign of dispersion between the predicted and the measured points.

The data were analysed for estimates of the response function values of

$$
\begin{aligned}
q_i(t) &= \sum_{n=1}^{N}\frac{1}{n!}\sum_{r_1=1}^{I}\cdots\sum_{r_n=r_{n-1}}^{I}\sum_{\tau_1=0}^{\mu}\cdots\sum_{\tau_n=0}^{\mu} J_{q_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)\prod_{k=1}^{n}\theta_{r_k}(t-\tau_k) \\
\varepsilon_i(t) &= \sum_{n=1}^{N}\frac{1}{n!}\sum_{r_1=1}^{I}\cdots\sum_{r_n=r_{n-1}}^{I}\sum_{\tau_1=0}^{\mu}\cdots\sum_{\tau_n=0}^{\mu} J_{\varepsilon_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)\prod_{k=1}^{n}\theta_{r_k}(t-\tau_k)
\end{aligned}
\qquad (18)
$$

for N=1 and 2.

Linear and mixed linear-quadratic analyses yielded the response function values of the thermoviscoelastic process. The Student's t-test values for the difference between the predicted and measured stress values are given in table 1 below.

Table 1: Student's t-test value for the 0°-90° GRP sample

Force Applied @ 1Hz	Linear vector analysis	Linear-quadratic vector analysis
$1 \le f \le 20$ kN	3.5	0.42
$1 \le f \le 40$ kN	1.4	0.35
$1 \le f \le 60$ kN	2.42	0.53
$-5 \le f \le 25$ kN	0.52	0.40
$-5 \le f \le 45$ kN	2.84	0.79
$-5 \le f \le 65$ kN	2.68	-0.43

The sample statistics for the mixed linear-non-linear analyses lie within the acceptance region of the two tailed Student's t-test, whilst the linear hypothesis is rejected four out of six times. The acceptance range being $-1.96 \le t_{95\%} \le 1.96$. Hence only the vector mixed linear-non-linear representation accurately characterise the observed combined behaviour of the mechanical and thermal forces. The actual test statistic values could be used as strength of evidence for the mixed linear-non-linear representations. However, a more sensible approach would be to analyse a statistically significant number of samples under a variety of different loading conditions.

Integrating each kernel function yields the linear and non-linear gain between the dependent and independent variables, and these were used to generate the surfaces shown in figures 3 and 4 using

$$
\begin{aligned}
q_i &= \sum_{n=1}^{N}\frac{1}{n!}\sum_{r_1=1}^{I}\cdots\sum_{r_n=r_{n-1}}^{I}\left\{\prod_{k=1}^{n}\theta_{r_k}\right\}\left\{\sum_{\tau_1=0}^{\mu}\cdots\sum_{\tau_n=0}^{\mu} J_{q_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)\right\} \\
\varepsilon_i &= \sum_{n=1}^{N}\frac{1}{n!}\sum_{r_1=1}^{I}\cdots\sum_{r_n=r_{n-1}}^{I}\left\{\prod_{k=1}^{n}\theta_{r_k}\right\}\left\{\sum_{\tau_1=0}^{\mu}\cdots\sum_{\tau_n=0}^{\mu} J_{\varepsilon_i\theta_{r_1}\ldots\theta_{r_n}}(\tau_1,\ldots,\tau_n)\right\}
\end{aligned}
\qquad (19)
$$

respectively, where N=1 and 2 are the order of truncation the system in each case, and where μ is the finite memory of the process.

Until the present work, no simultaneous estimates of Young's modulus and the thermoviscoelastic transport coefficient have been made, for this reason only estimates of the mechanical transport coefficient are compared with previous estimates. This is reasonable, because it is clear from figures 3 and 4 that the force due to the temperature gradient force relatively small compared with the mechanical force.

Effective values for the Young's modulus were then determined from the estimated response function values. These values are given in table 2 below.

Table 2: Effective Young's modulus for the 0°-90° GRP sample

Force Applied @ 1Hz	Linear vector analysis	Linear-quadratic vector analysis
$1 \leq f \leq 20$ kN	23400 ± 400 N mm^{-2}	23580 ± 400 N mm^{-2}
$1 \leq f \leq 40$ kN	24260 ± 400 N mm^{-2}	23580 ± 400 N mm^{-2}
$1 \leq f \leq 60$ kN	24300 ± 400 N mm^{-2}	23560 ± 400 N mm^{-2}
$-5 \leq f \leq 25$ kN	24540 ± 400 N mm^{-2}	24320 ± 400 N mm^{-2}
$-5 \leq f \leq 45$ kN	24260 ± 400 N mm^{-2}	23920 ± 400 N mm^{-2}
$-5 \leq f \leq 65$ kN	24340 ± 400 N mm^{-2}	23860 ± 400 N mm^{-2}

The Young's modulus values agree within the experimental uncertainties and are self consistent. In addition they lie close to the manufacturers quoted value for the Young's modulus of their typical glass fibre reinforced resins for which E_{11} =22270 N mm^{-2} for the curved plate sample used in the present work.

The results for repeated experiments on the curved plate for a given uniaxial compressive-tensile loadings are presented. As before, sample estimates of the response function values were obtained in sample. Out of sample rolling predictions were obtained from the response function values and force values. These predicted values were then compared with the measured data values. The Student's t-test values determined in that analysis are presented in table 3 below and the effective Young's modulus values are given in table 4 below.

Table 3: Student's t-test value for the curved plate sample

Force Applied @ 1Hz	Linear vector analysis	Linear-quadratic vector analysis
$-15 \le f \le 5$ kN	1.49	0.33
$-15 \le f \le 5$ kN	-4.3	-1.54
$-15 \le f \le 5$ kN	0.76	0.39
$-15 \le f \le 5$ kN	3.5	0.87
$-15 \le f \le 5$ kN	2.43	0.21
$-15 \le f \le 5$ kN	0.13	-0.70

Table 4: Effective Young's modulus for the curved plate specimen

Force Applied @ 1Hz	Linear analysis	Linear-quadratic vector analysis
$-15 \le f \le 5$ kN	27100 ± 700 N mm^{-2}	28860 ± 700 N mm^{-2}
$-15 \le f \le 5$ kN	28400 ± 700 N mm^{-2}	29240 ± 700 N mm^{-2}
$-15 \le f \le 5$ kN	24560 ± 700 N mm^{-2}	30020 ± 700 N mm^{-2}
$-15 \le f \le 5$ kN	28480 ± 400 N mm^{-2}	28700 ± 700 N mm^{-2}
$-15 \le f \le 5$ kN	28440 ± 700 N mm^{-2}	29060 ± 700 N mm^{-2}
$-15 \le f \le 5$ kN	28460 ± 700 N mm^{-2}	31340 ± 700 N mm^{-2}

The sample statistics for the mixed linear-non-linear analyses lie within the acceptance region of the two tailed Students t-test, the linear hypothesis being rejected two out of six times. Hence only the vector mixed linear-non-linear representation accurately and consistently characterise the observed combined behaviour of the mechanical and thermal forces. Again the Young's modulus values agree within the experimental uncertainties and are self consistent and lie reasonably close to the manufacturers quoted value for the Young's modulus of E_{11} =22270 N mm^{-2} for the curved plate sample used in the present work.

Conclusions

The results presented in this paper can be summarised as follows. A theoretical description of thermoviscoelasticity based on a vector form of the truncated Volterra functional expansion and the Onsager equations was developed. A tractable set simultaneous mixed linear-non-linear integral equations were obtained by the use of averaging operators. The resulting moment hierarchy was used to characterise the thermoviscoelastic process in complex materials. Values of the kernel, or response, functions of the Volterra series were estimated from the experimental data using the moment hierarchy.

The moment hierarchy was used to determine the properties of a composite solid under a range of applied loads. The first and second order response functions were estimated from the time series data collected from the heat flux, applied force, strain gauge and temperature gradient values. These estimated response functions were then used to predict the out of sample stress field values. These predictions demonstrated that the response functions provided a good, locally time invariant, representation of the thermoviscoelastic process.

The analysis has demonstrated that the moment hierarchy can extract and isolate linear and ascending order non-linear response functions when the input data are drawn from a stochastic process. The Young's modulus values obtained from the fatigue load data agree with those obtained in static testing procedures. The thermal conductivities obtained in the analysis were typically 50% larger than those obtained in standard hot box measurements. This is due to the continuum of thermal source terms that exist in the mechanically forced solid and a further study into this effect is underway. The stress-strain-temperature gradient and heat flux-strain-temperature gradient surfaces illustrate the Curies principle for thermodynamic processes does not hold for the thermoviscoelastic case.

The nature of one dimensional thermoviscoelasticity was considered. Linear and mixed linear-quadratic non-linear local constitutive representations were used to characterise the thermoviscoelastic process.

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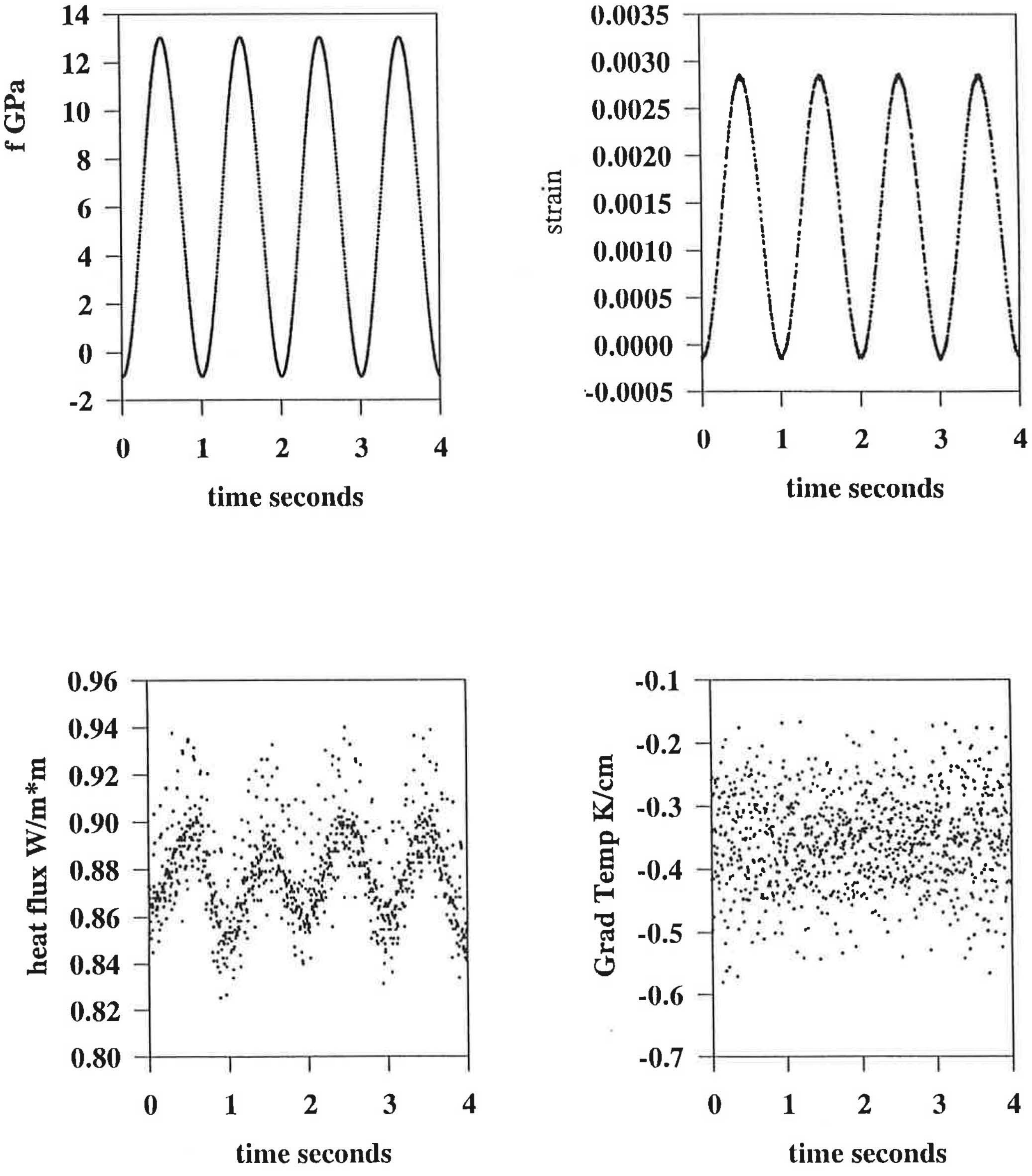
Figure captions

1. Sample of the time series sequences of applied force, the strain, temperature gradient and heat flux measurements in the thermoviscoelastic experiments.

2. Sample of the observed and the predicted stress values.

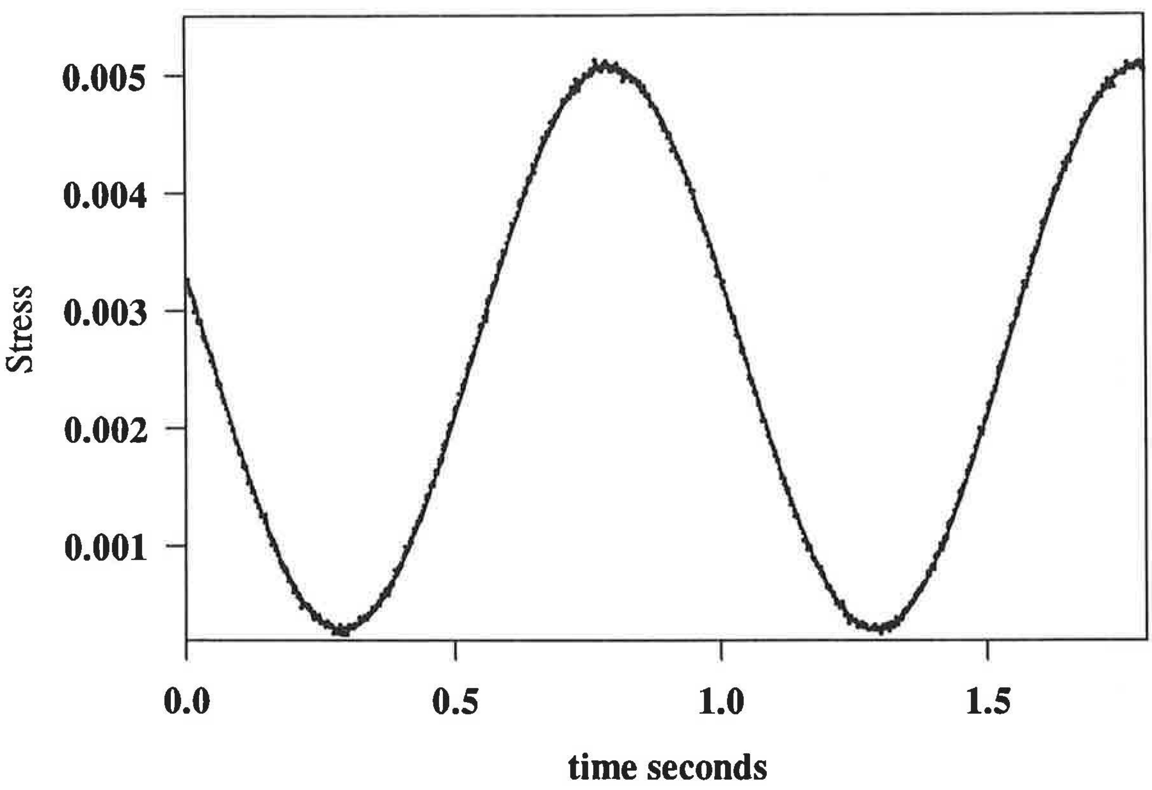
3. The heat flux-applied force-temperature gradient surface generated using the response function values estimated from the thermoviscoelastic data.

4. The stress-applied force-temperature gradient surface generated using the response function values estimated from the thermoviscoelastic data.

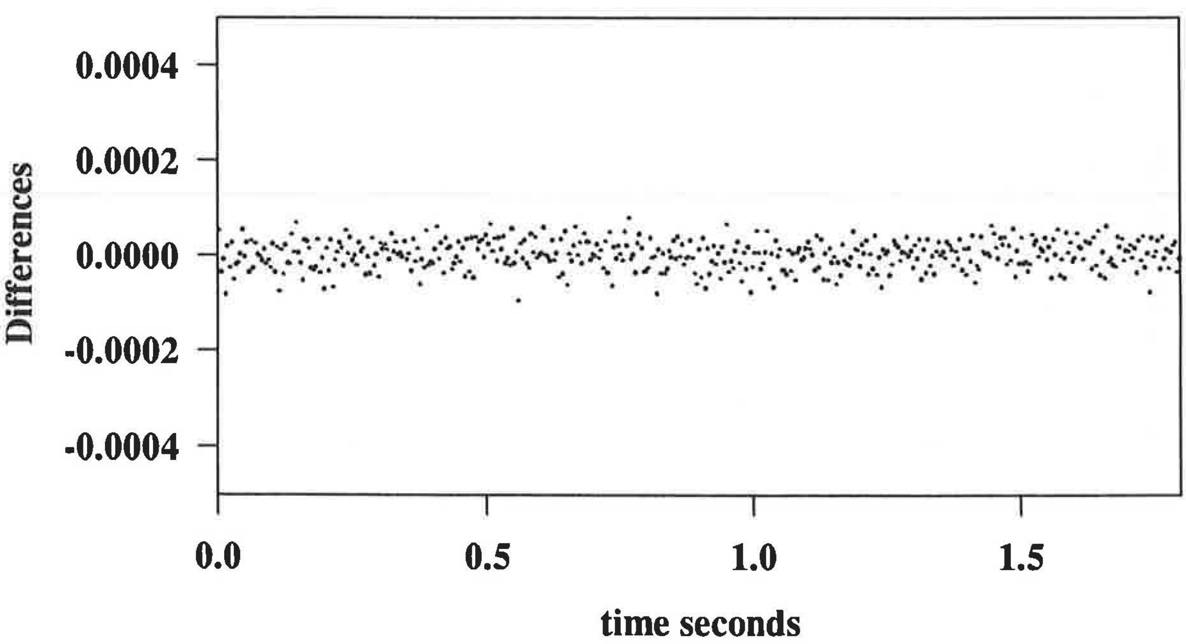


f GPa
time seconds
strain
time seconds
heat flux W/m*m
time seconds
Grad Temp K/cm
time seconds

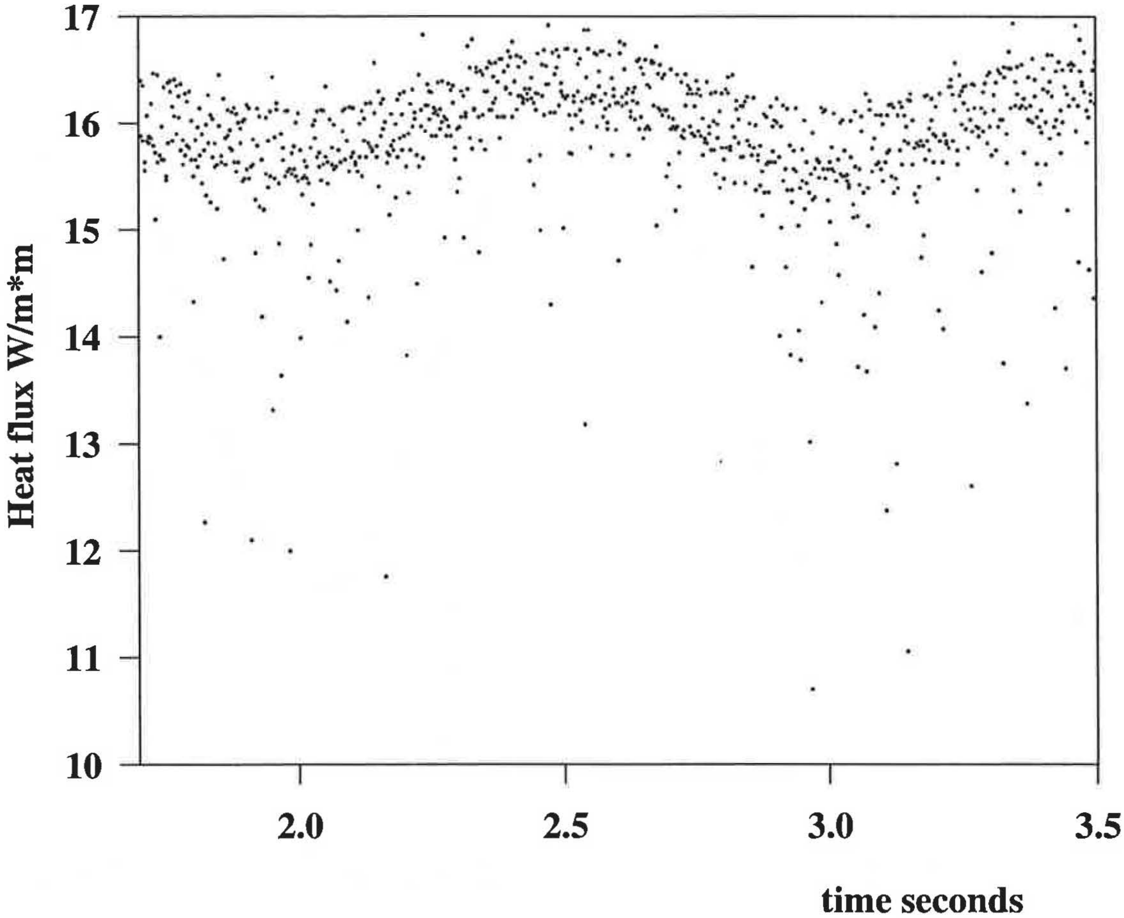
Measured with predicted stress values



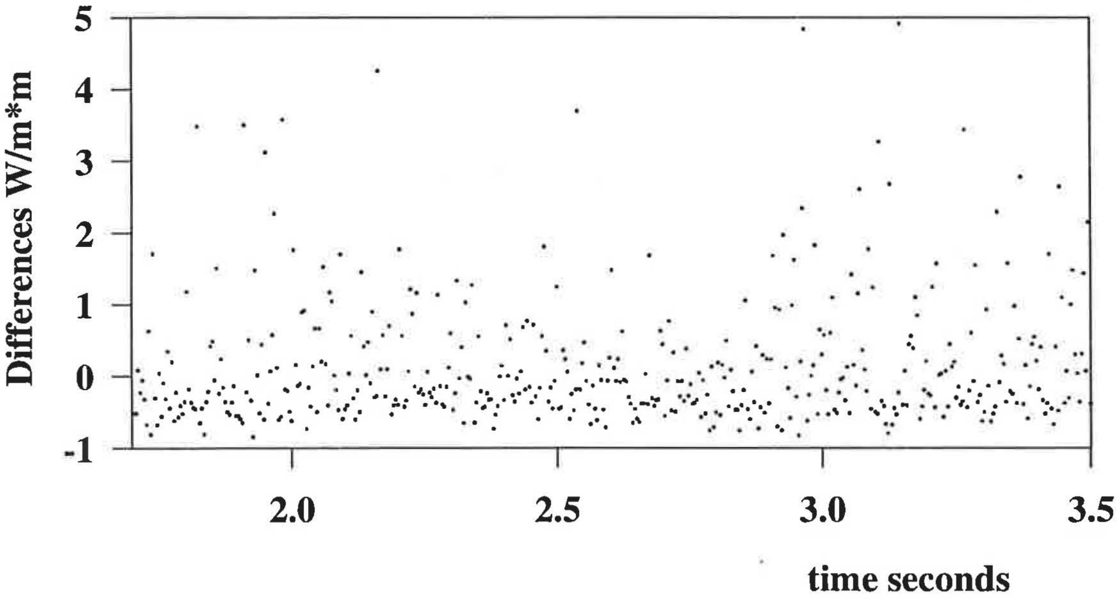
(Measured-predicted) stress difference values

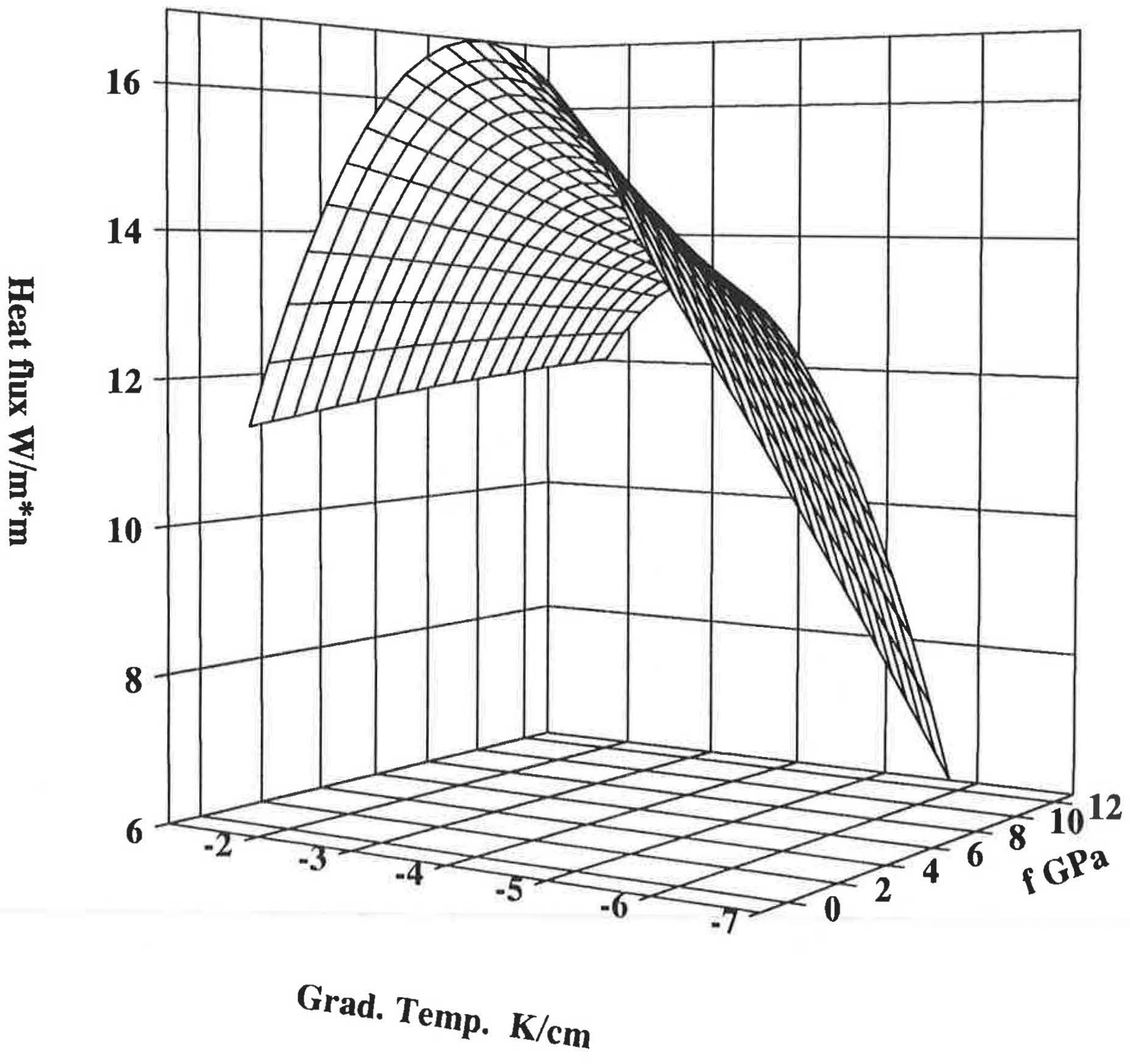


Measured and predicted heat flux values

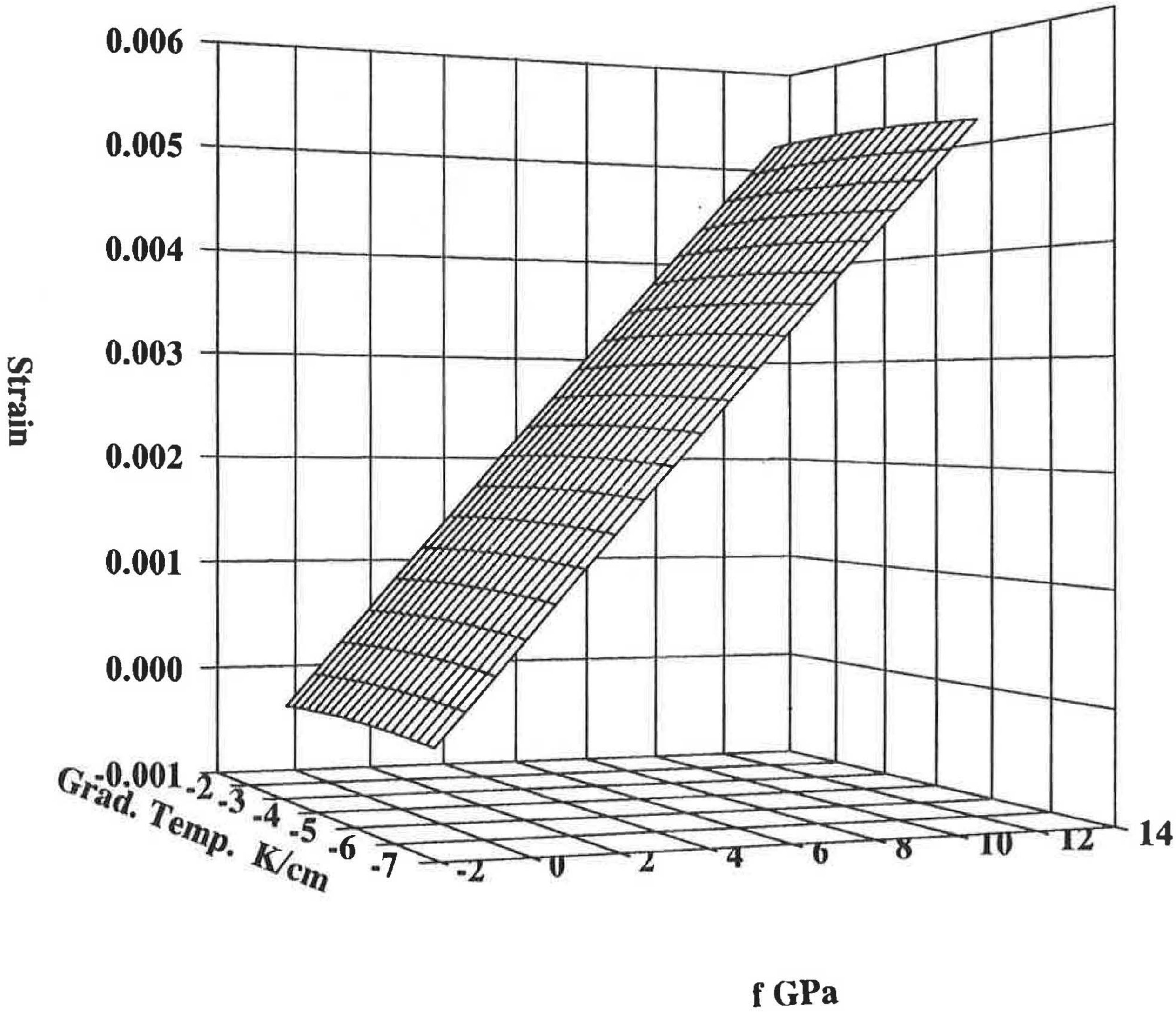


(Measured-predicted) difference values





16
14
12
10
8
6
Heat flux W/m*m
-2
-3
-4
-5
-6
-7
0
2
4
6
8
10
12
f GPa
Grad. Temp. K/cm



0.006
0.005
0.004
0.003
0.002
0.001
0.000
-0.001
Strain
-2 -3 -4 -5 -6 -7
Grad. Temp. K/cm
-2 0 2 4 6 8 10 12 14
f GPa