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Projected Krylov methods for general saddle-point systems

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$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Assumptions:

- B full rank
- saddle point matrix is invertible

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Assumptions:

- B full rank
- saddle point matrix is invertible

Want to use a constraint preconditioner

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix}$$

with a Krylov method

We know that:

- If A is spd on $\ker(B)$, we can use projected conjugate gradients. [Lukšan, Vlcek, 1998] [Gould, Hribar, Nocedal, 2001]
- Otherwise use a non-symmetric solver, e.g. GMRES/biCGstab [Rozloznìk, Simoncini, 2002]

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Q: Can we use the projected 'trick' in other cases?

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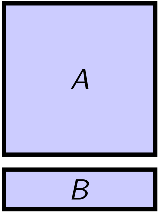
A class of indefinite problems

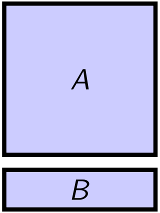
$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$



A class of indefinite problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x + x_0 \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

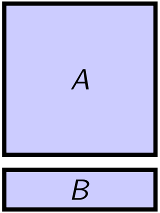
$$\begin{aligned} \hat{x} &= x + x_0 \\ Bx_0 &= b \end{aligned}$$



A class of indefinite problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$c = a - Ax_0$$



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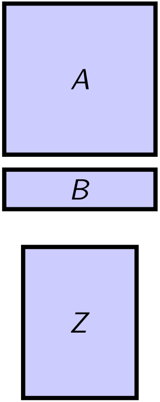
Let Z span the nullspace of B (i.e. $BZ = 0$)

$$BZ = 0 \Rightarrow x = Z\bar{x}$$



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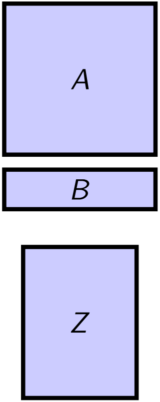

$$BZ = 0 \Rightarrow x = Z\bar{x}$$

$$AZ\bar{x} + \quad B^T y = \quad c$$

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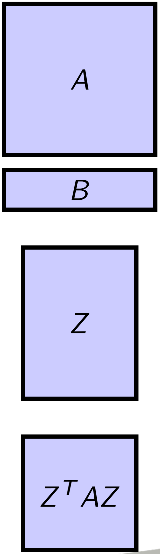
$$Z^T A Z \bar{x} + Z^T B^T y = Z^T c$$



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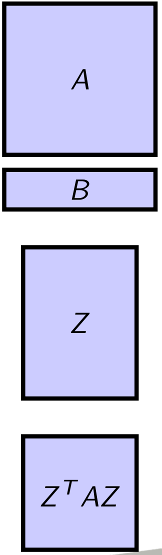
$$Z^T A Z \bar{x} = Z^T c$$

If A symmetric, $Z^T A Z$ symmetric, but (possibly) indefinite - apply MINRES!

Use as a preconditioner $Z^T G Z$, where G is positive definite on the null space of B.



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MINRES applied to the reduced system

$\bar{\mathbf{v}}_1 = Z^T\mathbf{c} - Z^TAZ\bar{\mathbf{x}}_0$
$\bar{\mathbf{z}}_1 = (Z^TGZ)^{-1}\bar{\mathbf{v}}_1$
$\beta_1 = \sqrt{\bar{\mathbf{z}}_1^T\bar{\mathbf{v}}_1}$
$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$
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$\bar{\mathbf{z}}_j = \bar{\mathbf{z}}_j/\beta_j$
$\alpha_j = \bar{\mathbf{z}}_j^T Z^TAZ\bar{\mathbf{z}}_j$
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$$
\begin{aligned}
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&\qquad \mathbf{v}_{j+1} = A Z \bar{\mathbf{z}}_j - \frac{\alpha_j}{\beta_j} \mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}} \mathbf{v}_{j-1} \\
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$$

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- $\mathbf{w}_k = Z\bar{\mathbf{w}}_k$

MINRES applied to the reduced system

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$
$\mathbf{z}_1 = Z(Z^T G Z)^{-1} Z^T \mathbf{v}_1$
$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$
$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$
for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$
$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$
$\mathbf{v}_{j+1} = A\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1}$
$\mathbf{z}_{j+1} = Z(Z^T G Z)^{-1} Z^T \mathbf{v}_{j+1}$
$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$
$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$
$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$
$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$
$\gamma_3 = s_{j-1} \beta_j$
$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$
$\mathbf{w}_{j+1} = (\mathbf{z}_j - \gamma_3 \mathbf{w}_{j-1} - \gamma_2 \mathbf{w}_j) / \gamma_1$
$\mathbf{x}_j = \mathbf{x}_{j-1} + c_{j+1} \eta \mathbf{w}_{j+1}$
$\eta = -s_{j+1} \eta$

- $\mathbf{x}_k = Z\bar{\mathbf{x}}_k$
- $Z^T \mathbf{v}_k = \bar{\mathbf{v}}_k$
- $\mathbf{z}_k = Z\bar{\mathbf{z}}_k$
- $\mathbf{w}_k = Z\bar{\mathbf{w}}_k$

MINRES applied to the reduced system

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$
$\mathbf{z}_1 = Z(Z^TGZ)^{-1}Z^T\mathbf{v}_1$
$\beta_1 = \sqrt{\mathbf{z}_1^T\mathbf{v}_1}$
$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$
for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j$
$\alpha_j = \mathbf{z}_j^T A\mathbf{z}_j$
$\mathbf{v}_{j+1} = A\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1}$
$\mathbf{z}_{j+1} = Z(Z^TGZ)^{-1}Z^T\mathbf{v}_{j+1}$
$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T\mathbf{v}_{j+1}}$
$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$
$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$
$\gamma_2 = s_j\alpha_j + c_{j-1}c_j\beta_j$
$\gamma_3 = s_{j-1}\beta_j$
$c_{j+1} = \gamma_0/\gamma_1;\ s_{j+1} = \beta_{j+1}/\gamma_1$
$\mathbf{w}_{j+1} = (\mathbf{z}_j - \gamma_3\mathbf{w}_{j-1} - \gamma_2\mathbf{w}_j)/\gamma_1$
$\mathbf{x}_j = \mathbf{x}_{j-1} + c_{j+1}\eta\mathbf{w}_{j+1}$
$\eta = -s_{j+1}\eta$

$$\mathbf{z} = Z(Z^TGZ)^{-1}Z^T\mathbf{v}$$
$$\Rightarrow \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}$$

- $\mathbf{x}_k = Z\bar{\mathbf{x}}_k$
- $Z^T\mathbf{v}_k = \bar{\mathbf{v}}_k$
- $\mathbf{z}_k = Z\bar{\mathbf{z}}_k$
- $\mathbf{w}_k = Z\bar{\mathbf{w}}_k$

$$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$$
$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{0} \end{bmatrix}$$
$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$$
$$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$$

for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j / \beta_j$$
$$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$$
$$\mathbf{v}_{j+1} = A\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1}$$
$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{0} \end{bmatrix}$$
$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$$
$$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$$
$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$
$$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$$
$$\gamma_3 = s_{j-1} \beta_j$$
$$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$$
$$\mathbf{w}_{j+1} = (\mathbf{z}_j - \gamma_3 \mathbf{w}_{j-1} - \gamma_2 \mathbf{w}_j) / \gamma_1$$
$$\mathbf{x}_j = \mathbf{x}_{j-1} + c_{j+1} \eta \mathbf{w}_{j+1}$$
$$\eta = -s_{j+1} \eta$$

Projected MINRES

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$$

$$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$$

$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$

$$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$$

$$\gamma_3 = s_{j-1} \beta_j$$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$$\eta = -s_{j+1} \eta$$

MINRES with (indefinite) constraint preconditioner

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0,\ \mathbf{u}_1 = -B\mathbf{x}_0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j,\ \mathbf{g}_j = \mathbf{g}_j/\beta_j$

$\alpha_j = \mathbf{z}_j^T(A\mathbf{z}_j + B^T\mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j\alpha_j + c_{j-1}c_j\beta_j$

$\gamma_3 = s_{j-1}\beta_j$

$c_{j+1} = \gamma_0/\gamma_1;\ s_{j+1} = \beta_{j+1}/\gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1}\left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1}\eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1}\eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0,\ \mathbf{u}_1 = 0$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + \mathbf{g}_1^T \mathbf{u}_1}$

$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j,\ \mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$

$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$

$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1 + 0}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:

If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Claim:
If x_0 chosen so that $Bx_0 = 0$, then both algorithms are identical

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j/\beta_j, \quad \mathbf{g}_j = \mathbf{g}_j/\beta_j$$

$$\alpha_j = \mathbf{z}_j^T(A\mathbf{z}_j + B^T\mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$$

$$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$$

$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$

$$\gamma_2 = s_j\alpha_j + c_{j-1}c_j\beta_j$$

$$\gamma_3 = s_{j-1}\beta_j$$

$$c_{j+1} = \gamma_0/\gamma_1; \; s_{j+1} = \beta_{j+1}/\gamma_1$$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1}\left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1}\eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$$\eta = -s_{j+1}\eta$$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j/\beta_j$, $\mathbf{g}_j = \mathbf{g}_j/\beta_j$

$\alpha_j = \mathbf{z}_j^T(A\mathbf{z}_j + B^T\mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j\alpha_j - c_{j-1}s_j\beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j\alpha_j + c_{j-1}c_j\beta_j$

$\gamma_3 = s_{j-1}\beta_j$

$c_{j+1} = \gamma_0/\gamma_1$; $s_{j+1} = \beta_{j+1}/\gamma_1$

$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1}\left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$

$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1}\eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$

$\eta = -s_{j+1}\eta$

$$\mathbf{z}_j^T A\mathbf{z}_j + \mathbf{z}_j^T B^T \mathbf{g}_j + \mathbf{y}_j^T B\mathbf{z}_j$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T (A\mathbf{z}_j + B^T \mathbf{g}_j) + \mathbf{y}_j^T B\mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

$$\mathbf{z}_j^T A\mathbf{z}_j + \mathbf{z}_j^T B^T \mathbf{g}_j + \mathbf{y}_j^T B\mathbf{z}_j$$
$$= \mathbf{z}_j^T A\mathbf{z}_j$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\quad \mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\quad \alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\quad \beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\quad \gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\quad \gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\quad \gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\quad \gamma_3 = s_{j-1} \beta_j$

$\quad c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\quad \eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0, \ \mathbf{u}_1 = 0$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$$

$$\eta = \beta_1, \ s_0 = s_1 = 0, \ c_0 = c_1 = 1$$

for $j = 1, 2 \ldots$ **until** convergence

$$\mathbf{z}_j = \mathbf{z}_j / \beta_j, \ \mathbf{g}_j = \mathbf{g}_j / \beta_j$$

$$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ B\mathbf{z}_j \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ \mathbf{u}_j \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ \mathbf{u}_{j-1} \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$$

$$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$$

$$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$$

$$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$$

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$$c_{j+1} = \gamma_0 / \gamma_1; \ s_{j+1} = \beta_{j+1} / \gamma_1$$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$$\eta = -s_{j+1} \eta$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + \mathbf{g}_{j+1}^T \mathbf{u}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

  $\mathbf{u}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0,\ \mathbf{u}_1 = 0$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j,\ \mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1} + 0}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$

$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$

$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$

$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0,\ \mathbf{u}_1 = 0$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1,\ s_0 = s_1 = 0,\ c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j,\ \mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$

$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1;\ s_{j+1} = \beta_{j+1} / \gamma_1$

$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$

$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$

$\eta = -s_{j+1} \eta$

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

  $\mathbf{u}_{j+1} = 0$

  $B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T \mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

problem?

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

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$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1} \eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1} \eta$

problem?

unchanged

Assume, $(k \leq j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

$\mathbf{v}_1 = \mathbf{c} - A\mathbf{x}_0$, $\mathbf{u}_1 = 0$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{g}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$$

$\beta_1 = \sqrt{\mathbf{z}_1^T \mathbf{v}_1}$

$\eta = \beta_1$, $s_0 = s_1 = 0$, $c_0 = c_1 = 1$

for $j = 1, 2 \ldots$ **until** convergence

$\mathbf{z}_j = \mathbf{z}_j / \beta_j$, $\mathbf{g}_j = \mathbf{g}_j / \beta_j$

$\alpha_j = \mathbf{z}_j^T A \mathbf{z}_j$

$$\begin{bmatrix} \mathbf{v}_{j+1} \\ \mathbf{u}_{j+1} \end{bmatrix} = \begin{bmatrix} A\mathbf{z}_j + B^T\mathbf{g}_j \\ 0 \end{bmatrix} - \frac{\alpha_j}{\beta_j} \begin{bmatrix} \mathbf{v}_j \\ 0 \end{bmatrix} - \frac{\beta_j}{\beta_{j-1}} \begin{bmatrix} \mathbf{v}_{j-1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{j+1} \\ \mathbf{g}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{j+1} \\ 0 \end{bmatrix}$$

$\beta_{j+1} = \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}}$

$\gamma_0 = c_j \alpha_j - c_{j-1} s_j \beta_j$

$\gamma_1 = \sqrt{\gamma_0^2 + \beta_{j+1}^2}$

$\gamma_2 = s_j \alpha_j + c_{j-1} c_j \beta_j$

$\gamma_3 = s_{j-1} \beta_j$

$c_{j+1} = \gamma_0 / \gamma_1$; $s_{j+1} = \beta_{j+1} / \gamma_1$

$$\begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix} = \frac{1}{\gamma_1} \left(\begin{bmatrix} \mathbf{z}_j \\ \mathbf{g}_{j+1} \end{bmatrix} - \gamma_3 \begin{bmatrix} \mathbf{w}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} - \gamma_2 \begin{bmatrix} \mathbf{w}_j \\ \mathbf{p}_j \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{j-1} \\ \mathbf{y}_{j-1} \end{bmatrix} + c_{j+1}\eta \begin{bmatrix} \mathbf{w}_{j+1} \\ \mathbf{p}_{j+1} \end{bmatrix}$$

$\eta = -s_{j+1}\eta$

$$\begin{aligned} \beta_{j+1} &= \sqrt{\mathbf{z}_{j+1}^T \left(A\mathbf{z}_j + B^T\mathbf{g}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1} \right)} \\ &= \sqrt{\mathbf{z}_{j+1}^T \left(A\mathbf{z}_j - \frac{\alpha_j}{\beta_j}\mathbf{v}_j - \frac{\beta_j}{\beta_{j-1}}\mathbf{v}_{j-1} \right)} \\ &= \sqrt{\mathbf{z}_{j+1}^T \mathbf{v}_{j+1}^{\text{PPMINRES}}} \end{aligned}$$

Assume, $(k \le j)$:

- $\mathbf{u}_k = 0$
- $B\mathbf{z}_k = 0$

$\mathbf{u}_{j+1} = 0$

$B\mathbf{z}_{j+1} = 0$

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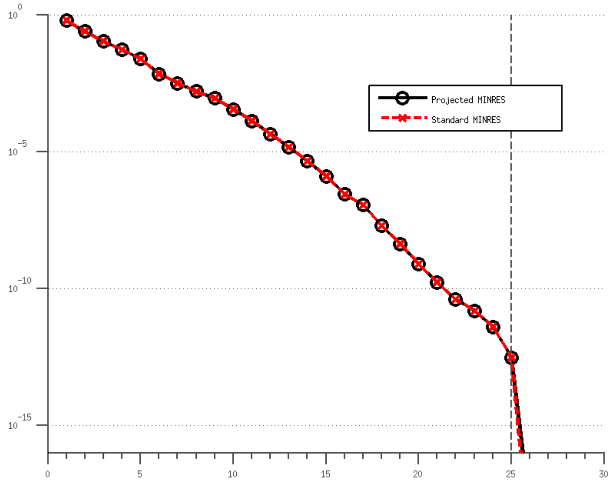
Numerical Comparison



Krylov-Subspace methods

$$\mathcal{A}x = c$$

Petrov-Galerkin approximation:

- trial space S_R
- test space S_L

Find $x_k = S_R z$ where

$$S_L^T \mathcal{A} S_R z = S_L^T c$$

or equivalently

$$r_k = c - \mathcal{A}x_k \perp S_L$$

Choosing S_R/S_L

S_R/S_L based on the Krylov subspace

$$\mathcal{K}_k := \text{span}\{r_0, \mathcal{A}r_0, \ldots, \mathcal{A}^{k-1}r_0\}$$

and the related

$$\mathcal{K}_k^T := \text{span}\{r_0, \mathcal{A}^T r_0, \ldots, (\mathcal{A}^T)^{k-1}r_0\}$$

have proved effective.

We want these bases orthogonalized (or bi-orthogonalized) by Arnoldi/Lanczos.

Lanczos Bi-Orthogonalization

Finds $T_{k,k}$ tridiagonal such that

$$\mathcal{A}V_k = V_{k+1}T_{k+1,k}, \qquad W_k^T V_k = I, \qquad W_k^T \mathcal{A} V_k = T_{k,k}.$$

Set $v_0 = w_0 = 0,\ t_{0,1} = t_{1,0} = 1$
Set $v_1 = c - \mathcal{A}x_0$. Set w_1 such that $\langle v_1, w_1 \rangle = 1$.
for $k = 1, 2 \ldots$ **until** convergence
    $v_{k+1} = \mathcal{A}v_k,\ v_{k+1} = \mathcal{A}^T w_k$
    $t_{k,k} = \langle v_{k+1}, w_k \rangle$
    $v_{k+1} = v_{k+1} - t_{k,k}v_k - t_{k-1,k}v_{k-1}$
    $w_{k+1} = w_{k+1} - t_{k,k}w_k - w_{k-1}$
    $t_{k,k+1} = \langle v_{k+1}, w_{k+1} \rangle$
    if $t_{k,k+1} \neq 0$
        $w_{k+1} = w_{k+1}/t_{k,k+1}$
    end
    $t_{k+1,k} = 1$
end

$$\mathcal{A}x = c$$

Lanczos Bi-Orthogonalization

Finds $T_{k,k}$ tridiagonal such that

$$\mathcal{A}V_k = V_{k+1}T_{k+1,k}, \qquad W_k^T V_k = I, \qquad W_k^T \mathcal{A} V_k = T_{k,k}.$$

Set $v_0 = w_0 = 0,\ t_{0,1} = t_{1,0} = 1$
Set $v_1 = c - \mathcal{A}x_0$. Set w_1 such that $\langle v_1, w_1 \rangle = 1$.
for $k = 1, 2 \ldots$ **until** convergence
    $v_{k+1} = \mathcal{A}v_k,\ v_{k+1} = \mathcal{A}^T w_k$
    $t_{k,k} = \langle v_{k+1}, w_k \rangle$
    $v_{k+1} = v_{k+1} - t_{k,k}v_k - t_{k-1,k}v_{k-1}$
    $w_{k+1} = w_{k+1} - t_{k,k}w_k - w_{k-1}$
    $t_{k,k+1} = \langle v_{k+1}, w_{k+1} \rangle$
    if $t_{k,k+1} \neq 0$
        $w_{k+1} = w_{k+1}/t_{k,k+1}$
    end
    $t_{k+1,k} = 1$
end

Use a pos. def. preconditioner $\mathcal{P} = \mathcal{L}\mathcal{L}^T$.

$$\mathcal{L}^{-1}\mathcal{A}\mathcal{L}^{-T}(\mathcal{L}^T x) = \mathcal{L}^{-1}c$$

Lanczos Bi-Orthogonalization

Finds $T_{k,k}$ tridiagonal such that

$$\mathcal{A}V_k = V_{k+1}T_{k+1,k}, \qquad W_k^T V_k = I, \qquad W_k^T \mathcal{A} V_k = T_{k,k}.$$

Set $v_0 = w_0 = 0$, $t_{0,1} = t_{1,0} = 1$
Set $v_1 = \mathcal{L}^{-1}c - \mathcal{L}^{-1}\mathcal{A}\mathcal{L}^{-T}x_0$. Set w_1 such that $\langle v_1, w_1 \rangle = 1$.
for $k = 1, 2 \ldots$ **until** convergence
  $v_{k+1} = \mathcal{L}^{-1}\mathcal{A}\mathcal{L}^{-T}v_k$, $v_{k+1} = \mathcal{L}^{-1}\mathcal{A}\mathcal{L}^{-T}w_k$
  $t_{k,k} = \langle v_{k+1}, w_k \rangle$
  $v_{k+1} = v_{k+1} - t_{k,k}v_k - t_{k-1,k}v_{k-1}$
  $w_{k+1} = w_{k+1} - t_{k,k}w_k - w_{k-1}$
  $t_{k,k+1} = \langle v_{k+1}, w_{k+1} \rangle$
  if $t_{k,k+1} \neq 0$
    $w_{k+1} = w_{k+1}/t_{k,k+1}$
  end
  $t_{k+1,k} = 1$
end

Use a pos. def. preconditioner $\mathcal{P} = \mathcal{L}\mathcal{L}^T$.

$$\mathcal{L}^{-1}\mathcal{A}\mathcal{L}^{-T}(\mathcal{L}^T x) = \mathcal{L}^{-1}c$$

Lanczos Bi-Orthogonalization

Finds $T_{k,k}$ tridiagonal such that

$$\mathcal{A}V_k = V_{k+1}T_{k+1,k}, \qquad W_k^T V_k = I, \qquad W_k^T \mathcal{A} V_k = T_{k,k}.$$

Set $v_0 = w_0 = 0$, $t_{0,1} = t_{1,0} = 1$
Set $s = c - \mathcal{A}x_0$. Set w_1 such that $\langle s, w_1 \rangle = 1$.
Solve $\mathcal{P}v_1 = s$ for v_1
for $k = 1, 2 \ldots$ **until** convergence
  $s = \mathcal{A}v_k$, $u = \mathcal{A}^T w_k$
  Solve $\mathcal{P}v_{k+1} = s$ and $\mathcal{P}w_{k+1} = u$
  $t_{k,k} = \langle s, w_k \rangle$
  $v_{k+1} = v_{k+1} - t_{k,k}v_k - t_{k-1,k}v_{k-1}$
  $w_{k+1} = w_{k+1} - t_{k,k}w_k - w_{k-1}$
  $t_{k,k+1} = \langle s, w_{k+1} \rangle$
  if $t_{k,k+1} \neq 0$
    $w_{k+1} = w_{k+1}/t_{k,k+1}$
  end
  $t_{k+1,k} = 1$
end

$$v_j = \mathcal{L}^{-T} v_j$$

$$w_j = \mathcal{L}^{-T} w_j$$

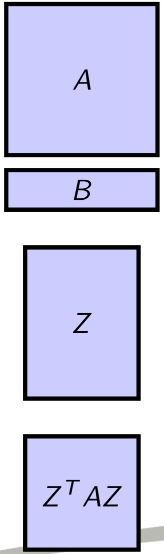
Apply to the null space

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

Let Z span the nullspace of B (i.e. $BZ = 0$)

$$Z^T A Z \bar{x} = Z^T c$$

Use $Z^T G Z$ as a preconditioner, where G is positive definite on the null space of B.



Preconditioned Lanczos Bi-Orthogonalization

Set $v_0 = w_0 = 0$ $t_{0,1} = t_{1,0} = 1$
Set $s = Z^T c - Z^T AZx_0$. Set w_1 such that $\langle s, w_1 \rangle = 1$.
Solve $Z^T GZv_1 = s$ for v_1
for $k = 1, 2 \ldots$ **until** convergence

$s = Z^T AZv_k$ and $u = Z^T AZ^T w_k$
Solve $Z^T GZv_{k+1} = s$ and $Z^T GZw_{k+1} = u$
$t_{k,k} = \langle s, w_k \rangle$
$v_{k+1} = v_{k+1} - t_{k,k} v_k - t_{k-1,k} v_{k-1}$
$w_{k+1} = w_{k+1} - t_{k,k} w_k - w_{k-1}$
$t_{k,k+1} = \langle s, w_{k+1} \rangle$
if $t_{k,k+1} \neq 0$

$w_{k+1} = w_{k+1}/t_{k,k+1}$

end
$t_{k+1,k} = 1$

end

Preconditioned Lanczos Bi-Orthogonalization

Set $v_0 = w_0 = 0$ $t_{0,1} = t_{1,0} = 1$
Set $s = c - Ax_0$. Set w_1 such that $\langle s, w_1 \rangle = 1$.
Set $v_1 = Z(Z^T GZ)^{-1} Z^T s$
for $k = 1, 2...$**until** convergence
  $s = Av_k$ and $u = A^T w_k$
  $v_{k+1} = Z(Z^T GZ)^{-1} Z^T s$ and $w_{k+1} = Z(Z^T GZ)^{-1} Z^T u$
  $t_{k,k} = \langle s, w_k \rangle$
  $v_{k+1} = v_{k+1} - t_{k,k} v_k - t_{k-1,k} v_{k-1}$
  $w_{k+1} = w_{k+1} - t_{k,k} w_k - w_{k-1}$
  $t_{k,k+1} = \langle s, w_{k+1} \rangle$
  if $t_{k,k+1} \neq 0$
    $w_{k+1} = w_{k+1} / t_{k,k+1}$
  end
  $t_{k+1,k} = 1$
end

$$Z^T s = s$$
$$w_j = Z\, w_j$$
$$v_j = Z\, v_j$$

Preconditioned Lanczos Bi-Orthogonalization

Set $v_0 = w_0 = 0$ $t_{0,1} = t_{1,0} = 1$
Set $s = c - Ax_0$. Set w_1 such that $\langle s, w_1 \rangle = 1$.
Set $v_1 = Z(Z^T GZ)^{-1} Z^T s$
for $k = 1, 2 \ldots$ **until** convergence
 $s = Av_k$ and $u = A^T w_k$
 $v_{k+1} = Z(Z^T GZ)^{-1} Z^T s$ and $w_{k+1} = Z(Z^T GZ)^{-1} Z^T u$
 $t_{k,k} = \langle s, w_k \rangle$
 $v_{k+1} = v_{k+1} - t_{k,k} v_k - t_{k-1,k} v_{k-1}$
 $w_{k+1} = w_{k+1} - t_{k,k} w_k - w_{k-1}$
 $t_{k,k+1} = \langle s, w_{k+1} \rangle$
 if $t_{k,k+1} \neq 0$
  $w_{k+1} = w_{k+1}/t_{k,k+1}$
 end
 $t_{k+1,k} = 1$
end

$$Z^T s = s$$
$$w_j = Z \, w_j$$
$$v_j = Z \, v_j$$

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Preconditioned Lanczos Bi-Orthogonalization

Set $v_0 = w_0 = 0$ $t_{0,1} = t_{1,0} = 1$
Set $s = c - Ax_0$. Set w_1 such that $\langle s, w_1 \rangle = 1$.
Set $v_1 = Z(Z^T GZ)^{-1} Z^T s$
for $k = 1, 2 \ldots$ **until** convergence
  $s = Av_k$ and $u = A^T w_k$
  $v_{k+1} = Z(Z^T GZ)^{-1} Z^T s$ and $w_{k+1} = Z(Z^T GZ)^{-1} Z^T u$
  $t_{k,k} = \langle s, w_k \rangle$
  $v_{k+1} = v_{k+1} - t_{k,k} v_k - t_{k-1,k} v_{k-1}$
  $w_{k+1} = w_{k+1} - t_{k,k} w_k - w_{k-1}$
  $t_{k,k+1} = \langle s, w_{k+1} \rangle$
  if $t_{k,k+1} \neq 0$
    $w_{k+1} = w_{k+1} / t_{k,k+1}$
  end
  $t_{k+1,k} = 1$
end

$$Z^T s = s$$
$$w_j = Z\, \mathrm{w}_j$$
$$v_j = Z\, \mathrm{v}_j$$

$$\mathbf{v} = Z(Z^T GZ)^{-1} Z^T \mathbf{s}$$
$$\Rightarrow \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \end{bmatrix}$$

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Conclusions

- Any well-defined Krylov method can be converted to a projected form
- There is an equivalence between projected methods and Krylov subspace methods applied with a constraint preconditioner
- Known methods such as MINRES, SYMMLQ, are well-defined when used with a constraint preconditioner

Gould, N.I.M., Orban, D. and Rees, T., *Projected Krylov Methods for Saddle Point Systems*, (RAL-P-2013-006)