RAL

...tory
...n Laboratory

RAL Report
RAL-94-057

comp
math

N=1 Superconformal Minimal Model Correlation Functions on the Torus

A Abdurrahman F Anton M A Namazie and C Núñez

September 1994

Rutherford Appleton Laboratory Chilton DIDCOT Oxfordshire OX11 0QX

ROM2F/93/44
RAL-94-057

N=1 SUPERCONFORMAL MINIMAL MODEL CORRELATION FUNCTIONS ON THE TORUS

*A. Abdurrahman**, *F. Anton*†, *M.A. Namazie*,
C. Núñez‡

Dipartimento di Fisica
Università di Roma, Tor Vergata
I.N.F.N., Sezione di Roma II
Via della Ricerca Scientifica 1
00133 Roma, Italy.

Abstract

The Coulomb gas formalism is employed to construct contour integral representations of two-point correlation functions on the torus for the N=1 superconformal unitary discrete series, characterized by the single integer p. (For the particular case of the Tricritical Ising Model, these include the energy and vacancy density operators.) Modular and monodromy properties of the superconformal blocks are examined and the generalization to superconformal theories of Verlinde's results on modular transformations and the fusion algebra discussed in some detail. For p odd the relevant modular matrix is (with respect to a particular basis) symmetric and unitary, as in ordinary rational conformal theory. However for p even, there appears to be an obstruction due to the Ramond vacuum state.

December 1993

*Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, OX11 0QX.
†Dept. of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP.
‡Inst. de Astro. y. Fis. del Espacio, 1428 Buenos Aires, Argentina.

1 Introduction

The Dotsenko-Fateev Coulomb gas[1)] technique is one of the more efficient ways of explicitly computing correlation functions of primary fields in two-dimensional conformal field theory, particularly on higher genus Riemann surfaces.

There is of course by now a considerable body of important work on the basic properties and structure of rational conformal field theories [2–10)]. As far as explicit results on the torus are concerned, minimal model, [11,12)] $SU(2)$ WZWN and parafermion models [13)] and superconformal theories [14)] have all been considered using the Coulomb gas prescription. In particular, in ref. 14), hereinafter referred to as (I), some examples of two-point conformal blocks in both the $N = 1$ and $N = 2$ discrete unitary series were constructed and their monodromy and modular properties briefly discussed.

As is well-known, introducing Feigin-Fuks screening operators to make correlation functions background charge neutral (on the torus), one obtains contour integral representations of the conformal blocks. In the $N = 1$ superconformal case, one-contour examples corresponding to Ramond primary field correlation functions $< \phi_{1,2}\phi_{1,2} >$ with the four possible spin structures were considered in (I). For the special case of the Tricritical Ising Model (TIM), i.e. $c = 7/10$, this is the leading magnetic spin operator.

In the present paper we wish, firstly, to complete and extend some of the considerations contained in (I) to include correlation functions involving double contour integrals, such as the Neveu-Schwarz primary $\phi_{1,3}$ and superdescendent $\phi^S_{1,3}$. (Again specializing to the TIM, these are the energy and vacancy density operators respectively.) The results presented here are obtained in the same way as in (I), the details however are rather more involved as one might have anticipated. Secondly, in the context of these $N = 1$ superconformal models, we shall discuss the generalization of Verlinde's results on the connection between fusion rules and modular transformations. The organization of this paper is as follows.

In the next section, expressions for the conformal blocks corresponding to $< \phi_{1,3}\phi_{1,3} >$ are given. It is perhaps to be emphasized that the Coulomb gas method remains somewhat at the level of a prescription (although, as has been demonstrated in some conformal theories at least, the Felder cohomological construction [10)] provides a derivation); it is therefore neccessary to check null-state decoupling, that the conformal blocks on the torus behave correctly when the latter degenerates, as also in the factorization limit, and that one indeed has consistency with the fusion rules.

In section 3, the monodromy properties are established and the Verlinde operators [2] acting on two-point functions are constructed. The modular transformations are obtained in section 4 and as a check, the Verlinde operators around the $a-$ and $b-$ homology cycles are seen to be related under conjugation by the matrix of S-transformations.

General results regarding the relation between modular matrices and the fusion algebra [2,5] have been obtained under the simplifying technical assumption that the (extended) left and right chiral algebras consist of generators with integer conformal weight only. Nevertheless, in the superconformal case, as we shall discuss, the Verlinde conjecture continues to be satisfied (although a proof, along the lines of say ref.2), is beyond the scope of this paper). This in itself is probably not particularly surprising since essentially the proof of the conjecture utilises rather general conformal and duality properties of a *canonical* basis constructed from characters, or more generally from conformal blocks (and the construction indeed generalises to the superconformal models considered here). The conformal blocks provide, in the language of vector bundles, a basis for holomorphic sections of a vector bundle $V_{g,n}$ over moduli space $\mathcal{M}_{g,n}$ of a genus g surface with n punctures (for the most part $g = 1$ and $n = 2$ in this paper). In the superconformal case because of the different possible boundary conditions around the $a-$ and $b-$cycles, it is as usual necessary to work with a spin covering of moduli space and some care has to be exercised that one is in fact working with a complete canonical basis. Given that the $b-$cycle monodromy operator $\phi(b)$ is in fact the fusion matrix N_{IJ}^{K} (generalised now to take into account the Neveu-Schwarz and Ramond sector dependence), conjugation of $\phi(b)$ gives the $a-$cycle monodromy operator $\phi(a)$ and consequently leads to the fusion rules being diagonalised by the modular matrix S - more or less as in the usual conformal situation. However, in superconformal models the question of whether the modular transformation matrix S is symmetric (and unitary) is somewhat more delicate (whereas in usual rational conformal theory it follows as a simple corollary of the diagonalisation). In fact the modular matrix can fail to be symmetric and unitary essentially because of the Ramond vacuum state.

By considering some examples and employing different bases of conformal blocks, we argue as just mentioned that the Verlinde conjecture extends to the $N = 1$ superconformal unitary discrete series characterized by the single integer p. Furthermore, for p odd, the existence of a canonical or Verlinde basis (to be defined below) appears to be a *sufficient* condition for the modular matrix S acting on the space of superconformal characters to be symmetric (and unitary). However, as we shall discuss in detail, for p even there appears to be an obstruction due to the Ramond vacuum: the modular matrix is not unitary and symmetric (in the Verlinde basis) because of the existence – for even p only – of the Ramond vacuum state at the self-symmetric point of the Kac table $(\frac{p}{2}, \frac{p+2}{2})$. In section 5 we give a generalization of the Verlinde formula to superconformal models. By way of clarification we briefly discuss the $p = 4$ case specifically in the light of our results. This is known to correspond to special points on the critical line of the Ashkin-Teller model or equivalently a Z_2 orbifold of the $c = 1$ rational Gaussian model at radius $\sqrt{3}$ and $\frac{\sqrt{3}}{2}$. Recently, a proof of the Verlinde formula generalised to fermionic rational conformal field theories has been advanced[20]; to the extent that these authors also conclude that the modular matrix is not in general unitary we are in agreement.

Finally monodromy and modular invariance are used to construct correlation functions for the diagonal $N = 1$ superconformal invariant of the discrete unitary series in section 6. Although several computational details are relegated to appendices, this paper is not meant to be self-contained in the sense that we rely on (I) for notation and indeed for many other details only alluded to below. For readers interested specifically in the discussion (sections 4 and 5) of the generalization of Verlinde's results, a cursory reading of section 3 should suffice whereas section 2 (apart perhaps from notational orientation) can be omitted altogether.

2 Two-Point Conformal Blocks

Conformal blocks for the Ramond correlation function $< \phi_{1,2}\phi_{1,2} >$ were obtained in (I). Expressed as a single contour integral in a $s-$channel basis, they are

$$G_i^\nu(r,s) = \oint_{C_i} dz \left(\frac{\theta_1(z_1-z_2)}{\theta_1'(0)}\right)^{\frac{3}{8}+\frac{\alpha_-^2}{2}} \left(\frac{\theta_1(z_1-z)\theta_1(z-z_2)}{\theta_1'(0)^2}\right)^{-\frac{1}{2}-\alpha_-^2} \frac{\theta_\nu(z-\frac{z_1+z_2}{2})}{\theta_\nu(\frac{z_1-z_2}{2})^{\frac{1}{2}}} \left\{\Gamma^\nu(r,s) - (-)^{r\delta_{1,\nu}+rs\delta_{4,\nu}}\Gamma^\nu(-r,s)\right\} \quad (2.1)$$

where $\alpha_\pm$ are the 'charges' defined in (I) and $\nu = 1,2,3,4$ denotes the spin structure or sector of states labelled by (r,s) propagating around the torus. The contours $C_i, i = 1, 2$ provide a linearly independent $s-$channel basis of conformal blocks. C_1 encircles the point z_1 and the inner radius (representing the torus as an annulus) while C_2 encircles z_2 and winds around the outer radius. The contours are closed (by ensuring they wind around as many times as necessary) curves on a branched cover of the torus. The one-dimensional lattice difference part $\Gamma^\nu(r,s) \equiv \Gamma^\nu(\lambda)$ and $\Gamma^\nu(-r,s) \equiv \Gamma^\nu(\tilde{\lambda})$, is given by

$$\Gamma^\nu(\lambda) = \frac{1}{\eta(\tau)^{3/2}} e^{i\pi(\frac{\lambda^2}{2N} - \frac{1}{12})\delta_{4,\nu}} \sum_{n\in\mathbf{Z}} (-)^{np(\delta_{1,\nu}+\delta_{4,\nu})} q^{\frac{(\lambda+nN)^2}{4N}} e^{2\pi i \frac{(\lambda+nN)}{\sqrt{N}}(-\frac{\alpha_-}{2}z_1 - \frac{\alpha_-}{2}z_2 + \alpha_- z)}$$
$$\Gamma^\nu(\tilde{\lambda}) = \frac{1}{\eta(\tau)^{3/2}} e^{i\pi(\frac{\lambda^2}{2N} - \frac{1}{12})\delta_{4,\nu}} \sum_{n\in\mathbf{Z}} (-)^{np(\delta_{1,\nu}+\delta_{4,\nu})} q^{\frac{(\tilde{\lambda}+nN)^2}{4N}} e^{2\pi i \frac{(\tilde{\lambda}+nN)}{\sqrt{N}}(-\frac{\alpha_-}{2}z_1 - \frac{\alpha_-}{2}z_2 + \alpha_- z)} \tag{2.2}$$

with $\lambda = r(p+2) - sp$ and $\tilde{\lambda} = -r(p+2) - sp$ and $N = 2p(p+2)$. The integer p labels the discrete unitary superconformal series ($p = 3$ corresponds to the TIM). The range of (r,s) in equation (2.1) is $0 \le s \le r \le p-1$ with $r - s \in 2\mathbf{Z}$ in the $N-S$ sector ($\nu = 3, 4$) while in the R sector ($\nu = 1, 2$) it is $0 \le s \le r-1$ for $0 \le r \le \left[\frac{p-1}{2}\right]$ and $0 \le s \le r+1$ for $\left[\frac{p+1}{2}\right] \le r \le p-1$ with $r - s \in 2\mathbf{Z}+1$. As mentioned in (I), it is necessary to allow $r, s = 0$ because of mixing of the blocks under modular transformations; one has to show that the additional conformal blocks that this gives rise to actually do vanish.

Turning to the field $\phi_{1,3}$ we shall regard the components of the supermultiplet *i.e.* primaries and superdescendents (which recall are in fact primary under the Virasoro sub-algebra) separately and thus have to consider three possible cases, namely

(a) $<\phi(1,3)\phi(1,3)>$, (b) $<\phi^S(1,3)\phi^S(1,3)>$ and (c) $<\phi(1,3)\phi^S(1,3)>$.

Since $\alpha_{1,3} = -\alpha_-$, two screening operators have to be inserted to satisfy charge neutrality of the correlation function. We may immediately write the expression for the conformal block corresponding to case (a) in the $s-$channel basis as (only even sectors $\nu = 2, 3, 4$ are non-vanishing)

$$\begin{aligned} G^\nu_{ij}(r,s) &= \oint_{C_i} dz \oint_{C_j} dw < \psi(z)\psi(w) >_\nu < e^{-i\alpha_-\varphi(z_1)} e^{-i\alpha_-\varphi(z_2)} e^{i\alpha_-\varphi(z)} e^{i\alpha_-\varphi(w)} > \\ &= \oint_{C_i} dz \oint_{C_j} dw \left(\frac{\theta_1(z_1 - z_2)\theta_1(z-w)\theta_1'(0)^2}{\theta_1(z_1 - z)\theta_1(z_1 - w)\theta_1(z - z_2)\theta_1(w - z_2)} \right)^{2\alpha_-^2} \\ & \frac{\theta_1'(0)}{\theta_1(z-w)} \frac{\theta_\nu(z-w)}{\theta_\nu(0)} \theta_\nu(0)^{1/2} \left\{ \Gamma^\nu(r,s) - (-)^{rs\delta_{4,\nu}} \Gamma^\nu(-r,s) \right\} \end{aligned} \tag{2.3}$$

where $\Gamma^\nu(\lambda)$ is given by equation (2.2) except that the coupling to the charge is now

$$exp[2\pi i \frac{(\lambda + nN)}{\sqrt{N}}(-\alpha_- z_1 - \alpha_- z_2 + \alpha_- z + \alpha_- w)]. \tag{2.4}$$

An elementary check that equation (2.3) has at least the correct structure is provided by the degeneration limit, $q = e^{2\pi i\tau} \to 0$, of the two-point function in which a four-point function on the sphere should be recovered. Indeed, perform the conformal transformation to plane co-ordinates $z_i = \frac{1}{2\pi i} ln x_i, \quad z = \frac{1}{2\pi i} ln x$ and $w = \frac{1}{2\pi i} ln y$. Including the Jacobian factors associated with the external lines $(\frac{\partial z_1}{\partial x_1}\frac{\partial z_2}{\partial x_2})^{\Delta_{1,3}}$ where the conformal dimension $\Delta_{1,3} = 2\alpha_-^2 - 1/2$, it is a simple matter to obtain the result

$$\begin{aligned} G^2_{i,j}(r,s) \;\to\;& q^{\Delta^R_{r,s} - \frac{\hat{c}}{16}} (x_1 x_2)^{-a} (x_1 - x_2)^{2\alpha_-^2} \\ & \oint_{C_i} dx x^{a-1/2} (x - x_2)^{b'} (x_1 - x)^{c'} \\ & \oint_{C_j} dy y^{a-1/2} (y - x_2)^{b'} (x_1 - y)^{c'} (x - y)^d (x + y) \end{aligned} \tag{2.5}$$

where $\qquad a(r,s) = [(r+1)\alpha_+ + (s+1)\alpha_-]\alpha_-$

and $\qquad b' = c' = -2\alpha_-^2, \quad d = 2\alpha_-^2 - 1.$

The torus contours $C_i C_j$ degenerate in this limit to Pochhammer contours on the plane. Further, it can be verified that this expression coincides with

$$< R_{\Delta_{2\alpha_0 - \beta_{r,s}}}(0) N_{\Delta_{1,3}}(x_1) N_{\Delta_{1,3}}(x_2) R_{\Delta_{\beta_{r,s}}}(\infty) >$$

computed on the sphere. To see this note that (up to a constant)

$$< \sigma(0)\psi(x)\psi(y)\sigma(\infty) > = \frac{x+y}{(x-y)\sqrt{xy}} \quad ,$$

a fact which may be deduced either from the operator product expansions, or by computing this four-point function itself employing the Coulomb gas formalism for the $c = 1/2$ Ising model.

In a similar fashion one gets for the remaining combinations:

$$\begin{aligned} G^3_{ij}(r,s) + e^{-i\pi(\frac{\lambda^2}{2N} - \frac{1}{12})} G^4_{ij}(r,s) \to & q^{\Delta^{N-S}_{r,s} - \frac{\hat{c}}{16}} (x_1 x_2)^{-a} (x_1 - x_2)^{2\alpha_-^2} \\ & \oint_{C_i} dx x^a (x - x_2)^{b'} (x_1 - x)^{c'} \\ & \oint_{C_j} dy y^a (y - x_2)^{b'} (x_1 - y)^{c'} (x - y)^d \end{aligned} \tag{2.6}$$

which is to be compared with $< N_{\Delta_{2\alpha_0 - \beta_{r,s}}}(0) N_{\Delta_{1,3}}(x_1) N_{\Delta_{1,3}}(x_2) N_{\Delta_{\beta_{r,s}}} >$. The result for $G^3_{ij}(r,s) - G^4_{ij}(r,s)$ is the same as the last equation except for the replacement $q^{\Delta^{N-S}_{r,s} + \frac{1}{2} - \hat{c}/16}$ (implying half odd integral states) and the inclusion of an additional factor $1 + (x-y)^2/xy$ in the integrand. The form of the four-point function

$$< \psi(0)\psi(x)\psi(y)\psi(\infty) > = \frac{1}{(x-y)}[1 + \frac{(x-y)^2}{xy}]$$

enables one to recognise this as

$$< N^S_{\Delta_{2\alpha_0-\beta_{r,s}}}(0) N_{\Delta_{1,3}}(x_1) N_{\Delta_{1,3}}(x_2) N^S_{\Delta_{\beta_{r,s}}}(\infty) > \ .$$

Note that we are working with the particular combinations

$$G^3(r,s) \pm e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})} G^4(r,s)$$

so that in $q-$expansions the descendent states above each primary (or superdescendent) always have integer level spacing. This will be important later when we discuss the extension of Verlinde's results to the superconformal situation.

Next, we turn to the counting of the number of conformal blocks. Since there is only one type of screening operator Q_- occuring in the present two contour problem, there are three independent configurations of $s-$channel contours possible: C_1C_1, C_2C_2, or C_1C_2. In fact from $a-$cycle monodromy considerations (see fig.1 and section 3) it is clear that blocks $G^\nu_{ij}(r,s)$ with contours C_1C_1 (*i.e.* $i=j=1$) are associated with the $(r,s+2)$ intermediate states. On the other hand the C_2C_2 blocks have as intermediate states $(r,s-2)$, whereas the C_1C_2 blocks have (r,s) intermediate states. We have to show that the only non-vanishing blocks are in accord with the number of intermediate states predicted by the fusion rules [1]

$$\begin{aligned} N_{r,s} \times N_{1,3} &\sim N_{r,s-2} + N^S_{r,s} + N_{r,s+2} \\ N^S_{r,s} \times N_{1,3} &\sim N^S_{r,s-2} + N_{r,s} + N^S_{r,s+2} \\ R_{r,s} \times N_{1,3} &\sim R_{r,s-2} + R_{r,s} + R_{r,s+2} \end{aligned} \tag{2.7}$$

The arguments for this are similar to those given in (I) for the one contour case. Firstly note that all $G^\nu(r=0,s)$ blocks vanish trivially. To see that the blocks $G^\nu(r,s=0)$ also vanish it is easier to work instead in a t-channel basis defined by contours $\tilde{C}_i\tilde{C}_j$ (this is clearly sufficient because the contours C_iC_j and $\tilde{C}_i\tilde{C}_j$ each comprise a linearly independent set). $\tilde{C}_1$ encircles the points z_1 and z_2 while $\tilde{C}_2$ winds around the inner and outer radius of the annulus such that a closed curve results. The essential point is that for $s=0$, there are competing contributions in $\Gamma^\nu(r,0)$ and $\Gamma^\nu(-r,0)$ which cancel at each order in a q-expansion. This is easy to see in the degeneration limit. For instance for $\tilde{G}^{\nu=3}_{ij}(r,0)$ at leading order in the $q \to 0$ limit

$$\tilde{G}^3_{ij}(r,0) \ \sim \ \oint_{\tilde{C}_i} dx x^{a(r,0)} (x-1)^{b'} (z-x)^{c'}$$

[1] As written, equation (2.7) holds for $s \neq 1,2$ on the L.H.S. If $s=1$, only the $(r,s+2)$ terms on the R.H.S. are present and for $s=2$ the $(r,s-2)$ terms are absent.

$$\oint_{\tilde{C}_j} dy y^{a(r,0)}(y-1)^{b'}(z-y)^{c'}(x-y)^d$$
$$\left[\left(\frac{xy}{z}\right)^{-r/2} - \left(\frac{xy}{z}\right)^{+r/2}\right]. \tag{2.8}$$

Performing the change of variables $x \to z/x$ and $y \to z/y$ in the first term in square brackets above, the factor $(xy/z)^{-r/2} \to (xy/z)^{+r/2}$ while the remaining factors in the integrand are invariant. Since under this transformation $\tilde{C}_1 \to \tilde{C}_1$ and $\tilde{C}_2 \to \tilde{C}_2$ the integral of the sum of the two terms in square brackets vanishes. More generally, the same conclusion is reached directly on the torus by performing the change of variables $z = z_1 + z_2 - u$ and $w = z_1 + z_2 - v$ in $\tilde{G}_{ij}(r,0)$ and observing that $\Gamma^\nu(r,0) \to \Gamma^\nu(-r,0)$ while the rest of the integrand and the contours $\tilde{C}_i$ and $\tilde{C}_j$ remain unchanged.

Consider the $s-$channel $G_{ij}^{\nu=3,4}(r,s=1)$ blocks next, again in the degeneration limit. Examining (2.6), one sees that there is no branch or pole at ∞ since $2 - a(r,1) - b' - c' - d$ is a positive integer for either the x or y integral; hence the C_1C_2 and C_2C_2 blocks vanish. A similar result holds for the $G^{\nu=2}_{\substack{22\\12}}(r,s=1)$ blocks as can be seen from equation(2.5).

For consistency with the fusion rules equation (2.6), we require to show that the $G_{22}^\nu(r,s=2)$ blocks are zero. One may convince oneself that this is indeed the case by examining the singularity structure of the degeneration limit expressions; consider for instance the $\nu = 2$ sector, equation (2.4). This is the sum of two terms one of which is the following (the other is given by $x \leftrightarrow y$)

$$\oint_{C_i} dx \oint_{C_j} dy x^{a(r,2)+1/2}(x-z)^{b'}(1-x)^{c'} \quad y^{a(r,2)-1/2}(y-z)^{b'}(1-y)^{c'}$$
$$x^{\frac{d}{2}} y^{\frac{d}{2}} (1+\frac{x}{y})^{\frac{d}{2}} (1+\frac{y}{x})^{\frac{d}{2}} \tag{2.9}$$

where we have simply rewritten the $(x-y)^d$ factor as indicated. Expand the last two factors of this expression (we ignore questions of convergence which amounts as usual to assuming that one may analytically continue in the parameter d to take care of this). Then the singularity at x or $y = \infty$ of a generic term in the double integral is

$$2 - a(r,2) - \frac{1}{2} - b' - c' - \frac{d}{2} + m - n$$

for x, and

$$2 - a(r,2) + \frac{1}{2} - b' - c' - \frac{d}{2} - m + n$$

for y, with $m, n \in \mathbf{Z}$. If the x-integral has a branch or pole at ∞ the y-integral does not and *vice versa*. The C_2C_2 blocks therefore vanish as expected.

There is one remaining consistency check, at least as far as the number of blocks is concerned. This is for the particular block $G^3_{11}(1,1) - G^4_{11}(1,1)$ at order $q^{\Delta_{1,1}+\frac{1}{2}-\frac{\hat{c}}{16}}$. This is the superdescendent of the identity and is the null state $G_{-\frac{1}{2}}|0>$ and should therefore decouple. Using a hypergeometric representation $\mathcal{F}_i(a, b', c', d; z)$ canonical for the point $z = 0$, for the double integral (equation(5.17) of ref.1) we have checked that this combination vanishes as $z \to 0$.

We now turn to case (b). The conformal blocks corresponding to $< \phi^S_{1,3}\phi^S_{1,3} >$ contain a further two ψ's arising from the $N-S$ vertex operator $N_{\Delta+1/2}(z)$ and are thus given by the expression

$$\oint\oint < \psi(z_1)\psi(z_2)\psi(z)\psi(w) >_\nu < e^{-i\alpha_-\varphi(z_1)}e^{-i\alpha_-\varphi(z_2)}e^{i\alpha_-\varphi(z)}e^{i\alpha_-\varphi(w)} >$$

which reduces, upon Wick contracting and including the sum over winding modes, to (for even spin structure $\nu = 2, 3, 4$)

$$\begin{aligned} G^\nu_{ij}(r,s) = &\oint_{C_i} dz \oint_{C_j} dw \\ &\left[\frac{\theta_\nu(z_1-z_2)\theta_\nu(z-w)}{\theta_1(z_1-z_2)\theta_1(z-w)} + \frac{\theta_\nu(z_1-z)\theta_\nu(w-z_2)}{\theta_1(z_1-z)\theta_1(w-z_2)} - \frac{\theta_\nu(z_1-w)\theta_\nu(z-z_2)}{\theta_1(z_1-w)\theta_1(z-z_2)}\right] \\ &\left[\frac{\theta_1'(0)}{\theta_\nu(0)}\right]^2 \left[\frac{\theta_1(z_1-z_2)\theta_1(z-w)\theta_1'(0)^2}{\theta_1(z_1-z)\theta_1(z_1-w)\theta_1(z-z_2)\theta_1(w-z_2)}\right]^{2\alpha_-^2} \\ &\left\{\Gamma^\nu(r,s) - (-)^{rs\delta_{4,\nu}}\Gamma^\nu(-r,s)\right\} \end{aligned} \tag{2.10}$$

where the coupling to the charge is again given by equation (2.4). The $q \to 0$ limit of the conformal block can again be straightforwardly obtained; now however one has to multiply $G(r,s)$ by a Jacobian factor of $(\frac{\partial z_1}{\partial x_1}\frac{\partial z_2}{\partial x_2})^{\Delta_{1,3}+1/2}$ to get in the Ramond sector

$$\begin{aligned} G^2_{ij} \to &\, q^{\Delta^R_{r,s}-\hat{c}/16}(x_1x_2)^{-a-1/2}(x_1-x_2)^{2\alpha_-^2} \\ &\oint_{C_i} dx \oint_{C_j} dy x^{a-1/2}(x-x_2)^{b'}(x_1-x)^{c'}y^{a-1/2}(y-x_2)^{b'}(x_1-y)^{c'}(x-y)^{2\alpha_-^2} \\ &\left[\frac{(x_1+x_2)(x+y)}{(x_1-x_2)(x-y)} + \frac{(x_1+x)(y+x_2)}{(x_1-x)(y-x_2)} - \frac{(x_1+y)(x-x_2)}{(x_1-y)(x-x_2)}\right] \end{aligned} \tag{2.11}$$

This agrees with $< R_{\Delta_{2\alpha_0-\beta}}(0)N^S_{\Delta_{1,3}}(x_1)N^S_{\Delta_{1,3}}(x_2)R_{\Delta_\beta}(\infty) >$ on using for the fermionic part

$$< \sigma(0)\psi(x_1)\psi(x_2)\psi(x)\psi(y)\sigma(\infty) >= \frac{1}{\sqrt{x_1x_2xy}}[.....]$$

where the expression in square brackets is the same as that in the previous equation (2.11). The latter equation can once again be verified to be consistent with the operator product expansion.

Similarly,

$$G^3_{ij}(r,s) + e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}G^4_{ij}(r,s) \to q^{\Delta^{N-S}_{r,s}-\hat{c}/16}(x_1x_2)^{-a}(x_1-x_2)^{2\alpha^2_- -1}$$
$$\oint_{C_i} dx \oint_{C_j} dy x^a(x-x_2)^{b'-1}(x_1-x)^{c'-1}y^a(y-x_2)^{b'-1}(x_1-y)^{c'-1}(x-y)^d$$
$$\Big[(x_1-x)^2(y-x_2)^2 - (x_1-x_2)(x_1-y)(x_2-x)(x-y)\Big] \qquad (2.12)$$

coinciding with $< N_{\Delta_{2\alpha_0-\beta}}(0)N^S_{\Delta_{1,3}}(x_1)N^S_{\Delta_{1,3}}(x_2)N_{\Delta_\beta}(\infty) >$ as may be seen by using the fact that the correlation function $< \psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4) >$ on the sphere is given by $Pf\left[\frac{1}{x_i-x_j}\right]$.

The remaining combination in the degeneration limit, $G^3_{ij}(r,s) - G^4_{ij}(r,s)$, which now has superdescendents as each of its four external lines, reproduces $< N^S_{\Delta_{2\alpha_0-\beta}}N^S_{\Delta_{1,3}}N^S_{\Delta_{1,3}}N^S_\beta >$ as expected. However, in as much as it involves a 6ψ correlator, its algebraic structure is complicated and we refrain from giving it here.

Counting the number of conformal blocks proceeds as in the previous case and makes use of the same arguments. When the trace over states propagating around the torus are $N-S$ primaries the number of blocks $G^3(r,s)+G^4(r,s)$ is in agreement with the second of equations (2.7). For $G^3(r,s)-G^4(r,s)$ and $G^2(r,s)$ respectively the number of non-vanishing blocks is as predicted by [2]

$$\begin{aligned} N^S_{r,s}\times N^S_{1,3} &\sim N_{r,s-2}+N^S_{r,s}+N_{r,s+2} \\ R_{r,s}\times N^S_{1,3} &\sim R_{r,s-2}+R_{r,s}+R_{r,s+2} \end{aligned} \qquad (2.13)$$

Lastly, consider the mixed two-point function case (c) $< \phi_{1,3}\phi^S_{1,3} >$; clearly it is only different from zero in the odd spin structure sector and the corresponding conformal block is given by

$$\oint dz \oint dw < \psi(z)\psi(w)\psi(z_2) >_{\nu=1} < e^{-i\alpha_-\varphi(z_1)}e^{-i\alpha_-\varphi(z_2)}e^{i\alpha_-\varphi(z)}e^{i\alpha_-\varphi(w)} >$$

[2] The previous footnote on page 6 pertains here as well.

The bosonic part is of course common to the two other cases (a) and (b). An expression for the fermionic part can formally be obtained by using a 'regulated' propagator in the $\nu = 1$ sector

$$< \psi(z)\psi(w) >_{\delta} \;=\; \frac{\theta_1'(0)}{\theta_1(z-w)} \frac{\theta_1(z-w+\sqrt{\delta})}{\theta_1(\delta)} - \frac{1}{\delta}$$

The methods of (I) then allow one to write the conformal block, well-defined in the limit $\delta \to 0$, and given by

$$G_{ij}^{\nu=1}(r,s) = \oint_{C_i} dz \oint_{C_j} dw \left\{ \frac{\theta_1'(z-w)}{\theta_1(z-w)} - \frac{\theta_1'(z-z_2)}{\theta_1(z-z_2)} + \frac{\theta_1'(w-z_2)}{\theta_1(w-z_2)} \right\}$$

$$\left[\frac{\theta_1(z_1-z_2)\theta_1(z-w)\theta_1'(0)^2}{\theta_1(z_1-z)\theta_1(z_1-w)\theta_1(z-z_2)\theta_1(w-z_2)} \right]^{2\alpha_-^2} \left\{ \Gamma^{\nu=1}(\lambda) - (-)^r \Gamma^{\nu=1}(\tilde{\lambda}) \right\} \tag{2.14}$$

where

$$\Gamma^{\nu=1}(\lambda) = \sum_{n \in \mathbf{Z}} (-)^{np} q^{\frac{(nN+\lambda)^2}{4N}} e^{2\pi i \frac{(nN+\lambda)}{\sqrt{N}}(z+w-z_1-z_2)\alpha_-}$$

(It is perhaps worth mentioning that in the first line of (2.14) the expression for the 3ψ correlator within braces is by itself monodromy and modular covariant.) In the degeneration limit equation (2.14) becomes

$$G_{ij}^{\nu=1}(r,s) \to q^{\Delta^R_{r,s} - \hat{c}/16} x_1^{-a} x_2^{-a-1/2} (x_1 - x_2)^{2\alpha_-^2}$$

$$\oint_{C_i} dx \oint_{C_j} dy x^{a-1/2}(x-x_2)^{b'}(x_1-x)^{c'} y^{a-1/2}(y-x_2)^{b'}(x_1-y)^{c'}(x-y)^{2\alpha_-^2}$$

$$\left[\frac{x+y}{x-y} - \frac{x+x_2}{x-x_2} + \frac{y+x_2}{y-x_2} \right] \tag{2.15}$$

Noting that the five-point function

$$< \sigma(0)\psi(x)\psi(y)\psi(x_2)\sigma(\infty) > \;=\; \frac{1}{\sqrt{xyx_2}}[......]$$

where again the expression in square brackets is as in equation (2.14), one can check that (2.14) indeed coincides with

$$< R_{\Delta_{2\alpha_0-\beta}}(0) N_{\Delta_{1,3}}(x_1) N^S_{\Delta_{1,3}}(x_2) R_{\Delta_\beta}(\infty) >$$

Furthermore, the number of blocks and intermediate states are consistent with the relevant fusion rules.

As another check on our two-point functions $< \phi_{1,3}(z_1)\phi_{1,3}(z_2) >$, it is instructive to consider the limit $z_1 \to z_2$ in which one expects the conformal blocks to factorize. This is most conveniently analysed in a t-channel basis of contours $\tilde{C}_1\tilde{C}_1, \tilde{C}_1\tilde{C}_2, \tilde{C}_2\tilde{C}_2$ (see fig.2). In particular one would like to verify that the intermediate states occuring in the factorized conformal block are precisely those dictated by the fusion rules. Once again we treat the three cases (a), (b) and (c) in turn

$$\tilde{G}^{\nu}_{ij}(r,s) \quad = \quad \oint_{\tilde{C}_i} dz \oint_{\tilde{C}_j} dw I(r,s) \tag{2.16}$$

where $I(r,s)$ denotes the integrand in each of the three expressions (2.4), (2.10) and (2.13) above.

In the first place consider $\tilde{C}_1\tilde{C}_1$, in which both the z and w contours encirle the points z_1 and z_2. Set $z_1 - z_2 = \epsilon$ and change variables $z - z_2 = \epsilon u, w - z_2 = \epsilon v$ *etc.*, then as $\epsilon \to 0$ we have for any of the even sectors $\nu = 2, 3, 4$

$$\tilde{G}^{\nu}_{11}(r,s) \quad \sim \quad \epsilon^{-2\Delta_{1,3}} \int_0^1 du \int_0^1 dv \, [uv(1-u)(1-v)]^{-2\alpha_-^2} (u-v)^{2\alpha_-^2-1} \tag{2.17}$$

$$\left[\frac{\theta_\nu(0)}{\eta(\tau)}\right]^{1/2} \left[K(\lambda) - (-)^{rs\delta_{4\nu}} K(\tilde{\lambda})\right] \tag{2.18}$$

where $K(\lambda) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbf{Z}} (-)^{np\delta_{4\nu}} q^{\frac{(\lambda+nN)^2}{4N}}$. This is the situation in which both screening contours are shrunk to a point and there are no screening charges left on the torus. The intermediate state is the identity channel as expected from the fusion rules with the character (up to a τ independent factor) as residue. If we shrink instead only one of the contours, by setting $z_1 - z_2 = \epsilon, z - z_2 = \epsilon u$ but $w - z_2 = v$ one obtains

$$\tilde{G}^{\nu}_{12}(r,s) \quad \sim \quad \epsilon^{-2\Delta_{1,3}+(\Delta_{1,3}+1/2)}$$

$$\oint_{\tilde{C}_2} dv \frac{\theta_1'(0)}{\theta_1(v)} \frac{\theta_\nu(v)}{\theta_\nu(0)} \left(\frac{\theta_1(v)}{\theta_1'(0)}\right)^{-2\alpha_-^2} \left[\Gamma(\lambda) - \Gamma(\tilde{\lambda})\right] \tag{2.19}$$

where $\Gamma(\lambda)$ has a charge coupling of $e^{2\pi i \frac{(\lambda+nN)}{\sqrt{N}} \alpha_- v}$. In this limit we recover therefore the superdescendent $\phi^S_{1,3}$ channel with $< \phi^S_{1,3}(0) >$ as residue. Finally, with both screening operators remaining on the torus, one obtains

$$\tilde{G}^{\nu}_{22}(r,s) \quad \sim \quad \epsilon^{-2\Delta_{1,3}+\Delta_{1,5}} < \phi_{1,5}(0) >$$

Case (b) is similarly handled: we simply state the results (ν is again any of the even sectors)

$$\tilde{G}^{\nu}_{11}(r,s) \quad \sim \quad \epsilon^{-2(\Delta_{1,3}+1/2)} \left[\frac{\theta_\nu(0)}{\eta(\tau)}\right]^{1/2} \left[K(\lambda) - (-)^{rs\delta_{4\nu}} K(\tilde{\lambda})\right],$$

$$\tilde{G}^{\nu}_{12}(r,s) \quad \sim \quad \epsilon^{-2(\Delta_{1,3}+1/2)+\Delta_{1,3}+1/2} < \phi^{S}_{1,3}(0) >$$

and

$$\tilde{G}^{\nu}_{22}(r,s) \quad \sim \quad \epsilon^{-2(\Delta_{1,3}+1/2)+\Delta_{1,5}} < \phi_{1,5}(0) >$$

As mentioned the remaining case (c) $< \phi_{1,3}\phi^{S}_{1,3} >$ occurs only for $\nu = 1$. Here, shrinking both contours leads to the expression

$$\tilde{G}^{\nu=1}_{11}(r,s) \quad \sim \quad \epsilon^{-(2\Delta_{1,3}+1/2)+1/2}\left[K(\lambda) - K(\tilde{\lambda})\right]$$

$$\int_1^\infty du \int_1^\infty dv u^{2\alpha_- -1}(1-u)^{-2\alpha_-^2} v^{2\alpha_-^2-1}(1-v)^{-2\alpha_-^2}(u-v)^{2\alpha_-^2}\left(\frac{u-v}{uv} + \frac{1}{u-v}\right) .$$

This corresponds to a trivial null state with conformal weight 1/2 and it is easy to show (using for instance formulas (5.17) and (5.18) of ref.1) that the double integral in fact vanishes. Keeping the next order in the ϵ expansion gives the expectation value $< G_{-3/2} >$ on the torus.

Shrinking $\tilde{C}_1$ but not $\tilde{C}_2$ yields

$$\tilde{G}^{\nu=1}_{12}(r,s) \sim \epsilon^{-(2\Delta_{1,3}+1/2)+\Delta_{1,3}} < \phi_{1,3}(0) >$$

in which the intermediate state is $\phi_{1,3}$ itself. The remaining situation, with both screening operators on the torus, results in

$$\tilde{G}^{\nu=1}_{22}(r,s) \sim \epsilon^{-(2\Delta_{1,3}+1/2)+(\Delta_{1,5}+1/2)} < \phi^{S}_{1,5}(0) >$$

These results are fully consistent with the fusion rules (2.7) and (2.13). In the next section we shall consider some other properties of the correlation functions, specifically their transformations under a and $b-$cycle monodromy.

3 Monodromy and the Verlinde operators

In this section we wish to discuss the properties of the conformal blocks under monodromy transformations: the point z_1 (or z_2) is transported once around either an $a-$ or $b-$cycle on the torus. This defines the Verlinde operators $\phi(a)$ and $\phi(b)$ which implement these transformations on two-point functions. The conformal case has been discussed in detail in ref.11).

Consider first the blocks corresponding to the one-contour Ramond correlation function

$$G^{\nu}_{i}(r,s) \quad = \quad \oint_{C_i} dz I^{\nu}(r,s) \tag{3.1}$$

given explicitly by equation (2.1), with $\nu = 1,2,3,4$ and $i = 1,2$ labelling the $s-$channel contours C_1 and C_2. The $a-$cycle monodromy transformation consists of shifting $z_1 \to z_1 + 1$ and $z \to z + 1$ when the contour is C_1 (since C_1 encircles z_1, the contour itself is dragged onto the second sheet under the transformation), while for C_2 we only need to shift $z_1 \to z_1 + 1$. The contours do not mix of course under the $a-$cycle transformation, but the spin structures $\nu = 3$ and 4, and $\nu = 1$ and 2 respectively are interchanged while the conformal block itself acquires a phase. However the combinations [3]

$$G_i^{\nu=3}(r,s) \pm e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})} G_i^{\nu=4}(r,s) \tag{3.2}$$

and

$$G_i^{\nu=2}(r,s) \pm e^{\frac{i\pi}{4}} G_i^{\nu=1}(r,s) \tag{3.3}$$

are in fact diagonal. Acting on the + combination of equation (3.2) for instance, the $s-$channel Verlinde operator is then

$$\phi^s(a)_{r,r';s,s'} \quad = \quad e^{2\pi i\left[-\Delta^{N-S}_{r,s} + \Delta^R_{intermediate}\right]} \quad \delta_{rr'}\delta_{ss'} \tag{3.4}$$

where the intermediate state has a conformal weight of either $\Delta^R_{r,s+1}$ or $\Delta^R_{r,s-1}$ depending on whether the contour is taken to be C_1 or C_2 respectively.

For later purposes, it will be useful to transform the operators $\phi^s(a)$ to a t-channel basis. Again consider the + combination of equation (3.2). The operator $\phi^t(a)$ acts on the basis $\tilde{G}_i^{\nu=3}(r,s) + e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}_i^{\nu=4}(r,s)$, where $i = 1,2$ refers now to $\tilde{C}_1$ and $\tilde{C}_2$ respectively. To compute it, we can open the contours, in which case up to irrelevant prefactors (and setting $x_1 = x, x_2 = 1$) $C_1 \to \int_0^x$ and $C_2 \to \int_1^\infty$ while $\tilde{C}_1 \to \int_x^1$ and $\tilde{C}_2 \to \int_{-\infty}^0$. Using formulas (4.13) and (4.14) of ref.1 one can relate the s and t-channel bases of open contours, and in this way obtain $\phi^t(a)$ as a 2×2 non-diagonal (in the space of contours) matrix with elements

$$\begin{aligned}
\phi^t(a)_{\tilde{C}_1\tilde{C}_1} &= -ie^{\frac{i\pi}{8}} e^{\frac{3}{2}i\pi\alpha_-^2} \frac{cos\pi(s\alpha_-^2 - r/2)}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'} \\
\phi^t(a)_{\tilde{C}_1\tilde{C}_2} &= 2ie^{i\frac{\pi}{8}} e^{i\frac{\pi}{2}\alpha_-^2} \frac{cos\pi((s+1)\alpha_-^2 - r/2)cos\pi((s-1)\alpha_-^2 - r/2)}{sin2\pi\alpha_-^2}\delta_{rr'}\delta_{ss'} \\
\phi^t(a)_{\tilde{C}_2\tilde{C}_1} &= -ie^{i\frac{\pi}{8}} e^{i\frac{\pi}{2}\alpha_-^2} \frac{cos\pi\alpha_-^2}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'} \\
\phi^t(a)_{\tilde{C}_2\tilde{C}_2} &= ie^{\frac{i\pi}{8}} e^{-i\frac{\pi}{2}\alpha_-^2} \frac{cos\pi(s\alpha_-^2 - r/2)}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'}
\end{aligned} \tag{3.5}$$

[3]For details see (I), in which the phase $e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}$ was omitted.

On the other hand, acting on the combination $G_i^{\nu=2}(r,s)+e^{\frac{i\pi}{4}}G_i^{\nu=1}(r,s)$ one obtains

$$
\begin{aligned}
\phi^t(a)_{\bar{C}_1\bar{C}_1} &= e^{\frac{-i\pi}{8}}e^{\frac{3}{2}i\pi\alpha_-^2}\frac{sin\pi(s\alpha_-^2-r/2)}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'}\\
\phi^t(a)_{\bar{C}_1\bar{C}_2} &= -2e^{-i\frac{\pi}{8}}e^{i\frac{\pi}{2}\alpha_-^2}\frac{sin\pi((s+1)\alpha_-^2-r/2)sin\pi((s-1)\alpha_-^2-r/2)}{sin2\pi\alpha_-^2}\delta_{rr'}\delta_{ss'}\\
\phi^t(a)_{\bar{C}_2\bar{C}_1} &= e^{-i\frac{\pi}{8}}e^{i\frac{\pi}{2}\alpha_-^2}\frac{cos\pi\alpha_-^2}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'}\\
\phi^t(a)_{\bar{C}_2\bar{C}_2} &= -e^{\frac{-i\pi}{8}}e^{-i\frac{\pi}{2}\alpha_-^2}\frac{sin\pi(s\alpha_-^2-r/2)}{sin\pi\alpha_-^2}\delta_{rr'}\delta_{ss'}
\end{aligned}
\tag{3.6}
$$

The two-contour examples, for instance case (a) $<\phi_{1,3}(z_1)\phi_{1,3}(z_2)>$, can be examined in the same manner. Isolating the phases that arise in the $s-$channel conformal block equation (2.3), under $z_1\to z_1+1$ *etc.* allows one to establish the precise intermediate states corresponding to the three respective configurations of double contours and these are in agreement with the fusion rules listed in the previous section. We shall simply state the results for case (a). When the torus trace is over $N-S$ primaries

$$
\begin{aligned}
G^3_{\substack{11\\22}}(r,s)+G^4_{\substack{11\\22}}(r,s) &\to e^{2\pi i[-\Delta^{N-S}_{r,s}+\Delta^{N-S}_{r,s\pm2}]}\left[G^3_{\substack{11\\22}}(r,s)+G^4_{\substack{11\\22}}(r,s)\right]\\
G^3_{12}(r,s)+G^4_{12}(r,s) &\to e^{2\pi i[-\Delta^{N-S}_{r,s}+(\Delta^{N-S}_{r,s}+1/2)]}\left[G^3_{12}(r,s)+G^4_{12}(r,s)\right]
\end{aligned}
\tag{3.7}
$$

while for the superdescendents

$$
\begin{aligned}
G^3_{\substack{11\\22}}(r,s)-G^4_{\substack{11\\22}}(r,s) &\to e^{2\pi i[-(\Delta^{N-S}_{r,s}+1/2)+(\Delta^{N-S}_{r,s\pm2}+1/2)]}\left[G^3_{\substack{11\\22}}(r,s)-G^4_{\substack{11\\22}}(r,s)\right]\\
G^3_{12}(r,s)-G^4_{12} &\to e^{2\pi i[-(\Delta^{N-S}_{r,s}+1/2)+\Delta^{N-S}_{r,s}]}\left[G^3_{12}(r,s)-G^4_{12}(r,s)\right]
\end{aligned}
\tag{3.8}
$$

In equations (3.7) and (3.8) we have suppressed the phase $e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}$ multiplying $G^{\nu=4}(r,s)$. Although, unlike the one-contour problem, the two-contour blocks are diagonal in spin structure under the action of $\phi(a)$, we continue to use the particular combinations above so that the intermediate states are always conformal primaries with respect to the Virasoro sub-algebra.

With Ramond spin structure,

$$
\begin{aligned}
G^2_{\substack{11\\22}}(r,s) &\to e^{2\pi i[-\Delta^R_{r,s}+\Delta^R_{r,s\pm2}]}G^2_{\substack{11\\22}}(r,s)\\
G^2_{12}(r,s) &\to e^{2\pi i[-\Delta^R_{r,s}+\Delta^R_{r,s}]}G^2_{12}(r,s)\,.
\end{aligned}
\tag{3.9}
$$

Similar expressions hold for the other two cases (b) and (c). Unlike $< \phi_{1,2}\phi_{1,2} >$, the $a-$cycle monodromy in a $s-$channel basis is diagonal within each sector (essentially because of the absence of spin fields in the correlator):

$$\phi^s(a)^{\nu\nu'}_{r,r';s,s'} = e^{2\pi i[-\Delta^{(\nu)}_{r,s}+\Delta_{intermediate}]} \quad \delta_{rr'}\delta_{ss'}\delta^{\nu\nu'} \tag{3.10}$$

where $\Delta^{(\nu)}_{r,s}$ stands for one of the three possibilities: $\Delta^{N-S}_{r,s}$, $\Delta^{N-S}_{r,s}+1/2$ or $\Delta^{R}_{r,s}$ and $\phi^s(a)$ is to be regarded as a diagonal 3×3 matrix on the space of blocks $(G_{1,1}, G_{2,2}, G_{1,2})$.

We can transform these operators to a $t-$channel basis of contours as well (using this time the double contour formulae equations (5.8), (5.14) of ref.1) and cite here the results for case (a) $< \phi_{1,3}\phi_{1,3} >$ only.

$$\left(\phi^t(a)^{\nu\nu'}\right)_{n,m} = \delta_{r,r'}\delta_{s,s'}\delta^{\nu\nu'}$$

$$\left\{\tilde{\alpha}_{n,1}\alpha_{1,m}e^{-2\pi i(a+c'+\frac{1}{2})} - e^{i\pi\delta_{\nu 2}}\tilde{\alpha}_{n,2}\alpha_{2,m} + \tilde{\alpha}_{n,3}\alpha_{3,m}e^{2\pi i(a+\frac{1}{2})}\right\} \tag{3.11}$$

where the trigonometric coefficients $\tilde{\alpha}$ and α ($n, m = 1, 2, 3$ denote the $t-$channel contour combinations $\tilde{C}_2\tilde{C}_2, \tilde{C}_1\tilde{C}_2$ and $\tilde{C}_1\tilde{C}_1$ respectively)

$$\alpha_{n,m}(a - \frac{1}{2}\delta_{\nu 2}, b', c', d)$$

$$\tilde{\alpha}_{n,m} \equiv \alpha_{n,m}(b', a - \frac{1}{2}\delta_{\nu 2}, c', d)$$

are given in equation (5.11) of ref.1), but for completeness we rederive them here as well in appendix A (correcting a minor sign error in the previous reference).

Next we turn to a discussion of $b-$cycle monodromy, starting again with a discussion of the simpler one-contour Ramond correlator to illustrate the arguments involved. The point z_1 is transported once around the $b-$cycle: $z_1 \to z_1 + \tau$. Following ref.11), it is convenient to split this transformation into two successive steps (fig.3) in which z_1 is first shifted to a point z_1' on the second sheet with $Im z_1' < Im(z_2 + \tau)$, and then z_1' is shifted to $z_1 + \tau$. Performing this transformation in a $s-$channel basis the (r, s) state propagating around the torus is interchanged with the intermediate state $(r, s+1)$ corresponding to the contour C_1, or $(r, s-1)$ with $s > 1$ corresponding to C_2, as suggested by fig.1. This can be seen explicitly by performing for the C_1 case the shifts $z_1 \to z_1 + \tau$ and $z \to z + \tau$ (since as before the contour C_1 which encircles z_1 is dragged along) in the integrand of equations (3.1), (2.1) whence

$$I^{\nu=3}(r,s) + e^{-i\pi(\frac{\lambda^2(r,s)}{2N}-\frac{1}{12})}I^{\nu=4}(r,s) \to$$

$$e^{\mp\frac{i\pi}{8}}e^{\mp\frac{i\pi}{2}\alpha_-^2}\left[I^{\nu=2}(r,s+1)+e^{\pm\frac{i\pi}{2}}e^{\frac{i\pi}{4}}I^{\nu=1}(r,s+1)\right] \tag{3.12}$$

and

$$I^{\nu=2}(r,s)+e^{\frac{i\pi}{4}}I^{\nu=1}(r,s)\rightarrow$$
$$e^{\mp\frac{i\pi}{8}}e^{\mp\frac{i\pi}{2}\alpha_-^2}\left[I^{\nu=3}(r,s+1)+e^{-i\pi(\frac{\lambda^2(r,s+1)}{2N}-\frac{1}{12})}I^{\nu=4}(r,s+1)\right] \tag{3.13}$$

We have to consider as well the action of the transformation on the contour itself. To start with, we may take C_1 to encircle clockwise the inner radius (representing the torus as an annulus) once and to go around the point z_1 $(s+1)$ times clockwise. This is a closed contour, and in the degeneration limit ($x_i = e^{2\pi i z_i}$) is as shown in fig.4a. By deforming the circle around the origin to one with a very large radius, we unwrap one of the circles around x_1 so that now C_1 winds around this point s times as depicted in fig.4b. Then as $x_1 \rightarrow x_1'$, the contour is dragged along onto the second sheet and becomes C_1' which encircles x_1' anticlockwise s times and ∞ once clockwise as in fig.4c.

Denoting the branch of $I(r,s)$, in the degeneration limit, at x_1 as b and that at x_2 as c, recall that in (I) it was shown that $b=c=-\alpha_-^2-1/2$,whereas the branch of $I(r,s)$ at the origin is

$$a(r,s)=[(r+1)\alpha_+ +(s+1)\alpha_-]\,\alpha_- - \left\{\begin{array}{lll} mod\mathbf{Z}, & \nu=3,4 & r-s\in 2\mathbf{Z} \\ \frac{1}{2}+mod\mathbf{Z}, & \nu=1,2 & r-s\in 2\mathbf{Z}+1 \end{array}\right.$$

The integrand $I^{\nu}(r,s)$ has been transformed under monodromy [4] to $I^{\nu'}(r,s+1)$; therefore the contour C_1' is still closed as the branch at x_1' is equal and opposite ($mod\ \mathbf{Z}$) to the transformed branch at ∞.

For definiteness consider the $G^{\nu=3}(r,s)+G^{\nu=4}(r,s)$ combination, equation (3.2). The contour C_1 is

$$\oint_{C_1} = e^{-i\pi a(r,s)}\int_0^{x_1} - e^{i\pi a(r,s)}\int_0^{x_1} = -2isin\pi a(r,s)\int_0^{x_1}$$

whereas after the monodromy operation, the contour C_1' is:

$$\oint_{C_1'} = e^{-i\pi(-a(r,s+1)-b-c+\frac{1}{2})}\int_\infty^{x_1'} - e^{i\pi(-a(r,s+1)-b-c+\frac{1}{2})}\int_\infty^{x_1'} = 2isin\pi(a(r,s)+b)\int_{x_1'}^\infty$$

[4]As is clear from (3.11) and (3.12), under $b-$cycle monodromy $\nu=3,4$ is transformed into $\nu'=2,1$ respectively and *vice versa*. The $e^{\pm\frac{i\pi}{2}}$ in (3.12) is cancelled by a phase $e^{\mp\frac{i\pi}{2}}$ arising from the $\theta_1(\frac{z_1-z_2}{2})^{-\frac{1}{2}}$ factor of the $\nu=1$ block only due to the change in time ordering implicit in $z_1\rightarrow z_1'$.

Hence

$$\int_0^{x_1} dx x^{a(r,s)}(x_1-x)^b(x-x_2)^c \rightarrow \frac{sin\pi(a+b)}{sin\pi a}\int_{x_1'}^{\infty} dx x^{a(r,s+1)-\frac{1}{2}}(x-x_1')^b(x_2-x)^c \ldots$$

(where only factors in the degeneration limit integrands which determine the singularity structure have been made explicit). In the second step, one has to complete the monodromy transformation by shifting z_1' to $z_1+\tau$ (or equivalently on the plane, x_1' back to x_1) as shown in fig.4d. This entails going around the singularity at x_2 either in a clockwise or anticlockwise sense, and thus

$$\int_{x_1'}^{\infty} \rightarrow e^{\mp i\pi c}\int_{x_1}^{x_2} + \int_{x_2}^{\infty}$$

A similar prescription applies to C_2, which encloses z_2 and the outer radius (of the annulus). We need only shift $z_1 \rightarrow z_1+\tau$, or what is the same thing, both $z_2 \rightarrow z_2-\tau$ and $z \rightarrow z-\tau$ together. Now the state (r,s) is interchanged instead with the intermediate state $(r,s-1)$ and the right hand sides of equations (3.12) and (3.13) are replaced by the combinations

$$e^{\mp\frac{i\pi}{8}}e^{\mp\frac{i\pi}{2}\alpha_-^2}\left[I^{\nu=2}(r,s-1) - e^{\pm\frac{i\pi}{2}}e^{\frac{i\pi}{4}}I^{\nu=1}(r,s-1)\right] \qquad (3.12')$$

and

$$e^{\mp\frac{i\pi}{8}}e^{\mp\frac{i\pi}{2}\alpha_-^2}\left[I^{\nu=3}(r,s-1) - e^{-i\pi(\frac{\lambda^2(r,s-1)}{2N}-\frac{1}{12})}I^{\nu=4}(r,s-1)\right] \qquad (3.13')$$

respectively. In this case we take the closed contour C_2 (in the degeneration limit) to wind $(s-1)$ times $(s>1)$ around x_2 in an anticlockwise sense and once around ∞ clockwise. Then under $x_2 \rightarrow x_2'$, the new contour C_2' winds s times anticlockwise around x_2' while winding once around the origin also in an anticlockwise sense: it is still closed by the previous arguments. Therefore (for $\nu=3,4$ and $\nu'=2,1$)

$$\oint_{C_2} = e^{-i\pi(a(r,s)+b+c)}\int_{x_2}^{\infty} - e^{i\pi(a(r,s)+b+c)}\int_{x_2}^{\infty} = -2isin\pi(a(r,s)+b+c)\int_{x_2}^{\infty}$$

$$\oint_{C_2'} = e^{i\pi(a(r,s-1)-1/2)}\int_0^{x_2'} - e^{-i\pi(a(r,s-1)-1/2)}\int_0^{x_2'} = 2isin\pi(a(r,s)+b)\int_0^{x_2'}$$

and hence

$$\int_{x_2}^{\infty} dx x^{a(r,s)}(x_1-x)^b(x-x_2)^c \rightarrow -\frac{sin\pi(a+b)}{sin\pi(a+b+c)}\int_0^{x_2'} dx x^{a(r,s-1)-\frac{1}{2}}(x_1-x)^b(x_2'-x)^c \ldots$$

In the second stage, we require to shift x_2' back to x_2 and again in performing this operation one has to go past x_1 either in a clockwise or anticlockwise sense resulting in

$$\int_0^{x_2'} \rightarrow \int_0^{x_1} + e^{\mp i\pi b}\int_{x_1}^{x_2}$$

Since after both steps the contour is now in a mixed basis, it is convenient to transform all these formulae to the t-channel using as before equations (3.13) and (3.14) of ref.1) which give

$$\int_{x_1}^{x_2} dx x^a (x_2 - x)^b (x - x_1)^c =$$

$$\frac{1}{sin\pi(a(r,s)+c)} \left(-sin\pi a(r,s) e^{\pm i\pi c} \int_0^{x_1} - sin\pi(a(r,s)+b+c) e^{\pm i\pi b} \int_{x_2}^{\infty} \right)$$

and

$$\int_{-\infty}^{0} dx (-x)^a (x_2 - x)^b (x_1 - x)^c =$$

$$\frac{sin\pi b}{sin\pi(a(r,s)+b)} \left(-e^{\pm i\pi a} \int_0^{x_1} + e^{\pm i\pi(a+b+c)} \int_{x_2}^{\infty} \right)$$

Using these relations, the action of $\phi^t(b)$ is:

$$\tilde{G}_i^{\nu=3}(r,s) + e^{-i\pi(\frac{\lambda^2}{2N} - \frac{1}{12})} \tilde{G}_i^{\nu=4}(r,s) \to \sum_{\tilde{C}_1,\tilde{C}_2} \sum_{r',s'} [\phi^t(b)_{\tilde{C}_i\tilde{C}_j}]_{r,s}^{r',s'} [\tilde{G}_j^{\nu=2}(r',s') + (-)^{\delta_{s',s-1}} e^{\frac{i\pi}{4}} \tilde{G}_j^{\nu=1}(r',s')]$$

is given by the matrix $\phi^t(b)$ with matrix elements [5]

$$\phi^t(b)_{\tilde{C}_1\tilde{C}_1} = e^{\mp\frac{i\pi}{8}} e^{\pm\frac{3}{2}i\pi\alpha_-^2} \frac{cos\pi\alpha_-^2}{sin2\pi\alpha_-^2} \delta_{r',r} (\delta_{s',s+1} + \delta_{\substack{s',s-1 \\ s>1}})$$

$$\phi^t(b)_{\tilde{C}_1\tilde{C}_2} = -e^{\mp\frac{i\pi}{8}} e^{\mp\frac{i\pi}{2}\alpha_-^2} \frac{1}{sin2\pi\alpha_-^2}$$

$$\delta_{r',r} \left(sin\pi(\alpha_-^2(s'+1) - r/2)\delta_{s',s+1} + sin\pi(\alpha_-^2(s'-1) - r/2)\delta_{\substack{s',s-1 \\ s>1}} \right)$$

$$\phi^t(b)_{\tilde{C}_2\tilde{C}_1} = e^{\mp\frac{i\pi}{8}} e^{\pm\frac{3}{2}i\pi\alpha_-^2} \frac{cos\pi\alpha_-^2}{2sin\pi\alpha_-^2 cos\pi(s'\alpha_-^2 - r/2)} \delta_{r',r} (\delta_{s',s+1} + \delta_{\substack{s',s-1 \\ s>1}})$$

$$\phi^t(b)_{\tilde{C}_2\tilde{C}_2} = -e^{\mp\frac{i\pi}{8}} e^{\mp\frac{i\pi}{2}\alpha_-^2} \frac{cos\pi\alpha_-^2}{sin2\pi\alpha_-^2 cos\pi(s'\alpha_-^2 - \frac{r}{2})}$$

$$\delta_{r',r} \left(sin\pi((s'+1)\alpha_-^2 - r/2)\delta_{s',s+1} + sin\pi((s'-1)\alpha_-^2 - r/2)\delta_{\substack{s',s-1 \\ s>1}} \right) \qquad (3.14)$$

The matrices acting on the other combinations of conformal blocks have a similar structure. In particular one observes that the first of these equations for $\phi^t(b)_{\tilde{C}_1\tilde{C}_1}$ is, up to a (r,s) independent normalization factor, the superconformal fusion algebra

$$R_{1,2} \times N_{r,s} \sim R_{r,s-1} + R_{r,s+1}$$

[5] For details see ref.17).

in accord with Verlinde's result[2)]. The normalization factor can of course be eliminated by a change in normalization of the basis, the new basis being defined by requiring that the action of the monodromy operator on the identity character reproduce the $R_{1,2}$ character.

The discussion of $b-$cycle monodromy for the two-contour problem proceeds in the same way but is somewhat more involved in detail. As mentioned above, under $\phi^s(b)$ the puncture z_1 is taken to $z_1+\tau$, and it is convenient to split this operation into two consecutive steps $z_1 \to z_1'$, and then $z_1' \to z_1+\tau$. From fig.1, or indeed the fusion rule equations (2.7) and (2.13), it follows there are (in general) three possible intermediate states depending on which of the three $s-$channel contour combinations C_1C_1, C_1C_2 or C_1C_2 are chosen. Explicitly, under[6] $z_1 \to z_1+\tau$ as can be seen from equation (2.3), (2.10) or (2.14) corresponding to cases (a), (b) or (c) respectively,

$$e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})\delta_{4,\nu}}I^\nu(r,s) \to e^{\mp 2\pi i\alpha_-^2}e^{-i\pi(\frac{\tilde{\lambda}^2}{2N}-\frac{1}{12})\delta_{4,\nu}}I^\nu(r,\tilde{s})$$

where $\tilde{s}=s+2$ for C_1C_1; $\tilde{s}=s,\ s>1$ for C_1C_2 and $\tilde{s}=s-2,\ s>2$ for C_2C_2.

Consider now the configurations of double contours involved. It is sufficient to work in the degeneration limit where one has integrals of the generic type

$$\oint_{C_x} dx \oint_{C_y} dy x^a(x_1-x)^{c'}(x-x_2)^{b'}y^a(x_1-y)^{c'}(y-x_2)^{b'}(x-y)^d\,.$$

The double integral is well defined when C_x and C_y are not both C_1 or C_2, but ambiguity arises when C_x and C_y are both together either C_1 or C_2. To avoid this we adopt the prescription that in the complex $y-$plane, the Pochammer contour $C_y(=C_1$ or $C_2)$ is taken to exclude the singular point $x=y$ (see fig.5a) which becomes a branch point for the remaining integral specified by the contour C_x. For definiteness let us take both $C_x=C_1$ and $C_y=C_1$. Then in the first step of the monodromy operation, the double contour is shifted onto the second sheet as illustrated in fig.6a where C_y' encircles now x_1' and ∞ (excluding the point x').

We would like to rewrite these closed contours as definite line integrals. Before performing the $b-$cycle monodromy operation one can write $\oint_{C_x}\oint_{C_y}\dots$ as

$$-4e^{i\pi a}sin\pi a\ sin\pi c \oint_{C_1} dx \int_0^{x_1} dy$$

[6]Shifting as well $z \to z+\tau$ for the C_1C_2 combination, and both $z\to z+\tau$ and $w \to w+\tau$ for C_1C_1.

$$\times x^a(x_1-x)^{c'}(x-x_2)^{b'}y^a(y-x_1)^{c'}(y-x_2)^{b'}(x-y)^d$$

and then as

$$16e^{2\pi ia}sin^2\pi a\ \ sin^2\pi c\left\{e^{\pm\frac{i\pi}{2}d}2cos\frac{\pi}{2}d\int_0^{x_1}dx\int_0^x dyI(x,y)\right\} \tag{3.15}$$

where

$$I(x,y)=x^a(x-x_1)^{c'}(x-x_2)^{b'}y^a(y-x_1)^{c'}(y-x_2)^{b'}(x-y)^d$$

and the + sign in $e^{\pm i\frac{\pi}{2}d}$ corresponds to choosing the contour C_x to be above the point y in the complex $x-$plane as drawn in fig.5a. The expression in braces above can of course be rewritten as the line integrals[1] $\int_0^{x_1}\int_0^{x_1}$, with the $x-$contour drawn above that of y as in fig.5b. The other combinations of contours are obtained similarly. After performing the first step of the monodromy transformation, the resulting contours are

$$\oint_{C_1'}dx'\oint_{C_1'}dy'x'^{a'}(x_1'-x)^{c'}(x'-x_2)^{b'}y'^{a'}(x_1-y')^{c'}(y'-x_2)^{b'}(x'-y')^d$$

$$=16e^{2\pi i(a'+b'+c'+d)}sin^2\pi(a'+b'+c'+d)sin^2\pi ce^{\mp\frac{i\pi}{2}d}\ \ 2cos\frac{\pi}{2}d\int_{x_1'}^{\infty}dx'\int_{x_1'}^{x'}dy'I(y',x') \tag{3.16}$$

where

$$I(x',y')=x'^{a'}(x'-x_1')^{c'}(x'-x_2')^{b'}y^a(y'-x_1')^{c'}(y'-x_2')^{b'}(y'-x')^d$$

$a'=a(r',s')\equiv a(r,s+2)$. Therefore one has that

$$\int_0^{x_1}dx\int_0^x dyI(x,y)\rightarrow e^{\mp\pi id}e^{2\pi i(b'+c'+d)}\frac{sin^2\pi(a+c'+d)}{sin^2\pi a}\int_{x_1'}^{\infty}dx'\int_{x_1'}^{x'}dy'I(y',x') \tag{3.17}$$

The second step involves shifting z_1' to $z_1+\tau$ which in the degeneration limit corresponds to analytically continuing the open contours in fig.6 from x_1' back to x_1 in the process of which we have to go around the point x_2 in either a clockwise or anticlockwise sense (as illustrated in fig.6b). Thus

$$\int_{x_1'}^{\infty}dx'\int_{x_1'}^{x'}dy'\rightarrow e^{\mp 2\pi ib'}\int_{x_1}^{x_2}dx\int_{x_1}^{x}dy+e^{\mp i\pi b'}\int_{x_2}^{\infty}dx\int_{x_1}^{x_2}dy+\int_{x_2}^{\infty}dx\int_{x_2}^{x}dy \tag{3.18}$$

On the right hand side of equation (3.18) the first term is in a $t-$channel basis, the third is in a $s-$channel basis while the second term is mixed. Combining all these transformations *i.e.* (3.15)-(3.18) and taking into account the phase coming from the transformation of the integrand itself, we obtain (employing (A.6)-(A.8) to rewrite the whole expression in an $s-$channel basis) the following matrix elements of the operator $\phi^s(b)$ in an $s-$channel basis.

$$
\begin{aligned}
\phi^s(b)^{C_1C_1}_{C_1C_1} &= -e^{\pm 3\pi i b'}\frac{sin\pi(a'+d/2)}{sin\pi(a'+c'+d/2)}\delta_{r',r}\delta_{s',s+2} \\
\phi^s(b)^{C_2C_1}_{C_1C_1} &= e^{\pm 3\pi i b'}e^{\pm i\pi a'}\frac{sin\pi(b')}{sin\pi(a'+c')}\delta_{r',r}\delta_{s',s+2} \\
\phi^s(b)^{C_2C_2}_{C_1C_1} &= e^{\pm 3\pi i b'}e^{\pm i\pi(a+d/2)}\frac{sin\pi(b')sin\pi(b'+d/2)}{sin\pi(a'+c')sin\pi(a'+c'+d/2)}\delta_{r',r}\delta_{s',s+2}
\end{aligned}
\tag{3.19}
$$

As in the one contour case one can elect to rewrite these in a $t-$channel basis instead, using again the results of ref.1) for the change of bases. We shall not give the explicit expressions here, but simply note that in this example as well Verlinde's conjecture is satisfied (see appendix D).

In the next section we go on to examine the modular transformations of the various two-point functions introduced so far.

4 Modular Transformations

We now wish to establish the transformation properties of the two-point conformal blocks under the modular group, generated by S and T. Again, the conformal case has been dealt with in detail in ref.11). Consider first the Ramond sector correlator $<\phi_{1,2}\phi_{1,2}>$ in the t-channel basis

$$\tilde{G}^{\nu}_i(r,s) \quad = \quad \oint_{\tilde{C}_i} dz I^{\nu}(r,s) \tag{4.1}$$

Under T *i.e.* $\tau \to \tau + 1$ the spin structures $\nu = 3$ and 4 are interchanged while $\nu = 1$ and 2 are unchanged. Explicitly, from equations (2.1) and (2.2) one sees that under T

$$\tilde{G}^{\nu=1,2}_i \to e^{2\pi i(\Delta_{r,s}-\frac{c}{24})}\tilde{G}^{\nu=1,2}_i(r,s)$$

and

$$\tilde{G}^{\nu=3}_i(r,s) \to e^{2\pi i(\Delta_{r,s}-\frac{c}{24})}e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}^{\nu=4}_i(r,s)$$

while

$$e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}^{\nu=4}_i(r,s) \to e^{2\pi i(\Delta_{r,s}-\frac{c}{24})}\tilde{G}^{\nu=3}_i(r,s) \tag{4.2}$$

The modular transformation S maps the $a-$ $-$cycle into b and takes the $b-$cycle to $-a$. The spin structures $\nu = 2$ and 4 are interchanged under S while $\nu = 1$ and 3 are left invariant. Let us consider the action of S on the integrand $I^{\nu}(r,s)$ in (4.1). Perform the shifts $\tau \to -1/\tau, z_{\frac{1}{2}} \to z'_{\frac{1}{2}} = -z_{\frac{1}{2}}/\tau$, and in addition the change of variable $z = -z'/\tau$. (We shall assume that $Im(-z_1/\tau) > Im(-z_2/\tau)$ so that S does not change the time ordering of the correlation function.)

When $\nu = 3$, combining the factors of τ (and the phase $e^{-i\pi\frac{\alpha_-^2}{2\tau}(z_1+z_2-2z')^2}$) that arise from modular transforming the θ-functions in equation (2.1), with that from the Jacobian of the change of variable, and finally from the modular transformation of the lattice part equation (2.2) itself, the latter becomes after a Poisson resummation

$$\left(\frac{dz'_1}{dz_1}\frac{dz'_2}{dz_2}\right)^{-\Delta_{1,2}} \frac{\epsilon}{\sqrt{p(p+2)}}\frac{1}{\eta(\tau)^{3/2}}\sum_{n\in\mathbf{Z}} q^{\frac{n^2}{N}} e^{2\pi i n\left[\frac{\lambda}{N}-\frac{\alpha_-}{\sqrt{N}}(z_1+z_2-2z')\right]} \tag{4.3}$$

where ϵ is a λ-independent phase that will be fixed subsequently. For $\nu = 4$, the result is a little more complicated and reads

$$\left(\frac{dz'_1}{dz_1}\frac{dz'_2}{dz_2}\right)^{-\Delta_{1,2}} \frac{\epsilon}{\sqrt{p(p+2)}} e^{\pm i\pi p\left[\frac{\lambda}{N}-\frac{\alpha_-}{\sqrt{N}}(z_1+z_2-2z')\right]}$$

$$\frac{1}{\eta(\tau)^{3/2}}\sum_{n\in\mathbf{Z}} q^{\frac{(n\pm p/2)^2}{N}} e^{2\pi i n\left[\frac{\lambda}{N}-\frac{\alpha_-}{\sqrt{N}}(z_1+z_2-2z')\right]} \tag{4.4}$$

Equations (4.3) and (4.4) may be cast into the usual form by making the shift $n = \frac{1}{2}(\lambda' + Nk)$ in equation (4.3) where $\lambda' = r'(p+2) - s'p$ with $r' - s' \in 2\mathbf{Z}$, and setting in equation (4.4) $n = \frac{1}{2}(\lambda' + Nk) \mp p/2$ with now $r' - s' \in 2\mathbf{Z}+1$. Dropping the irrelevant Jacobian factors in the last two equations (they are a conformal transformation on the external lines of the two-point function), gives the S transformation of the integrands

$$I^{\nu=3}(r,s) \to \frac{\epsilon}{\sqrt{p(p+2)}}\sum_{r'=1}^{p}\sum_{\substack{s'=1 \\ r'-s'\in 2\mathbf{Z}}}^{2(p+2)} e^{i\pi(r\alpha_+ + s\alpha_-)(r'\alpha_+ + s'\alpha_-)} I^{\nu'=3}(r',s') \tag{4.5}$$

and

$$I^{\nu=4}(r,s) \to e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\frac{\epsilon}{\sqrt{p(p+2)}}\sum_{r'=1}^{p}\sum_{\substack{s'=1 \\ r'-s'\in 2\mathbf{Z}+1}}^{2(p+2)} e^{i\pi(r\alpha_+ + s\alpha_-)(r'\alpha_+ + s'\alpha_-)} I^{\nu'=2}(r',s') \tag{4.6}$$

It is somewhat more convenient for what follows to invert these relations by multiplying by $e^{-i\pi(r\alpha_++s\alpha_-)(r''\alpha_++s''\alpha_-)}$ and summing over (r,s) obtaining thereby the action of S^{-1} on the integrands.

$$I^{\nu=3}(r,s|\tau)=\frac{\epsilon^{-1}}{\sqrt{p(p+2)}}\sum_{r'=1}^{p}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}}}^{2(p+2)} e^{-i\pi(r\alpha_++s\alpha_-)(r'\alpha_++s'\alpha_-)}I^{\nu'=3}(r',s'|-\frac{1}{\tau}) \tag{4.5'}$$

and

$$I^{\nu=4}(r,s|\tau)=\frac{\epsilon^{-1}}{\sqrt{p(p+2)}}e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}$$
$$\sum_{r'=1}^{p}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}+1}}^{2(p+2)} e^{-i\pi(r\alpha_++s\alpha_-)(r'\alpha_++s'\alpha_-)}I^{\nu'=2}(r',s'|-\frac{1}{\tau}) \tag{4.6'}$$

where we have suppressed dependence on the arguments z_i. Conversely, starting from $I^{\nu=2}(r,s)$ one obtains

$$e^{-i\pi(\frac{\lambda'^2}{2N}-\frac{1}{12})}I^{\nu'=4}(r',s') \qquad \text{whereas} \qquad I^{\nu=1}(r,s)\rightarrow I^{\nu'=1}(r',s')\,,$$

with phases ϵ^{-1} and η^{-1} and the same summations as in (4.5′) and (4.6′) above, respectively.

Now consider the action of S on the contours $\tilde{C}_1$ and $\tilde{C}_2$ which do not mix under the modular transformation. In fact for $\tilde{C}_1$, corresponding to the identity intermediate state, $\tilde{C}_1\rightarrow\tilde{C}_1'$ with the latter encircling the points $-z_1/\tau$ and $-z_2/\tau$; therefore one has that

$$\tilde{G}_1^{\nu}(r,s)=\frac{\epsilon^{-1}}{\sqrt{p(p+2)}}e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})\delta_{\nu 4}}\sum_{r'=1}^{p}\sum_{s'=1}^{2(p+2)} e^{-i\pi(r\alpha_++s\alpha_-)(r'\alpha_++s'\alpha_-)}$$
$$\oint_{\tilde{C}_1'} dz' e^{-i\pi(\frac{\lambda'^2}{2N}-\frac{1}{12})\delta_{\nu'4}}I^{\nu'}(r',s'\mid -\frac{1}{\tau}) \tag{4.7}$$

For $\nu'=3,4$ (*i.e.* $\nu=3,2$ and $r'-s'\in 2\mathbf{Z}$) and $\nu'=1,2$ (*i.e.* $\nu=1,4$ and $r'-s'\in 2\mathbf{Z}+1$) respectively, one can reduce the range of summations (r',s') to the standard ones in the following way. In the $\nu' = 3, 4$ case this is achieved by first splitting the s' sum into two parts[7]:

$$\sum_{r'=1}^{p} \left(\sum_{s'=0}^{p+1} + \sum_{s'=p+2}^{2(p+2)-1} \right)$$

with $r' - s' \in 2\mathbf{Z}$, and shifting $r' = p - r'', s' = p + 2 + s''$ in the second term. Under the shift $\Gamma(\pm r', s') = \Gamma(\mp r'', s'')$, hence the lattice difference parts of the conformal block are interchanged. For $\nu' = 4$ there is also an overall factor $(-1)^{r''s''}$, coming from a similar factor in front of the second lattice term of equation (2.1). The resulting expressions are

$$\frac{\epsilon^{-1}}{\sqrt{p(p+2)}} \sum_{r'=0}^{p-1} \sum_{s'=0}^{p+1} e^{\frac{i\pi}{2}(rs'+sr')} e^{-i\pi ss'\alpha_-^2} \left(e^{-i\pi rr'\alpha_+{}^2} - (-)^{rr'} e^{i\pi rr'\alpha_+^2} \right)$$

$$\oint_{\tilde{C}_1'} dz' e^{-i\pi(\frac{\lambda'^2}{2N} - \frac{1}{12})\delta_{\nu',4}} I^{\nu'}(r', s')$$

Next the s' sum is split once more, now into

$$\sum_{r'=0}^{p-1} \left(\sum_{s'=0}^{r'} + \sum_{s'=r'+1}^{p+1} \right)$$

and this time the shift $r' = p - r'', s' = p + 2 - s''$ performed in the second term. Further, changing variables $z \to z_1 + z_2 - z$ in the second term, one sees that $\Gamma(\pm r', s') = \Gamma(\pm r'', s'')$ while the other factors are left invariant. Recombining the sums one obtains the action of S^{-1} on the conformal blocks

$$\tilde{G}_1^{\nu=3,2}(r, s|z_1, z_2, \tau) = \sum_{r'=1}^{p-1} \sum_{s'=1}^{r'} e^{-i\pi(\frac{\lambda'^2}{2N} - \frac{1}{12})\delta_{\nu'4}}$$

$$(S^{-1}_{\nu'=3,4})^{r's'}_{r,s} \tilde{G}_1^{\nu'=3,4}(r', s'| -\frac{z_1}{\tau}, -\frac{z_2}{\tau}, -\frac{1}{\tau}) \tag{4.8}$$

where the matrix $S^{-1}_{\nu'=3} = (S^{-1}_{\nu'=4})$ is

$$-\frac{4\epsilon^{-1}}{\sqrt{p(p+2)}} sin\pi(rr'\alpha_+^2 - \frac{rs'}{2}) sin\pi(ss'\alpha_-^2 - \frac{sr'}{2}) \tag{4.9}$$

with $r' - s' \in 2\mathbf{Z}$. When $\nu' = 1, 2$ (*i.e.* $\nu = 1, 4$) one splits the sum into four terms:

$$\left(\sum_{r'=1}^{[\frac{p-1}{2}]} + \sum_{r'=[\frac{p+1}{2}]}^{p} \right) \left(\sum_{s'=0}^{p+1} + \sum_{s'=p+2}^{2(p+2)-1} \right)$$

[7]The only restriction on the s' sum is that it span an interval of $2(p+2)$ units (satisfying $r' - s' \in 2\mathbf{Z}$ of course). Thus it may be rewritten as indicated, recalling the fact that the $s' = 0$ block vanishes.

with the restriction $r' - s' \in 2\mathbf{Z}+1$. Implementing the same shifts from (r', s') to (r'', s'') in successive steps as before, one recovers the standard range.[8] The final result is

$$\tilde{G}_1^{\nu=4}(r,s|z_1,z_2,\tau) = e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})\delta_{\nu'2}} \left(\sum_{r'=1}^{[\frac{p-1}{2}]} \sum_{s'=1}^{r'-1} + \sum_{r'=[\frac{p+1}{2}]}^{p-1} \sum_{s'=1}^{r'+1} \right) (S_{\nu'=2}^{-1})_{r,s}^{r's'} \quad \tilde{G}_1^{\nu'=2}(r',s'|-\frac{z_1}{\tau},-\frac{z_2}{\tau},-\frac{1}{\tau}) . \quad (4.10)$$

The matrix $S_{\nu'=2}^{-1}$ in this case is

$$-\frac{4\epsilon^{-1}\gamma_{r',s'}}{\sqrt{p(p+2)}} sin\pi(rr'\alpha_+^2 - \frac{sr'}{2}) sin\pi(ss'\alpha_-^2 - \frac{rs'}{2}). \quad (4.11)$$

where $\gamma_{r',s'} = (1 - \frac{1}{2}\delta_{p,2\mathbf{Z}}\delta_{r',\frac{p}{2}}\delta_{s',\frac{p+2}{2}})$ and $r' - s' \in 2\mathbf{Z}+1$. For the remaining spin structure $\nu' = 1$ the modular matrix, $S_{\nu'=1}^{-1}$ is, for odd p, symmetric and equal to

$$-\frac{4\eta^{-1}}{\sqrt{p(p+2)}} sin\pi(rr'{\alpha_+}^2 - \frac{sr'}{2} + \frac{r'}{2}) cos\pi(ss'{\alpha_-}^2 - \frac{rs'}{2} - \frac{r'}{2}) \quad (4.12)$$

with η an (r,s)–independent phase.

Equations (4.8), (4.9) and (4.10), (4.11) are essentially the modular transformation of superconformal characters[15,16)]. In the ensuing discussion it is convenient to deal with the p odd and p even cases separately. Calling $\mathcal{S}$ the 4×4 matrix that acts on the following vector of conformal blocks

$$\mathbf{G}_1(r,s) = \begin{pmatrix} \tilde{G}_1^{\nu=1} & \tilde{G}_1^{\nu=2} & \tilde{G}_1^{\nu=3} & \tilde{G}_1^{\nu=4} \end{pmatrix} \quad (4.13)$$

one has that

$$\mathbf{G}_1(r,s|\tau) = \sum_{r',s'} (\mathcal{S}_{\tilde{C}_1\tilde{C}_1})_{r,s}^{r',s'} \mathbf{G}_1(r',s'|-\frac{1}{\tau})$$

with the above matrices as entries *i.e.*

$$\mathcal{S}_{\tilde{C}_1\tilde{C}_1} = \begin{pmatrix} (S_{\nu'=1}^{-1})_{r,s}^{r',s'} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi(\frac{\lambda'^2}{2N}-\frac{1}{12})}(S_{\nu'=4}^{-1})_{r,s}^{r',s'} \\ 0 & 0 & (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & 0 \\ 0 & e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}(S_{\nu'=2}^{-1})_{r,s}^{r',s'} & 0 & 0 \end{pmatrix} .$$

[8] After the second shift for p even, there is an extra term with $r' = \frac{p}{2}$, $s' = r'+1$ (corresponding to the Ramond vacuum state) which can be incorporated into the range above by introducing a factor $\frac{1}{2}$ in front of the corresponding matrix element. For details see ref.18).

(4.14)

One can check by direct computation that $\mathcal{S}^{-2} = diag(\eta^2, \epsilon^2, \epsilon^2, \epsilon^2)$ (see appendix B). Using the fact that in the $z_1 \to z_2$ limit the $\tilde{C}_1$ block factorizes for even spin structures as $(z_1 - z_2)^{-2\Delta_{1,2}} \times \chi_{r,s}(\tau)$ with the character as residue[9], and that $S^2\tau = \tau$ while $S^2(z_1 - z_2) = -(z_1 - z_2)$, the action of S^2 on blocks can only be a phase. In particular ϵ is fixed to be $\epsilon = e^{\pm i\pi\Delta_{1,2}}$ by this argument. When $\nu = 1$, as was obtained in (I), the leading term in the factorization limit has vanishing residue and one is left instead with the vacuum expectation value of the supersymmetry current $G_{-\frac{3}{2}}$ on the torus (a function of τ only by virtue of translation invariance on the torus). The resulting expression $(z_1 - z_2)^{-2\Delta_{1,2}+\frac{3}{2}} \times < G_{-\frac{3}{2}}(0) >$ fixes η to be $e^{\pm i\pi(\Delta_{1,2}-\frac{3}{4})}$.

Furthermore, defining $\overline{S}_{\nu'=2} = e^{i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})} S^{-1}_{\nu'=2}$ and $\overline{S}_{\nu'=4} = e^{-i\pi(\frac{\lambda'^2}{2N}-\frac{1}{12})} S^{-1}_{\nu'=4}$, for p odd one observes that the matrix $\overline{S}_{\nu'=2} = \epsilon^{-2}(\overline{S}_{\nu'=4})^{\dagger}$ whereas $S_{\nu'=1,3} = (S_{\nu'=1,3})^T$. The unitarity of the 4×4 matrix $\mathcal{S}_{\tilde{C}_1\tilde{C}_1}$ follows directly from the result that $\mathcal{S}^{-2} = diag(\eta^2, \epsilon^2, \epsilon^2, \epsilon^2)$ and the last two properties. In addition, note that clearly it is possible to make $\mathcal{S}$ symmetric (while maintaining unitarity) simply by redefining the $\tilde{G}^{\nu=4}(r,s)$ block, multiplying it by the phase $e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}$.

For even p the situation is a little different for two reasons. Firstly, the modular transformation of the $\nu = 1$ or odd spin structure sector is, instead of equation (4.12), found to be

$$\tilde{G}^{\nu=1}_{i=1}(r,s|z_1,z_2,\tau) = \left(\sum_{r'=1}^{\frac{p}{2}-1} \sum_{s'=1}^{r'-1} + \sum_{r'=\frac{p}{2}+1}^{p-1} \sum_{s'=\frac{p+2}{2}+1}^{r'+1} \right) (S^{-1}_{\nu'=1})^{r',s'}_{r,s} \tilde{G}^{\nu'=1}_{i=1}(r',s'|-\frac{z_1}{\tau},-\frac{z_2}{\tau},-\frac{1}{\tau})$$
$$+ (1-(-)^r)\tilde{G}^{\nu'=1}_{i=1}(\frac{p}{2},\frac{p+2}{2}|-\frac{z_1}{\tau},-\frac{z_2}{\tau},-\frac{1}{\tau}) \qquad (4.15)$$

Notice that for $\nu = 1$ (p even), equation (4.12), and hence the S^{-1} terms in (4.15), vanish when either (r,s) or (r',s') are equal to $(\frac{p}{2},\frac{p+2}{2})$. This is also consistent with the conjugation relation equation (4.16) below, which when acting on the block $\tilde{G}^{\nu=1}_{i=1}(r,s)$, has a vanishing $L.H.S.$ for this value of (r,s) because $\phi^t(a)_{\tilde{C}_1\tilde{C}_1} \sim sin\pi(s\alpha_-^2 - \frac{r}{2})$ as is evident from equation (3.6). Now, if one computes $(S^{-1}_{\nu=1})^2$ for p odd one obtains η^{-2} times the unit matrix as one ought. On the contrary, for p even, $[S^{-1}_{\nu=1} + (1-(-)^r)\delta_{r,r'}\delta_{s,s'}\delta_{r',\frac{p}{2}}\delta_{s',\frac{p+2}{2}}]^2 \neq \eta^{-2}\mathbf{1}$. This

[9]To be precise there is a numerical factor, independent of (even) spin structure, $B(\frac{1}{2}-\alpha_-{}^2, \frac{1}{2}-\alpha_-{}^2)$ omitted here. This can be absorbed in the overall normalisation of the conformal blocks $\tilde{G}(r,s)$, to be fixed in section 6, such that the factorization limit is in fact as stated above.

suggests that the $\tilde{G}^{\nu=1}_{i=1}(\frac{p}{2},\frac{p+2}{2})$ block is zero and should in fact be excluded from the state space. Indeed, we have checked that this particular block vanishes at leading and first sub-leading order of a $q-$expansion in the degeneration limit (transforming to a $s-$channel basis). The computation is straightforward but tedious (requiring several delicate $O(q)$ cancellations between various terms) and we refrain from reproducing it here. Unfortunately, we do not have a general argument for this at arbitrary order in q, but shall nonetheless assume in the following that the $\tilde{G}^{\nu=1}_{i=1}(\frac{p}{2},\frac{p+2}{2})$ block is identically zero. Taking only the first line of equation (4.15) to be the correct modular transformation of the $\nu=1$ sector (and excluding the $(r,s)=(r',s')=(\frac{p}{2},\frac{p+2}{2})$ element) one then indeed has $(S^{-1}_{\nu=1})^2=\eta^{-2}\mathbf{1}$.

Secondly, there is the presence of the factor $\gamma_{r',s'}$ in equations (4.11) and (4.14). The physical origin of this factor is the fact that Ramond states, apart from the vacuum $(r',s')=(\frac{p}{2},\frac{p+2}{2})$, are doubly degenerate. On computation one obtains[10], as in the p odd situation, that $\mathcal{S}^2=diag(\eta^2,\epsilon^2,\epsilon^2,\epsilon^2)$. However, the matrix $\mathcal{S}$ is no longer unitary. With the normalization of Ramond conformal blocks used here (see also ref.15), the unitarity relations assume the form

$$\sum_{r,s}\gamma_{r,s}(\overline{S}^{\nu'=4})^{r',s'}_{r,s}(\overline{S}^{\nu'=4})^{*\,r'',s''}_{r,s}=\delta_{r',r''}\delta_{s',s''}$$

$$\sum_{r,s}(\overline{S}^{\nu'=2})^{r',s'}_{r,s}(\overline{S}^{\nu'=2})^{*\,r'',s''}_{r,s}=\gamma_{r',s'}\delta_{r',r''}\delta_{s',s''}$$

or in terms of the 4×4 matrix $\mathcal{S}$

$$\mathcal{S}^{\dagger}\gamma\mathcal{S}=\gamma \tag{4.16}$$

where γ is the diagonal unit matrix except for a $\frac{1}{2}$ at the $r=\frac{p}{2}, s=\frac{p+2}{2}$ entry *i.e.* $\gamma=diag(1,1,...,1,\frac{1}{2},1...)$. Essentially the same result has also been obtained by the authors of ref. 20).

Since the transformation S interchanges the $a-$ and $b-$ cycles, there exists a conjugation relation between the monodromy operators

$$\phi^t(a)=S^{-1}\phi^t(b)S \tag{4.17}$$

As a check on the above computations we have verified that the matrix elements of this operator equation are indeed satisfied and in appendix C the calculation for one of these (the $\tilde{C}_2\tilde{C}_2$ element) is demonstrated. Incidentally, one expects

[10] The factor $\gamma_{r',s'}$ in equation (4.11) is in fact essential for this property to hold

that both equation (4.17) above and the fact that $S^{-2} \sim diag(\eta^2, \epsilon^2, \epsilon^2, \epsilon^2)$ are of course basis independent and must be satisfied simply for reasons of algebraic consistency.

At this point it is perhaps worth making some comparison with the structure expected from more abstract considerations of rational conformal field theory[2),5)]. The first point to note is that refs.2) and 5) make the technical assumption that left and right extended chiral algebras consist only of generators with integral conformal weight (thereby evidently excluding the superconformal case). In the superconformal models considered here at least, we have constructed a basis – the *Verlinde basis* – proportional to

$$E_{\pm}(r,s) \equiv \frac{1}{2}\left[\tilde{G}_1^{\nu=3}(r,s) \pm e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}_1^{\nu=4}(r,s)\right]$$

and

$$O_{\pm}(r,s) \equiv \frac{1}{2}\left[\tilde{G}_1^{\nu=2}(r,s) \pm e^{\frac{i\pi}{4}}\tilde{G}_1^{\nu=1}(r,s)\right] \tag{4.18}$$

with respect to which both (i) the Verlinde conjecture (that the $b-$cycle monodromy on characters yield, up to an overall normalization, the fusion rule coefficients) is satisfied and which in addition (ii) is an eigenstate of $\phi^t(a)$; or more precisely in superconformal models, transforms irreducibly under the $\phi^t(a)$ algebra. The proof [2)] of the Verlinde conjecture relies only on conformal and duality properties (consistency with the fusion rules) under certain manipulations of conformal blocks in their degeneration and factorization limits. Since in the superconformal case we have already explicitly checked that these limits are indeed consistent with the fusion rules and as already emphasized, we are using bases in which the descendents in a $q-$expansion are always at integer level spacing above the highest weight state (regarding superdescendents as Virasoro primaries), the present example suggests that by working in the appropriate basis as above the proof can always be carried through. However we have no general proof of this statement in superconformal models, even less for an arbitrary rational conformal theory including generators of non-integer conformal weight. Now, in the conformal case both the above properties (i) and (ii) of the basis, together with the (basis independent) conjugation equation (4.17) are sufficient to demonstrate that the fusion rules are diagonalised by the modular matrix S. The (basis dependent) symmetry and unitarity of S follow as a corollary.[11]

[11] Assuming in addition permutation symmetry of the fusion coefficients N_{ijk}, a consequence of the associativity of the fusion algebra (the latter itself following from the fact that both $s-$ and $t-$channel conformal blocks provide a complete basis).

In the next section we shall demonstrate, assuming that the matrix elements of $\phi(b)$ are indeed always the fusion coefficients N_{IJ}^{K} (generalized to account for Neveu-Schwarz and Ramond sector dependence), that the fusion rules are diagonalised in superconformal models as well. However, the question of whether the diagonalising modular matrix is symmetric and unitary appears to be more delicate and in particular depends on whether p is odd or even. Acting on a *complete* Verlinde basis of conformal blocks:

$$\overline{\mathbf{G}}_1 = \begin{pmatrix} E_+ & E_- & O_+ & O_- \end{pmatrix}$$

and denoting as before the action of the modular transformation by

$$\overline{\mathbf{G}}_1(r, s \mid \tau) = \sum_{r',s'} (\mathcal{S}_{\bar{C}_1 \bar{C}_1})_{r,s}^{r',s'} \overline{\mathbf{G}}_1(r', s' \mid -\frac{1}{\tau}) \ ,$$

the modular matrix is instead

$$\mathcal{S}_{\bar{C}_1 \bar{C}_1} = \frac{1}{2} \begin{pmatrix} S^{-1}_{\nu'=3} & S^{-1}_{\nu'=3} & S^{-1}_{\nu'=2} & S^{-1}_{\nu'=2} \\ S^{-1}_{\nu'=3} & S^{-1}_{\nu'=3} & -S^{-1}_{\nu'=2} & -S^{-1}_{\nu'=2} \\ S^{-1}_{\nu'=4} & -S^{-1}_{\nu'=4} & S^{-1}_{\nu'=1} & -S^{-1}_{\nu'=1} \\ S^{-1}_{\nu'=4} & -S^{-1}_{\nu'=4} & -S^{-1}_{\nu'=1} & S^{-1}_{\nu'=1} \end{pmatrix} \tag{4.19}$$

and is symmetric and unitary for p odd. This is in accord with the well-known result that for $p = 3$ in particular, there are two equivalent representations of the TIM – either as a $p = 4$ conformal model or as the $N = 1$ superconformal model discussed above – and hence Verlinde's results must hold for this case at least. In fact for $p = 3$ the combinations of blocks (4.18) in the factorization limit are of course nothing but the Virasoro characters:

$$E_+(1,1) \sim \chi_0^{Vir},\ E_-(1,1) \sim \chi_{\frac{3}{2}}^{Vir},\ E_+(1,3) \sim \chi_{\frac{1}{10}}^{Vir},\ E_-(1,3) \sim \chi_{\frac{3}{5}}^{Vir}$$

and

$$O_\pm(2,3) \sim \chi_{\frac{3}{80}}^{Vir}, \qquad O_\pm(2,1) \sim \chi_{\frac{7}{16}}^{Vir}.$$

However acting on a different basis in general will not lead to a symmetric matrix $\mathcal{S}_{\bar{C}_1 \bar{C}_1}$ as we have just seen in equation (4.14) above (although it does happen to be unitary in that particular case). For that matter, neither is it sufficient to act only on a subset of the Verlinde basis (by for instance taking just the + signs in (4.18) above) because the action of S will take one out of the basis (for example $G^3 + G^4 \to G^3 + G^2$ under a S-transformation). Despite the fact that the odd spin structure ($\nu = 1$) sector does not mix under modular transformations with the even sectors, it is still necessary to work with a complete 4−dimensional basis.[12] Therefore, for p odd, the Verlinde basis is a sufficient condition to obtain a symmetric and unitary modular matrix (though not it would seem strictly a necessary condition, since we have constructed as well a basis in which the action of $\phi(a)$ is not diagonal but S is nonetheless symmetric and unitary).

For p even, working in the Verlinde basis as above, the factor $\gamma_{r,s}$ in the matrix $S^{-1}_{\nu'=2}$ (equation 4.11) associated with the Ramond vacuum state appears to prevent $\mathcal{S}$ from being either symmetric or unitary. It is however possible to transform the Ramond sector ($\nu = 2$) blocks to a basis with respect to which the modular matrix is symmetric (and unitary) in the following way (see also the second of refs.16). Define a new basis for $(r, s) = (\frac{p}{2}, \frac{p+2}{2})$ by

$$\tilde{G}^{\nu=2}_{i=1}(r,s) = \sum_{r',s'} \delta_{r,r'}\delta_{s,s'}[1 + (\sqrt{2}-1)\delta_{r,\frac{p}{2}}\delta_{s,\frac{p+2}{2}}]\tilde{G}^{\nu=2}_{i=1}(r',s')' \tag{4.20}$$

Then with respect to either of the two following bases the modular matrix $\mathcal{S}$ is seen to be symmetric (and unitary):

$$\begin{aligned} &(i) \quad \mathbf{G}(r,s)' = \left(\tilde{G}^{\nu=1}_1(r,s) \;\; \tilde{G}^{\nu=2}_1(r,s)' \;\; \tilde{G}^{\nu=3}_1(r,s) \;\; e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}^{\nu=4}_1(r,s) \right) \\ &(ii) \quad \overline{\mathbf{G}}' = \left(E_+ \;\; E_- \;\; O'_+ \;\; O'_- \right) \end{aligned} \tag{4.21}$$

where $O'_\pm$ refer to the new basis (4.20) above. Now we should also consider the monodromy operations with respect to these new bases. The first of these, *viz.* (i), is obviously not a Verlinde basis as defined previously. As for (ii), consider the $\phi^t(b)$ action on the $\overline{\mathbf{G}}'$ basis. One observes from equation (3.14) that the $\tilde{C}_1\tilde{C}_1$ matrix element of a b−cycle monodromy transformation of $E_+(r,s)$ is

$$\sum_{r',s'} e^{\mp\frac{i\pi}{8}} e^{\pm\frac{3}{2}i\pi\alpha_-^2} \frac{cos\pi\alpha_-^2}{sin2\pi\alpha_-^2}\delta_{r',r}(\delta_{s',s+1} + \delta_{\substack{s',s-1\\ s>1}})(1 + (\sqrt{2}-1)\delta_{r',\frac{p}{2}}\delta_{s',\frac{p+2}{2}})$$

Therefore the Verlinde conjecture itself is not satisfied by this particular choice of basis.

We also have to consider the situation where the contour is $\tilde{C}_2$, in which case the intermediate state is either (for even spin structure $\nu = 2, 3, 4$) the superdescendent $\phi^S_{1,3}$ or (for odd spin structure $\nu = 1$) the primary $\phi_{1,3}$ itself. Representing the torus as a parallelogram, $\tilde{C}_2$ is illustrated in fig.7a. On performing the modular transformation S, $\tilde{C}_2$ is rotated anti-clockwise, preserving

[12] We thank A.Sagnotti for a useful discussion on this point; see for instance ref.19).

its orientation in the z-plane leading to the contour $\tilde{C}_2'$ shown in fig.7c. To work out the action of S on $\tilde{C}_2$ it is sufficient to compare $\tilde{C}_2$ and $\tilde{C}_2'$ in the degeneration limit, where they are both Pochammer contours on the complex plane encircling the singular points 0 and ∞. On transforming to annular coordinates $\xi = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$, the contour $\tilde{C}_2$ (see fig. 7b) in the $q \to 0$ limit is

$$\oint_{\tilde{C}_2} dx x^{\tilde{a}}(z-x)^b(x-1)^c =$$

$$-4 sin\pi\tilde{a} sin\pi(\tilde{a}+b+c)e^{-i\pi(\tilde{a}+b+c)}\int_0^\infty dx(-x)^{\tilde{a}}(z-x)^b(x-1)^c \quad (4.22)$$

where $\tilde{a} = a(r,s) - \frac{1}{2}(\delta_{\nu,1}+\delta_{\nu,2})$. To compare $\tilde{C}_2'$ with this we need to write it also as an integral from 0 to ∞. In doing this note that on the first sheet (upper parallelogram of fig.7c) the branch at the origin (in the degeneration limit) after the modular transformation is given by

$$\tilde{a}' = [(r'+1)\alpha_+ + (s'+1)\alpha_-]\alpha_- - \frac{1}{2}(\delta_{\nu',1}+\delta_{\nu',2})$$

as obtained above. However on the second sheet (lower parallelogram in fig. 7c) we have to shift $z = -z'/\tau$ to $-z'/\tau + 1/\tau$ in addition to the modular transformation $z_i' = -z_i/\tau$ of z_1 and z_2 in the integrand (equation 2.1). After Poisson resumming *etc.* as before one observes that the branch at the origin is now in fact

$$\tilde{a}'' = [(r'+1)\alpha_+ + ((s'-2)+1)\alpha_-]\alpha_- - \frac{1}{2}(\delta_{\nu',1}+\delta_{\nu',2})$$

Therefore one has that[13]

$$\oint_{\tilde{C}_2'} dx x^{\tilde{a}'}(z-x)^b(x-1)^c$$

$$= e^{i\pi\tilde{a}'}\int_0^\infty dx(-x)^{\tilde{a}'}(z-x)^b(x-1)^c$$

$$+ \quad e^{i\pi(\delta_{\nu',3}+\delta_{\nu',2})}e^{i\pi\tilde{a}'}e^{-2\pi i(\tilde{a}'+b+c)}\int_0^\infty dx(-x)^{\tilde{a}''}(z-x)^b(x-1)^c$$

$$+ \quad e^{i\pi(\delta_{\nu',3}+\delta_{\nu',2})}e^{i\pi\tilde{a}'}e^{-2\pi i(\tilde{a}'+b+c)}e^{2\pi i(\tilde{a}''+b+c)}\int_\infty^0 dx(-x)^{\tilde{a}''}(z-x)^b(x-1)^c$$

$$+ \quad e^{i\pi\tilde{a}'}e^{-2\pi i(\tilde{a}'+b+c)}e^{2\pi i(\tilde{a}''+b+c)}e^{-2\pi i\tilde{a}''}\int_\infty^0 dx(-x)^{\tilde{a}'}(z-x)^b(x-1)^c \quad (4.23)$$

Combining this with the modular transformation of the integrands, equations (4.5′) and (4.6′), gives the transformation of the conformal blocks $\tilde{G}_{i=2}^{\nu=3,2}(r,s)$

[13]Notice that when one goes to the second sheet (effectively by letting $z \to z-\tau$) the integrand transforms to $e^{i\pi(\delta_{\nu',3}+\delta_{\nu',2})}I^{\nu'}(r',s'-2)$

in the $\tilde{C}_2$ channel. Once again the summations are brought into the standard range by implementing the two successive shifts in the dummy indices (r', s') described above. (Note that in implementing the second shift, $r' = p - r''$ and $s' = p+2-s''$, one performs a change of variables $z \to z_1 + z_2 - z$ in the integrand of the closed contour integral over $\tilde{C}_2$, whereas eqns. (4.22) and (4.23) above are in terms of open contours in the degeneration limit. (This introduces some additional phases which must be included to get the final result.) For $\nu' = 3, 4$ respectively one obtains (p either odd or even):

$$(S^{-1}_{\nu'=3,4})^{r's'}_{rs} = \frac{2\epsilon^{-1}}{\sqrt{p(p+2)}}(-)^{(r-s+\delta_{\nu',4}+(r'-s')\delta_{\nu',3})/2} e^{-i\pi\alpha_-^2} \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})$$

$$\left[\frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} - \frac{\sin\pi(s'(s-1)\alpha_-^2 - \frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2 - \frac{(s-1)}{2})}\right] . \tag{4.24}$$

A similar expression is obtained for $\nu' = 2$ (p odd):

$$(S^{-1}_{\nu'=2})^{r's'}_{rs} = \frac{2\epsilon^{-1}}{\sqrt{p(p+2)}}(-)^{((r'-s'+1)/2} e^{-i\pi\alpha_-^2} \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})$$

$$\left[\frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} - \frac{\sin\pi(s'(s-1)\alpha_-^2 - \frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2 - \frac{(s-1)}{2})}\right] . \tag{4.25}$$

One should remark here that when p is an even integer, there appear to be the following additional terms in $S^{-1}_{\nu'=2}$ *viz.*

$$-i\frac{\epsilon^{-1}}{\sqrt{p(p+2)}}(-)^{((r'-s'+1)/2} e^{-i\pi\alpha_-^2} \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})$$

$$\left[\frac{\cos\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} - \frac{\cos\pi(s'(s-1)\alpha_-^2 - \frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2 - \frac{(s-1)}{2})}\right] \delta_{r',\frac{p}{2}}\delta_{s',\frac{p+2}{2}} \tag{4.26}$$

However, the same argument that led to the vanishing, in a s−channel q-expansion, of the $\tilde{G}^{\nu=1}(\frac{p}{2}, \frac{p+2}{2})$ block for the closed contour $\tilde{C}_1$, implies that for the contour $\tilde{C}_2$ instead it is now the block $\tilde{G}^{\nu=2}(\frac{p}{2}, \frac{p+2}{2})$ that vanishes. Therefore in the present case we exclude the $r', s' = \frac{p}{2}, \frac{p+2}{2}$ state in the $\nu' = 2$ sector and hence the terms in (4.26) above do not contribute and equation (4.25) in fact holds for p even as well. Finally, the modular matrix in the $\nu = 1$ sector is

$$(S^{-1}_{\nu'=1})^{r's'}_{rs} = \frac{2i\eta^{-1}}{\sqrt{p(p+2)}}\gamma_{r',s'} e^{-i\pi\alpha_-^2} \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})$$

$$\left[\frac{\sin\pi(s'(s+1)\alpha_-^2-\frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2-\frac{(s+1)}{2})}-\frac{\sin\pi(s'(s-1)\alpha_-^2-\frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2-\frac{(s-1)}{2})}\right]. \qquad (4.27)$$

These matrices vanish for $s'=1$, corresponding to the fact that they are square matrices (there is no $s=1$ conformal block for $\tilde{C}_2$). As in the identity $\tilde{C}_1$ channel, the square of the $\tilde{C}_2$ channel modular matrix $\mathcal{S}$ (4×4 in spin structure space) is a phase [14] : $diag(\eta^{-2}\epsilon^{-2}\epsilon^{-2}\epsilon^{-2})e^{-2\pi i\alpha_-^2}$ (see appendix B). Comparing this with the action of $\mathcal{S}^{-2}$ in the factorization limit where the conformal block (for even spin structures) decomposes as $(z_1-z_2)^{-2\Delta_{1,2}+\Delta_{1,3}+\frac{1}{2}} < N_{1,3}^S(0) >$, and for odd spin structure as $(z_1-z_2)^{-2\Delta_{1,2}+\Delta_{1,3}} < N_{1,3}(0) >$, one is able to completely fix ϵ to be $e^{-i\pi\Delta_{1,2}}$ and $\eta=e^{i\pi(\Delta_{1,2}-\frac{3}{4})}$. As in conformal models, the $\tilde{C}_2$ channel matrix S is in general not symmetric for either odd or even p. The unitarity relations take the form:

$$\begin{aligned}
\sum_{r,s} M_{r,s}(S^{-1}_{\nu'=1})^{r',s'}_{r,s}(S^{-1}_{\nu'=1})^{*r'',s''}_{r,s} &= M_{r',s'}\delta_{r',r''}\delta_{s',s''}\\
\sum_{r,s} N_{r,s}(S^{-1}_{\nu'=2})^{r',s'}_{r,s}(S^{-1}_{\nu'=2})^{*r'',s''}_{r,s} &= M_{r',s'}\delta_{r',r''}\delta_{s',s''}\\
\sum_{r,s} N_{r,s}(S^{-1}_{\nu'=3})^{r',s'}_{r,s}(S^{-1}_{\nu'=3})^{*r'',s''}_{r,s} &= N_{r',s'}\delta_{r',r''}\delta_{s',s''}\\
\sum_{r,s} M_{r,s}(S^{-1}_{\nu'=4})^{r',s'}_{r,s}(S^{-1}_{\nu'=4})^{*r'',s''}_{r,s} &= N_{r',s'}\delta_{r',r''}\delta_{s',s''} \qquad (4.28)
\end{aligned}$$

where

$$\begin{aligned}
N_{r,s} &= \cos(\pi(s+1)\alpha_-^2-r/2)\cos(\pi(s-1)\alpha_-^2-r/2)\\
M_{r,s} &= \gamma_{r,s}\sin(\pi(s+1)\alpha_-^2-r/2)\sin(\pi(s-1)\alpha_-^2-r/2) \qquad (4.29)
\end{aligned}$$

The proof of these goes as follows. For example consider the third relation $\nu'=3$ in equation (4.28): assume this relation is not true and multiply both sides from the right by $(S_{\hat{\nu}=3})^{\hat{r};\hat{s}}_{r'',s''}$. Summing over (r'',s'') we get

$$N_{\hat{r},\hat{s}}S^{r',s'}_{\hat{r},\hat{s}}-N_{r',s'}S^{\hat{r},\hat{s}}_{r',s'}\neq 0$$

which is a contradiction because $N_{\hat{r},\hat{s}}S^{r',s'}_{\hat{r},\hat{s}}$ is symmetric as can be easily shown by direct substitution. The other relation is proved in a similar way.

The modular transformation for the double contour correlators can be obtained similarly. In particular, for the two cases $< \phi_{1,3}\phi_{1,3} >$ and $< \phi^S_{1,3}\phi^S_{1,3} >$ which involve only even spin structures the modular transformation matrices

[14]Note that when p is even, for this to be the case the extra terms in equation (4.26) should not contribute as we have argued to be the case. Likewise, the factor $\gamma_{r',s'}$ in (4.27) is crucial.

$S_{\nu'}^{-1}$, when the contours are both taken as $\tilde{C}_1\tilde{C}_1$, are identical to equations (4.9) and (4.11) as has to be the case since we know that, in the factorization limit, this combination of contours reproduces the superconformal characters.

A Verlinde basis of conformal blocks is provided by:

$$\mathbf{G}(r,s) = \left(\tilde{G}_{11}^{\nu=2}(r,s) \quad \tilde{G}_{11}^{\nu=3}(r,s) \quad e^{-i\pi(\frac{\lambda^2}{2N}-\frac{1}{12})}\tilde{G}_{11}^{\nu=4}(r,s) \right) \tag{4.30}$$

and again, for p odd, the 3×3 modular matrix

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & (S_{\nu'=4}^{-1})_{r,s}^{r',s'} \\ 0 & (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & 0 \\ (S_{\nu'=2}^{-1})_{r,s}^{r',s'} & 0 & 0 \end{pmatrix} \tag{4.31}$$

with entries given by equations (4.9) and (4.11) is symmetric and unitary.

On the other hand, one could also work with another (Verlinde) basis:

$$\left(E_+(r,s)\; E_-(r,s)\; \tilde{G}_{11}^{\nu=2}(r,s) \right) \tag{4.32}$$

where $E_\pm(r,s)$ are defined as in equation (4.18). With respect to this basis

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & (S_{\nu'=2}^{-1})_{r,s}^{r',s'} \\ (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & (S_{\nu'=3}^{-1})_{r,s}^{r',s'} & -(S_{\nu'=2}^{-1})_{r,s}^{r',s'} \\ 2(S_{\nu'=4}^{-1})_{r,s}^{r',s'} & -2(S_{\nu'=4}^{-1})_{r,s}^{r',s'} & 0 \end{pmatrix} \tag{4.33}$$

which is in fact not symmetric, even for p odd. This can be understood by observing that, although both bases are eigenstates of $\phi^t(a)$, the basis (4.30) is in fact an irreducible representation of the $\phi^t(a)$ algebra whereas the other basis (4.32) is not (since it is a sum of irreducible representations). (In the one contour example discussed previously, the basis (4.18) was both an eigenstate of $\phi^t(a)$ and also irreducible under this algebra.) One concludes that the Verlinde basis appears to be a sufficient condition to obtain a symmetric modular matrix $\mathcal{S}$ for p odd.

In addition note that both (4.31) and (4.33) satisfy $\mathcal{S}^2 = 1$ when acting on characters - for p odd as well as p even.

For the other combinations of contours one obtains the modular transformation of various one-point functions. The remaining two-point function case (c)

$< \phi_{1,3}\phi^S_{1,3} >$, with contours $\tilde{C}_1\tilde{C}_1$, has the modular transformation appropriate to $< G_{-\frac{3}{2}} >$ *i.e.* equation (4.12). With the other combinations of contours, $\tilde{C}_1\tilde{C}_2$ and $\tilde{C}_2\tilde{C}_2$ one gets the modular transformations of the torus one-point functions $< \phi_{1,3}(0) >$ and $< \phi^S_{1,5}(0) >$ respectively.

5 Superconformal Verlinde Formulae

In previous sections we have checked in several examples that, in an appropriate basis, the $b-$cycle monodromy operator in the $t-$channel, $\phi^t(b)$, yields the expected superconformal fusion rules up to an r, s independent factor; that is to say, the Verlinde conjecture appears to hold also in superconformal models. Let us assume that this is in fact true. The factor can of course be eliminated by an overall (r, s independent) change of normalization of the blocks, and the matrix elements of $\phi^t(b)$ defined to be precisely the fusion coefficients N^K_{IJ} occuring in

$$\varphi_I \times \varphi_J = N^K_{IJ}\varphi_K \tag{5.1}$$

Here, φ_I is one of the operators $N_{r,s}$, $N^S_{r,s}$ or $R_{r,s}$ and the upper case indices denote both r, s as well as spin structure sector ν (or rather the appropriate combinations of spin structures discussed before). Since we are interested in the action of $\phi_I(b)$ on characters rather than two-point blocks, the change in normalization may be effected by factorizing the $t-$channel blocks on the identity intermediate state and (for example in the one contour example considered above, which fixes the index I in $\phi_I(b)$ to be the Ramond state $R_{1,2}$) requiring that

$$\phi^t(b)_{\tilde{C}_1\tilde{C}_1}\ lim_{z_1\to z_2}E_+(1,1) = lim_{z_1\to z_2}O_+(1,2) \tag{5.2}$$

(equating terms proportional to $(z_1 - z_2)^{-2\Delta_{1,2}}$). In other words rescaling

$$O_+(r,s) \to \left[e^{\mp\frac{i\pi}{8}}e^{\pm\frac{3}{2}i\pi\alpha_-{}^2}\frac{cos\pi\alpha_-{}^2}{sin2\pi\alpha_-{}^2}\right]^{-1} O_+(r,s)\,.$$

In general then, as in ref.2), one has an equation of the form

$$\phi_I(b)\chi_J = \sum_K N^K_{IJ}\chi_K \tag{5.3}$$

where χ_I denotes a particular character (or rather combination of characters). The normalization condition equation (5.2) above is then equivalent to setting

$$N^K_{I0} = \delta^K_I \tag{5.4}$$

where $J = 0$ denotes the identity or Neveu-Schwarz vacuum character. Furthermore, with respect to the same basis of conformal blocks, one has under $\phi^t(a)$:

$$\phi^t(a)\ lim_{z_1 \to z_2} E_+(r,s) = 2sin\pi(s\alpha_-{}^2 - \frac{r}{2})\ lim_{z_1 \to z_2} E_+(r,s)$$

Or again more generally

$$\phi_I(a)\chi_J = \lambda_I^{(J)}\chi_J \tag{5.5}$$

Working now in the context of the two contour example with basis (4.30), one may proceed exactly as in ref.2), using the conjugation relation between $a-$ and $b-$ cycle monodromy, eqn. (4.17), to express the fusion coefficients in terms of the modular matrices and the eigenvalue of $\phi_I(a)$:

$$N_{IJ}^K = \sum_L S_J^L \lambda_I^{(L)} (S^{-1})_L^K \tag{5.6}$$

where the modular matrices S_I^K are given by equation (4.31) of the previous section. Using the normalization condition equation (5.4) allows the eigenvalue to be expressed as

$$\lambda_I^{(L)} = \frac{S_I^L}{S_{I=0}^L}$$

whence

$$N_{IJ}^K = \sum_L \frac{S_I^L S_J^L (S^{-1})_L^K}{S_{I=0}^L} \tag{5.7}$$

In the last section, we have shown that (acting on characters) $S^2 = 1$ in our chosen basis and therefore multiplying by S^2 from the right yields a result for the (integer) number of couplings between the three operators labelled by I, J, K:

$$N_{IJK} = \sum_L \frac{S_{I,L} S_{J,L} S_{L,K}}{S_{I=0,L}} \tag{5.8}$$

Finally taking into account the sectorial superselection rules:

$$NS \times NS \sim NS \quad R \times R \sim NS \text{ and } NS \times R \sim R$$

one has the formulas

$$\begin{aligned}
(N_{NS\,NS}^{NS})_{r',s';r'',s'';\bar{r},\bar{s}} &= \sum_{r-s\ even} \frac{(S^{NS\to NS})_{r',s';r,s}(S^{NS\to NS})_{r'',s'';r,s}(S^{NS\to NS})_{r,s;\bar{r},\bar{s}}}{(S^{NS\to NS})_{1,1;r,s}} \\
(N_{R\,R}^{NS})_{r',s';r'',s'';\bar{r},\bar{s}} &= \sum_{r-s\ even} \frac{(S^{R\to NS})_{r',s';r,s}(S^{R\to NS})_{r'',s'';r,s}(S^{NS\to NS})_{r,s;\bar{r},\bar{s}}}{(S^{NS\to NS})_{1,1;r,s}} \\
(N_{R\,NS}^{R})_{r',s';\bar{r},\bar{s};r'',s''} &= \sum_{r-s\ even} \frac{(S^{R\to NS})_{r',s';r,s}(S^{NS\to NS})_{\bar{r},\bar{s};r,s}(S^{NS\to R})_{r,s;r'',s''}}{(S^{NS\to NS})_{1,1;r,s}}
\end{aligned}$$

$$(5.9)$$

where the modular matrices on characters are:

$$(S^{NS\to NS})_{r,s;r',s'} = (S^{R\to NS})_{r,s;r',s'} = -\frac{4}{\sqrt{p(p+2)}} sin\pi(rr'\alpha_+^2 - \frac{rs'}{2}) sin\pi(ss'\alpha_-^2 - \frac{sr'}{2})$$

with both $(r-s)$ and $(r'-s')$ even in $S^{NS\to NS}$; but $(r-s)$ odd and $(r'-s')$ even in $S^{R\to NS}$. While

$$(S^{NS\to R})_{r,s;r',s'} = -\frac{4\gamma_{r',s'}}{\sqrt{p(p+2)}} sin\pi(rr'\alpha_+^2 - \frac{sr'}{2}) sin\pi(ss'\alpha_-^2 - \frac{rs'}{2})$$

with $(r-s)$ even and $(r'-s')$ odd. Indices are to be raised and lowered with

$$(N_{NS\,NS}^{NS})^{1,1}_{r',s';r'',s''} = (S^{NS\to NS})^2 = \delta_{r',r''}\delta_{s',s''} \tag{5.10}$$

and

$$(N_{R\,R}^{NS})^{1,1}_{r',s';r'',s''} = (S^{R\to NS})(S^{NS\to R}) = \delta_{r',r''}\delta_{s',s''} \tag{5.11}$$

The equations (5.9) are essentially those already obtained by the authors of ref.20), apart from their explicit 'degeneracy' factors (which in our choice of basis is encoded in $S^{NS\to R}$).

Only the first equation has full permutation symmetry of the fusion coefficient $N_{NS\,NS}^{NS}$ and it follows that $S^{NS\to NS}$ is a symmetric matrix – both for odd and even values of p – as is indeed the case. Note also that since $S^{NS\to NS}$ and $S^{R\to NS}$ are identical matrices (disregarding the parity of $r-s$), one has that

$$(N_{R\,NS}^{R})_{r',s';\bar{r},\bar{s};r'',s''} = (N_{NS\,R}^{R})_{\bar{r},\bar{s};r',s';r'',s''}\ , \tag{5.12}$$

again for p either odd or even. Finally, for p odd, $(S^{R\to NS})^T = S^{NS\to R}$ implies the relation

$$(N_{R\,R}^{NS})_{r',s';\bar{r},\bar{s};r'',s''} = (N_{R\,NS}^{R})_{r',s';\bar{r},\bar{s};r'',s''} \tag{5.13}$$

Conversely equation (5.13) requires that $(S^{R\to NS})^T = S^{NS\to R}$. This is clearly true in the TIM fusion algebra where one has for example

$$\frac{7}{16}\times\frac{3}{80} = (\frac{1}{10})_{NS} \qquad \text{and} \qquad \frac{7}{16}\times(\frac{1}{10})_{NS} = \frac{3}{80}\,.$$

In p even models however, equation (5.13) is violated whenever a Ramond vacuum operator occurs on the RHS of the fusion algebra (see Appendix E).

In Appendix E, the fusion rules reproduced by the above equations for $p = 4$ are listed. This case corresponds to two particular points on the critical line of the Ashkin-Teller model which describes two independent Ising spins coupled by a four-spin interaction. The fusion coefficients are all integer. For purposes of comparison and by way of 'normalization' of these coefficients with respect to the TIM fusion algebra, we include the latter as obtained from equations (5.8).

In view of our previous remarks concerning the absence of symmetry and unitarity of the modular matrix S in the superconformal Verlinde basis for even p, some clarification is called for. It is well established[21),22)]that the $p = 4$ case or Ashkin-Teller model at criticality can also be described as a Z_2 orbifold of a $c = 1$ Gaussian model with radius $\sqrt{3}$ or $\frac{\sqrt{3}}{2}$ *i.e.* an ordinary rational conformal field theory to which Verlinde's original results ought to apply. The apparent contradiction is removed by noting that the identification of characters of the Ashkin-Teller model partition function with those in the $N = 1$ superconformal $p = 4$ diagonal invariant is in fact the following. As shown in ref. 21), the $N = 1$ partition function in terms of the relevant 12 characters

$$
\begin{aligned}
Z^{N=1}(\tau;\alpha) = \frac{1}{2}\Big\{ &|\chi^{NS}(1,1)|^2 + |\chi^{NS}(3,1)|^2 + |\chi^{NS}(2,2)|^2 + |\chi^{NS}(3,3)|^2 \\
&+ \quad (NS \to \widehat{NS}) \\
&+|\chi^{R}(2,1)|^2 + |\chi^{R}(3,4)|^2 + |\chi^{R}(3,2)|^2 + \frac{1}{2}|\chi^{R}(2,3)|^2 + \alpha\, Tr_{[Ramond]}(-)^F \Big\} ,
\end{aligned}
\tag{5.14}
$$

where $\hat{NS}$ denotes the $\nu = 4$ sector. This can be re-expressed as the Ashkin-Teller partition function, written in terms of the 6 characters of the (P, P) sector ($U^{\pm}$, U_2^{+}, $U_3^{\pm}$, U_6^{+}) and 3 characters (W , $W^{\pm}$) of the (P, A), (A, P) and (A, A) sectors of the twisted Gaussian model:

$$
\begin{aligned}
Z^{A-T}(\tau;\mp) = &\frac{1}{4}\left(|U^{+}|^2 + |U^{-}|^2 + |U_2^{+}|^2 + 2|U_3^{+}|^2 + 2|U_3^{-}|^2 + 2|U_6^{+}|^2\right) \\
&+ \frac{1}{2}|W|^2 + |W^{+}|^2 + |W^{-}|^2 \mp \frac{1}{2}|U_6^{-}|^2
\end{aligned}
\tag{5.15}
$$

In the last equation, (5.15), the final term $U_6^{-} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbf{Z}}(\pm)^n q^{\frac{3}{2}(n+\frac{1}{6})^2} = 1$, and the $-$ sign corresponds to radius $\frac{\sqrt{3}}{2}$ and the $+$ sign to radius $\sqrt{3}$ respectively of the Gaussian model. In terms of the characters themselves the relations are as follows:

$$
\begin{aligned}
U_2^{+} &= \sqrt{2}\chi^{R}(2,1) \\
\sqrt{2}U_6^{+} &= \chi^{R}(2,3) \\
U^{-} &= e^{-\frac{i\pi}{24}}(\chi^{\widehat{NS}}(1,1) + \chi^{\widehat{NS}}(3,1))
\end{aligned}
$$

$$\begin{aligned}\sqrt{2}U_3^- &= \sqrt{2}e^{-i\pi\frac{7}{24}}\\ U^+ &= \chi^{NS}(1,1)+\chi^{NS}(3,1)\\ \sqrt{2}U_3^+ &= \sqrt{2}\chi^{NS}(3,3)\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}W &= \chi^{NS}(1,1)-\chi^{NS}(3,1)=e^{\frac{i\pi}{24}}(\chi^{\widehat{NS}}(1,1)-\chi^{\widehat{NS}}(3,1))\\ W^+ &= \chi^{NS}(2,2)=\frac{1}{\sqrt{2}}(\chi^R(3,4)+\chi^R(3,2))\\ W^- &= e^{-\frac{i\pi}{12}}\chi^{\widehat{NS}}(2,2)=\frac{1}{\sqrt{2}}(\chi^R(3,4)-\chi^R(3,2))\end{aligned} \tag{5.17}$$

With respect to the U, W characters the modular matrices, as shown in ref. 21), are in fact unitary and symmetric (in addition satisfying $S^2 = 1$). However the U, W basis is not a complete superconformal Verlinde basis in the sense discussed in the previous section.

There is one problematic feature, specific to our Coulomb gas construction for even p, remaining. The $N = 1$ superconformal partition function, equation (5.14), can always be obtained by factorizing the modular and monodromy invariant two-point correlation function (given in the next section) on the identity intermediate state. Indeed this is how we shall fix all remaining arbitrary constants. However, as will be discussed more generally in the following section, this leads in the present $p = 4$ context to $\alpha = 0$ in equation (5.14) above. Now a matching of the Ashkin-Teller model with the $N = 1$ superconformal modular invariant above requires an identification of the term $\mp\frac{1}{2}|U_6^-|^2 = \mp\frac{1}{2}$ with $\frac{\alpha}{2}Tr_{[Ramond]}(-)^F$ which is clearly not possible if $\alpha = 0$.

6 Invariant Correlation Functions

One may now combine the left and right moving blocks to obtain modular and monodromy invariant two-point correlation functions. As an example consider the superconformal diagonal (A_{p-1}, A_{p+1}) discrete series[15] (for p odd this is in fact the only invariant) in a $t-$channel basis. It is clear that requiring S and T modular invariance implies the invariant correlator $<\phi_{1,2}(z_1,\bar{z}_1)\phi_{1,2}(z_2,\bar{z}_2)>$ has the following form

$$\alpha_1\left\{\sum_{\substack{r,s\\ r-s\in 2\mathbf{Z}}}\left(|\tilde{G}_1^{\nu=3}(r,s)|^2+|\tilde{G}_1^{\nu=4}(r,s)|^2\right)+\sum_{\substack{r,s\\ r-s\in 2\mathbf{Z}+1}}\gamma_{r,s}|\tilde{G}_1^{\nu=2}(r,s)|^2\right\}$$

$$+ \ \alpha_2 \sum_{\substack{r,s \\ r-s\in 2\mathbf{Z}+1}} |\tilde{G}_1^{\nu=1}(r,s)|^2$$

$$+\alpha_3 \left\{ \sum_{\substack{r,s \\ r-s\in 2\mathbf{Z}}} N_{r,s} \left(|\tilde{G}_2^{\nu=3}(r,s)|^2 + |\tilde{G}_2^{\nu=4}(r,s)|^2 \right) + \sum_{\substack{r,s \\ r-s\in 2\mathbf{Z}+1}} M_{r,s} |\tilde{G}_2^{\nu=2}(r,s)|^2 \right\}$$

$$+ \ \alpha_4 \sum_{\substack{r,s \\ r-s\in 2\mathbf{Z}+1}} M_{r,s} |\tilde{G}_2^{\nu=1}(r,s)|^2$$

Since S and T are diagonal in $(\tilde{C}_1, \tilde{C}_2)$ space, the relative weights of the $\tilde{G}_1$ and $\tilde{G}_2$ blocks remain arbitrary. These are fixed by implementing invariance under the monodromy operation $\phi^t(a)$ defined above. In particular, $\phi^t(a)$ acting on the expression above generates cross-terms of the form $\tilde{G}_1^\nu(\tilde{G}_2^\nu)^*$. Requiring the vanishing of such terms provides a set of linear simultaneous equations for the coefficients α_i with a solution:

$$\alpha_1 = \alpha_2 = -\alpha_3 \cos^2 \pi\alpha_-^2 = -\alpha_4 \cos^2 \pi\alpha_-^2.$$

Notice that due to the conjugation relation $\phi(b) = S\phi(a)S^{-1}$, the expression above is automatically invariant under $\phi(b)$-monodromy as it should be.

Finally, the remaining overall constant may be determined by demanding that the factorization limit $z_1 \to z_2$ reproduces the correctly normalized partition function on the torus. This fixes α_1 to be $\alpha_1 = \frac{1}{2}B(\frac{1}{2} - \alpha_-{}^2, \frac{1}{2} - \alpha_-{}^2)^{-2}$, where $B(a,b)$ is the usual beta function $\Gamma(a)\Gamma(b)/\Gamma(a+b)$.

It is important to realize that requiring both modular and monodromy invariance fixes the structure of the invariant correlator as a function of z_1, z_2 and τ completely: there are no further terms that can be appended. In particular for p even models in the $z_1 \to z_2$ factorization limit, the constant term $Tr_{[Ramond]}(-)^F$ contributed by the Ramond vacuum state of conformal weight $\frac{c}{24}$ which is expected in the usual operator formulation is in fact missing. This is a simple consequence of the following zero mode argument already mentioned in (I). In the $z_1 \to z_2$ limit the screening contour $\tilde{C}_1$, encircling z_1 and z_2, is shrunk to a point and there is no fermion field from the screening operator left on the torus. When $\nu = 1$ therefore, the one point function residue of the factorized conformal block will always be that of a single fermionic line (and certainly not the identity intermediate state) in order to absorb the fermionic zero mode on the torus with odd spin structure. Therefore it cannot contribute to the partition function.

In fact in this paper we have argued that, for consistency of the modular transformations, the blocks $\tilde{G}_1^{\nu=1}(\frac{p}{2}, \frac{p+2}{2})$ and $\tilde{G}_2^{\nu=2}(\frac{p}{2}, \frac{p+2}{2})$ identically vanish, at least in a q−expansion. This appears to be related to the particular form of the spin field we have used in our Ramond vertex operator (and consequently the three point function $< \sigma\sigma\psi >_\nu$). The o.p.e. of the supercurrent $G(z)$ with this operator vanishes for the state with weight $\Delta^R_{\frac{p}{2},\frac{p+2}{2}} = \frac{c}{24}$ [23]. This is a rather puzzling feature (for even p) of the superconformal Coulomb gas construction presented in (I) for the $(\frac{p}{2}, \frac{p+2}{2})$ Ramond block, suggesting possibly an incompleteness of the particular vertex operator used, and remains to be understood in terms of the Felder cohomological construction of the $N = 1$ superconformal Coulomb gas.

The two contour case, because of the presence of the additional channel $\phi_{1,5}$ (corresponding to the contour combination $\tilde{C}_2\tilde{C}_2$), has rather more terms in the expression for the invariant correlation function. However, again modular invariance and $\phi(a)$ monodromy suffice to fix all constants up to an overall normalization. One may also construct, for even p, off-diagonal two-point function modular invariants and then proceed to fix the remaining undetermined coefficients as above.

Awknowledgements

It is a pleasure to thank Chan Hong Mo, K.S. Narain, H. Sarmadi and Florence Tsou for countless discussions. Conversations on specific points in the above with M. Bianchi, J. Cardy, P. Di Vecchia, V. Dotsenko, E. Gava, A. Hodges, D. Lancaster, G. Pradisi and A. Sagnotti are gratefully awknowledged. One of us (M.A.N.) awknowledges financial support from the Commission of European Communities, Directorate-General XII for Science, Research and Development under contract no. SCI-0394-C.

Appendix A

In this appendix we give the computation of the coefficients $\bar{\alpha}_{mn}$ relating the t−channel basis functions $\bar{I}_m(x_2 - x_1)$ canonical for the point $x_1 = x_2$ to the s-channel functions $I_n(x_1)$ canonical for $x_1 = 0$. Formally this is written (equation (5.14) of ref.1) as

$$\tilde{I}_m(x_2-x_1) = \tilde{\alpha}_{mn} I_n(x_1) \tag{A.1}$$

where

$$I_1(x_1) = \int_{x_2}^{\infty} dx \int_{x_2}^{x} dy x^a (x-x_1)^c (x-x_2)^b y^a (y-x_1)^c (y-x_2)^b (x-y)^d \tag{A.2}$$

$$I_2(x_1) = \int_{x_2}^{\infty} dx \int_{0}^{x_1} dy x^a (x-x_1)^c (x-x_2)^b y^a (x_1-y)^c (x_2-y)^b (x-y)^d \tag{A.3}$$

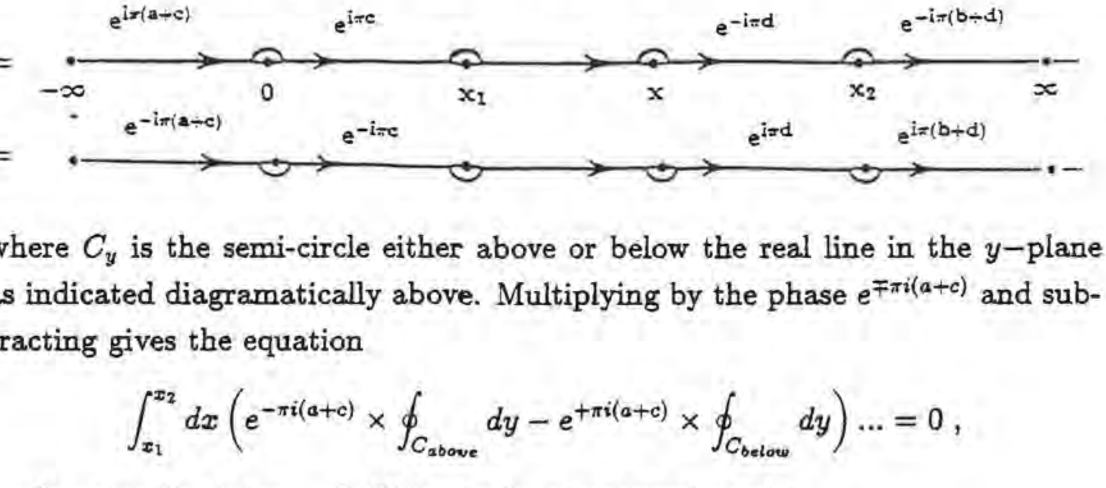
$$I_3(x_1) = \int_{0}^{x_1} dx \int_{0}^{x} dy x^a (x_1-x)^c (x_2-x)^b y^a (x_1-y)^c (x_2-y)^b (x-y)^d \tag{A.4}$$

and $\tilde{I}_m(x_2-x_1)$ is obtained by replacing x_1 with (x_2-x_1) in $I_m(x_1)$. The inverse relation to (A.1) is given by

$$I_m(x_1) = \tilde{\alpha}^{-1}_{mn} \tilde{I}_n(x_2-x_1) \tag{A.5}$$

and the calculation of the coefficients $\tilde{\alpha}^{-1}_{mn}$ is given in ref.1). Here we focus on the calculation of the coefficients occuring in (A.1), which are in fact related by a permutation of their arguments to the $\tilde{\alpha}^{-1}_{mn}$. Consider

$$0 = \int_{x_1}^{x_2} dx \oint_{C_y} dy\, x^a (x-x_1)^c (x_2-x)^b y^a (y-x_1)^c (x_2-y)^b (y-x)^d$$



where C_y is the semi-circle either above or below the real line in the y-plane as indicated diagramatically above. Multiplying by the phase $e^{\mp\pi i(a+c)}$ and subtracting gives the equation

$$\int_{x_1}^{x_2} dx \left(e^{-\pi i(a+c)} \times \oint_{C_{above}} dy - e^{+\pi i(a+c)} \times \oint_{C_{below}} dy \right) ... = 0 \,,$$

... denoting the integrand. This can be rearranged to give:

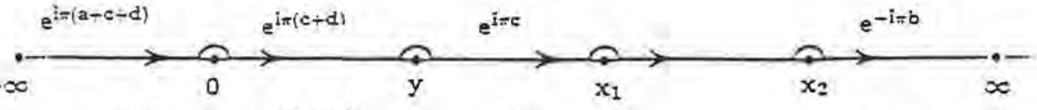
$$\int_{x_1}^{x_2} dx \int_{x_1}^{x} dy x^a (x-x_1)^c (x_2-x)^b y^a (y-x_1)^c (x_2-y)^b (y-x)^d$$

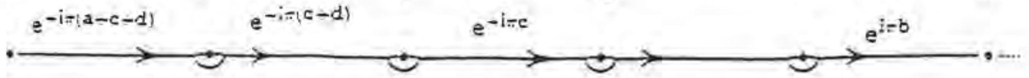
$$= -\frac{sin\pi a}{sin\pi(a+c+d/2)\, 2cos\pi(d/2)} \int_{x_1}^{x_2} dx \int_0^{x_1} dy x^a (x-x_1)^c (x_2-x)^b y^a (x_1-y)^c (x_2-y)^b (y-x)^d$$

$$-\frac{sin\pi(a+b+c+d)}{sin\pi(a+c+d/2)\,2cos\pi(d/2)}\int_{x_1}^{x_2}dx\int_{x_2}^{\infty}dyx^a(x-x_1)^c(x_2-x)^b y^a(y-x_1)^c(y-x_2)^b(x-y)^d \tag{A.6}$$

The contours on the RHS of (A.6) are in a mixed basis and we wish to transform them into purely $s-$channel contours, starting with the first term. Thus consider

$$\oint_{C_x}dx\int_0^{x_1}dyx^a(x-x_1)^c(x_2-x)^b y^a(x_1-y)^c(x_2-y)^b(y-x)^d$$

=


=


Now

$$\left(e^{-\pi i(a+c+d)}\times\oint_{C_{above}}dx-e^{+\pi i(a+c+d)}\times\oint_{C_{below}}dx\right)\int_0^{x_1}dy...=0$$

yields

$$\int_{x_1}^{x_2}dx\int_0^{x_1}dyx^a(x-x_1)^c(x_2-x)^b y^a(x_1-y)^c(x_2-y)^b(y-x)^d$$

$$=-\frac{sin\pi(a+d/2)\,2cos\pi(d/2)}{sin\pi(a+c+d)}\int_0^{x_1}dx\int_0^{x}dyx^a(x_1-x)^c(x_2-x)^b y^a(x_1-y)^c(x_2-y)^b(y-x)^d$$

$$-\frac{sin\pi(a+b+c+d)}{sin\pi(a+b+d)}\int_{x_2}^{\infty}dx\int_0^{x_1}dyx^a(x-x_1)^c(x-x_2)^b y^a(x_1-y)^c(x_2-y)^b(y-x)^d\,. \tag{A.7}$$

Next we consider

$$\oint_{C_x}dx\int_{x_2}^{\infty}dyx^a(x-x_1)^c(x_2-x)^b y^a(y-x_1)^c(y-x_2)^b(x-y)^d$$

=


=


Therefore

$$\left(e^{-\pi i(a+c)}\times\oint_{C_{above}}dx-e^{+\pi i(a+c)}\times\oint_{C_{below}}dx\right)\int_{x_2}^{\infty}dy...=0$$

This gives now

$$\int_{x_1}^{x_2} dx \int_{x_2}^{\infty} dy x^a (x-x_1)^c (x_2-x)^b y^a (y-x_1)^c (y-x_2)^b (x-y)^d$$

$$= -\frac{sin\pi(a+b+c+d/2)\, 2cos\pi(d/2)}{sin\pi(a+c)} \int_{x_2}^{\infty} dx \int_{x_2}^{x} dy x^a (x-x_1)^c (x-x_2)^b y^a (y-x_1)^c (y-x_2)^b (y-x)^d$$

$$-\frac{sin\pi a}{sin\pi(a+c)} \int_{x_2}^{\infty} dx \int_{0}^{x_1} dy x^a (x-x_1)^c (x-x_2)^b y^a (x_1-y)^c (x_2-y)^b (y-x)^d \qquad (A.8)$$

Substituting (A.7) and (A.8) into (A.6) one obtains $\bar{I}_3(x_2-x_1)$ and in particular the $\tilde{\alpha}_{nm}$ coefficients

$$\tilde{\alpha}_{31} = \frac{sin\pi(a+b+c+d/2)\, sin\pi(a+b+c+d)}{sin\pi(a+c)\, sin\pi(a+c+d/2)}$$

$$\tilde{\alpha}_{32} = \frac{sin\pi a\, sin\pi(a+b+c+d)}{sin\pi(a+c)\, sin\pi(a+c+d)}$$

$$\tilde{\alpha}_{33} = \frac{sin\pi a\, sin\pi(a+d/2)}{sin\pi(a+c+d/2)\, sin\pi(a+c+d)}$$

In a similar fashion one can compute the remaining coefficients

$$\tilde{\alpha}_{11} = \frac{sin\pi b\, sin\pi(b+d/2)}{sin\pi(a+c)\, sin\pi(a+c+d/2)}$$

$$\tilde{\alpha}_{12} = -\frac{sin\pi b\, sin\pi c}{sin\pi(a+c)\, sin\pi(a+c+d)}$$

$$\tilde{\alpha}_{13} = \frac{sin\pi c\, sin\pi(c+d/2)}{sin\pi(a+c+d/2)\, sin\pi(a+c+d)}$$

$$\tilde{\alpha}_{21} = -\frac{sin\pi(a+b+c+d/2)\, sin\pi(b+d/2)}{sin\pi(a+c)\, sin\pi(a+c+d/2)}\, 2cos\pi d/2$$

$$\tilde{\alpha}_{22} = \frac{sin\pi(a+b+c+d/2)\, sin\pi c}{sin\pi(a+c)\, sin\pi(a+c+d/2)} - \frac{sin\pi(a+d/2)\, sin\pi b}{sin\pi(a+c+d/2)\, sin\pi(a+c+d)}$$

$$\tilde{\alpha}_{23} = \frac{sin\pi(a+d/2)\, sin\pi(c+d/2)}{sin\pi(a+c+d/2)\, sin\pi(a+c+d)}\, 2cos\pi d/2$$

The inverse matrix elements $\tilde{\alpha}_{ij}^{-1}$ are given by

$$\tilde{\alpha}_{mn}^{-1}(a,b,c;d) \equiv \alpha_{mn}(a,b,c;d) = \tilde{\alpha}_{mn}(b,a,c;d)$$

which may be compared to equation (5.11) of ref.1).

Appendix B

In this appendix we show that $\mathcal{S}^{-2} = diag(\eta^2, \epsilon^2, \epsilon^2, \epsilon^2)$.

Consider first the identity channel modular matrices; we only give the details for one of the entries of the matrix $\mathcal{S}_{\tilde{C}_1\tilde{C}_1}$:

$$\left(\sum_{r'=1}^{[\frac{p-1}{2}]}\sum_{s'=1}^{r'-1} + \sum_{r'=[\frac{p+1}{2}]}^{p-1}\sum_{s'=1}^{r'+1}\right)(\overline{S}^{-1}_{\nu'=2})^{r',s'}_{r,s}(\overline{S}^{-1}_{\nu'=4})^{r'',s''}_{r',s'} = \epsilon^{-2}\delta_{r,r''}\delta_{s,s''}, \qquad (B.1)$$

where $r'-s'$ is odd and (r,s) and (r'',s'') are in the NS range. It is more convenient to start with the sum

$$\sum_{r'=1}^{4p}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}+1}}^{4(p+2)}(\widetilde{S}^{-1}_{\nu'=2})^{r',s'}_{r,s}(\overline{S}^{-1}_{\nu'=4})^{r'',s''}_{r',s'}, \qquad (B.2)$$

where

$$(\overline{S}^{-1}_{\nu'=2})^{r',s'}_{r,s} = \gamma_{r',s'}(\widetilde{S}^{-1}_{\nu'=2})^{r',s'}_{r,s}. \qquad (B.3)$$

Bringing this range back to the standard range in the same way as before we find that this sum reduces to

$$2^5\left(\sum_{r'=1}^{[\frac{p-1}{2}]}\sum_{s'=1}^{r'-1} + \sum_{r'=[\frac{p+1}{2}]}^{p-1}\sum_{s'=1}^{r'+1}\right)(\overline{S}^{-1}_{\nu'=2})^{r',s'}_{r,s}(\overline{S}^{-1}_{\nu'=4})^{r'',s''}_{r',s'}. \qquad (B.4)$$

Substituting the explict expressions for $(\widetilde{S}^{-1}_{\nu'=2})^{r',s'}_{r,s}$ and $(S^{-1}_{\nu'=4})^{r'',s''}_{r',s'}$ in (B.2), and using the trigonometric identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$, and then summing over r', we obtain $4p\delta_{r,r''}$ (ignoring the prefactor $\frac{16}{p(p+2)}$). The sum over s' gives $4(p+2)\delta_{s,s''}$. However there is a factor of $\frac{1}{2^2}$ coming from the trigonometric identity and another factor of $1/2$ from the constraint that (r',s') is odd. Altogether we get $2^5\delta_{r,r''}\delta_{s,s''}$ and equation (B.1) follows.

We also demonstrate the proof for one of the entries of the matrix $\mathcal{S}_{\tilde{C}_2\tilde{C}_2}$, and show that

$$\sum_{r'=1}^{p-1}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}}}^{r'}(S^{-1}_{\nu'=3})^{r',s'}_{r,s}(S^{-1}_{\nu'=3})^{r'',s''}_{r',s'} = \epsilon^{-2}\delta_{r,r''}\delta_{s,s''} \qquad (B.5)$$

where now $S^{-1}_{\nu'=3}$ is given by eqn.(4.24) instead. Using this equation one has:

$$\sum_{r'=1}^{p-1}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}}}^{r'}(S^{-1}_{\nu'=3})^{r',s'}_{r,s}(S^{-1}_{\nu'=3})^{r'',s''}_{r',s'} = \sum_{r'=1}^{p-1}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}}}^{r'}\frac{4\epsilon^{-2}}{p(p+2)}(-)^{(r-s+r''-s'')/2}$$

$$\times \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})\sin\pi(r'r''\alpha_+^2 - \frac{r'r''}{2})$$
$$\times\left[\frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} - \frac{\sin\pi(s'(s-1)\alpha_-^2 - \frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2 - \frac{(s-1)}{2})}\right]$$
$$\times\left[\frac{\sin\pi(s''(s'+1)\alpha_-^2 - \frac{s''(s'+1)}{2})}{\sin\pi((s'+1)\alpha_-^2 - \frac{(s'+1)}{2})} - \frac{\sin\pi(s''(s'-1)\alpha_-^2 - \frac{s''(s'-1)}{2})}{\sin\pi((s'-1)\alpha_-^2 - \frac{(s'-1)}{2})}\right] \tag{B.6}$$

As in the previous case we can extend the range of r' and s' to $(1,4p)$ and $(1,4(p+2)-1)$ respectively, by inserting a factor of 2^{-5}. Summing over r', we obtain $4p\ \delta_{r,r''}$ (ignoring for the moment the prefactor $\frac{4}{p(p+2)}$). However there is a factor of $\frac{1}{2}$ coming from the use of the identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ and another factor of $\frac{1}{2}$ from the constraint that (r', s') is even. The sum over s' has four terms

$$[+]\times[+],\quad [+]\times[-],\quad [-]\times[+] \text{ and } [-]\times[-].$$

Consider the sum over s' of the first of these terms

$$\sum_{\substack{s'=0\\ r'-s'\in 2\mathbf{Z}}}^{4(p+2)-1} \frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} \times \frac{\sin\pi(s''(s'+1)\alpha_-^2 - \frac{s''(s'+1)}{2})}{\sin\pi((s'+1)\alpha_-^2 - \frac{(s'+1)}{2})} \tag{B.7}$$

where we have added $s' = 0$ to the sum (which as mentioned previously amounts to adding zero to the sum). Rewriting (B.7) as

$$\sum_{\substack{s'=0\\ s'\in 2\mathbf{Z}}}^{4(p+2)-1} \frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\cos\pi((s+1)\alpha_-^2 - \frac{s}{2})} \times \frac{\sin\pi(s''(s'+1)\alpha_-^2 - \frac{s''s'}{2})}{\cos\pi((s'+1)\alpha_-^2 - \frac{s'}{2})}$$
$$+i\times \sum_{\substack{s'=0\\ s'\in 2\mathbf{Z}-1}}^{4(p+2)-1} \frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\cos\pi((s+1)\alpha_-^2 - \frac{s}{2})} \times \frac{\cos\pi(s''(s'+1)\alpha_-^2 - \frac{s''s'}{2})}{\cos\pi((s'+1)\alpha_-^2 - \frac{s'}{2})} \tag{B.7'}$$

and using the following identities:

$$\sum_{k=1}^{n}(-1)^k \sin\pi(2k-1)x = (-1)^n \frac{\sin\pi(2nx)}{2\cos\pi x},$$
$$\sum_{k=1}^{n}(-1)^k \cos\pi kx = -\frac{1}{2} + (-1)^n \frac{\cos\pi((n+\frac{1}{2})x)}{2\cos\pi\frac{x}{2}}$$

we express

$$\frac{\sin\pi(s''(s'+1)\alpha_-^2 - \frac{s''s'}{2})}{\cos\pi((s'+1)\alpha_-^2 - \frac{s'}{2})},$$

and

$$\frac{\cos\pi(s''(s'+1)\alpha_-^2 - \frac{s''s'}{2})}{\cos\pi((s'+1)\alpha_-^2 - \frac{s'}{2})}$$

in terms of sums over k of sines and cosines respectively. Then performing the sum over s' first we obtain

$$-4(p+2)\left[\sum_{\substack{k=1\\ s''\in 2\mathbf{Z}}}^{\frac{s''}{2}} \delta_{s,2k-2} + \sum_{\substack{k=1\\ s''\in 2\mathbf{Z}-1}}^{\frac{s''-1}{2}} \delta_{s,2k-1}\right] \qquad (B.7'')$$

where we have made use of

$$\sum_{k=1}^{n-1} \sin\pi(x+ky) = \frac{\sin\pi(x+\frac{n-1}{2}y)\sin\pi(x+\frac{ny}{2})}{\sin\pi(\frac{y}{2})}$$

and

$$\sum_{k=1}^{n-1} \cos\pi(x+ky) = \frac{\cos\pi(x+\frac{n-1}{2}y)\sin\pi(x+\frac{ny}{2})}{\sin\pi(\frac{y}{2})}$$

to perform the sum over s'. Similarly, the other terms give

$$4(p+2)\left[\sum_{\substack{k=1\\ s''\in 2\mathbf{Z}}}^{\frac{s''}{2}} \delta_{s,2k-2} + \sum_{\substack{k=1\\ s''\in 2\mathbf{Z}-1}}^{\frac{s''-1}{2}} \delta_{s,2k-1}\right],$$

$$-4(p+2)\left[\sum_{\substack{k=1\\ s''\in 2\mathbf{Z}}}^{\frac{s''}{2}} \delta_{s,2k} + \sum_{\substack{k=1\\ s''\in 2\mathbf{Z}-1}}^{\frac{s''-1}{2}} \delta_{s,2k+1}\right],$$

and

$$4(p+2)\left[\sum_{\substack{k=1\\ s''\in 2\mathbf{Z}}}^{\frac{s''}{2}} \delta_{s,2k} + \sum_{\substack{k=1\\ s''\in 2\mathbf{Z}-1}}^{\frac{s''-1}{2}} \delta_{s,2k+1}\right] \qquad (B.8)$$

respectively. From $(B.7'')$ and $(B.8)$ (ignoring again the prefactor $\frac{4}{p(p+2)}$), the sum over s' gives

$$8(p+2)\left[\sum_{\substack{k=1\\ s''\in 2\mathbf{Z}}}^{\frac{s''}{2}} (\delta_{s,2k}-\delta_{s,2k-2}) + \sum_{\substack{k=1\\ s''\in 2\mathbf{Z}-1}}^{\frac{s''-1}{2}} (\delta_{s,2k+1}-\delta_{s,2k-1})\right] = 8(p+2)\delta_{s,s''}$$

Collecting everything, including the 2^{-5}, factor we arrive at $(B.5)$.

Appendix C

We wish, as a check on the various computations of monodromy and modular transformation matrices in the text, to verify the conjugation relation (4.17) in the one-contour $< R_{1,2}R_{1,2} >$ example. Starting with

$$\phi^t(b) = S\phi^t(a)S^{-1} \,, \qquad (C.1)$$

consider its action on $\tilde{G}^{\nu=2}_{i=2}(r,s)$ and in particular the $\tilde{C}_2\tilde{C}_2$ matrix element of this operator equation, which is one of the more difficult cases.

$$\left(\phi^t(b)^{\nu''=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r'',s''}_{r,s} = \sum_{r',s'} \left(S^{-1\,\nu'=4}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r,s} \left(\phi^t(a)^{\nu'=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r',s'} \left(S^{\nu''=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r'',s''}_{r',s'} \qquad (C.2)$$

In writing (C.2) the fact that $\phi^t(a)$ is diagonal in r,s and that S is diagonal in the space of contours has been utilised. Using the methods detailed in section 3 one can compute $\phi^t(b)$ directly and the final result (for the matrix element of interest here) is:

$$\phi^t(b)_{\tilde{C}_2\tilde{C}_2} = e^{\mp\frac{i\pi}{8}} e^{\mp\frac{i\pi}{2}\alpha_-^2} \frac{cos\pi\alpha_-^2}{sin2\pi\alpha_-^2 sin\pi(s'\alpha_-^2 - \frac{r}{2})}$$

$$\delta_{r',r}\left(cos\pi((s'+1)\alpha_-^2 - r/2)\delta_{s',s+1} + cos\pi((s'-1)\alpha_-^2 - r/2)\delta_{\substack{s',s-1\\ s>1}}\right) \qquad (C.3)$$

where the upper sign in the phases, corresponding to $z_1 + \tau$ going clockwise around $z_2 + \tau$, is chosen. Now, on substituting for the various expressions in the RHS of (C.2) one may verify that (C.3) is reproduced after performing the sum. However, given $\phi^t(b)$, there is a more direct way to see that this is so in this particular ($\nu' = 3$) case at least. Multiply both sides of (C.2) from the right by:

$$\left[(S^{-1}_{\tilde{C}_2\tilde{C}_2})^{\overline{\nu}=3}\right]^{\overline{r},\overline{s}}_{r'',s''} = \left[(S^{*}_{\tilde{C}_2\tilde{C}_2})^{\overline{\nu}=3}\right]^{\overline{r},\overline{s}}_{r'',s''}$$

Then summing over r'', s'' (where $r'' \in (1, p-1)$, $s'' \in (1, r'')$) we have that

$$\left(S^{-1\,\nu'=4}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r,s} = \sum_{r'',s''} \frac{\left(\phi^t(b)^{\nu''=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r'',s''}_{r,s} \left(S^{-1\,\nu'=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r'',s''}}{\left(\phi^t(a)^{\nu'=3}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r',s'}} \qquad (C.4)$$

where $(\overline{r}, \overline{s})$ has been relabelled (r', s'). The RHS of (C.4), on substituting explicitly for the various expressions, is:

$$\frac{2\epsilon^{-1}}{\sqrt{p(p+2)}}(-)^{(r-s+1)/2} e^{-i\pi\alpha_-^2} \sin\pi(rr'\alpha_+^2 - \frac{rr'}{2})$$

$$\times\frac{1}{2}\left(\left[\frac{\sin\pi(s'(s+1)\alpha_-^2 - \frac{s'(s+1)}{2})}{\sin\pi((s+1)\alpha_-^2 - \frac{(s+1)}{2})} - \frac{\sin\pi(s'(s-1)\alpha_-^2 - \frac{s'(s-1)}{2})}{\sin\pi((s-1)\alpha_-^2 - \frac{(s-1)}{2})}\right] + [.....]\right) \qquad (C.5)$$

where [.....] is given by

$$-\frac{\sin\pi(s'sα_-^2-\frac{s's}{2})\cos\pi α_-^2}{\cos\pi(s'α_-^2-\frac{s'}{2})\sin\pi(sα_-^2-\frac{s}{2})}\left(\frac{\sin\pi((s+1)α_-^2-\frac{s}{2})}{\cos\pi((s+1)α_-^2-\frac{s}{2})}+\frac{\sin\pi((s-1)α_-^2-\frac{s}{2})}{\cos\pi((s-1)α_-^2-\frac{s}{2})}\right)$$

$$-\frac{\sin\pi(s'α_-^2-\frac{s'}{2})}{\cos\pi(s'α_-^2-\frac{s'}{2})}\left(\frac{\cos\pi(s'(s+1)α_-^2-\frac{s'(s+1)}{2})}{\cos\pi((s+1)α_-^2-\frac{s}{2})}-\frac{\cos\pi(s'(s-1)α_-^2-\frac{s'(s-1)}{2})}{\cos\pi((s-1)α_-^2-\frac{s}{2})}\right)$$

Using various trigonometric identities one can show that the expression inside the second square bracket in (C.5) is equal to that in the first square bracket (an easy way to do this is to take the difference of the two expressions and see that it is zero). Thus we recover the expression, equation (4.24), obtained in section 4 for $\left(S^{-1^{\nu'=4}}_{\tilde{C}_2\tilde{C}_2}\right)^{r',s'}_{r,s}$, and hence (C.2) is verified.

Appendix D

In this appendix we verify the Verlinde conjecture in the double contour integral example $< \phi_{1,3}\phi_{1,3} >$ by computing the $b-$cycle monodromy operator, using the conjugation relation equation (4.17), and the $t-$channel $a-$cycle monodromy result equation (3.11).

$$\phi^t(b)^m_n = \sum_{p,q}(S)^p_n\phi^t(a)^q_p(S^{-1})^m_q \qquad (D.1)$$

where the indices $m, n, ... = 1, 2$ and 3 label the $t-$channel basis of contours $\tilde{C}_2\tilde{C}_2, \tilde{C}_1\tilde{C}_2$ and $\tilde{C}_1\tilde{C}_1$ respectively. As an example we consider the $\tilde{C}_1\tilde{C}_1, \tilde{C}_1\tilde{C}_1$ matrix element of the action of $\phi^t(b)$ on $G^{\nu=3}_{\tilde{C}_1\tilde{C}_1}(r,s)$. Then (D.1) reads

$$\left(\phi^t(b)^{\nu''=3}_{3,3}\right)^{r'',s''}_{r,s} = \sum_{r'=1}^{p-1}\sum_{s'=1}^{r'}\left(S^{-1^{\nu'=3}}_{3,3}\right)^{r',s'}_{r,s}\left(\phi^t(a)^{\nu'=3}_{3,3}\right)^{r',s'}_{r',s'}\left(S^{\nu''=3}_{3,3}\right)^{r'',s''}_{r',s'} \qquad (D.2)$$

where we have again used the fact that $\phi^t(a)$ is diagonal in r, s and that S is diagonal in the space of contours. The matrix elements on the RHS of (D.2) are given in the text: $\left(\phi^t(a)^{\nu'=3}_{3,3}\right)^{r',s'}_{r',s'}$ is given by equation (3.11), $\left(S^{-1^{\nu'=3}}_{3,3}\right)^{r',s'}_{r,s}$ is given by the same expression as in the one contour case eqn. (4.9), for which $S^{-1} = S^*$. Therefore (D.2) gives

$$\frac{16}{p(p+2)}e^{2i\pi α_-^2}\frac{\cos\pi α_-^2}{\cos\pi 3α_-^2}\sum_{r'=1}^{p-1}\sum_{\substack{s'=1\\ r'-s'\in 2\mathbf{Z}}}^{r'}\sin\pi(rr'α_+^2-\frac{rr'}{2})\sin\pi(r'r''α_+^2-\frac{r'r''}{2})$$

$$\times \sin\pi(ss'\alpha_-^2 - \frac{ss'}{2}) \sin\pi(s's''\alpha_-^2 - \frac{s's''}{2})$$

$$\times \left[e^{-2i\pi(s'\alpha_-^2 - \frac{s'}{2})} \frac{\sin\pi((s'-2)\alpha_-^2 - \frac{s'}{2})}{\sin\pi(s'\alpha_-^2 - \frac{s'}{2})} + e^{-2i\pi\alpha_-^2} + e^{+2i\pi(s'\alpha_-^2 - \frac{s'}{2})} \frac{\sin\pi((s'+2)\alpha_-^2 - \frac{s'}{2})}{\sin\pi(s'\alpha_-^2 - \frac{s'}{2})} \right] \quad (D.3)$$

where we used the fact that $r-s$, $r'-s'$ and $r''-s''$ are even to separate the r from the s in the arguments of the exponentials and the sines and cosines. As before we can extend the range of r' and s' to $(1, 4p)$ and $(1, 4(p+2))$ respectively, by inserting a factor of 2^{-5}. Summing over r', we obtain $4p\ \delta_{r,r''}$ (ignoring for the moment the prefactor $\frac{4}{p(p+2)}$). However there is a factor of $\frac{1}{2}$ coming from the use of the trigonometric identity $2\sin x \sin y = \cos(x-y) - \cos(x+y)$ and another factor of $\frac{1}{2}$ from the constraint that (r', s') is even. The sum over s' has three terms. The middle term has the same structure as the sum over r', giving $\frac{1}{2}4(p+2)\delta_{s,s''}$ (which is not surprising since the sum over this term, apart from the (r,s) independent factor $\frac{\cos\pi\alpha_-^2}{\cos\pi 3\alpha_-^2}$, is nothing but $S^{-1}S$). Next consider the sum involving the first term. Rewrite $e^{-2i\pi(s'\alpha_-^2 - \frac{s'}{2})}$ as $\cos\pi(2s'\alpha_-^2 - s') - i\sin\pi(2s'\alpha_-^2 - s')$ and perform the sum over s' as was done in Appendix B. This gives

$$(p+2)e^{2\pi i\alpha_-^2}\,(\delta_{s'',s-2} + \delta_{s'',s+2} - \delta_{s'',2-s}) + i2(p+2)\sin 2\pi\alpha_-^2\ \delta_{s'',s} - \sin\pi 2\alpha_-^2\,(.....) \quad (D.4)$$

where $(.....)$ is

$$\sum_{s'=1}^{4(p+2)} \frac{\cos\pi(s'\alpha_-^2 - \frac{s'}{2})}{\sin\pi(s'\alpha_-^2 - \frac{s'}{2})} \sin\pi(ss'\alpha_-^2 - \frac{ss'}{2}) \sin\pi(s's''\alpha_-^2 - \frac{s's''}{2}) \quad (D.5)$$

The sum in (D.5) can be carried out using the same techniques as those employed in Appendix B, however there is no need to do this since there is a contribution of opposite sign coming from the sum of the third term in (D.3). The sum over s' of this is given by (D.4) with $(.....) \to -(.....)$. Putting everything together (noting that the $\delta_{s'',2-s}$ term only contributes when $s=1$) we arrive at

$$\left(\phi^t(b)_{3,3}^{\nu''=3}\right)_{r,s}^{r'',s''} = e^{4i\pi\alpha_-^2} \frac{\cos\pi\alpha_-^2}{\cos\pi 3\alpha_-^2} \delta_{r,r''} \left(\delta_{\substack{s'',s-2\\ s>2}} + \delta_{\substack{s'',s\\ s>1}} + \delta_{s'',s+2} \right)$$

which is, up to a (r,s) independent normalization factor, precisely the appropriate N_{ij}^k of the superconformal fusion algebra.

Appendix E

The fusion rules obtained from the generalised Verlinde formulae, equations (5.8), for $p = 4$ *i.e.* $c = 1$ are listed below. In the Neveu-Schwarz sector the operators have conformal weights $\Delta_{1,1} = 0, \Delta_{2,2} = \frac{1}{16}, \Delta_{3,1} = 1$ and $\Delta_{3,3} = (\Delta_{1,3}) = \frac{1}{6}$. Below, $(\Delta_{r,s})_{NS}$ denotes as usual the pair of Virasoro primary and its superdescendent of weight $\Delta_{r,s} + \frac{1}{2}$. In the Ramond sector the operators are $\Delta_{2,1} = \frac{3}{8}$, the Ramond vacuum state $\Delta_{2,3} = \frac{1}{24}$, $\Delta_{3,2} = \frac{9}{16}$ and $\Delta_{3,4} = (\Delta_{1,2}) = \frac{1}{16}$.

$NS \times NS \to NS:$

$(\frac{1}{16})_{NS} \times (\frac{1}{16})_{NS} = (0)_{NS} + (1)_{NS} + 2(\frac{1}{6})_{NS}$

$(\frac{1}{16})_{NS} \times (\frac{1}{6})_{NS} = 2(\frac{1}{16})_{NS}$

$(\frac{1}{16})_{NS} \times (1)_{NS} = (\frac{1}{16})_{NS}$

$(1)_{NS} \times (\frac{1}{6})_{NS} = (\frac{1}{6})_{NS}$

$(1)_{NS} \times (1)_{NS} = (0)_{NS}$

$(\frac{1}{6})_{NS} \times (\frac{1}{6})_{NS} = (0)_{NS} + (1)_{NS} + (\frac{1}{6})_{NS}$

$R \times R \to NS:$

$\frac{3}{8}_R \times \frac{3}{8}_R = (0)_{NS} + (1)_{NS}$

$\frac{1}{24}_R \times \frac{1}{24}_R = 2(0)_{NS} + 2(\frac{1}{6})_{NS} + 2(1)_{NS}$

$\frac{9}{16}_R \times \frac{9}{16}_R = (0) + (\frac{1}{6})_{NS}$

$\frac{1}{16}_R \times \frac{1}{16}_R = (0) + (\frac{1}{6})_{NS}$

$\frac{3}{8}_R \times \frac{1}{24}_R = 2(\frac{1}{6})_{NS}$

$\frac{3}{8}_R \times \frac{9}{16}_R = (\frac{1}{16})_{NS}$

$\frac{3}{8}_R \times \frac{1}{16}_R = (\frac{1}{16})_{NS}$

$\frac{1}{24}_R \times \frac{9}{16}_R = 2(\frac{1}{16})_{NS}$

$\frac{1}{24}_R \times \frac{1}{16}_R = 2(\frac{1}{16})_{NS}$

$\frac{9}{16}_R \times \frac{1}{16}_R = (1)_{NS} + (\frac{1}{6})_{NS}$

$R \times NS \to R:$

$\frac{3}{8}_R \times (\frac{1}{16})_{NS} = \frac{9}{16}_R + \frac{1}{16}_R$

$\frac{3}{8}_R \times (1)_{NS} = \frac{3}{8}_R$

$\frac{3}{8}_R \times (\frac{1}{6})_{NS} = \frac{1}{24}_R$

$\frac{9}{16}_R \times (\frac{1}{16})_{NS} = \frac{3}{8}_R + \frac{1}{24}_R$

$\frac{9}{16}_R \times (1)_{NS} = \frac{1}{16}_R$

$\frac{9}{16}_R \times (\frac{1}{6})_{NS} = \frac{9}{16}_R + \frac{1}{16}_R$

$\frac{1}{24}_R \times (\frac{1}{16})_{NS} = 2\ \frac{9}{16}_R + 2\ \frac{1}{16}_R$

$\frac{1}{24}_R \times (1)_{NS} = \frac{1}{24}_R$

$\frac{1}{24}_R \times (\frac{1}{6})_{NS} = \frac{1}{24}_R + 2\frac{3}{8}_R$

$\frac{1}{16}_R \times (\frac{1}{16})_{NS} = \frac{3}{8}_R + \frac{1}{24}_R$

$\frac{1}{16}_R \times (1)_{NS} = \frac{9}{16}_R$

$\frac{1}{16}_R \times (\frac{1}{6})_{NS} = \frac{9}{16}_R + \frac{1}{16}_R$

For comparison, the $p = 3$ or TIM fusion rules as obtained from (5.8) are listed below.

$R \times R \to NS:$

$\frac{3}{80}_R \times \frac{3}{80}_R = (0)_{NS} + (\frac{1}{10})_{NS}$

$\frac{7}{16}_R \times \frac{7}{16}_R = (0)_{NS}$

$\frac{3}{80}_R \times \frac{7}{16}_R = (\frac{1}{10})_{NS}$

$R \times NS \to R:$

$\frac{3}{80}_R \times (\frac{1}{10})_{NS} = \frac{3}{80}_R + \frac{7}{16}_R$

$\frac{7}{16}_R \times (\frac{1}{10})_{NS} = \frac{3}{80}_R$

References

[1] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 [FS12](1984)312, B251 [FS13](1985)691.

[2] E. Verlinde, Nucl. Phys. B300 (1988) 360;
R. Dijkgraaf and E. Verlinde, Proc. of the Annecy Conf. on Conformal Field Theory (March 1988), Utrecht preprint THU-88/25.

[3] R. Brustein, S. Yankielowicz and J-B. Zuber, Nucl. Phys. B301 (1988) 224.

[4] C. Vafa, Phys. Lett.B206 (1988) 421.

[5] G. Moore and N. Seiberg, Phys. Lett. B212 (1988) 451; Nucl. Phys. B313 (1988) 16; Comm. Math. Phys. 117 (1988) 441, 123 (1989) 177.

[6] D. Friedan and S. Shenker, Nucl. Phys. B281 (1987) 509.

[7] C. Vafa, Phys. Lett. B199 (1987) 195.

[8] S.D. Mathur, S. Mukhi and A. Sen, Nucl. Phys. B312 (1989) 15, B318 (1989) 483; Phys. Lett. B213 (1988) 303;
S.D. Mathur and A. Sen, Phys. Lett. B218 (1989) 176; Nucl. Phys. B327 (1989) 725.

[9] T. Eguchi and H. Ooguri, Phys. Lett. B203 (1988) 44.

[10] G. Felder, Nucl. Phys. B317 (1989) 215.

[11] T. Jayaraman and K.S. Narain, Nucl. Phys. B331 (1990) 629.

[12] J. Bagger, D. Nemeschansky and J-B. Zuber, Phys. Lett.B216 (1989) 320;
O. Foda and B. Nienhuis, Nucl. Phys. B324 (1989) 643.

[13] T. Jayaraman, K.S. Narain and M.H. Sarmadi, Nucl. Phys.B343 (1990) 418.

[14] T. Jayaraman, M.A. Namazie, K.S. Narain, C. Núñez and M.H. Sarmadi, Nucl. Phys. B336 (1990) 610; referred to as (I).

[15] A. Cappelli, Phys. Lett. B185 (1987) 82.

[16] D. Kastor, Nucl. Phys. B280 (1987) 304;
Y. Matsuo and S. Yahikozawa, Phys. Lett. B178 (1986) 211.

[17] A. Abdurrahman, D.Phil. thesis, Oxford University (1991).

[18] F. Anton, D.Phil. thesis, Oxford University (1993).

[19] G. Pradisi and A. Sagnotti, Univ. of Rome preprint ROM2F-92/53.

[20] W. Eholzer and R. Hübel, Univ. Bonn preprint BONN-HE-93-05, hep-th/9307031.

[21] S-K. Yang and H.B. Zheng, Nucl. Phys. B285 (1987) 410;
S-K. Yang, Nucl. Phys. B285 (1987) 183.

[22] P. Ginsparg, in Fields, Strings and Critical Phenomena (Les Houches, Session XLIX, 1988), ed. by E. Brézin and J. Zinn-Justin (Elsevier Science Publishers, 1989).

[23] G. Mussardo, G. Sotkov and M. Stanishkov, Phys. Lett. B195 (1987) 397.

Figure Captions

Fig.1 $s-$channel conformal blocks for case (a) $<\phi_{1,3}\phi_{1,3}>$. Intermediate states correspond to the three configurations of contours C_1C_1, C_1C_2, C_2C_2 respectively. The $\times$ denotes the trace over descendent states propagating around the torus.

Fig.2 Factorization limit in a $t-$channel basis for case (a).

Fig.3 Transforming $z_1 \to z_1 + \tau$ in two stages, with $Im z' < Im(z_2 + \tau)$.

Fig.4 Contour transformations to compute $\phi^s(b)$.

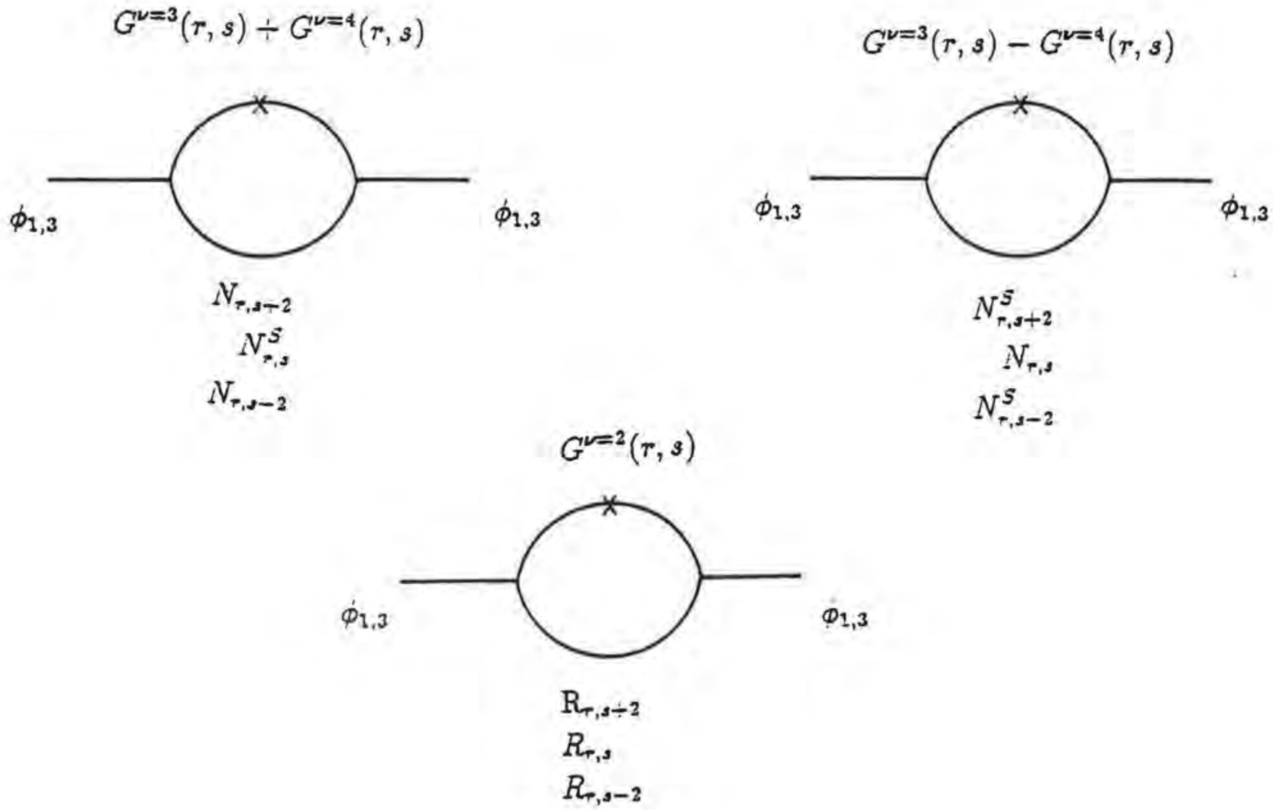
Fig.5 Prescription for the double contour integral.

Fig.6 Two contour $\phi^s(b)$ monodromy transformations.

Fig.7a The closed contour $\tilde{C}_2$ starts and ends at P and traverses the first and second sheets of the cut torus as indicated by the arrows. To facilitate comparison with fig.7b, markings 0 and ∞ indicate the respective sides of the parallelogram that are mapped onto 0 and ∞ in the $q \to 0$ limit.

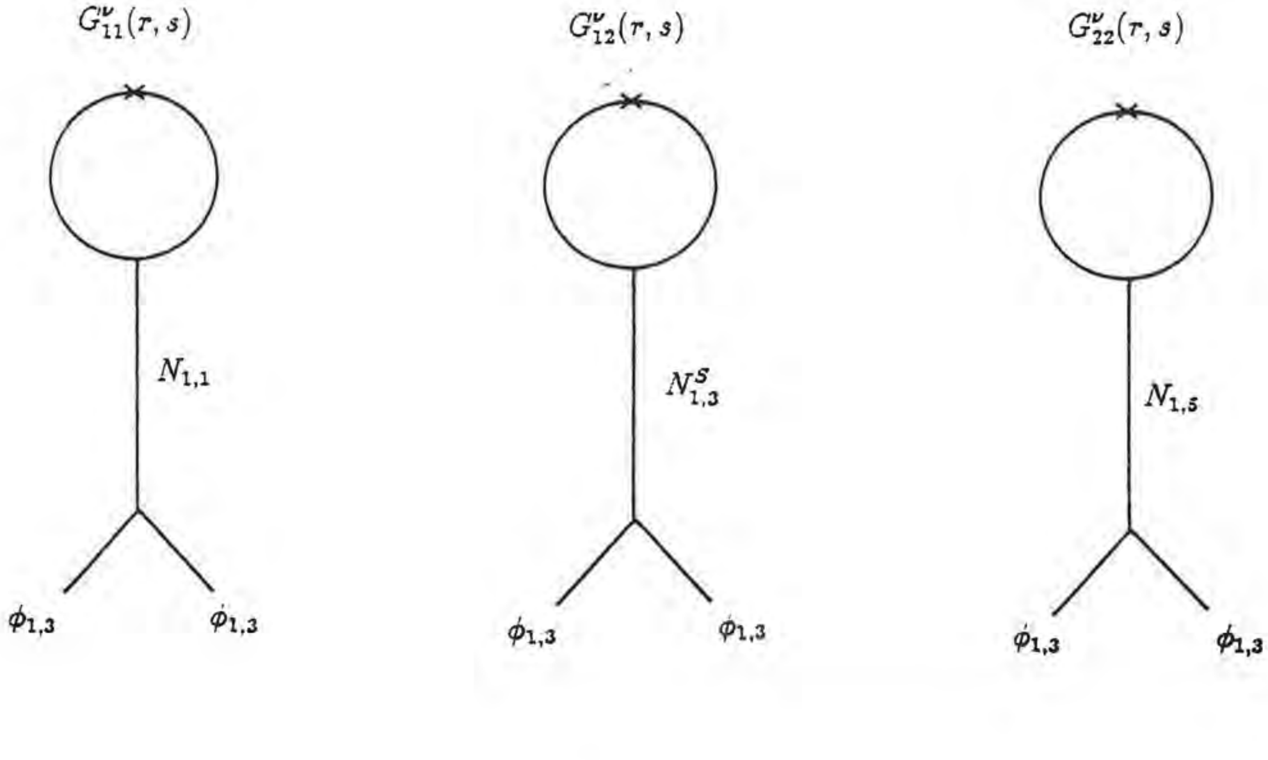
Fig.7b $\tilde{C}_2$ in the $q \to 0$ limit consists of an anti-clockwise rotation by π around the origin, going to ∞, an anti-clockwise rotation of 2π around ∞, going back to the origin, a clockwise rotation of 2π about the origin, going to ∞ again and finally making another clockwise rotation of 2π about ∞ before returning to the origin.

Fig.7c The contour $\tilde{C}_2'$; phases are measured from the point P. The second sheet is the lower parallelogram.



$G^{\nu=3}(r,s) + G^{\nu=4}(r,s)$
$\phi_{1,3}$
$\phi_{1,3}$
$N_{r,s+2}$
$N^S_{r,s}$
$N_{r,s-2}$
$G^{\nu=3}(r,s) - G^{\nu=4}(r,s)$
$\phi_{1,3}$
$\phi_{1,3}$
$N^S_{r,s+2}$
$N_{r,s}$
$N^S_{r,s-2}$
$G^{\nu=2}(r,s)$
$\phi_{1,3}$
$\phi_{1,3}$
$R_{r,s+2}$
$R_{r,s}$
$R_{r,s-2}$

Fig.1



$G^{\nu}_{11}(r,s)$
$N_{1,1}$
$\phi_{1,3}$
$\phi_{1,3}$
$G^{\nu}_{12}(r,s)$
$N^S_{1,3}$
$\phi_{1,3}$
$\phi_{1,3}$
$G^{\nu}_{22}(r,s)$
$N_{1,5}$
$\phi_{1,3}$
$\phi_{1,3}$

Fig.2

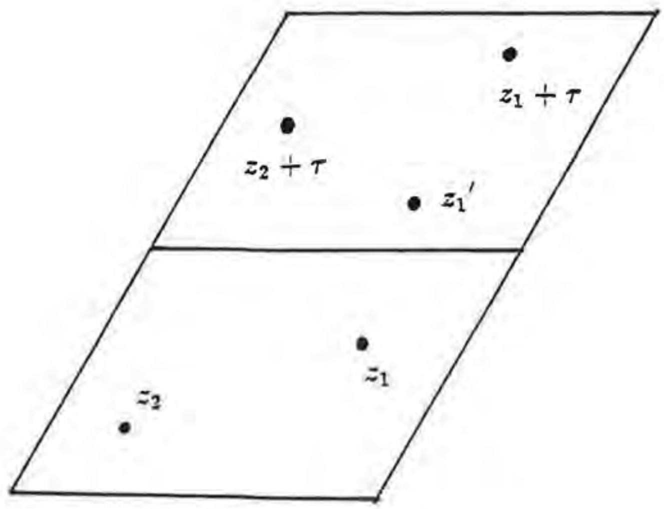


Fig.3

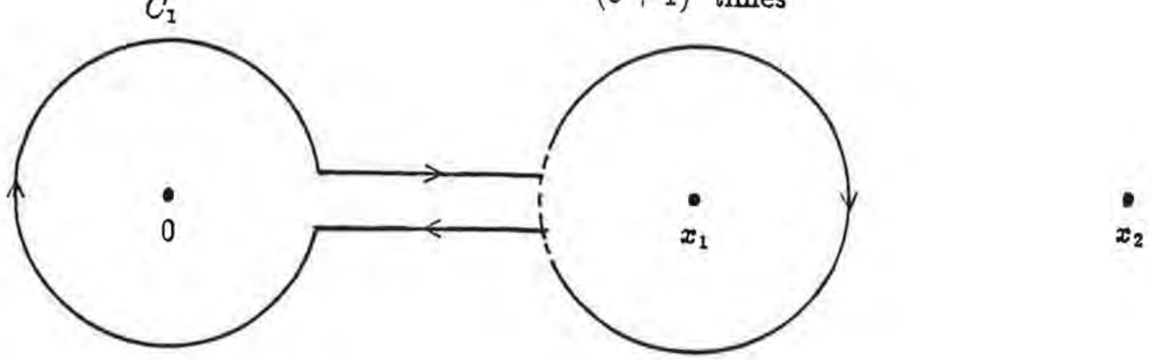
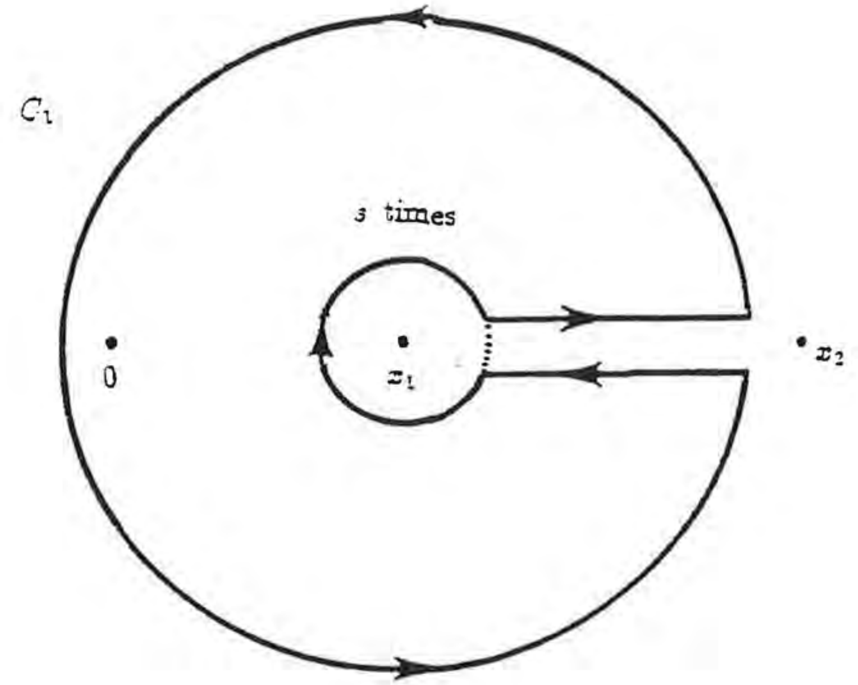
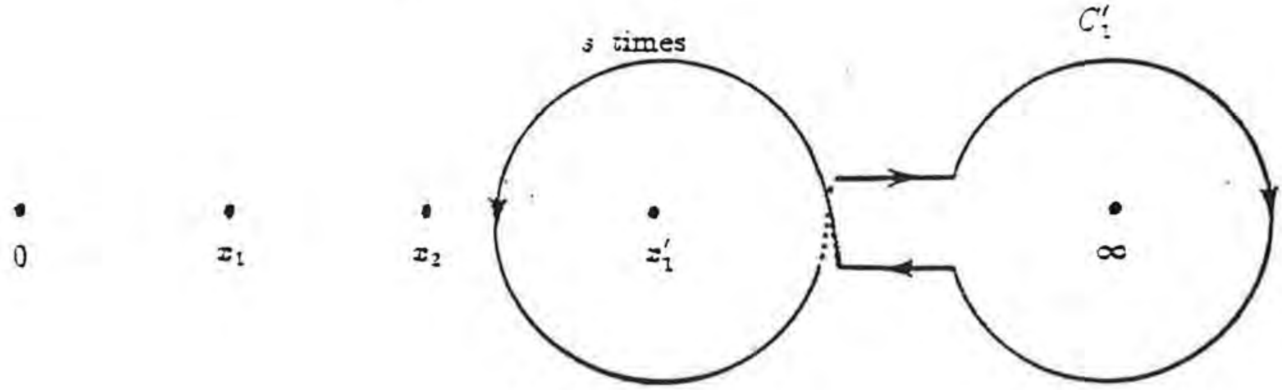


Fig.4 ***(a)***



(b)



(c)

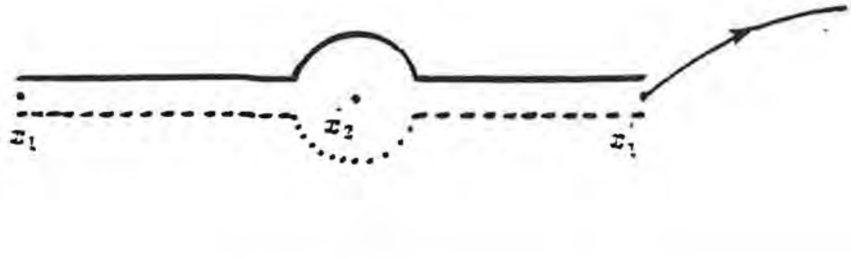
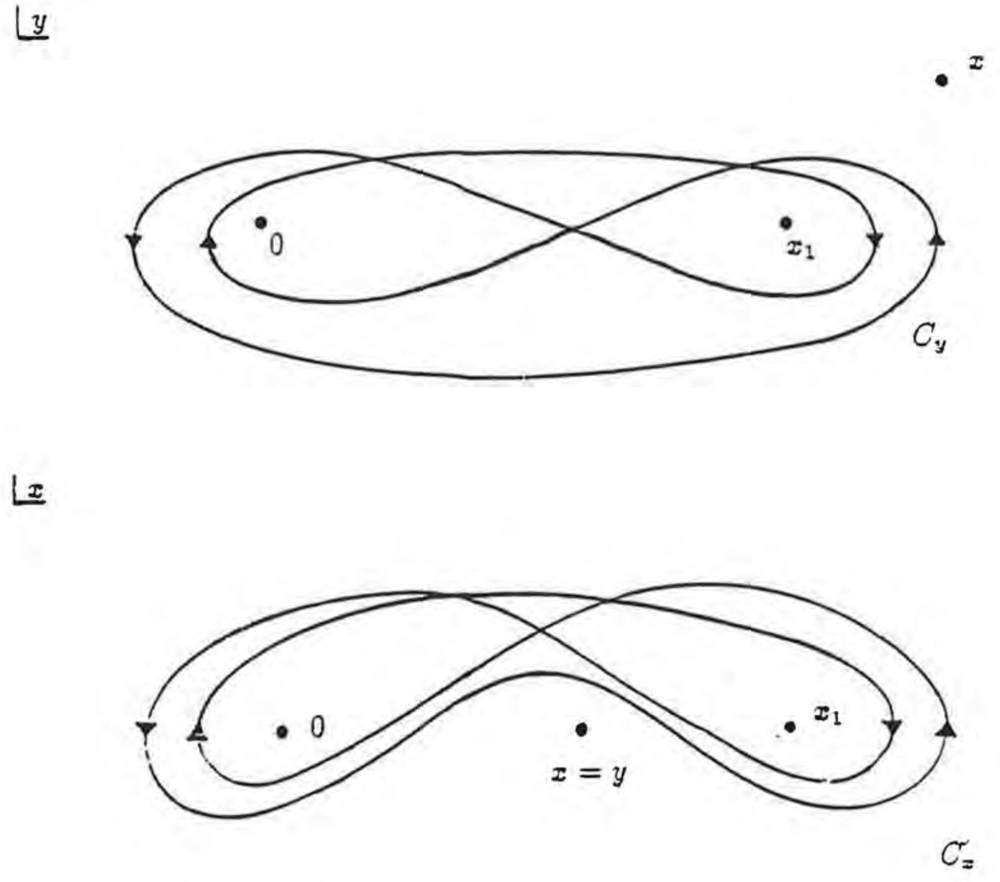
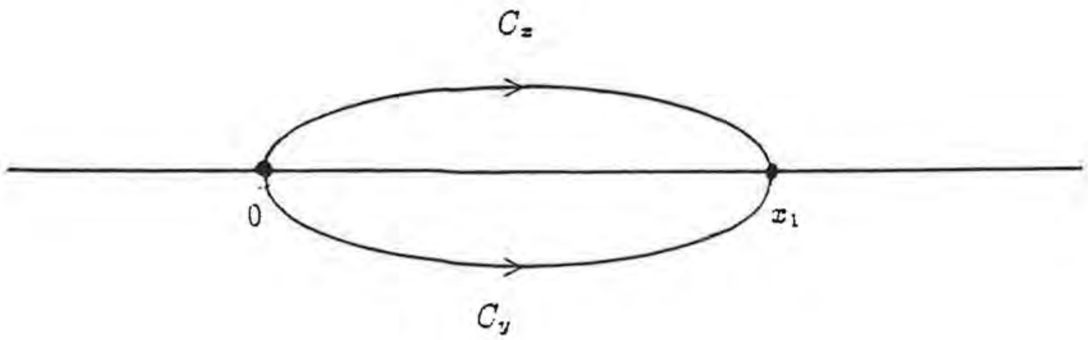


Fig.4 (*d*)

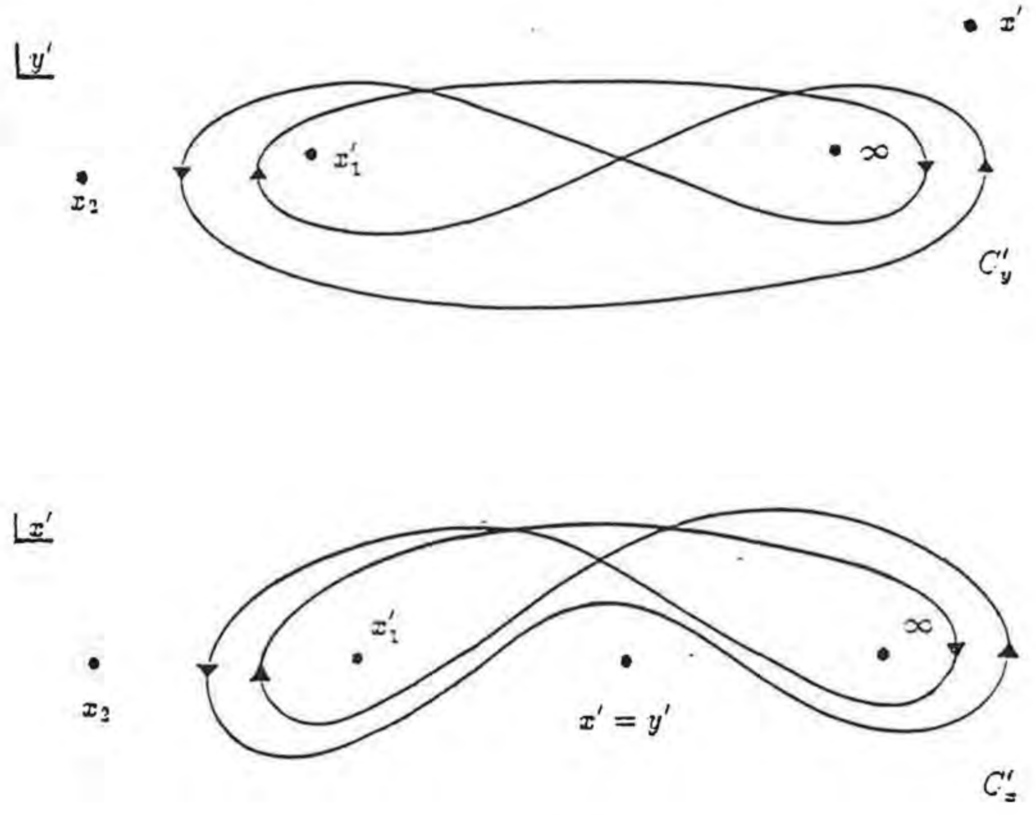


(a)

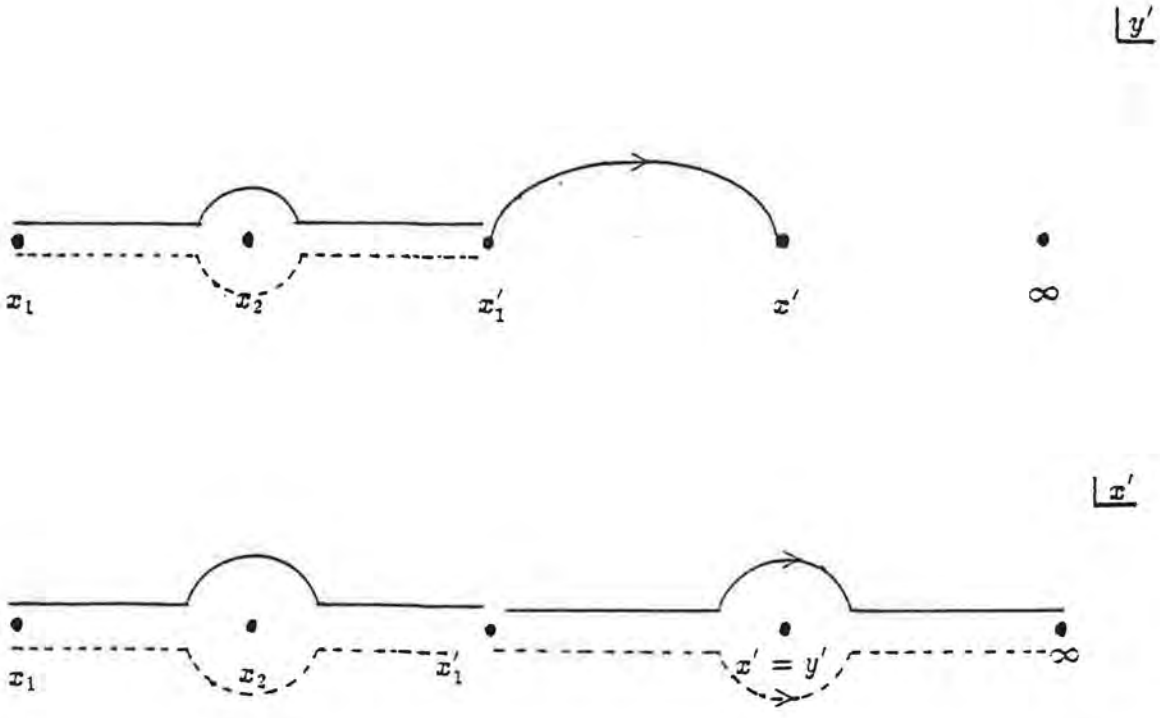


(b)

Fig.5


y'
x'
x_1'
∞
x_2
C_y'
x'
x_1'
∞
x_2
$x' = y'$
C_x'

(a)


y'
x_1
x_2
x_1'
x'
∞
x'
x_1
x_2
x_1'
$x' = y'$
∞

(b)

Fig.6

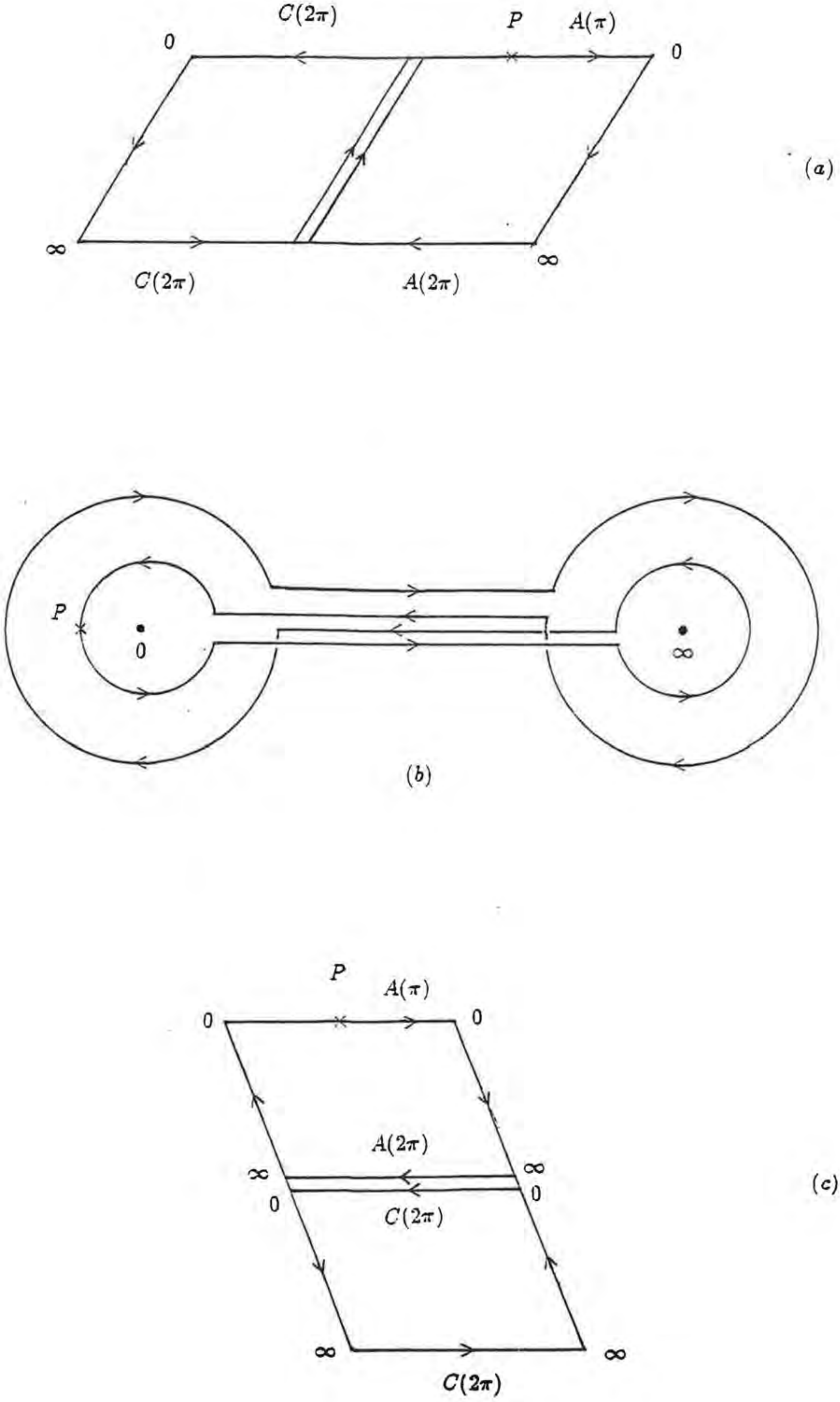


Fig.7